How To Spend It? Capital Accumulation in a Changing World

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Abstract

In a society characterized by a multitude of heterogeneous agents and a large number of possibly immaterial (i.e. cultural, educational, etc...) goods, each one having a distinct social (relative price) and personal value (individual preference), we study the impact of these relative values' evolution on capital accumulation, depending on economic and social parameters, such as capital mobility, productivity, and personal and social values discrepancies.

We consider an arbitrary number of agents, each endowed with a one-period production function and a two-period intertemporal utility. Agents live, produce and consume over one period, but optimize over two periods, so providing a stock of goods for the next generation. In period one, the inherited stock may be partly disposed of to produce alternate goods, depending on the agent individual preferences and on the present goods' social value, thus creating a dynamics in capital accumulation.

A phenomenon of threshold appears in the dynamics of the agent capital stock. Below this threshold, the initial stock will quickly fade away; above, capital accumulation is possible. The threshold strongly depends on both personal and social values volatilities. When social values vary strongly, the threshold increases, and stocks depreciate faster than they are replaced. Shocks on the goods' social values may drive stocks above or below the threshold, in turn inducing a reversal in dynamics.

For a precursor, i.e. an agent whose personal values will be next period social values, a strong mobility in capital will decrease the threshold: there is a gain to innovate. When social values are an average of several sectors' values, one sector will ultimately dominate. Its values will become social values, and its stock will appreciate at the expense of other sectors.

Key words: Capital theory, Capital accumulation, Investment allocation, Two-sector models, Disaggregated capital, Take-off, Threshold effect, Intergenerational models, Cambridge capital controversy.

JEL Classification: E22, O10, O30, O40.

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Introduction

Capital theory has been the subject of intense debates among economists since the early 19th century [31]. The latest of these debates, the Cambridge capital controversy, opposed so-called neo-Ricardian economists - such as Joan Robinson and Piero Sraffa at the University of Cambridge in England - to neoclassical economists such as Paul Samuelson and Robert Solow at the Massachusetts Institute of Technology, in Cambridge, Massachusetts. While neoclassicals defended the orthodox treatment of interest as determined by supply and demand, neo-Ricardians assumed it to be determined by the conjunction of technological conditions of production and by the distribution of income[29], [9], [8]. The resolution of the debate and its implications has not been agreed upon by economists.

The root of the debate lies in the aggregation problem: using microeconomic conceptions to understand the whole society's production could prove to be a fallacy of composition. Yet although general equilibrium models with heterogeneous labor and capital have been developed by the neoclassical school, most versions of neoclassical growth theory, and notably in the Solow growth model, assume a simple production function for the entire economy. The subsequent theories of endogenous growth and real business cycles likewise used aggregate production functions. The controversy was simply put aside [12], [13], [5][1].

The technical criticism of marginal productivity theory may have been blurred by ideological considerations. In a recent paper, Romer [28] considers Joan Robinson [26], [27] was engaged in academic politics when she waged her campaign against capital and the aggregate production function. The mathematisation of growth theories initiated by Solow [31] represented, by comparison, a simplification of the notion of capital, allowing for a clear formalisation of the problems raised by capital accumulation and its mechanisms. This advantage has largely compensated the flaws associated with this approach.

Without taking part to the debate, let us simply state that the problems and debate raised by Joan Robinson's criticism remain open. The complex nature of capital requires a richer formalism involving multiple agents and factors with environment-dependent productivities. The description of a disaggregated capital valorized by multiple factors would avoid the problem of price of capital, even if it must later be aggregated again to recover macro concepts.

This paper is a first step towards such a formalism. Our approach considers an economy with a large number of different agents and goods. Each good has a specific social value, and heterogeneous personal values. We call "value" of a good a price that can be either objective (exchange value) or subjective (moral value). Eventhough subjective, a value may nonetheless impact on the agent's utility by weighing on his consumption. These values can be personal (micro) or social (macro). Social or macro values are the environment where individual values are set and evolve, and under certain conditions, micro values may become macro values. The introduction of social and personal values allows to differentiate between the agent technological capacity and its social valorization. It allows to study differentiated and combined production sectors and the reciprocal impact of social environment and sector development.

This paper specifically studies personal and social values interactions and their impact on capital accumulation, consumption and production. We analyse how heterogeneity in personal values impacts social values and capital accumulation. Is innovative activity rewarded, and in what conditions? What are the conditions that favor capital accumulation for precursors? Can two sectors with distinct values coexist, and how are social values impacted by their competition?

To answer these questions, we consider a society characterized by a multitude of heterogeneous agents and a large number of possibly immaterial (i.e. cultural, educational, etc...) goods, each one having distinct social (relative price) and personal value (individual preference). We study the impact of these relative values' evolution on capital accumulation, depending on economic and social parameters, such as capital mobility, productivity, and personal and social values discrepancies.

There exist an arbitrary number of agents, endowed with a one-period production function and a two-period intertemporal utility. Agents live, produce and consume over one period, but optimize over two periods, so providing a stock of goods for the next generation. In period one, the inherited stock may be partly disposed of to produce alternate goods, depending on the agent individual preferences and on the present goods' social value, thus creating a dynamics in capital accumulation.

Each agent is characterized by two vectors of infinite dimensions. The first vector is a vector of goods whose coordinate i^{th} is the agent endowment in the good i. The second vector is a vector of so-called personal

[1] For more details on the controversies, see [32], [16], [24], [21], [17], [25], [4].

values, whose i^{th} coordinate represents the relative personal value of the i^{th} good. Besides, there exist a social value vector whose i^{th} coordinate represents the relative value bestowed by society on the i^{th} good, its price.

There is an infinite number of goods, each being characterized by three factors: its social or macro value, the agents' individual endowment in this good, and the good's personal or micro value for the agent. Let us precise that this vector χ contains all potential goods, whose values are set to 0 at time t. It is a grid of all past and future goods. Likewise, V contains all personal values attributed to these possible goods.

Agents can produce all goods, but have a specific advantage to produce within their respective personal value V. The quantity of all produced goods is a function of their respective personal and social values. Each good's individual value is associated with a marginal productivity, that is itself a function of the good's social value. Agents' productivity depends on the social value of their production, which also acts as an indicator of their technological environment. The two vectors, χ and V, are sufficient to characterize the production of all goods. This feature allows to treat factors of production as heterogeneous goods with independent or multiple valuations, without refering to any notion of profit or rate of return. Let us remark that this feature of the model encapsulates the interactions between social and personal values. It is the equivalent of neo-Ricardian determinants of capital, i.e. distribution of income and technological environment.

An important feature of the model is that the vector χ_t can be either endogeneous or exogeneous. When χ is exogeneous, it is the environment that determines ϕ. Conversely, when χ is endogeneous, agents may modify their environment. We will consider the implications of both possibilities in the following.

To do so, we consider four configurations of social and individual values, and their respective relevance for capital accumulation. In the benchmark case, the society is homogenous, and individual values perfectly match social values. Social values evolve exogenously through external stocks. A first departure from this benchmark, case two, considers individual values differing from social values by a stochastic term. Personal values are randomly distributed around social values. Although social values still evolve exogenously, and agents do not influence this evolution, the society is now heterogeneous. Some agents may significantly depart from the common social values, without however affecting future social values. The third case refines the relation between social and personal values. Some individual values may now anticipate future social values up to a random noise. Some agents may impact the evolution of future social values. The fourth and last case considers a two-sector economy with distinct sector values. Future social values are a weighted mean of present sector values, so that each sector will partly be precursor. It may impact future social values, and the weight attributed to each sector will be proportional to the social value of its accumulated stock.

We find that, in all of the above cases, a phenomenon of threshold appears in the dynamics of the agent capital stock. Below this threshold, the initial stock will quickly fade away; above, capital accumulation is possible. The threshold strongly depends on both personal and social values volatilities. When social values vary strongly, the threshold increases, and stocks depreciate faster than they are replaced. Shocks on the goods' social values may drive stocks above or below the threshold, in turn inducing a reversal in dynamics.

More specifically, in the first case, a strong volatility in social values prevents stock accumulation. Excessive exogenous innovation may impair capital accumulation. On the other hand, unexpected shocks in social values may drive capital stock value below, or above, the threshold and either deter or initiate capital accumulation. A standard example of this case would be the sudden appreciation in the price of a natural resource such as petrol.

The threshold decreases when productivity - the efficiency in capital stock transformation - increases. Productivity plays here the role of technology in growth models. Capital mobility, i.e. the proportion of capital stocks that may be transformed in alternate goods each period, is also an important parameter. Its increase has an ambiguous effect, depending on productivity: when productivity is high, the threshold decreases. Thus high capital mobility favors capital accumulation in a high capital efficiency environment. In a low productivity environment, the threshold increases: production barely covers consumption needs.

Heterogeneity in social values increases the threshold for capital accumulation. When social values and personal preferences are purely independent, the volatility of the latter impairs capital accumulation. Originality here is not productive. However, this result is modified when endogenity in social values' dynamics is introduced in the model. Indeed, the precursors' threshold decreases when the proportion of capital transformed is high: high capital mobility favors precursors.

When heterogeneous sectors are considered, the competition between sectors for capital stock accumulation impacts social values, leading to unstable dynamics. Capital accumulation discrepencies drive social values towards the most valued sector, appreciating its stock and in turn enhancing future stock accumulation, while the other sector value will tend to loose value, depreciating its stock accordingly. The first sector values and stock will ultimately dominate while the other sector stock will be depleted. A shock in social values may, as in case one, reverse the situation and each sector's role. A standard example would be the impact of Corn Laws in the competition between industrial and agricultural sectors in XIXth century England.

The rest of this paper is organized as follows. Section 2 presents the model. Section 3 presents the agent's optimization problem. Section 4 describes the four cases presented above and their results. Section 5 discusses the results and concludes.

1 The model

We consider a society with an inifinite number of heterogeneous agents and a large number of possibly immaterial goods. Each agent endowment in each good is encoded in a vector ϕ_t, where the i-th coordinate of ϕ_t, $(\phi_t)_i$, is the quantity of good i owned by the agent. Goods have distinct social values, or relative prices, encoded by a price vector χ_t

Agents have individual preferences over the entire set of goods, so that personal and social values may differ. These personal values are modeled by a vector of relative prices V_t.

It is important to note here that agents have a one-period lifespan and a one-period production function, but a two-period intertemporal utility. They live, produce and consume over one period, but optimize over two periods, so providing for a stock of goods for the next generation.

This feature gives its overlapping aspect to the model. Eventhough agents only live one period, their behaviors will impact their heirs through the level of capital transmitted. The model is therefore one of capital accumulation. This will appear clearly in the dynamics of ϕ_t.

1.1 Utility

An agent inherits in period t a stock of goods ϕ_t. He will sell part of this stock and use it as capital in his production Y_t. This production, along with the remaining stock, will either be consumed or passed on as ϕ_{t+1}, the bequest left to his heirs. The agent constraint is then:

$$\phi_{t+1} = \phi_t + Y_t - C_t$$

We assume $\phi_{t+1} \geq 0$ ("no-debt condition"). Each agent is described by a two-periods utility that is a sum of two terms characterizing the agent's tradeoff between consuming today and transmitting his wealth.We call $\bar{U}_t$ this intertemporal utility :

$$\bar{U}_t = U(V_t, \phi_t, \chi_t, C_t) + \rho \hat{U}\left(\phi_t + Y_t - C_t, E_t \chi_{t+1}\right)$$

The first term $U(V_t, \phi_t, \chi_t, C_t)$ is the agent's utility to consume. We choose:

$$U(V_t, \phi_t, \chi_t, C_t) = (V_t)^t C_t - \frac{1}{2}\alpha\left((V_t)^t C_t\right)^2 - \frac{1}{2}\beta\left((V_t - \chi_t)^t C_t\right)^2 - \frac{1}{2}\gamma (C_t - \phi_t)^t (C_t - \phi_t)$$

The contribution $(V_t)^t C_t - \frac{1}{2}\alpha\left((V_t)^t C_t\right)^2$ is the quadratic utility over N goods, each weighed by V_t. A consumption colinear to V_t maximizes this part of the utility when $\|C_t\|$ is constant.

This choice of utility function is in line with the Leontief paradox [18], stating that the country with the world's highest per worker-capital has a lower capital/labor ratio in exports than in imports. In that, we follow the Linder hypothesis [20] and consider that demand plays a more important role than comparative advantage as a determinant of trade, with the hypothesis that countries - and here agents - which share similar demands will be more likely to trade. This modeling stems from the assumption that an environment, to endure, needs to sustain the level of factor-endowment it has reached. It will therefore favor the very environment it has created itself.

Some exogenous evolution may radically change the goods' relative values, and trigger a depreciation of the agent's goods. The agent whose capital value has been eroded will no longer be able to accumulate. To consume and produce within his new environment, he will eat up his own capital, that will tend to disappear. The composition of its capital will evolve, but not quickly enough to avoid depletion.

The loss $-\frac{1}{2}\beta\left((V_t-\chi_t)^t C_t\right)^2$ is the new and most important term in this utility. It stems from discrepencies between personal and social values, V_t and χ_t. Actually, the scalar product $(V_t-\chi_t)^t C_t$ measures the part of C_t proportional to $V_t-\chi_t$, and thus the part of C_t transverse to χ_t. This term is the loss experienced by an agent when its consumption departs from χ_t. It reflects the model hypothesis of a link between goods and the environment: the environment valorizes goods, so that goods' utility depends on the environment.

We also assume the agent experiences a loss $-\frac{1}{2}\gamma(C_t-\phi_t)^t(C_t-\phi_t)$ when his consumption differs from his bequest. However this aspect is less important and will be neglected afterwards. Nonetheless it is an impediment to investment since it favors stock consumption rather than stock transformation, and represents an incentive to consume one's bequest.

The second term in the intertemporal utility is the contribution of the bequest to the agent's utility. Here, it is $\rho\hat{U}(\phi_{t+1},E_t\chi_{t+1})$, where ρ is the time discount factor and where we choose:

$$\begin{aligned}\hat{U}(\phi_{t+1},E_t\chi_{t+1}) &= V_t^t\phi_{t+1}-\frac{1}{2}\delta\left(V_t^t\phi_{t+1}\right)^2-\frac{1}{2}\varepsilon\left((V_t-E_t\chi_{t+1})^t\phi_{t+1}\right)^2\\ &= V_t^t(\phi_t+Y_t-C_t)-\frac{1}{2}\delta\left(V_t^t(\phi_t+Y_t-C_t)\right)^2-\frac{1}{2}\varepsilon\left((V_t-E_t\chi_{t+1})^t(\phi_t+Y_t-C_t)\right)^2\end{aligned}$$

the agent's utility at time t to leave a bequest ϕ_{t+1}. Note that this solely depends on the agent preferences at time t.

The term $\hat{U}(\phi_{t+1},E_t\chi_{t+1})$ encapsulates the overlapping character of the model. Non-overlapping models usually include infinitely-lived agents, each agent sharing the same objectives and values as his heirs do. On the contrary, we consider agents whose personal values may differ, at least partly, from their descendants. This does not necessarily prevent some continuity within personal values through time.

Let us recall that the main motivations for saving are saving for retirement, or "hump" saving, precautionary savings or "unintended" bequests, due to uncertainty about the length of life, and planned bequests (see [15]). In this paper, we strictly consider planned bequests. In so doing, we depart from Modigliani's life cycle hypothesis[22], whose theory would rather favor a combination of hump and precautionary saving. However these two motivations are irrelevant in our context, since agents live only one period.

The two first terms in $\hat{U}$ are similar to the two first terms in $U(V_t,\phi_t,\chi_t,C_t)$. They are optimal when ϕ_{t+1} is colinear to V_t. The last term is an anticipated loss when V_t differs from χ_{t+1}. Evolving social values induce a loss in the social value of the bequest. Agents are assumed to be myopic, so that $E_t\chi_{t+1}=\chi_t$. A fraction η of the inherited stock is, at no cost, disposed of in period one to enable an alternate production of goods. This dynamics in capital accumulation therefore depends on the evolution of social and individual values.

1.2 Production

We consider an economy in which agents are producers and there are no wages. In this context, we study the capital stock dynamics, transmission and transformation, under various combinations of social and personal preferences for bequest, production and consumption. A fraction η of ϕ_t is transformed to produce a set of goods Y_t, a vector in the same space as ϕ_t.The final goods instantaneous production function is:

$$Y_t=\eta(\phi_t.\chi_t)\left(\xi K_2(\xi,\eta,\phi_t.V_t)V_t+(1-\xi)K_1(\xi,\eta,\phi_t.\chi_t)\chi_t\right)-\eta(\phi_t)$$

where $\eta(\phi_t.\chi_t)$ is the social value, i.e. the monetary equivalent of the quantity transformed $\eta\phi_t$. The quantity $\eta\phi_t$ is itself transfomed into equipments, wages, etc. The agent produces along two axes: V_t and χ_t, up to a proportion ξ and $(1-\xi)$, respectively. The productivities $K_1(\xi,\eta,\phi_t.\chi_t)$ and $K_2(\xi,\eta,\phi_t.V_t)$ represent the quantity of goods produced along V_t and χ_t respectively, per unit of $\phi_t.\chi_t$ transformed, i.e. per unit of stock social value.

The productivity K_1 is a function of $\phi_t.\chi_t$: productivity depends on the technological environment valorization of capital. Similarly K_2 is a function of $\phi_t.V_t$: productivity also depends on agents' personal technology valorization of capital. The proportion ξ maximizes the function $Y_t.\chi_t$.

The productivity $K_2(\xi, \eta, \phi_t.V_t)$ is increasing in ξ[2] and its second derivative is assumed negative, to reflect the cost of departing from social values. We assume that there is an optimum for ξ between 0 and 1. The term $-\eta(\phi_t)$ reflects the part of ϕ_t used up in the producing process: we therefore consider a net production Y_t. This production can be derived from the optimization problem of $Y_t.\chi_t$.

To simplify the computation, we will suppose η, the fraction of goods transformed, to be constant across agents. Since η is fixed, we can dismiss the second order derivative of K with respect to η. We further suppose decreasing returns to scale in investment, so that the derivatives satisfy the following conditions : $K_1 > 0$, $K_{11} < 0$, $K_3 > 0$, $K_{13} > 0$, $K_4 > 0$, $K_{14} < 0$.

Let us remark that the traditional characterization of technological change - a change in the set of feasible production possibilities - is impossible here. Eventhough technical progress may be qualified as not neutral in our model, since the environment does influence factors' relative productivity, this influence can be either positive or negative, depending on the factor. This feature comes from the fact that our vector values are totally arbitrary, and that we avoided the aggregation of factors of production.

Let us also remark that the evolution of V_t relative to χ_t is similar to the disruptive force of creative destruction in the Schumpeterian growth theory[30], when V_t anticipates χ_t (case 3 of the model). In this case, each agent may impact and disrupt society. The introduction of new, innovative goods broadens the set of relative prices, thus further depreciating part of them. Therefore our agent can be seen, in some particular cases, as a Schumpeterian entrepreneur.

2 The optimization problem

2.1 Utility optimization

Let us recall that the agent optimizes the intertemporal utility:

$$Opt_{(C_t,\phi_{t+1})}\bar{U}_t = Opt_{(C_t,\phi_{t+1})}U(V_t, \phi_t, \chi_t, C_t) + \rho\hat{U}\left(\phi_{t+1}, E_t\chi_{t+1}\right) + \lambda\left(\phi_t + Y_t - C_t - \phi_{t+1}\right)$$

Where we assume $\phi_{t+1} \geq 0$. Although agents may leave a bequest, this bequest must necessarily be positive (no debt condition). Recall also that:

$$\begin{aligned} E_t\chi_{t+1} &= \chi_t \\ Y_t &= F\left(\phi_t, \chi_t - \chi_{t-1}, V_t - V_{t-1}\right) \\ \phi_{t+1} &= \phi_t + Y_t - C_t \end{aligned}$$

The optimization equations are straightforward. Replacing directly ϕ_{t+1} by the expression above allows to optimize on C_t only.

$$\frac{\partial}{\partial C_t}U(V_t, \phi_t, \chi_t, C_t) + \frac{\partial}{\partial C_t}\rho\hat{U}\left(\phi_t + Y_t - C_t, E_t\chi_{t+1}\right) = 0$$

That is:

$$\begin{aligned} 0 &= (1-\rho)V_t - \alpha V_t\left((V_t)^t C_t\right) - \beta(V_t - \chi_t)\left((V_t - \chi_t)^t C_t\right) - \gamma(C_t - \phi_t) \\ &\quad + \delta\rho V_t\left(V_t^t(\phi_t + Y_t - C_t)\right) + \varepsilon\rho\left(V_t - E_t\chi_{t+1}\right)\left(\left(V_t - E_t\chi_{t+1}\right)^t(\phi_t + Y_t - C_t)\right) \end{aligned}$$

Using $E_t\chi_{t+1} = \chi_t$, this can be rewritten in a more compact form as:

$$W_t = (\gamma + N)C_t$$

Where the vector W_t and the matrix N are defined by :

$$W_t = \left(1 - \rho + \delta\rho V_t^t(\phi_t + Y_t)\right)V_t + \gamma\phi_t + \varepsilon\rho\left((V_t - \chi_t)^t(\phi_t + Y_t)\right)(V_t - \chi_t)$$

$$N_{ij} = (\alpha + \delta\rho)(V_t)_i(V_t)_j + \beta(V_t - \chi_t)_i(V_t - \chi_t)_j + \varepsilon\rho(V_t - \chi_t)_i(V_t - \chi_t)_j$$

[2] Symetricaly $K_1(\xi, \eta, \phi_t.\chi_t)$ is decreasing in ξ.

2.2 Production optimization

The function Y_t is part of the optimization problem and as such, the parameter ξ must be identified. Recall the production function:

$$Y_t = \eta(\phi_t.\chi_t)(\xi K_2(\xi,\eta,\phi_t.V_t)V_t + (1-\xi)K_1(\xi,\eta,\phi_t.\chi_t)\chi_t) - \eta(\phi_t)$$

The agent will optimize $Y_t.\chi_t$ on ξ while taking other variables as given. As explained above, we assume that the agent is relatively more productive in the V_t, and as such $\xi > \frac{1}{2}$. However, due to the cost of differing from χ_t, the return from departing from χ_t decreases with ξ, so that the optimal ξ is lower than 1. To model the advantage to produce V_t we assume, for the sake of simplicity, that for the optimal ξ, productivities K_2 and K_1 are related by

$$K_2(\xi,\eta,\phi_t.V_t) = \tau K_1(\xi,\eta,\phi_t.\chi_t)$$

with $\tau \geqslant 1$. This ratio reflects the average gain in productivity of producing $\xi\eta(\phi_t.\chi_t)$ goods in the direction of V_t. The vector Y_t can then be rewritten:

$$Y_t = \eta(\phi_t.\chi_t)(\tau\xi K_1(\xi,\eta,\phi_t.V_t)V_t + (1-\xi)K_1(\xi,\eta,\phi_t.\chi_t)\chi_t) - \eta(\phi_t)$$

Depending on τ and the production function parameters, ξ can, as a second order approximation[3], be written : $\xi = \frac{1+\varphi z}{2}$, where $z = \sin^2(\arg(V_t,\chi_t))$ (note that $\xi = \frac{1}{2}$ when $V_t = \chi_t$ as needed). Moreover, remark that $\tau = 1$ when $V_t = \chi_t$. The parameters τ and φ depend on z, ϕ_t, χ_t and V_t. However they also depend on some unknown K_1 parameters, and can thus be seen (by a change in variables) as independent variables. Moreover, using the envelope theorem, their dependence on z, ϕ_t, χ_t and V_t will be discarded in first approximation. Ultimately we will approximate the productivity function $K_1(\xi,\eta,\phi_t.\chi_t)$ at the optimal ξ by a first order linear expansion:

$$K_1(\xi,\eta,\phi_t.\chi_t) = K_1 + \hat{K}_1\phi_t.\chi_t$$

so that:

$$K_1(\xi,\eta,\phi_t.V_t) = K_1 + \hat{K}_1\phi_t.V_t$$

where K_1 and $\hat{K}_1$ are constant. In the absence of a specific form for the productivity function, we can assume a valid first order expansion for a function depending weakly on $\phi_t.\chi_t$, or $\phi_t.V_t$. This implies that this approximation is valid for $\hat{K}_1 << K_1$.

3 Results

We study the impact of relative prices on intertemporal capital accumulation, depending on internal economic and social parameters, i.e. capital mobility, productivity, and personal and social values discrepancies. This amounts to studying the dynamic for the variable $\Phi_t.\chi_t$, i.e. the capital stock social value.

We solve the model for several cases, in which we consider capital accumulation under various configurations of social and individual values. Four cases are considered.

The benchmark case is an homogenous society where individual and social values match perfectly. Social values evolve exogenously through external shocks and follow an $AR(1)$ stochastic process. A first departure from this benchmark, case two, considers individual values differing from social values by a stochastic term. In this case, personal values are randomly distributed around social values. Social values evolve exogenously, and agents do not influence this evolution, but the society being heterogeneous, some agents may significantly depart from the common social values. Case three refines the relation between social and personal values. Some individual values may now anticipate future social values up to a random noise. These agents may be seen as precursors by impacting the evolution of future social values. The fourth and last case considers a two-sector economy, each with distinct sector values. The future social values will be a weighted mean of present sector values. Each sector will then partly be precursor, and the weight attributed to each sector will be proportional to the social value of its accumulated stock.

[3] See appendix.

3.1 Benchmark case : identical personal and social values

In the benchmark case, there are no discrepencies between the society and the agents that compose it. Individual and social values are strictly identical, and $V_t = \chi_t$. The society is entirely homogenous, and the capital accumulation dynamics is driven by the evolution of the variable χ_t[4], that follows the $AR(1)$ stochastic process:

$$\chi_{t+1} = \sqrt{1-\sigma^2}\chi_t + \varepsilon_{t+1}$$

where ε_{t+1} is a white noise of variance $\sigma^2 < 1$, and $\sqrt{1-\sigma^2}$ is a factor normalizing vector χ_{t+1} [5]. When σ is large (close to one), χ_{t+1} can depart widely from χ_t and social values may exhibit strong variations.

Under these assumptions, we show[6] that the dynamics for $\Phi_t.\chi_t$ is:

$$\Phi_{t+1}.\chi_{t+1} = \left(\left(\eta K_1 + \frac{\alpha+\delta\rho\eta(1-K_1)}{\gamma+\alpha+\delta\rho}\right)\Phi_t.\chi_t - \frac{1-\rho}{(\gamma+\alpha+\delta\rho)} + \eta\frac{\gamma+\alpha}{\gamma+\alpha+\delta\rho}\hat{K}_1(\Phi_t.\chi_t)^2\right)\chi_t.\chi_{t+1} - \eta\Phi_t.\chi_{t+1} \tag{1}$$

As already mentioned, the social value χ_t is exogenous and follows a stochastic process

$$\chi_{t+1} = \sqrt{1-\sigma^2}\chi_t + \varepsilon_{t+1} \tag{2}$$

where ε_{t+1} is a random noise[7] of variance σ^2. It follows from our assumption on the dynamics of χ_t that $\Phi_t.\chi_t$ is itself a stochastic process. Since at time t, there is no uncertainty on Φ_t and χ_t, then on average $\left\langle(\Phi_t.\chi_t)^2\right\rangle = \langle\Phi_t.\chi_t\rangle^2$, and equation 1 becomes:

$$\frac{\langle\Phi_{t+1}.\chi_{t+1}\rangle}{\sqrt{1-\sigma^2}} = \frac{(\alpha+\eta(\alpha+\gamma)(K_1-1))}{\alpha+\gamma+\delta\rho}\langle\Phi_t.\chi_t\rangle - \frac{1-\rho}{(\gamma+\alpha+\delta\rho)} + \eta\frac{\gamma+\alpha}{\gamma+\alpha+\delta\rho}\hat{K}_1\langle\Phi_t.\chi_t\rangle^2$$

Such an equation can be solved with a continuous time approximation:

$$\Phi(t).\chi(t) = \frac{\Phi^+ - \Phi^-\frac{\Phi_0.\chi_0-\Phi^+}{\Phi_0.\chi_0-\Phi^-}\exp(\Delta t)}{1-\frac{\Phi_0.\chi_0-\Phi^+}{\Phi_0.\chi_0-\Phi^-}\exp(\Delta t)}$$

with:

$$\begin{aligned}\Delta &= \sqrt{(a-1)^2+4bc}\\ \Phi^\pm &= \frac{-(a-1)\pm\Delta}{2c}\\ \Phi_0.\chi_0 &= \Phi(0).\chi(0)\end{aligned}$$

and:

$$\begin{aligned}a &= \left(\frac{(\alpha+\eta(\alpha+\gamma)(K_1-1))}{\alpha+\gamma+\delta\rho}\right)\sqrt{1-\sigma^2}\\ b &= \frac{1-\rho}{\gamma+\alpha+\delta\rho}\sqrt{1-\sigma^2}\\ c &= \eta\frac{\gamma+\alpha}{\gamma+\alpha+\delta\rho}\sqrt{1-\sigma^2}\hat{K}_1\end{aligned}$$

The dynamics for $\Phi(t).\chi(t)$ presents a threshold pattern. For $\Phi_0.\chi_0$ below the threshold Φ^+, $\Phi(t).\chi(t)$ will tend to zero. For $\Phi_0.\chi_0$ above the threshold Φ^+, $\Phi(t).\chi(t)$ will tend to $+\infty$. If the stock initial social value is low, the stock decreases. When its initial value is sufficiently large, an explosive accumulation occurs. The threshold Φ^+ is explicitely given by:

$$\Phi^+ = \frac{-(a-1)+\Delta}{2c}$$

[4] This assumption of χ_t exogenous will later be relaxed.
[5] See appendix for details.
[6] See appendix.
[7] See appendix for details.

This threshold varies along with the parameters $\hat{K}_1$, σ^2 and η. Variations with respect to $\hat{K}_1$ and σ^2 are unambiguous and satisfy $\frac{\partial \Phi^+}{\partial \sigma^2} > 0$ and $\frac{\partial \Phi^+}{\partial \hat{K}_1} < 0$.

The variance σ^2 measures the volatility of χ_t, as seen in equation 2. Since $\frac{\partial \Phi^+}{\partial \sigma^2}$ is positive, the threshold increases with volatility, hindering capital accumulation. Stocks may depreciate faster than the time it takes to rebuild them. A larger initial stock will be necessary to accumulate capital.

The second inequality $\frac{\partial \Phi^+}{\partial \hat{K}_1} < 0$ shows that when productivity increases, the threshold diminishes and capital may accumulate even whith a low initial capital stock.

Variations in the threshold with respect to η are more ambiguous and depend on the parameter values. When $2\eta > -\frac{b}{\sqrt{1-\sigma^2}} + \sqrt{\frac{b^2}{1-\sigma^2} + 2(1-\rho) K_1}$, the term $\frac{\partial \Phi^+}{\partial \eta}$ is positive. It is negative otherwise.

When the agent productivity K_1 is small, and when η, the fraction of capital devoted to his own production, is large, his production will be low, and lower than his consumption. There will be an incentive to consume the remaining stock of inherited capital, and the bequest will be depleted. As a consequence, the threshold Φ^+ will increase: a large initial capital Φ_0 will be necessary to accumulate further capital.

On the contrary, if K_1 is large enough, an increase in η will lower the threshold Φ^+. The more productive the agent, the larger the gain to transform his stock of capital.

We can now study the impact of a shock on χ_t, seen as a shift on $\phi_t.\chi_t$, followed by a resuming dynamics afterward. This shock can lead to a radical change in the dynamics, depending on the stock value $\phi_t.\chi_t$, moving above or below the threshold.

For example, a depreciating capital stock can suddenly appreciate, inducing a trend in capital accumulation. On the contrary, a sudden depreciation in an otherwise increasing capital stock may lead to a reversal in capital accumulation. This may illustrate the impact of technological changes rendering some older equipements obsolete, or inversely, the sudden wealth induced by the discovery - or revalorisation - of some natural resources.

Last but not least, the solution of our model shows the effects of specialization within a society. Supposing that χ_t is oriented along one single good, the slightest modification ε_{t+1} in whichever alternate direction will be transverse to χ_t, and will induce a noticable reduction in the value of accumulated stock. The fall of this value below the minimal accumulation threshold could lead to a reversal in dynamics and a shortage in stocks. Our results therefore support a strong heterogeneity among sectors within a single society. The milder the specialization in production, the greater the resilience towards headwinds or shocks.

3.2 Case Two : Heterogeneity in personal values

Consider now that personal values are related to social values by the following relation:

$$V_t = \sqrt{1-\tilde{\sigma}^2}\chi_t + \tilde{\varepsilon}_t$$

where $\tilde{\varepsilon}_t$ is a white noise of variance $\tilde{\sigma}^2$. The factor $\sqrt{1-\tilde{\sigma}^2}$ normalizes the vector V_t. The variance $\tilde{\sigma}^2$ therefore represents the dispersion of individual values within a society. Agents' individual values are spread around $\sqrt{1-\tilde{\sigma}^2}\chi_t$ following a gaussian variable.

The social value χ_t is the normalized average of individual values, i.e. $\chi_t = \sum \frac{V_t}{\sqrt{1-\tilde{\sigma}^2}}$ where the sum is peformed over agents. When $\tilde{\sigma}^2 = 0$, the society is entirely homogenous, as studied in our benchmark case. The dynamics followed by χ_t is identical to case one:

$$\chi_{t+1} = \sqrt{1-\sigma^2}\chi_t + \varepsilon_{t+1} \tag{3}$$

The dynamic equation for $\langle \phi_t.\chi_t \rangle$[8], where the bracket sign signifies the average over all agents of $\phi_t.\chi_t$, is :

$$\begin{aligned}\frac{\langle \phi_{t+1}.\chi_{t+1} \rangle}{\sqrt{1-\sigma^2}} &= \left((A\tau(1+\varphi z) + B(1-\varphi z)) \frac{\eta K_1}{2} + (1-\eta) \frac{\alpha + 2z\delta\rho}{\alpha + \delta\rho} \right) \langle \phi_t.\chi_t \rangle \\ &\quad + \frac{1}{2} (A(1-2z)\tau(1+\varphi z) + B(1-\varphi z)) \eta \hat{K}_1 \langle \phi_t.\chi_t \rangle^2 - \frac{(1-\rho)}{(\alpha + \delta\rho)}\end{aligned}$$

[8] Shown in the appendix.

where:

$$\begin{aligned} A &= \left((1-2z) - \frac{\delta\rho}{\alpha+\delta\rho}\right) \\ B &= \left(1 - \frac{\delta\rho\,(1-2z)}{\alpha+\delta\rho}\right) \end{aligned}$$

The solution of the dynamic equation is similar to the benchmark case:

$$\Phi\,(t)\,.\chi\,(t) = \frac{\Phi^{+} - \Phi^{-}\frac{\Phi_0.\chi_0-\Phi^{+}}{\Phi_0.\chi_0-\Phi^{-}}\exp\left(\Delta' t\right)}{1 - \frac{\Phi_0.\chi_0-\Phi^{+}}{\Phi_0.\chi_0-\Phi^{-}}\exp\left(\Delta' t\right)}$$

and presents a threshold:

$$\Phi^{+} = \frac{-\,(d-1) + \Delta'}{2f}$$

where:

$$\begin{aligned} \Delta' &= \sqrt{(d-1)^2 + 4ef} \\ \Phi^{-} &= \frac{-\,(d-1) - \Delta'}{2f} \end{aligned}$$

and

$$\begin{aligned} d &= \left((A\tau\,(1+\varphi z) + B\,(1-\varphi z))\,\frac{\eta K_1}{2} + (1-\eta)\,\frac{\alpha+2z\delta\rho}{\alpha+\delta\rho}\right)\sqrt{1-\sigma^2} \\ e &= \frac{(1-\rho)}{(\alpha+\delta\rho)}\sqrt{1-\sigma^2} \\ f &= \frac{1}{2}\,(A\,(1-2z)\,\tau\,(1+\varphi z) + B\,(1-\varphi z))\,\eta\hat{K}_1\sqrt{1-\sigma^2} \end{aligned}$$

The parameter τ measures the agent's relative gain in productivity in V_t. The parameter z measures the angle between personal and social values vectors V_t and χ_t. When $z = 0$, $V_t = \chi_t$ and when $z = 1$, V_t and χ_t are orthogonal. The parameter φ measures the extra weight attributed by each agent to the production of V_t compared to its production of χ_t.

When τ is close to 1, the agent is only marginaly more productive in V_t. An increase in z, φ or η increases the threshold Φ^{+}, since producing V_t does not compensate the cost of differing from χ_t.

When τ increases and is sufficiently large, Φ^{+} diminishes with z, φ or η. Actually, for an agent highly productive in V_t, producing V_t largely compensates differing from χ_t. Moreover, the farther from χ_t agent's values are, the higher the incentive to produce V_t will be, and the higher the agent's production, so that Φ^{+} diminishes.

When individual values do not impact future social values, differing from χ_t has an ambiguous effect. When relative productivity in V_t is small, producing χ_t favors capital accumulation. When this relative productivity is large, producing V_t favors capital accumulation. The more V_t differs from χ_t, the greater the capital accumulation.

3.3 Case Three: the dynamics of a precursor agent

In this case, the individual values of one agent at time t may influence social values at time $t+1$. Whereas the two previous cases considered societies with exogeneously evolving social values, this case endogeneizes χ_t by modeling the social impact of precursors. Alternatively, it could depict a society where aggregated indivual values V_t will become social values χ_{t+1}, thus modeling the internal evolution of social values. This situation can be described by the following dynamics on V_t and χ_t:

$$\begin{aligned} V_t &= \sqrt{1-\tilde{\sigma}^2}\chi_t + \tilde{\varepsilon}_t \\ \chi_{t+1} &= \sqrt{1-\sigma^2}V_t + \varepsilon_t \\ \chi_{t+1} &= \sqrt{(1-\sigma^2)\left(1-\tilde{\sigma}^2\right)}\chi_t + \varepsilon_t + \sqrt{1-\sigma^2}\tilde{\varepsilon}_t \end{aligned}$$

Here ε_t and $\tilde{\varepsilon}_t$ are white noises of variance σ^2 and $\tilde{\sigma}^2$, respectively.

At time t, V_t differs from χ_t by a white noise $\tilde{\varepsilon}_t$ and χ_{t+1} is determined by V_t up to some random noise modeling external shocks. The dynamics for $\langle \phi_t.\chi_t \rangle$ is computed in the appendix. Assuming the following normalizations, $\alpha = 1$, $\delta = 1$, $K_1 = K_2 = 1$, $\hat{K}_1 = \hat{K}_2$, and assuming also $\tau = 1$ to focus on the precursor effect, we have:

$$\frac{\langle \phi_{t+1}.\chi_{t+1} \rangle}{\sqrt{1-\sigma^2}} = -\frac{1-\rho}{1+\rho} + \frac{(z\eta(1+z\varphi)+1-2z)}{1+\rho}\langle \phi_t.\chi_t \rangle + \frac{(1-2z)\eta\hat{K}_1}{\rho+1}\langle \phi_t.\chi_t \rangle^2$$

that presents the same threshold pattern as in case two, with:

$$\Phi_+ = \frac{-\left((z\eta(1+z\varphi)+1-2z) - \frac{1+\rho}{\sqrt{1-\sigma^2}}\right) + \sqrt{\left((z\eta(1+z\varphi)+1-2z) - \frac{1+\rho}{\sqrt{1-\sigma^2}}\right)^2 + 4(1-\rho)(1-2z)\eta\hat{K}_1}}{2(1-2z)\eta\hat{K}_1}$$

The variation of the threshold depends on the parameters and is given by:

$$\frac{\partial \Phi_+}{\partial \sigma^2} > 0,\ \frac{\partial \Phi_+}{\partial \hat{K}_1} < 0,\ \frac{\partial \Phi_+}{\partial (1+z\varphi)} < 0,\ \frac{\partial \Phi_+}{\partial \eta} < 0,\ \frac{\partial \Phi_+}{\partial z} > 0$$

and

$$\frac{\partial \Phi_+}{\partial \rho} < 0 \text{ if } \eta\hat{K}_1 > \frac{2-((z\eta(1+z\varphi)+1-2z))\sqrt{1-\sigma^2}}{1-\sigma^2}$$
$$\frac{\partial \Phi_+}{\partial \rho} > 0 \text{ otherwise.}$$

The parameter σ^2 is the fluctuation of the exogenous part of χ_t. Here again, we find that the threshold is increased by this volatility. A changing environment tends to erode capital value and impair its accumulation, in turn increasing the accumulation threshold.

Greater productivity favors capital accumulation, and depresses the accumulation threshold. Thus $\frac{\partial \Phi_+}{\partial \hat{K}_1} < 0$.

Recall that $1 + z\varphi$ represents the relative supplement of production in V_t. Here, contrarily to case two, the result is unambiguous: $\frac{\partial \Phi_+}{\partial(1+z\varphi)} < 0$. Departing from the mainstream χ_t is overall positive, and favors accumulation. Since tomorrow's values will match individual values, the agent and his heirs will benefit from this decision. A capital accumulated today and positively valued tomorrow actually amounts to an investment. Along the same line, since the capital allocated to V_t will be positively valued tomorrow, the higher the proportion of capital allocated to production, the higher the capital accumulation, so that $\frac{\partial \Phi_+}{\partial \eta} < 0$.

Time preference has an ambiguous effect on the variation of the threshold depending on the agent's productivity $\eta\hat{K}_1$. When it is large enough, then $\frac{\partial \Phi_+}{\partial \rho} < 0$. The agent is productive enough to leave a bequest, that in turn favors accumulation.

When $\frac{\partial \Phi_+}{\partial z} > 0$, there is no specific gain to produce V_t different from χ_t, contrarily to case 2. Since the agent foresees χ_t, it would gain in producing the future environment. However it would loose by producing it beforehand, without benefiting from the actual environment. So that when z increases, so does the threshold.

3.4 Case Four: A Two-Sector Economy

This case models a society in which χ is endogeneous. The society is composed of two sectors of equal initial size and wealth. To simplify, we discard the two sectors common values to focus on their differences. To do so, we consider that the vectors of sectors values, V_1 et V_2 respectively, are strictly orthogonal, so that the goods they produce are radically different. At each moment in time, χ is a weighed value of V_1 et V_2, and weights are proportional to each sector goods' value. Wee show in the appendix that the dynamics of the stock of each sector is given by :

$$\phi_{t+1} = \left(\eta(\phi_t.\chi_t)\left(1+\hat{K}_1(\phi_t.V_t)\right)\right)V_t - \left(\frac{1-\rho+\delta\rho\left((1-\eta)\phi_t.V_t + \eta(\phi_t.\chi_t)\left(1+\hat{K}_1(\phi_t.V_t)\right)\right)}{(1+\rho)(1-z)}\right)\frac{\chi_t+V_t}{2} + (1-\eta)\phi_t$$

We call $y_t = (\phi_t.\chi_t)$ the social value of ϕ_t.

$$\phi_{t+1} = \left(\eta y_t \left(1 + \hat{K}_1 (\phi_t.V_t)\right)\right) V_t - \left(\frac{1 - \rho + \delta\rho \left((1-\eta)\phi_t.V_t + \eta y_t \left(1 + \hat{K}_1 (\phi_t.V_t)\right)\right)}{(1+\rho)(1-z)}\right) \frac{\chi_t + V_t}{2} + (1-\eta)\phi_t$$

Let us now consider the case of a two-sector economy, with stocks $\phi_t^{(1)}$ and $\phi_t^{(2)}$ and group values $V_t^{(1)}$ and $V_t^{(2)}$, respectively. We also assume that these values are time independent, such that $V_t^{(1)} = V_0^{(1)}$ and $V_t^{(2)} = V_0^{(2)}$. Besides $V_t^{(1)}$ and $V_t^{(2)}$ are transversal, $V_t^{(1)}.V_t^{(2)} = V_0^{(1)}.V_0^{(2)} = 0$.

We further assume that social values are endogenous and given by a weighted average of previous period personal values. The weights are the relative ratio of each groups' stock previous periods' social value.

$$\begin{aligned} \chi_{t+1} &= \frac{\left(\phi_t^{(1)}.\chi_t\right) V_t^{(1)} + \left(\phi_t^{(2)}.\chi_t\right) V_t^{(2)}}{\sqrt{\left(\phi_t^{(1)}.\chi_t\right)^2 + \left(\phi_t^{(2)}.\chi_t\right)^2 + 2\left(\phi_t^{(1)}.\chi_t\right)\left(\phi_t^{(2)}.\chi_t\right) V_t^{(1)}.V_t^{(2)}}} \\ &= \frac{\left(\phi_t^{(1)}.\chi_t\right) V_0^{(1)} + \left(\phi_t^{(2)}.\chi_t\right) V_0^{(2)}}{\sqrt{\left(\phi_t^{(1)}.\chi_t\right)^2 + \left(\phi_t^{(2)}.\chi_t\right)^2}} \\ &= \frac{y_t^{(1)} V_0^{(1)} + y_t^{(2)} V_0^{(2)}}{\sqrt{\left(y_t^{(1)}\right)^2 + \left(y_t^{(2)}\right)^2}} \end{aligned} \tag{4}$$

Let us also define $z^{(i)}$, $i = 1$ or 2

$$1 - 2z_t^{(i)} = \chi_t.V_0^{(i)}$$

$z^{(i)}$ measures the angle between χ_t and $V_0^{(i)}$. We define also:

$$\begin{aligned} y_t^{(i)} &= \phi_t^{(i)}.\chi_t \\ x_t^{(i,i)} &= \phi_t^{(i)}.V_0^{(i)} \end{aligned}$$

Respectively the social value of group i's stock, and the group value attributed by group i to it's own stock $\phi_t^{(i)}$.

We also define the following "crossed" values:

$$x_t^{(i,3-i)} = \phi_t^{(i)}.V_0^{(3-i)}$$

, i.e. the value attributed by a group to the stock of the other group.

We show in the appendix, that the values $x_t^{(i,i)}$ and $x_t^{(i,3-i)}$ satisfy the following dynamics:

$$x_{t+1}^{(i,j)} = \eta y_t^{(i)} \left(1 + \hat{K}_1 x_t^{(i,i)}\right) w_{i,j} - \left(\frac{1 - \rho + \delta\rho \left((1-\eta) x_t^{(i,i)} + \eta y_t^{(i)} \left(1 + \hat{K}_1 x_t^{(i,i)}\right)\right)}{(1+\rho)\left(1 - z_t^{(i)}\right)}\right) \frac{z_t^{(j)} + w_{i,j}}{2} + (1-\eta) x_t^{(i,j)}$$

with $w_{i,j} = \delta_{i,j}$.

The dynamic for sector 1 is explicitly given as:

$$\begin{aligned} \eta x_{t+1}^{(1,1)} &= \eta y^{(1)} \left(1 + \hat{K}_1 x^{(1,1)}\right) - \left(1 - \rho + \rho \left((1-\eta) x^{(1,1)} + \eta y^{(1)} \left(1 + \hat{K}_1 x^{(1,1)}\right)\right)\right) \frac{z^{(1)} + 1}{2(1+\rho)\left(1 - z^{(1)}\right)} \\ \eta x_{t+1}^{(1,2)} &= -\left(1 - \rho + \rho \left((1-\eta) x^{(1,1)} + \eta y^{(1)} \left(1 + \hat{K}_1 x^{(1,1)}\right)\right)\right) \frac{z^{(1)}}{2(1+\rho)\left(1 - z^{(1)}\right)} \end{aligned}$$

and the same for sector 2.

The dynamics for $y_t^{(i)}$ is then deduced from the equation for χ_{t+1} above.

$$\begin{aligned} y_{t+1}^{(i)} &= \frac{y_t^{(1)} x_{t+1}^{(1,1)} + y_t^{(2)} x_{t+1}^{(1,1)}}{\sqrt{\left(y_t^{(1)}\right)^2 + \left(y_t^{(2)}\right)^2 + 2\left(y_t^{(1)}\right)\left(y_t^{(2)}\right) w}} \\ &= \left(1 - 2z_t^{(i)}\right) x_{t+1}^{(1,1)} + \left(1 - 2z_{t+1}^{(3-i)}\right) x_{t+1}^{(1,1)} \end{aligned}$$

The appendix shows that no equilibrium point exist. However, it also shows there is an "equilibrium" dynamics $y_t^{(1)} = y_t^{(2)}$, $\left(1 - 2z_t^{(1)}\right) = \left(1 - 2z_t^{(2)}\right) = \frac{1}{\sqrt{2}}$. That is, both group values participate equaly to the social value $\chi_t = \frac{V_0^{(1)} + V_0^{(2)}}{\sqrt{2}}$. For $y_t^{(1)}$ and $y_t^{(2)}$ above a certain threshold, $y_t^{(1)}$ and $y_t^{(2)}$ goes to $+\infty$ and that $x_t^{(1,2)}$ and $x_t^{(2,1)}$ tend to 0. In other word, above this threshold, both group accumulate, but ultimately in the direction of their own sector values only. The appendix shows that this equilibrium dynamic is unstable. A small desequilibrium in the composition of $\chi_t = \frac{V_0^{(1)} + V_0^{(2)}}{\sqrt{2}}$, i.e. the overvaluation of one group value, leads to depart from the symetric situation presented above. One group will accumulate with a lower threshold, and the other group will experience a increase in his threshold, i.e. an increased difficulty to accumulate. Moreover this last group will specialize slower than the first group. Ultimately the accumulation of the second group will progressively be socialy undervalued and will tend to 0.

4 Discussion

What are the main contributions of this model? Let us first remark that the vector χ is similar to a vector of market prices for all - material and immaterial, potential or real, past, present, and future - goods. This vector therefore represents the global supply and demand within a society at a point in time. This feature makes it a technological indicator for the society it describes. In this model, the vector of prices reflects the period in which the society is studied, as well as its technicity level. An agricultural society will present a vector χ whose non-null coordinates are affected to agricultural goods. Inversely, a highly technological society has the bulk of its non-null coordinates concentrated in technologically-intensive goods. In other words, these coordinates represent the solvent demand of the society they describe.

One should not confuse solvent demand and the aggregate sum of personal values attributed to each good. Note also that potential goods - goods that are not available yet - may have a non-null personal value. In a slave society, slaves may find an interest in mecanization - and yet have no social value - the slave society may find no interest in it. In other words, we state that aggregate personal values are not necessarily reflected by the vector of market prices. The latter includes and depends on other social or external factors, be it historical or environmental.

If vector χ can be reduced to a vector of market price, the vector V, on the contrary, represents the real interest granted to a good by the agent, and is an indicator of the productivity, or technological capacity of the agent with respect to that good. The point of our formalisation is to have many types of valorization coexisting within a society with multiple agents.

It is straightforward to see that in our production function productivity in good i is a combination of V and χ. In other words, the agent's productivity, i.e. his personal technology, may, or may not, be valorized by his environment. The agent permanently faces a trade-off, in which he must confront the valorization granted by his environment to a certain good (the neoclassical "interest of capital", $dY/d\chi$), to his own personal interest (his own "productivity", dY/dV). This represents a cost for the agent that depends on the discrepencies between V and χ. To sum up this argument in one vivid picture, there would be no possibility to use a computer in the absence of electrical circuit or plug.

Inversely, the society as a whole may specialize only inasmuch it is totally homogeneous, or if there exist a massive constraint to set aside V in favor of χ. This is usually the case in economies of war, when national interests superseede all particular interests.

In neoclassical models, specialization is justified by the notion of comparative advantage. The Heckscher–Ohlin theorem[23], for example, considers production and consumption as two totally dissociated behaviors.

There is a gain to specialize in one's comparative advantage, since the decrease in prices brought about by specialization favors the consumption of differenciated goods. Agents are indifferent to the activity they practice : their environment is not affected by it, and their consumption is strictly identical and equally varied for all agents. The comparative advantage argument to justify specialization rests on opportunity costs. These costs determine in which sector an economy should specialize, but they do not allow to study the impact of specialization on society. In such a context, specialization utlimately benefits all agents, but only inasmuch there are no technological changes, and production and consumption do not depend on a close environment.

On the contrary, the present paper advocates that consumption and production are environment-dependent, and therefore is dependent on technological change. In our model, agents preferences in consumption are also their preferences in specialization, i.e their comparative advantage with respect to the vector of social values χ. Since χ acts as an environment there is a cost to consume and produce far from χ. This flows from the hypothesis that agents are consumers-producers. The loss in profit associated with the distance from χ is the agent opportunity cost that arises from technological differences between the society and the agent.

In the present model, labor-augmenting technology or "knowledge" is replaced by the environment χ and by the productivity associated to V. However, we do not consider the impact technological evolution may have in terms of growth. Contrarily to the Solow growth model, we do not consider the technological innovation impact in terms of global productivity, or the quantitative growth in technology*:* heterogeneity of goods excludes these aggregated quantities. In the same vein, we do not measure an aggregated growth rate. Rather, we study how the technological environnement evolution, be it endogeneous or exogeneous, modifies or allows capital accumulation and/or wealth distribution. What this models allows, is a better understanding of social inter-relationships.

For the same reasons, this paper must be distinguished from the investment specific technological change (hereafter ISTC) literature. Let us recall that ISTC may be defined as technological change embodied in the form of new equipment, for example advances in computer technology, robotization of assembly lines, etc[1].

We agree with the ISTC literature in that factors are differenciated. Indeed, this type of technological innovation is different from the usual changes in total factor productivity in which capital of different generations is thought of as being the same type of good, or having the same cost as previous vintages of capital (i.e. as measured in units of the consumption good). This view rejoins Robinson's view on capital. Besides, much work on the topic have shown the importance of ISTC in growth phenomena (see [11] , [10],Iglesias (2002)[14] and Licandro et al. (2002)[19], for the role of ISTC as an important source of long-term growth, [7], for the role of technological improvement in equipment and software in the productivity resurgence of the 1990s). It has been advocated that not only the investment, but also its allocation, play an important role in harvesting the benefits of information and communication technologies embodied in capital goods (see Collechia and Schreyer (2002) [6], Bose (1971) [3], Weitzman (1971) [33], Araujo and Teixeira (2002) [2] and Araujo (2004) [1]).

Our results underscore the importance of heterogeneity of factors of production and the value bestowed to them by technological environment as a source of capital accumulation. These results are therefore close to those of economic literature in this field. However they differ in that we do not consider growth as a global phenomenon, but we rather focus on sectorial and inter-sectorial development.

Our take-off threshold approach may also be viewed in relation to development economics and economic history. Our results concern the impact of parameters such as capital mobility, environment volatility and agents' relative productivity on sectors' development. To put it differently, our work focuses on the conditions allowing new technologies or environment - a change in V and in χ - to appear, impact other sectors, and disseminate to the entire society. Case one can accounts for the impact of exogenous technological or historical modifications on social capital accumulation. A resource curse would, in such a context, be modeled as a shock on capital stock value impacting on an economy take-off . Case three deals with the take-off of an innovative sector confronting its "established" technological environment. Case four studies the eviction effect of two competing sectors unstable dynamics. Finally, these two last cases combined describe internal mechanisms in industrial take-offs. The competition between the agriculture and industrial sectors in England at the turn of the XIXth century, for example, would fit in this framework. It took the implementation of Corn Laws to reverse a momentum that seemed, at first, favorable to the agricultural sector.

5 Conclusion

We have studied the interactions between social and individual values, their impact on capital accumulation, consumption and production. In a society characterized by a multitude of heterogeneous agents and a large number of possibly immaterial goods, each one having distinct social (relative price) and personal value (individual preference), we have studied the impact of these relative values' evolution on capital accumulation, depending on internal economic and social parameters, i.e. capital mobility, productivity, personal and social values discrepancies.

We have found that large variations in social values generally hinder accumulation: in a society of relatively homogeneous values, stocks tend to quickly depreciate, eventhough punctual strong appreciation are possible. This would tend to show that too much specialization may not be beneficial to a society. The same thing holds true for the individual.

In the same line, we have shown that highly differentiated individual and social values favor capital accumulation when this diversity reflects future changes in social values. However, investment capacity must be large enough to compensate for the cost of departing from actual values.

At the same time, heterogeneity can become a factor of inequality. When social values are weighed means of group values, competition leads to an unstable dynamics: there is a multiplier effect between value and accumulation. One group will increasingly accumulate and impose its values, to the detriment of other sectors' capital and values.

This leads to a problem of redistribution: if one sector's capital is not valorised, it will not have the means to diversify.

Future research includes the study of this framework in a growth model with disaggregated goods. Some production parameters such as capital mobility should be endogeneized, since the choice of producing depends on the environment. Such a model would account for the global wealth effects of sectors' dynamics and its implications for redistribution policies.

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Appendix

Write the optimization problem :

$$\begin{aligned}
& \left(1-\rho+\delta\rho V_t^t\left(\phi_t+Y_t\right)\right) V_t+\gamma\phi_t+\varepsilon\rho\left(\left(V_t-\chi_t\right)^t\left(\phi_t+Y_t\right)\right)\left(V_t-\chi_t\right) \\
= & \left(\alpha+\delta\rho\right) V_t\left(\left(V_t\right)^t C_t\right)+\beta\left(V_t-\chi_t\right)\left(\left(V_t-\chi_t\right)^t C_t\right)+\gamma C_t+\varepsilon\rho\left(V_t-\chi_t\right)\left(\left(V_t-\chi_t\right)^t C_t\right)
\end{aligned}$$

In a matricial form:

$$W_t=\left(\gamma+N\right) C_t \tag{5}$$

where we have defined:

$$W_t=\left(1-\rho+\delta\rho V_t^t\left(\phi_t+Y_t\right)\right) V_t+\gamma\phi_t+\varepsilon\rho\left(\left(V_t-\chi_t\right)^t\left(\phi_t+Y_t\right)\right)\left(V_t-\chi_t\right)$$

$$N_{ij}=\left(\alpha+\delta\rho\right)\left(V_t\right)_i\left(V_t\right)_j+\beta\left(V_t-\chi_t\right)_i\left(V_t-\chi_t\right)_j+\varepsilon\rho\left(V_t-\chi_t\right)_i\left(V_t-\chi_t\right)_j$$

Finding C_t and then Y_t amounts to invert the matrix $(\gamma+N)$. To do so, decompose N in several terms. First, we set:

$$\begin{aligned}
X_t &= \frac{\sqrt{\alpha} V_t}{\sqrt{\gamma}} \\
Y_t &= \frac{\sqrt{\beta+\varepsilon}\left(V_t-\chi_t\right)}{\sqrt{\gamma}}
\end{aligned}$$

and introduce the normalization $\left(V_t\right)^t\left(V_t\right)=\left(\chi_t\right)^t\left(\chi_t\right)=\left(E_t\chi_{t+1}\right)^t\left(E_t\chi_{t+1}\right)=1$
$\left(\chi_t\right)=\left(E_t\chi_{t+1}\right)$ $\sin y=\sin x$, $\sin z=0\sqrt{\gamma}=\delta$.
Then write N as the sum:

$$N=\sum_i M^{(i)}$$

where:

$$\begin{aligned}
M^{(i)} &= X^{(i)}\left(X^{(i)}\right)^t \\
P^{(il)} &= X^{(i)}\left(X^{(l)}\right)^t \\
\alpha^{(i)} &= \left(X^{(i)}\right)^t X^{(i)} \\
\gamma^{(il)} &= \left(X^{(i)}\right)^t X^{(l)}
\end{aligned}$$

To expand the inverse of $(\gamma+N)$ in powers of N, we need to compute $\left(\sum_i M^{(i)}\right)^k$. It is obtained by using the following relations:

$$\left(\sum_i M^{(i)}\right)^k=\sum_{i,l} c_k^{(il)} P^{(il)}$$

with:

$$c_{k+1}^{(il)}=\sum_m c_k^{(im)}\gamma^{(ml)} \tag{6}$$

Gathering the coefficients $c_k^{(im)}$ and $\gamma^{(ml)}$ in a matricial form:

$$\begin{aligned}
C_{k+1} &= \left(c_k^{(im)}\right), G=\left(\gamma^{(ml)}\right) \\
C_k &= C_1 G^{k-1}
\end{aligned}$$

so that the previous relation 6 rewrites:

$$C_{k+1} = C_k G$$

One can check that:

$$\begin{aligned} C_1 &= (\delta_{il}) \\ C_k &= G^{k-1} \end{aligned}$$

which allows to obtain $\left(\sum_i M^{(i)}\right)^k$:

$$\left(\sum_i M^{(i)}\right)^k = Tr\left(G^{k-1}P^t\right)$$

and utlimately:

$$\left(1+\sum_i M^{(i)}\right)^{-1} = 1+\sum_{k=1}^{\infty}\left(-\sum_i M^{(i)}\right)^k = 1+\sum_{k=1}^{\infty} Tr\left((-1)^k (G)^{k-1} P^t\right) = 1 - Tr\left((1+G)^{-1} P^t\right)$$

$$(\gamma+N)^{-1} = \gamma^{-1}\left(1+\frac{N}{\gamma}\right)^{-1} = \gamma^{-1}\left(1-Tr\left((1+G)^{-1}P^t\right)\right) \tag{7}$$

$(\gamma+N)^{-1}$ can thus be found straightforwardly by noticing that:

$$1+G = \begin{pmatrix} 1+X^tX & X^tY \\ X^tY & 1+Y^tY \end{pmatrix}$$

$$\begin{aligned} P &= \begin{pmatrix} XX^t & XY^t \\ YX^t & YY^t \end{pmatrix} \\ P^t &= \begin{pmatrix} XX^t & YX^t \\ XY^t & YY^t \end{pmatrix} \end{aligned}$$

and by using the identity:

$$\frac{1}{(Y^tY)(X^tX)-(X^tY)^2} Tr\begin{pmatrix} Y^tY & -X^tY \\ -X^tY & X^tX \end{pmatrix}\begin{pmatrix} XX^t & YX^t \\ XY^t & YY^t \end{pmatrix} = 1 \text{ as operator}$$

Actually, this last relation yields:

$$\begin{aligned} & \frac{1}{(1+Y^tY)(1+X^tX)-(X^tY)^2} Tr\begin{pmatrix} 1+Y^tY & -X^tY \\ -X^tY & 1+X^tX \end{pmatrix}\begin{pmatrix} XX^t & YX^t \\ XY^t & YY^t \end{pmatrix} \\ = & \frac{(Y^tY)(X^tX)-(X^tY)^2+XX^t+YY^t}{(1+Y^tY)(1+X^tX)-(X^tY)^2} \end{aligned}$$

and one is thus led to:

$$\gamma^{-1}\left(1-Tr\left((1+G)^{-1}P^t\right)\right) = \gamma^{-1}\frac{1+Y^tY+X^tX-XX^t-YY^t}{(1+Y^tY)(1+X^tX)-(X^tY)^2}$$

Ultimately, rescale the variables

$$\begin{aligned} X &= V_t \\ Y &= (V_t-\chi_t) \end{aligned}$$

so that the final result is:

$$\begin{aligned} \gamma^{-1}\left(1-Tr\left((1+G)^{-1}P^t\right)\right) &= \frac{\gamma+(\beta+\varepsilon)Y^tY+(\alpha+\delta\rho)-(\alpha+\delta\rho)XX^t-(\beta+\varepsilon)YY^t}{(\gamma+(\beta+\varepsilon)Y^tY)(\gamma+(\alpha+\delta\rho))-(\alpha+\delta\rho)(\beta+\varepsilon)(X^tY)^2} \\ &= M \end{aligned}$$

If we let

$$z = \sin^2 x$$

where

$$x = \frac{1}{2}\arg(V_t, \chi_t)$$

the scalar products Y^tY, X^tY and XX^t can be computed and give the following alternate expression for M:

$$M = \frac{\gamma + 4(\beta+\varepsilon)z + (\alpha+\delta\rho) - (\alpha+\delta\rho)XX^t - (\beta+\varepsilon)YY^t}{(\gamma+4(\beta+\varepsilon)z)(\gamma+(\alpha+\delta\rho)) - 4(\alpha+\delta\rho)(\beta+\varepsilon)(z)^2}$$

We are interested in the dynamics for ϕ_t and for $\phi_t.\chi_t$ the social value given to a certain stock of capital goods. Recall the form of the production function,

$$\begin{aligned} Y_t &= (\eta(\phi_t.\chi_t))^{1-a}\left(\frac{K_1(\xi,(\phi_t.V_t))\xi V_t + K_2(1-\xi,(\phi_t.\chi_t))(1-\xi)\chi_t}{\sqrt{1-4\xi(1-\xi)z^2}}\right) - \eta(\phi_t) \\ &\sim \eta(\phi_t.\chi_t)(K_1\xi V_t + K_2(1-\xi)\chi_t) - \eta(\phi_t) \end{aligned}$$

where K_1 and K_2 are the value of $K_1(\xi,(\phi_t.V_t))$ and $K_2(1-\xi,(\phi_t.\chi_t))$ at the optimal value of ξ, and define the following quantity:

$$\tilde{\phi}_t = \phi_t + Y_t = (1-\eta)\phi_t + \eta(\phi_t.\chi_t)(K_1\xi V_t + K_2(1-\xi)\chi_t)$$

and rewrite the vector W involved in the consumption as:

$$W = uX + vY + \gamma\phi_t$$

$$\begin{aligned} W_t &= \left(1-\rho+\delta\rho V_t^t(\phi_t+Y_t)\right)V_t + \gamma\phi_t + \varepsilon\left((V_t-\chi_t)^t(\phi_t+Y_t)\right)(V_t-\chi_t) \\ u &= 1-\rho+\delta\rho V_t^t(\phi_t+Y_t) = 1-\rho+\delta\rho X^t(\phi_t+Y_t) = 1-\rho+\delta\rho\left(X^t\tilde{\phi}_t\right) \\ &= 1-\rho+\delta\rho((1-\eta)\phi_t.V_t + \eta(\phi_t.\chi_t)(K_1\xi + (1-2z)K_2(1-\xi))) \\ v &= \varepsilon\left((V_t-\chi_t)^t(\phi_t+Y_t)\right) = \varepsilon Y^t(\phi_t+Y_t) = \varepsilon\left(Y^t\tilde{\phi}_t\right) \\ &= \varepsilon((1-\eta)\phi_t.(V_t-\chi_t) + 2z\eta(\phi_t.\chi_t)(K_1\xi - K_2(1-\xi))) \end{aligned}$$

The consumption is thus given by (5):

$$\begin{aligned} C &= MW = \frac{((\gamma+4(\beta+\varepsilon)z)u - 2(\alpha+\delta\rho)zv - (\alpha+\delta\rho)\gamma X^t\phi_t)}{(\gamma+4(\beta+\varepsilon)z)(\gamma+(\alpha+\delta\rho)) - 4(\alpha+\delta\rho)(\beta+\varepsilon)(z)^2}X \\ &+\frac{((\gamma+(\alpha+\delta\rho))v - 2(\beta+\varepsilon)zu - (\beta+\varepsilon)\gamma Y^t\phi_t)}{(\gamma+4(\beta+\varepsilon)z)(\gamma+(\alpha+\delta\rho)) - 4(\alpha+\delta\rho)(\beta+\varepsilon)(z)^2}Y \\ &+\gamma\frac{\gamma+4(\beta+\varepsilon)z+(\alpha+\delta\rho)}{(\gamma+4(\beta+\varepsilon)z)(\gamma+(\alpha+\delta\rho)) - 4(\alpha+\delta\rho)(\beta+\varepsilon)(z)^2}\phi_t \end{aligned}$$

And the dynamics for ϕ_t is deduced from the intertemporal constraint equation:

$$\begin{aligned} \phi_{t+1} &= \phi_t + Y_t - C_t \qquad (8) \\ &= \left(\eta(\phi_t.\chi_t)(K_1\xi + K_2(1-\xi)) - \frac{((\gamma+4(\beta+\varepsilon)z)u - 2\alpha zv - \alpha\gamma V_t.\phi_t)}{(\gamma+4(\beta+\varepsilon)z)(\gamma+\alpha) - 4\alpha(\beta+\varepsilon)(z)^2}\right)V_t \\ &-\left(\eta(\phi_t.\chi_t)K_2(1-\xi) + \frac{((\gamma+\alpha)v - 2(\beta+\varepsilon)zu - (\beta+\varepsilon)\gamma(V_t-\chi_t).\phi_t)}{(\gamma+4(\beta+\varepsilon)z)(\gamma+\alpha) - 4\alpha(\beta+\varepsilon)(z)^2}\right)(V_t-\chi_t) \\ &+\left((1-\eta) - \gamma\frac{\gamma+4(\beta+\varepsilon)z+\alpha}{(\gamma+4(\beta+\varepsilon)z)(\gamma+\alpha) - 4\alpha(\beta+\varepsilon)(z)^2}\right)\phi_t \end{aligned}$$

We also choose

$$\begin{aligned}
K_1\left(\xi,\left(\phi_t.V_t\right)\right) &= K_1+\left(\Phi_t.V_t\right)\hat{K}_1 \\
K_2\left(\xi,\left(\phi_t.V_t\right)\right) &= K_2+\left(\Phi_t.\chi_t\right)\hat{K}_2
\end{aligned}$$

in $K_1\left(\phi_t.\chi_t\right)\xi+K_2\left(\phi_t.\chi_t\right)(1-\xi)$, ξ depends on V_t. However, due to the envelop theorem, the production functions do not depend,to the first order, on the derivatives of ξ with respect to the other parameters. As a consequence we can, in first approximation consider ξ as a constant. It's dependence in the other parameters, V_t and χ_t is of second order only, that is $\xi=\frac{1+\varphi z}{2}$ as second order approximation, with φ a constant parameter.

Case 1

We start to solve the benchmark case $V_t=\chi_t$ so that $z=0$: The dynamics equation (8) reduces to:

$$\begin{aligned}
\phi_{t+1} &= \phi_t+Y_t-C_t \\
&= \left(\eta\left(\phi_t.\chi_t\right)\left(K_1\xi+K_2(1-\xi)\right)-\frac{\left(u-\alpha\chi_t.\phi_t\right)}{\gamma+(\alpha+\delta\rho)}\right)\chi_t-\eta\phi_t
\end{aligned}$$

and

$$\begin{aligned}
u &= 1-\rho+\delta\rho\left((1-\eta)+\eta\left(K_1\xi+(1-2z)K_2(1-\xi)\right)\right)\left(\phi_t.\chi_t\right) \\
v &= 0
\end{aligned}$$

multiplied by χ_t yields the following equation for the dynamics of the capital stock value:

$$\frac{\left\langle\Phi_{t+1}.\chi_{t+1}\right\rangle}{\sqrt{1-\sigma^2}}=\left(\eta\left(K_1-1\right)+\frac{-\delta\rho\left((1-\eta)+\eta K_1\right)+\alpha+\delta\rho}{\gamma+\alpha+\delta\rho}\right)\left\langle\Phi_t.\chi_t\right\rangle-\frac{1-\rho}{(\gamma+\alpha+\delta\rho)}+\eta\left(1-\frac{\delta\rho}{\gamma+\alpha+\delta\rho}\right)\hat{K}_1\left\langle\Phi_t.\chi_t\right\rangle^2$$

or equivalently:

$$\frac{\left\langle\Phi_{t+1}.\chi_{t+1}\right\rangle}{\sqrt{1-\sigma^2}}=\left(\left(\eta\left(K_1-1\right)+\frac{\alpha-\delta\rho\eta\left(K_1-1\right)}{\gamma+\alpha+\delta\rho}\right)\left\langle\Phi_t.\chi_t\right\rangle-\frac{1-\rho}{(\gamma+\alpha+\delta\rho)}+\eta\left(\frac{\gamma+\alpha}{\gamma+\alpha+\delta\rho}\right)\hat{K}_1\left\langle\Phi_t.\chi_t\right\rangle^2\right)$$

Let us consider the following exogenous dynamics for χ_t:

$$\chi_{t+1}=\chi_t+\delta_{t+1}$$

Normalizing the vector of values: $1=\chi_{t+1}^2=1+2\chi_t.\delta_{t+1}+\delta_{t+1}^2$ leads to the relation

$$\chi_t.\delta_{t+1}=-\frac{1}{2}\delta_{t+1}^2$$

so that δ_{t+1} is not orthogonal to χ_t. Statistically, we decompose χ_{t+1} as: $\chi_{t+1}=\chi_t+\delta_{t+1}=\chi_t+x\chi_t+\varepsilon_{t+1}$ with $\langle\varepsilon_{t+1}\rangle=0$, $\left\langle\varepsilon_{t+1}^2\right\rangle=\sigma^2$, $\langle\chi_t.\varepsilon_{t+1}\rangle=0$. The condition on the norm becomes:

$$1=\left\langle\left(\chi_t+x\chi_t+\varepsilon_{t+1}\right)^2\right\rangle=1+2x+x^2+\sigma^2$$

so that $x=-1+\sqrt{1-\sigma^2}$ and:

$$\begin{aligned}
\chi_{t+1} &= \sqrt{1-\sigma^2}\chi_t+\varepsilon_{t+1} \\
\left\langle\chi_{t+1}.\chi_t\right\rangle &= \sqrt{1-\sigma^2}=\left\langle 1-\frac{1}{2}\delta_{t+1}^2\right\rangle
\end{aligned}$$

In mean we thus have:

$$\frac{\left\langle\Phi_{t+1}.\chi_{t+1}\right\rangle}{\sqrt{1-\sigma^2}}=\left(\eta\left(K_1-1\right)+\frac{\alpha-\delta\rho\eta\left(K_1-1\right)}{\gamma+\alpha+\delta\rho}\right)\left\langle\Phi_t.\chi_t\right\rangle-\frac{1-\rho}{(\gamma+\alpha+\delta\rho)}+\eta\left(\frac{\gamma+\alpha}{\gamma+\alpha+\delta\rho}\right)\hat{K}_1\left\langle\left(\Phi_t.\chi_t\right)^2\right\rangle$$

since the expectation is taken on χ_{t+1}, at time t, $\left\langle (\Phi_t.\chi_t)^2 \right\rangle = \langle \Phi_t.\chi_t \rangle^2$, and one is led to:

$$\frac{\langle \Phi_{t+1}.\chi_{t+1} \rangle}{\sqrt{1-\sigma^2}} = \frac{(\alpha+\eta(\alpha+\gamma)(K_1-1))}{\alpha+\gamma+\delta\rho} \langle \Phi_t.\chi_t \rangle - \frac{1-\rho}{(\gamma+\alpha+\delta\rho)} + \eta\frac{\gamma+\alpha}{\gamma+\alpha+\delta\rho}\hat{K}_1 \langle \Phi_t.\chi_t \rangle^2$$

This is an equation of type:

$$\Phi_{t+1}.\chi_{t+1} = a\Phi_t.\chi_t - b + c(\Phi_t.\chi_t)^2$$

where:

$$\begin{aligned}
a &= \left(\frac{(\alpha+\eta(\alpha+\gamma)(K_1-1))}{\alpha+\gamma+\delta\rho} \right) \sqrt{1-\sigma^2} \\
b &= \frac{1-\rho}{\gamma+\alpha+\delta\rho}\sqrt{1-\sigma^2} \\
c &= \eta\frac{\gamma+\alpha}{\gamma+\alpha+\delta\rho}\sqrt{1-\sigma^2}\hat{K}_1
\end{aligned}$$

Such an equation can be solved by a continuous time approximation:

$$y' = (a-1)y - b + cy^2$$

whose solution is:

$$\Phi(t).\chi(t) = \frac{\Phi^+ - \Phi^- \frac{\Phi_0.\chi_0 - \Phi^+}{\Phi_0.\chi_0 - \Phi^-}\exp(\Delta t)}{1 - \frac{\Phi_0.\chi_0 - \Phi^+}{\Phi_0.\chi_0 - \Phi^-}\exp(\Delta t)}$$

with:

$$\begin{aligned}
\Delta &= \sqrt{(a-1)^2 + 4bc} \\
\Phi^{\pm} &= \frac{-(a-1) \pm \Delta}{2c} \\
\Phi_0.\chi_0 &= \Phi(0).\chi(0)
\end{aligned}$$

As explained in the text, the dynamics presents a threshold pattern. Since $\Phi^- < 0$,

If $\Phi_0.\chi_0 < \Phi^+$ then the dynamics converges, $\Phi(t).\chi(t) \to 0$.

If $\Phi^+ < \Phi_0.\chi_0$ then dynamics the explodes in finite time $\Phi(t).\chi(t) \to \frac{1}{a}\ln\left(\frac{\Phi^- - \Phi_0.\chi_0}{\Phi^+ - \Phi_0.\chi_0}\right)$.

If the initial social value of the stock is low, then decrease of the stock. For large enough initial value, explosive accumulation.

Case 2

To modify the solution of the previous case and take into account the heterogenity of agents with respect to the society as a whole, i.e. adding corrections due to $(V_t - \chi_t)$ we consider that V_t differs from χ_t in the following way:

$$V_t = \sqrt{1-\tilde{\sigma}^2}\chi_t + \tilde{\varepsilon}_t$$

where $\tilde{\varepsilon}_t$ is random gaussian, with mean zeo and variance $\tilde{\sigma}^2$ independant from ε_t.

We compute some expectations that are relevant to derive the dynamics for the social value of the capital stock.

$$\begin{aligned}
\langle (V_t - \chi_t).\chi_{t+1} \rangle &= \left\langle (V_t - \chi_t).\left(\sqrt{1-\sigma^2}\chi_t + \varepsilon_t\right) \right\rangle \\
&= \left\langle \left(\left(\sqrt{1-\tilde{\sigma}^2} - 1\right)\chi_t + \tilde{\varepsilon}_t\right).\left(\sqrt{1-\sigma^2}\chi_t + \varepsilon_t\right) \right\rangle \\
&= \left(\sqrt{1-\tilde{\sigma}^2} - 1\right)\sqrt{1-\sigma^2}
\end{aligned}$$

$$\begin{aligned}
\langle V_t.\chi_{t+1}\rangle &= \left\langle V_t.\left(\sqrt{1-\sigma^2}\chi_t+\varepsilon_t\right)\right\rangle \\
&= \left\langle\left(\left(\sqrt{1-\tilde{\sigma}^2}\right)\chi_t+\tilde{\varepsilon}_t\right).\left(\sqrt{1-\sigma^2}\chi_t+\varepsilon_t\right)\right\rangle \\
&= \sqrt{1-\tilde{\sigma}^2}\sqrt{1-\sigma^2}
\end{aligned}$$

and:

$$\begin{aligned}
\langle z\rangle &= \left\langle\frac{1-V_t.\chi_t}{2}\right\rangle=\frac{1-\langle V_t.\chi_t\rangle}{2} \\
&= \frac{1-\left\langle\left(\sqrt{1-\tilde{\sigma}^2}\chi_t+\tilde{\varepsilon}_t\right).\chi_t\right\rangle}{2} \\
&= \frac{1-\sqrt{\left(1-\tilde{\sigma}^2\right)}}{2}
\end{aligned}$$

We will also need the expectations of some squared terms, as $\left\langle\left(\Phi_t.V_t\right)^2\right\rangle$, $\langle\left(\Phi_t.V_t\right)\left(\chi_t.V_t\right)\rangle$, $\langle\left(\chi_t.V_t\right)\left(\chi_t.V_t\right)\rangle$ and $\langle\left(\Phi_t.V_t\right)\left(\Phi_t.\chi_{t+1}\right)\rangle$. Note that $\left\langle\left(\Phi_t.V_t\right)^2\right\rangle=\langle\left(\Phi_t.V_t\right)\left(\Phi_t.V_t\right)\rangle$. These terms involve contributions as $\sum\left\langle\left(\Phi_t\right)_i.\left(\tilde{\varepsilon}_t\right)_i\left(\Phi_t\right)_i.\left(\tilde{\varepsilon}_t\right)_j\right\rangle$ or $\sum\left\langle\left(\Phi_t\right)_i.\left(\tilde{\varepsilon}_t\right)_i\left(\Phi_t\right)_i.\left(\varepsilon_{t+1}\right)_j\right\rangle$. The second term is nul, given the independance of $\tilde{\varepsilon}_t$ and ε_{t+1}. The other terms involve contributions like $\left\langle\left(\phi_t.\tilde{\varepsilon}_t\right)^2\right\rangle$, $\langle\left(\phi_t.\tilde{\varepsilon}_t\right)\left(\chi_t.\tilde{\varepsilon}_t\right)\rangle$, $\left\langle\left(\chi_t.\tilde{\varepsilon}_t\right)^2\right\rangle$.

Let us consider the first term only, the reasonning is similar for the others.

Since we are working with vectors with large number of components, and given that the random terms $(\tilde{\varepsilon}_t)$ and (ε_{t+1}) are gaussian and isotropic, the expectations can be computed to get: $\sum\left\langle\left(\Phi_t\right)_i\left(\tilde{\varepsilon}_t\right)_i\left(\Phi_t\right)_i\left(\tilde{\varepsilon}_t\right)_j\right\rangle=\sum\left(\Phi_t\right)_i\left(\Phi_t\right)_i\left\langle\left(\tilde{\varepsilon}_t\right)_i\left(\tilde{\varepsilon}_t\right)_j\right\rangle=\sum\left(\Phi_t\right)_i\left(\Phi_t\right)_i\frac{\tilde{\sigma}^2}{N}=\frac{\phi_t.\phi_t}{N}\tilde{\sigma}^2$.

As a consequence, all variance terms arising in the mean of squared values are of order $\frac{1}{N}$ and arenegligible.

We will thus write $\left(\Phi_t.V_t\right)^2=\langle\Phi_t.V_t\rangle^2$ and the same for other quantities.

For the same reasons, and for the sake of simplicity, in the sequel we will implicitely understand z, everywhere it appears, as it's mean value, that is $\frac{1-\sqrt{(1-\tilde{\sigma}^2)}}{2}$.

We will also assume $K_1=K_2$, $\hat{K}_1=\hat{K}_2$. We also write $\xi=\frac{1}{2}\left(1+\varphi z\right)$ (the optimal choice is to produce toward V_t, since the agent has a comparative advantage in this direction. This choice of parametrization is justified by the envelop th. ensuring that ξ should be of second order in $V_t-\chi_t$. As shown before the correction is actually proportional to z, and in first approximation it's value depends on the derivatives of the productivity that are incuded in the factor φ).

$$\begin{aligned}
\left(K_1\xi+K_2\left(1-\xi\right)\right) &= K_1+\frac{1}{2}\left(1+\sqrt{1-\tilde{\sigma}^2}+\varphi z\left(\sqrt{1-\tilde{\sigma}^2}-1\right)\right)\hat{K}_1\left(\phi_t.\chi_t\right)+\frac{1}{2}\left(1+\varphi z\right)\hat{K}_1\left(\phi_t.\tilde{\varepsilon}_t\right) \\
\left(K_1\xi-K_2\left(1-\xi\right)\right) &= \varphi zK_1+\frac{1}{2}\hat{K}_1\left(\sqrt{1-\tilde{\sigma}^2}-1+\varphi z\left(1+\sqrt{1-\tilde{\sigma}^2}\right)\left(\phi_t.\chi_t\right)\right)+\frac{1}{2}\left(1+\varphi z\right)\hat{K}_1\left(\phi_t.\tilde{\varepsilon}_t\right)
\end{aligned}$$

$$\begin{aligned}
\langle\left(K_1\xi+K_2\left(1-\xi\right)\right)\rangle &= K_1+\frac{1}{2}\left(1+\sqrt{1-\tilde{\sigma}^2}+\varphi z\left(\sqrt{1-\tilde{\sigma}^2}-1\right)\right)\hat{K}_1\left(\phi_t.\chi_t\right) \\
\langle\left(K_1\xi-K_2\left(1-\xi\right)\right)\rangle &= \varphi zK_1+\frac{1}{2}\hat{K}_1\left(\sqrt{1-\tilde{\sigma}^2}-1+\varphi z\left(1+\sqrt{1-\tilde{\sigma}^2}\right)\right)\left(\phi_t.\chi_t\right)
\end{aligned}$$

$$\begin{aligned}
\langle K_1\xi\rangle &= \frac{1+\varphi z}{2}\left(K_1+\hat{K}_1\left(\phi_t.\chi_t\right)\sqrt{1-\tilde{\sigma}^2}\right) \\
\langle K_2\xi\rangle &= \frac{1-\varphi z}{2}\left(K_1+\hat{K}_1\left(\phi_t.\chi_t\right)\right)
\end{aligned}$$

One is lead to the following dynamic for $\langle \phi_t . \chi_t \rangle$:

$$\begin{aligned}\left\langle \phi_{t+1} . \chi_{t+1} \right\rangle &= \eta \left\langle \phi_t . \chi_t \right\rangle \left\langle K_1 \xi + K_2 \left(1-\xi\right) \right\rangle \left\langle V_t . \chi_{t+1} \right\rangle \\ &-\frac{\left(\left(\gamma + 4\left(\beta+\varepsilon\right) z\right) \langle u \rangle - 2\left(\alpha + \rho\delta\right) z \langle v \rangle - \left(\alpha+\rho\delta\right)\gamma \left\langle V_t . \phi_t \right\rangle\right)}{\left(\gamma + 4\left(\beta+\varepsilon\right) z\right)\left(\gamma+\alpha+\rho\delta\right) - 4\left(\alpha+\rho\delta\right)\left(\beta+\varepsilon\right)\left(z\right)^2} \left\langle V_t . \chi_{t+1} \right\rangle \\ &-\eta \left\langle \phi_t . \chi_t \right\rangle \left\langle K_2 \left(1-\xi\right) \right\rangle \left\langle \left(V_t - \chi_t\right) . \chi_{t+1} \right\rangle \\ &-\frac{\left(\left(\gamma+\alpha+\rho\delta\right)\langle v \rangle - 2\left(\beta+\varepsilon\right) z \langle u \rangle - \left(\beta+\varepsilon\right)\gamma \left\langle \left(V_t - \chi_t\right) . \phi_t \right\rangle\right)}{\left(\gamma + 4\left(\beta+\varepsilon\right) z\right)\left(\gamma+\alpha+\rho\delta\right) - 4\left(\alpha+\rho\delta\right)\left(\beta+\varepsilon\right)\left(z\right)^2} \left\langle \left(V_t - \chi_t\right) . \chi_{t+1} \right\rangle \\ &+\left(\left(1-\eta\right) - \gamma \frac{\gamma + 4\left(\beta+\varepsilon\right) z + \alpha + \rho\delta}{\left(\gamma + 4\left(\beta+\varepsilon\right) z\right)\left(\gamma+\alpha+\rho\delta\right) - 4\left(\alpha+\rho\delta\right)\left(\beta+\varepsilon\right)\left(z\right)^2}\right) \left\langle \phi_t . \chi_{t+1} \right\rangle\end{aligned}$$

In the sequel, we will consider the simplification $\gamma = 0$. It makes the computation more tractable and will not impair the arguments. The previous equation reduces to:

$$\begin{aligned}\left\langle \phi_{t+1} . \chi_{t+1} \right\rangle &= \left(\eta \left\langle \phi_t . \chi_t \right\rangle \left\langle K_1 \xi + K_2 \left(1-\xi\right) \right\rangle - \frac{4\left(\beta+\varepsilon\right) z \langle u \rangle - 2\left(\alpha+\rho\delta\right) z \langle v \rangle}{\left(4\left(\beta+\varepsilon\right) z\right)\left(\alpha+\rho\delta\right) - 4\left(\alpha+\rho\delta\right)\left(\beta+\varepsilon\right)\left(z\right)^2}\right) \left\langle V_t . \chi_{t+1} \right\rangle \\ &-\left(\eta \left\langle \phi_t . \chi_t \right\rangle \left\langle K_2 \left(1-\xi\right) \right\rangle + \frac{\left(\left(\alpha+\rho\delta\right)\langle v \rangle - 2\left(\beta+\varepsilon\right) z \langle u \rangle\right)}{4\left(\beta+\varepsilon\right) z \left(\alpha+\rho\delta\right) - 4\left(\alpha+\rho\delta\right)\left(\beta+\varepsilon\right)\left(z\right)^2}\right) \left\langle \left(V_t - \chi_t\right) . \chi_{t+1} \right\rangle \\ &+\left(1-\eta\right) \left\langle \phi_t . \chi_{t+1} \right\rangle\end{aligned}$$

Now, compute some relevant quantities given the chosen assumptions.

$$\begin{aligned}\langle K_1 \xi \rangle &= \frac{1+\varphi z}{2}\left(K_1 + \hat{K}_1 \left(\phi_t . \chi_t\right) \sqrt{1-\tilde{\sigma}^2}\right) \\ \langle K_2 \xi \rangle &= \frac{1-\varphi z}{2}\left(K_1 + \hat{K}_1 \left(\phi_t . \chi_t\right)\right)\end{aligned}$$

$$\begin{aligned}\left\langle \sqrt{1-\tilde{\sigma}^2} K_1 \xi + K_2 \left(1-\xi\right) \right\rangle &= \frac{\sqrt{1-\tilde{\sigma}^2}\left(1+\varphi z\right) + \left(1-\varphi z\right)}{2} K_1 + \frac{\left(1-\tilde{\sigma}^2\right)\left(1+\varphi z\right) + \left(1-\varphi z\right)}{2} \hat{K}_1 \left(\phi_t . \chi_t\right) \\ &= \frac{\left(1-2z\right)\left(1+\varphi z\right) + \left(1-\varphi z\right)}{2} K_1 + \frac{\left(1-2z\right)^2\left(1+\varphi z\right) + \left(1-\varphi z\right)}{2} \hat{K}_1 \left(\phi_t . \chi_t\right)\end{aligned}$$

$$\left\langle \left(V_t - \chi_t\right) . \chi_{t+1} \right\rangle = \left(\sqrt{1-\tilde{\sigma}^2} - 1\right) \sqrt{1-\sigma^2}$$

$$\left\langle V_t . \chi_{t+1} \right\rangle = \sqrt{1-\tilde{\sigma}^2}\sqrt{1-\sigma^2}$$

$$\left\langle \phi_t . \chi_{t+1} \right\rangle = \left\langle \phi_t . \chi_t \right\rangle \sqrt{1-\sigma^2}$$

which allows to rewrite:

$$\begin{aligned}\frac{\left\langle \phi_{t+1} . \chi_{t+1} \right\rangle}{\sqrt{1-\sigma^2}} &= \eta \left(\phi_t . \chi_t\right) \left\langle \sqrt{1-\tilde{\sigma}^2} K_1 \xi + K_2 \left(1-\xi\right) \right\rangle \\ &-\frac{2z\left(\beta+\varepsilon\right)\left(1+\sqrt{1-\tilde{\sigma}^2}\right)\langle u \rangle + \left(\left(1-2z\right)\sqrt{1-\tilde{\sigma}^2} - 1\right)\alpha \langle v \rangle}{4z\alpha\left(\beta+\varepsilon\right)\left(1-z\right)} \\ &+\left(1-\eta\right)\left\langle \phi_t . \chi_t \right\rangle\end{aligned}$$

The relevant quantities $\langle u\rangle$ and $\langle v\rangle$ can be expanded as:

$$\begin{aligned}
\langle u\rangle &= 1-\rho \\
&\quad +\rho\delta\left(\left(1-\eta\right)\left(1-2z\right)+\eta\left(\tau\frac{1+\varphi z}{2}\left(K_1+\hat{K}_1\left(\phi_t{\cdot}\chi_t\right)\left(1-2z\right)\right)+\left(1-2z\right)\frac{1-\varphi z}{2}\left(K_1+\hat{K}_1\left(\phi_t{\cdot}\chi_t\right)\right)\right)\right)\left(\phi_t{\cdot}\chi_t\right) \\
&= 1-\rho \\
&\quad +\rho\delta\left(1-\eta\right)\left(1-2z\right)\left(\phi_t{\cdot}\chi_t\right) \\
&\quad +\eta\rho\delta\left(\frac{K_1}{2}\left(-2z+\tau-z\varphi+2z^2\varphi+z\tau\varphi+1\right)+\frac{\left(1-2z\right)}{2}\left(\tau-z\varphi+z\tau\varphi+1\right)\hat{K}_1\left(\phi_t{\cdot}\chi_t\right)\right)\left(\phi_t{\cdot}\chi_t\right) \\
&= 1-\rho \\
&\quad +\rho\delta\left(\left(1-\eta\right)\left(1-2z\right)+\eta\left(\frac{K_1}{2}\left(-2z+\tau-z\varphi+2z^2\varphi+z\tau\varphi+1\right)\right)\right)\left(\phi_t{\cdot}\chi_t\right) \\
&\quad +\rho\delta\left(\frac{\left(1-2z\right)}{2}\left(\tau-z\varphi+z\tau\varphi+1\right)\hat{K}_1\left(\phi_t{\cdot}\chi_t\right)^2\right)
\end{aligned}$$

$$\begin{aligned}
\langle v\rangle &= \varepsilon\left(\left(1-\eta\right)\phi_t.\left(V_t-\chi_t\right)+2z\eta\left(\phi_t{\cdot}\chi_t\right)\left(K_1\xi-K_2\left(1-\xi\right)\right)\right) \\
&= \varepsilon 2z\left\langle-\left(1-\eta\right)+\eta\left(\frac{1+\varphi z}{2}\tau\left(K_1+\left(1-2z\right)\hat{K}_1\left(\phi_t{\cdot}\chi_t\right)\right)-\frac{1-\varphi z}{2}\left(K_1+\hat{K}_1\left(\phi_t{\cdot}\chi_t\right)\right)\right)\right\rangle\left(\phi_t{\cdot}\chi_t\right)
\end{aligned}$$

And the following usefull combination of these terms are then:

$$\begin{aligned}
&-\frac{2z\left(\beta+\varepsilon\right)\left(1+\sqrt{1-\tilde{\sigma}^2}\right)\langle u\rangle+\left(\left(1-2z\right)\sqrt{1-\tilde{\sigma}^2}-1\right)\left(\alpha+\rho\delta\right)\langle v\rangle}{4z\left(\alpha+\rho\delta\right)\left(\beta+\varepsilon\right)\left(1-z\right)} \\
&= -\frac{4z\left(\beta+\varepsilon\right)\left(1-z\right)\langle u\rangle+\left(\left(1-2z\right)^2-1\right)\left(\alpha+\rho\delta\right)\langle v\rangle}{4z\left(\alpha+\rho\delta\right)\left(\beta+\varepsilon\right)\left(1-z\right)} \\
&= -\frac{4z\left(\beta+\varepsilon\right)\left(1-z\right)\langle u\rangle-4z\left(1-z\right)\left(\alpha+\rho\delta\right)\langle v\rangle}{4z\left(\alpha+\rho\delta\right)\left(\beta+\varepsilon\right)\left(1-z\right)} \\
&= -\frac{\left(\beta+\varepsilon\right)\langle u\rangle-\left(\alpha+\rho\delta\right)\langle v\rangle}{\left(\alpha+\rho\delta\right)\left(\beta+\varepsilon\right)}
\end{aligned}$$

$$\begin{aligned}
&\left(\beta+\varepsilon\right)\langle u\rangle-\alpha\langle v\rangle \\
&= \left(\beta+\varepsilon\right)\left(1-\rho+\delta\left(\left(1-\eta\right)\left(1-2z\right)+\eta\left(K_1\left(1-z+z^2\varphi\right)\right)\right)\left(\phi_t{\cdot}\chi_t\right)+\delta\eta\left(\left(1-2z\right)\hat{K}_1\left(\phi_t{\cdot}\chi_t\right)^2\right)\right) \\
&\quad -\alpha 2z\varepsilon\left(-\left(1-\eta\right)+\eta\left(\varphi zK_1+z\left(\varphi-z\varphi-1\right)\hat{K}_1\left(\phi_t{\cdot}\chi_t\right)\right)\right)\left(\phi_t{\cdot}\chi_t\right) \\
&= \left(\beta+\varepsilon\right)\left(1-\rho\right)+\left(\left(\beta+\varepsilon\right)\left(\delta\left(\left(1-\eta\right)\left(1-2z\right)+\eta\left(K_1\left(1-z+z^2\varphi\right)\right)\right)\right)-\alpha 2z\varepsilon\left(-\left(1-\eta\right)+\eta\varphi zK_1\right)\right)\left(\phi_t{\cdot}\chi_t\right) \\
&\quad +\left(\left(\beta+\varepsilon\right)\left(\delta\eta\left(\left(1-2z\right)\right)\right)-\alpha 2z\varepsilon\left(\eta\left(z\left(\varphi-z\varphi-1\right)\hat{K}_1\right)\right)\right)\hat{K}_1\left(\phi_t{\cdot}\chi_t\right)^2
\end{aligned}$$

So that the dynamics in it's expanded form is thus:

$$\begin{aligned}
\frac{\langle\phi_{t+1}{\cdot}\chi_{t+1}\rangle}{\eta\sqrt{1-\sigma^2}} &= \left(\frac{\left(1-2z\right)\tau\left(1+\varphi z\right)+\left(1-\varphi z\right)}{2}K_1\right)\langle\phi_t{\cdot}\chi_t\rangle \\
&\quad -\left(\frac{\left(\delta\rho\left(\frac{(1-\eta)(1-2z)}{\eta}+\frac{K_1}{2}\left(-2z+\tau-z\varphi+2z^2\varphi+z\tau\varphi+1\right)\right)\right)}{\alpha+\delta\rho}-\frac{\left(1-\eta\right)}{\eta}\right)\langle\phi_t{\cdot}\chi_t\rangle \\
&\quad +\left(\frac{\left(1-2z\right)^2\tau\left(1+\varphi z\right)+\left(1-\varphi z\right)}{2}-\frac{\delta\rho\left(1-2z\right)\left(\tau-z\varphi+z\tau\varphi+1\right)}{2\left(\alpha+\delta\rho\right)}\right)\hat{K}_1\langle\phi_t{\cdot}\chi_t\rangle^2 \\
&\quad -\frac{\left(1-\rho\right)}{\left(\alpha+\delta\rho\right)\eta}
\end{aligned}$$

Reordering the various terms and applying the same treatment as for case 1 leads directly to the results presented in text.

Case 3

The dynamical system describing the evolution of V_t and χ_t is now:

$$\begin{aligned} V_t &= \sqrt{1-\tilde{\sigma}^2}\chi_t + \tilde{\varepsilon}_t \\ \chi_{t+1} &= \sqrt{1-\sigma^2}V_t + \varepsilon_t \end{aligned}$$

as explained in the text, it describes the evolution in social values by a group of precursor agents. Note that the second equation can be rewritten as:

$$\chi_{t+1} = \sqrt{\left(1-\sigma^2\right)\left(1-\tilde{\sigma}^2\right)}\chi_t + \varepsilon_t + \sqrt{1-\sigma^2}\tilde{\varepsilon}_t$$

By the same techniques described in the previous case, we find the dynamics for the stock of capital goods:

$$\begin{aligned} \phi_{t+1} &= \eta\left(\phi_t.\chi_t\right) K_1 \xi \left(1 - \frac{\rho\delta}{2\left(1-z\right)\left(\alpha+\rho\delta\right)}\right) V_t \\ &\quad - \left(\frac{1-\rho+\rho\delta\left(1-\eta\right)\phi_t.V_t}{2\left(1-z\right)\left(\alpha+\rho\delta\right)} + \frac{\rho\delta\eta\left(\phi_t.\chi_t\right)\left(\left(1-2z\right)K_2\left(1-\xi\right)\right)}{2\left(1-z\right)\left(\alpha+\rho\delta\right)}\right) V_t \\ &\quad + \eta\left(\phi_t.\chi_t\right) K_2\left(1-\xi\right)\left(1 - \frac{\rho\delta\left(1-2z\right)}{2\left(1-z\right)\left(\alpha+\rho\delta\right)}\right)\chi_t \\ &\quad - \left(\frac{1-\rho+\rho\delta\left(1-\eta\right)\phi_t.V_t}{2\left(1-z\right)\left(\alpha+\rho\delta\right)} + \frac{\rho\delta\eta\left(\phi_t.\chi_t\right)K_1\xi}{2\left(1-z\right)\left(\alpha+\rho\delta\right)}\right)\chi_t + \left(1-\eta\right)\phi_t \\ &= \eta\left(\phi_t.\chi_t\right)\left(\frac{1+\varphi z}{2}\right)\left(K_1 + \hat{K}_1\phi_t.V_t\right)\left(1 - \frac{\rho\delta}{2\left(1-z\right)\left(\alpha+\rho\delta\right)}\right) V_t \\ &\quad - \left(\frac{1-\rho+\rho\delta\left(1-\eta\right)\phi_t.V_t}{2\left(1-z\right)\left(\alpha+\rho\delta\right)} + \frac{\rho\delta\eta\left(\phi_t.\chi_t\right)\left(\left(1-2z\right)\left(\frac{1-\varphi z}{2}\right)\left(K_2+\hat{K}_2\phi_t.\chi_t\right)\right)}{2\left(1-z\right)\left(\alpha+\rho\delta\right)}\right) V_t \\ &\quad + \eta\left(\phi_t.\chi_t\right)\left(\frac{1-\varphi z}{2}\right)\left(K_2+\hat{K}_2\phi_t.\chi_t\right)\left(1 - \frac{\rho\delta\left(1-2z\right)}{2\left(1-z\right)\left(\alpha+\rho\delta\right)}\right)\chi_t \\ &\quad - \left(\frac{1-\rho+\rho\delta\left(1-\eta\right)\phi_t.V_t}{2\left(1-z\right)\left(\alpha+\rho\delta\right)} + \frac{\rho\delta\eta\left(\phi_t.\chi_t\right)\left(\frac{1+\varphi z}{2}\right)\left(K_1+\hat{K}_1\phi_t.V_t\right)}{2\left(1-z\right)\left(\alpha+\rho\delta\right)}\right)\chi_t + \left(1-\eta\right)\phi_t \end{aligned}$$

Define $y_t = \phi_t.\chi_t$ and set $\tau = 1$, so that we will focus only on the precursor effect, that is the consequence of endogenizing χ_t, disgarding the influence of an advantage in productivity for V_t. We also normalize the coefficients to $\alpha = 1$, $\delta = 1$, $K_1 = K_2 = 1$, $\hat{K}_1 = \hat{K}_2$.

One is led to:

$$\frac{y_{t+1}}{\sqrt{1-\sigma^2}} = -\frac{1-\rho}{\rho+1} + \frac{z\eta - 2z + z^2\eta\varphi + 1}{\rho+1}y_t + \frac{\left(1-2z\right)\eta\hat{K}_1}{\rho+1}y_t^2$$

Once again, the dynamics presents a threshold pattern, with

$$\Phi_+ = \frac{-\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right) - \frac{1+\rho}{\sqrt{1-\sigma^2}}\right) + \sqrt{\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right) - \frac{1+\rho}{\sqrt{1-\sigma^2}}\right)^2 + 4\left(1-\rho\right)\left(1-2z\right)\eta\hat{K}_1}}{2\left(1-2z\right)\eta\hat{K}_1}$$

The variations of the threshold with respect to some of the parameters are computed straightforwardly:

$$\begin{aligned}
\frac{\partial\Phi_+}{\partial\sigma^2} &> 0\\
\frac{\partial\Phi_+}{\partial\hat{K}_1} &< 0\\
\frac{\partial\Phi_+}{\partial\left(1+z\varphi\right)} &< 0\\
\frac{\partial\Phi_+}{\partial\eta} &= -\frac{z\left(1+z\varphi\right)}{2\left(1-2z\right)\eta\hat{K}_1}\left(1-\frac{\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right)-\frac{1+\rho}{\sqrt{1-\sigma^2}}\right)}{\sqrt{\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right)-\frac{1+\rho}{\sqrt{1-\sigma^2}}\right)^2+4\left(1-\rho\right)\left(1-2z\right)\eta\hat{K}_1}}\right)\\
&\quad+\frac{\hat{K}_1}{\eta}\frac{\partial\Phi_+}{\partial\hat{K}_1}
\end{aligned}$$

and thus:

$$\frac{\partial\Phi_+}{\partial\eta}<0$$

Concerning $\frac{\partial\Phi_+}{\partial\rho}$, remark that

$$2\left(1-2z\right)\eta\hat{K}_1\frac{\partial\Phi_+}{\partial\rho}=\frac{1}{\sqrt{1-\sigma^2}}+\frac{-\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right)-\frac{1+\rho}{\sqrt{1-\sigma^2}}\right)-2\sqrt{1-\sigma^2}\left(1-2z\right)\eta\hat{K}_1}{\sqrt{1-\sigma^2}\sqrt{\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right)-\frac{1+\rho}{\sqrt{1-\sigma^2}}\right)^2+4\left(1-\rho\right)\left(1-2z\right)\eta\hat{K}_1}}$$

As a consequence, $\frac{\partial\Phi_+}{\partial\rho}>0$ if

$$\begin{aligned}
0 &< \sqrt{\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right)-\frac{1+\rho}{\sqrt{1-\sigma^2}}\right)^2+4\left(1-\rho\right)\left(1-2z\right)\eta\hat{K}_1}\\
&\quad-\left(\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right)-\frac{1+\rho}{\sqrt{1-\sigma^2}}\right)+2\sqrt{1-\sigma^2}\left(1-2z\right)\eta\hat{K}_1\right)
\end{aligned}$$

Given that,

$$\begin{aligned}
&\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right)-\frac{1+\rho}{\sqrt{1-\sigma^2}}\right)^2+4\left(1-\rho\right)\left(1-2z\right)\eta\hat{K}_1\\
&-\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right)-\frac{1+\rho}{\sqrt{1-\sigma^2}}+2\sqrt{1-\sigma^2}\left(1-2z\right)\eta\hat{K}_1\right)^2\\
=\; &\left(1-\rho\right)-\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right)-\frac{1+\rho}{\sqrt{1-\sigma^2}}\right)\sqrt{1-\sigma^2}-\left(1-\sigma^2\right)\left(1-2z\right)\eta\hat{K}_1\\
=\; &2-\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right)\right)\sqrt{1-\sigma^2}-\left(1-\sigma^2\right)\left(1-2z\right)\eta\hat{K}_1
\end{aligned}$$

one obtains:

$$\begin{aligned}
\frac{\partial\Phi_+}{\partial\rho} &< 0 \text{ if } \eta\hat{K}_1>\frac{2-\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right)\right)\sqrt{1-\sigma^2}}{1-\sigma^2}\\
\frac{\partial\Phi_+}{\partial\rho} &> 0 \text{ otherwise.}
\end{aligned}$$

For the last parameter, z, let us call

$$a=\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right)-\frac{1+\rho}{\sqrt{1-\sigma^2}}\right)$$

and note that:

$$\frac{\partial a}{\partial z} = \eta\left(1 + z\varphi\right) - 2 < 0$$

We can thus write:

$$\begin{aligned}
\frac{\partial \Phi_+}{\partial z} &= \frac{1}{2\left(1-2z\right)\eta \hat{K}_1}\frac{\partial a}{\partial z}\left(-1 + \frac{a}{\sqrt{\left(\left(z\eta\left(1+z\varphi\right)+1-2z\right) - \frac{1+\rho}{\sqrt{1-\sigma^2}}\right)^2 + 4\left(1-\rho\right)\left(1-2z\right)\eta \hat{K}_1}}\right) \\
&\quad -2\frac{\hat{K}_1}{\left(1-2z\right)}\frac{\partial \Phi_+}{\partial \hat{K}_1}
\end{aligned}$$

so that:

$$\frac{\partial \Phi_+}{\partial z} > 0$$

Case 4

In that case, we consider again the normalization $\alpha = 1$, $\delta = 1$ and we assume that each agent produces only it's own good, that is, his production is proportionnal to V_t. In other word, we assume that the productivity of agent i is $K_i = K_i + \hat{K}_i\left(\phi_t.V_t\right)$. We also assume the normalization $K_i = 1$, that is $K_i = 1 + \hat{K}_i\left(\phi_t.V_t\right)$, and $\hat{K}_1 = \hat{K}_2$.

Given these assumptions, the dynamics for ϕ_t of any group is then:

$$\begin{aligned}
\phi_{t+1} &= \left(\eta\left(\phi_t.\chi_t\right)\left(1+\hat{K}_1\left(\phi_t.V_t\right)\right)\right)V_t \\
&\quad -\left(\frac{1-\rho+\delta\rho\left(\left(1-\eta\right)\phi_t.V_t + \eta\left(\phi_t.\chi_t\right)\left(1+\hat{K}_1\left(\phi_t.V_t\right)\right)\right)}{\left(1+\rho\right)\left(1-z\right)}\right)\frac{\chi_t+V_t}{2} \\
&\quad +\left(1-\eta\right)\phi_t
\end{aligned}$$

or, setting $y_t = \left(\phi_t.\chi_t\right)$

$$\phi_{t+1} = \left(\eta y_t\left(1+\hat{K}_1\left(\phi_t.V_t\right)\right)\right)V_t - \left(\frac{1-\rho+\delta\rho\left(\left(1-\eta\right)\phi_t.V_t + \eta y_t\left(1+\hat{K}_1\left(\phi_t.V_t\right)\right)\right)}{\left(1+\rho\right)\left(1-z\right)}\right)\frac{\chi_t+V_t}{2} + \left(1-\eta\right)\phi_t$$

Now use the hypothesis described in the text. The values $V_t^{(i)}$ are assumed, for the sake of simplicity to be constant and orthogonals.

$$\begin{aligned}
V_t^{(1)} &= V_0^{(1)} \\
V_t^{(2)} &= V_0^{(2)} \\
V_t^{(1)}.V_t^{(2)} &= V_0^{(1)}.V_0^{(2)} = 0
\end{aligned}$$

The social value at time $t+1$ is a weighted sum of the 2 groups values, the weight beeing given by the relative social value of the stock of each group at time t. In other words, the social value evolves endogeneously according to the relative evolution of the two groups.

$$\begin{aligned}
\chi_{t+1} &= \frac{\left(\phi_t^{(1)}.\chi_t\right)V_t^{(1)} + \left(\phi_t^{(2)}.\chi_t\right)V_t^{(2)}}{\sqrt{\left(\phi_t^{(1)}.\chi_t\right)^2 + \left(\phi_t^{(2)}.\chi_t\right)^2 + 2\left(\phi_t^{(1)}.\chi_t\right)\left(\phi_t^{(2)}.\chi_t\right)V_t^{(1)}.V_t^{(2)}}} \qquad (9) \\
&= \frac{y_t^{(1)}V_0^{(1)} + y_t^{(2)}V_0^{(2)}}{\sqrt{\left(y_t^{(1)}\right)^2 + \left(y_t^{(2)}\right)^2 + 2\left(y_t^{(1)}\right)\left(y_t^{(2)}\right)w}}
\end{aligned}$$

As before, we define, for each group the measure of the angle between χ_t and $V_0^{(i)}$ as:

$$1 - 2z^{(i)} = \chi_t . V_0^{(i)}$$

Inserting those notations yields the dynamics for $\phi_t^{(i)}$ and $y_t^{(i)}$.

$$\begin{aligned}
\phi_{t+1}^{(i)} &= \eta y_t^{(i)} \left(1 + \hat{K}_1 \left(\phi_t^{(i)} . V_0^{(i)}\right)\right) V_0^{(i)} \\
&\quad - \left(\frac{1 - \rho + \delta\rho \left((1-\eta)\, \phi_t^{(i)} . V_0^{(i)} + \eta y_t^{(i)} \left(1 + \hat{K}_1 \left(\phi_t^{(i)} . V_0^{(i)}\right)\right)\right)}{(1+\rho)\left(1 - z^{(i)}\right)}\right) \frac{\chi_t + V_0^{(i)}}{2} \\
&\quad + (1-\eta)\, \phi_t^{(i)}
\end{aligned}$$

$$\begin{aligned}
y_{t+1}^{(i)} &= \eta y_t^{(i)} \left(1 + \hat{K}_1 \left(\phi_t^{(i)} . V_0^{(i)}\right)\right) \frac{y_t^{(i)}}{\sqrt{\left(y_t^{(1)}\right)^2 + \left(y_t^{(2)}\right)^2}} \\
&\quad - \left(\frac{1 - \rho - \delta \left((1-\eta)\, \phi_t^{(i)} . V_0^{(i)} + \eta y_t^{(i)} \left(1 + \hat{K}_1 \left(\phi_t^{(i)} . V_0^{(i)}\right)\right)\right)}{1 - z^{(i)}}\right) \frac{y_t^{(i)} \left(1 - z^{(i)}\right) + y_t^{(3-i)} \left(\frac{1 - 2z^{(3-i)}}{2}\right)}{\sqrt{\left(y_t^{(1)}\right)^2 + \left(y_t^{(2)}\right)^2}} \\
&\quad + (1-\eta) \frac{y_t^{(i)} \phi_t^{(i)} . V_0^{(i)} + y_t^{(3-i)} \phi_t^{(i)} . V_0^{(3-i)}}{\sqrt{\left(y_t^{(1)}\right)^2 + \left(y_t^{(2)}\right)^2}}
\end{aligned}$$

However, we will not describe the dynamics in terms of $y_t^{(i)}$. We will rather focus on the dynamics of two other variables, and deduce the evolution of $y_t^{(i)}$ from these equations.

To do so, we will need the evolution of the personnal value of the stock of agent i:

$$\begin{aligned}
\phi_{t+1}^{(i)} . V_0^{(i)} &= \eta y_t^{(i)} \left(1 + \hat{K}_1 \left(\phi_t^{(i)} . V_0^{(i)}\right)\right) \\
&\quad - \left(\frac{1 - \rho + \delta\rho \left((1-\eta)\, \phi_t^{(i)} . V_0^{(i)} + \eta y_t^{(i)} \left(1 + \hat{K}_1 \left(\phi_t^{(i)} . V_0^{(i)}\right)\right)\right)}{(1+\rho)\left(1 - z^{(i)}\right)}\right) \frac{z^{(i)} + 1}{2} \\
&\quad + (1-\eta)\, \phi_t^{(i)} . V_0^{(i)}
\end{aligned}$$

as well as the evolution of the value attributed by agent $3-i$ to the stock of agent i:

$$\phi_{t+1}^{(i)} . V_0^{(3-i)} = - \left(\frac{1 - \rho + \delta\rho \left((1-\eta)\, \phi_t^{(i)} . V_0^{(i)} + \eta y_t^{(i)} \left(1 + \hat{K}_1 \left(\phi_t^{(i)} . V_0^{(i)}\right)\right)\right)}{(1+\rho)\left(1 - z^{(i)}\right)}\right) \frac{z^{(3-i)}}{2} + (1-\eta)\, \phi_t^{(i)} . V_0^{(3-i)}$$

We will denote these expressions by a letter x, that is:

$$\begin{aligned}
x_t^{(i,i)} &= \phi_t^{(i)} . V_0^{(i)} \\
x_t^{(i,3-i)} &= \phi_t^{(i)} . V_0^{(3-i)}
\end{aligned}$$

We thus have the relations

$$\phi_{t+1}^{(i)} . \chi_{t+1} = \frac{\left(\phi_t^{(i)} . \chi_t\right) \phi_{t+1}^{(i)} . V_0^{(i)} + \left(\phi_t^{(3-i)} . \chi_t\right) \phi_{t+1}^{(i)} . V_0^{(3-i)}}{\sqrt{\left(\phi_t^{(1)} . \chi_t\right)^2 + \left(\phi_t^{(2)} . \chi_t\right)^2}}$$

or,

$$y_{t+1}^{(i)} = \frac{\sum_j y_t^{(j)} x_{t+1}^{(i,j)}}{\sqrt{\sum_j y_t^{(j)} y_t^{(j)}}} \tag{10}$$

which expresses $y_{t+1}^{(i)}$ as a function of the $x_{t+1}^{(i,j)}$. The dynamics system will then rather be expressed in terms of the variables $x_t^{(i,j)}$, where i and j run from 1 to 2. It is straightforward to get:

$$\begin{aligned} x_{t+1}^{(i,j)} &= \eta y_t^{(i)} \left(1 + \hat{K}_1 x_t^{(i,i)}\right) \delta_{i,j} \\ &- \left(\frac{1 - \rho + \delta\rho \left((1-\eta) x_t^{(i,i)} + \eta y_t^{(i)} \left(1 + \hat{K}_1 x_t^{(i,i)}\right)\right)}{(1+\rho)\left(1 - z_t^{(i)}\right)} \right) \frac{z_t^{(j)} + \delta_{i,j}}{2} + (1-\eta) x_t^{(i,j)} \end{aligned} \tag{11}$$

where $\delta_{i,j}$ is the Kronecker symbol, and $y_{t+1}^{(i)}$ is deduced from (yi).

These equations have to be supplemented with the dynamics for the variables $z_{t+1}^{(i)}$. They are derived from the definition of (9)

$$1 - 2z_{t+1}^{(i)} = \chi_{t+1}.V_0^{(i)} = \frac{y_t^{(i)}}{\sqrt{\sum_j y_t^{(j)} y_t^{(j)}}} \tag{12}$$

We are first interested in the equilibrium of the system. We only look for feasible equilibria, i.e., equilibria such that $y^{(i)}$, $x^{(i,j)}$, and $z^{(i)}$ have positive values. It corresponds to an equilibrium for which the equilibrium social value is located between $V_0^{(1)}$ and $V_0^{(2)}$.

Proposition 1 *For generic values of the parameters, there is no equilibrium for the system given by the equations (10), (11) and (12).*

Proof. The equations for a possible equilibrium are:

$$y^{(i)} = \frac{\sum_j y^{(j)} x^{(i,j)}}{\sqrt{\sum_{i,j} y^{(i)} y^{(j)} w_{i,j}}}$$

$$\begin{aligned} x^{(i,j)} &= \eta y^{(i)} \left(1 + \hat{K}_1 x^{(i,i)}\right) \delta_{i,j} \\ &- \left(\frac{1 - \rho - \delta \left((1-\eta) x^{(i,i)} + \eta y^{(i)} \left(1 + \hat{K}_1 x^{(i,i)}\right)\right)}{(1+\rho)\left(1 - z^{(i)}\right)} \right) \frac{z^{(j)} + \delta_{i,j}}{2} + (1-\eta) x^{(i,j)} \end{aligned}$$

$$1 - 2z^{(i)} = \frac{y^{(i)}}{\sqrt{\sum_j y^{(j)} y^{(j)}}} \tag{13}$$

We focus on the equation for $i = 1$, the proof is symetric for $i = 2$.

$$\begin{aligned} \eta x^{(1,1)} &= \eta y^{(1)} \left(1 + \hat{K}_1 x^{(1,1)}\right) - \left(1 - \rho + \rho \left((1-\eta) x^{(1,1)} + \eta y^{(1)} \left(1 + \hat{K}_1 x^{(1,1)}\right)\right)\right) \frac{z^{(1)} + 1}{2(1+\rho)\left(1 - z^{(1)}\right)} \\ \eta x^{(1,2)} &= -\left(1 - \rho + \rho \left((1-\eta) x^{(1,1)} + \eta y^{(1)} \left(1 + \hat{K}_1 x^{(1,1)}\right)\right)\right) \frac{z^{(1)}}{2(1+\rho)\left(1 - z^{(1)}\right)} \end{aligned}$$

Which leads to express $x^{(1,2)}$ as a function of $x^{(1,1)}$, $z^{(1)}$ and $y^{(1)}$

$$x^{(1,2)} = \frac{z^{(1)} x^{(1,1)} \left(1 - \hat{K}_1 y^{(1)}\right) - y^{(1)} z^{(1)}}{z^{(1)} + 1}$$

and $x^{(1,1)}$, as a function of $z^{(1)}$ and $y^{(1)}$:

$$x^{(1,1)} = \frac{2(\rho+1)\left(z^{(1)}-1\right)\eta y^{(1)} + \left(1-\rho+\eta\rho y^{(1)}\right)\left(z^{(1)}+1\right)}{2(\rho+1)\left(z^{(1)}-1\right)\eta\left(1-y^{(1)}\hat{K}_1\right) - \left(z^{(1)}+1\right)\rho\left(y^{(1)}\eta\hat{K}_1+1-\eta\right)}$$

The condition $x^{(1,2)} > 0$ implies that $y^{(1)} < \frac{1}{\hat{K}_1}$ and $x^{(1,1)} > 0$ leads to

$$y^{(1)} > \frac{(1-\rho)(z+1)}{\eta(\rho-2z-3z\rho+2)}$$

Inserting $x^{(1,1)}$ in the expression for $x^{(1,2)}$ allows to find an other condition for $x^{(1,2)} > 0$:

$$(\rho-(1-\rho)K_1) < 0 \text{ and } y^{(1)} > \frac{1-\rho}{(1-\rho)K_1-\rho} \tag{14}$$

so that, combined with $y^{(1)} < \frac{1}{\hat{K}_1}$, yields:

$$\frac{1}{K_1} > y^{(1)} > \frac{1-\rho}{(1-\rho)K_1-\rho}$$

which is satisfied only if

$$(\rho-(1-\rho)K_1) > 0$$

a contradiction with the first inequality of 14. ■

Proposition 2 *a) If $y_0^{(1)} = y_0^{(2)}$ and $x_0^{(i,3-i)} = x^{0\,(3-i,i)}$ there is a solution to the system with $1-2z_t^{(i)} = \frac{1}{\sqrt{2}}$, $x_t^{(i,3-i)} \to 0$ below a certain threshold (a lower bound for this threshold is given below), and $x_t^{(i,i)} \to \infty$ otherwise. This solution corresponds to a symetric situation in which $\chi_t = \frac{V_0^{(1)}+V_0^{(2)}}{\sqrt{2}}$, $y_t^{(1)} = y_t^{(2)}$. This dynamic is unstable. If $1-2z_0^{(1)} > \frac{1}{\sqrt{2}}$, then $1-2z_t^{(1)} \to 1$*

Proposition 3 *b) If $1-2z_0^{(1)} >\neq \frac{1}{\sqrt{2}}$, then $1-2z_t^{(1)} \to 1$ and $1-2z_t^{(1)} \to 0$*

Proof. For $1-2z_t^{(i)} = \frac{1}{\sqrt{2}}$, the system becomes:

$$\begin{aligned}
x_{t+1}^{(1,1)} &= \eta\frac{x^{(1,1)}+x^{(1,2)}}{\sqrt{2}}\left(1+\hat{K}_1x^{(1,1)}\right) \\
&\quad -\left(1-\rho+\rho\left((1-\eta)x^{(1,1)}+\eta\frac{x^{(1,1)}+x^{(1,2)}}{\sqrt{2}}\left(1+\hat{K}_1x^{(1,1)}\right)\right)\right)\frac{3-2\sqrt{2}}{2(1+\rho)}+(1-\eta)x^{(1,1)} \\
x_{t+1}^{(1,2)} &= -\left(1-\rho+\rho\left((1-\eta)x^{(1,1)}+\eta\frac{x^{(1,1)}+x^{(1,2)}}{\sqrt{2}}\left(1+\hat{K}_1x^{(1,1)}\right)\right)\right)\frac{5-2\sqrt{2}}{34(1+\rho)}+(1-\eta)x_t^{(1,2)}
\end{aligned}$$

From the second equation, one deduces that:

$$x_{t+1}^{(1,2)} - x_t^{(1,2)} < 0$$

Since there is no equilibrium with $x_t^{(1,2)} > 0$, then $x_t^{(1,2)} \to 0$.

From the first equation $x_{t+1}^{(1,1)} - x_t^{(1,1)} > 0$ only if

$$\eta\frac{x^{(1,1)}+x^{(1,2)}}{\sqrt{2}}\left(1-\frac{3-2\sqrt{2}}{2(1+\rho)}\rho\right)\left(1+\hat{K}_1x^{(1,1)}\right)-\left(1-\rho+\rho\left((1-\eta)x^{(1,1)}\right)\right)\frac{3-2\sqrt{2}}{2(1+\rho)}-\eta x^{(1,1)} > 0$$

that is if

$$x_0^{(1,1)} > \frac{-a+\sqrt{a^2+b}}{c}$$

where:

$$\begin{aligned}
a &= \left(\frac{1}{2}\sqrt{2}\eta\left(1-\frac{3-2\sqrt{2}}{2\left(1+\rho\right)}\rho\right)\left(1+x^{(1,2)}K_1\right)-\rho\frac{1-\eta}{2\rho+2}\left(3-2\sqrt{2}\right)-\eta\right)\\
b &= -2\eta K_1\left(1-\frac{3-2\sqrt{2}}{2\left(1+\rho\right)}\rho\right)\left(x^{(1,2)}\eta\left(1-\frac{3-2\sqrt{2}}{2\left(1+\rho\right)}\rho\right)-\frac{1-\rho}{2\left(1+\rho\right)}\left(3-2\sqrt{2}\right)\right)\\
c &= \sqrt{2}\eta K_1\left(\frac{\rho}{2\rho+2}\left(2\sqrt{2}-3\right)+1\right)
\end{aligned}$$

Thus, above a certain threshold depending on the initial conditions for $x^{(1,1)}$ and $x^{(1,2)}$, the value of $x^{(1,1)}$ will go to ∞, and $x^{(1,1)} \to 0$ below this threshold. Set $x^{(1,2)} = 0$, then a lower bound for the threshold is,

$$x_0^{(1,1)} > \frac{-a' + \sqrt{a'^2 + b'}}{c}$$

with:

$$\begin{aligned}
a' &= \left(\frac{1}{2}\sqrt{2}\eta\left(1-\frac{3-2\sqrt{2}}{2\left(1+\rho\right)}\rho\right)-\rho\frac{1-\eta}{2\rho+2}\left(3-2\sqrt{2}\right)-\eta\right)\\
b' &= 2\eta K_1\left(\left(1-\frac{3-2\sqrt{2}}{2\left(1+\rho\right)}\rho\right)\right)\left(\frac{1-\rho}{2\left(1+\rho\right)}\left(3-2\sqrt{2}\right)\right)
\end{aligned}$$

Since the equations are symetric between the 2 agents, the dynamics ae identical if the initial conditions are equal, and then $1 - 2z_t^{(i)} = \frac{1}{\sqrt{2}}$.

b) Let us compute the variation of the dynamics with respect to a small change $\delta z^{(1)}$ in $z^{(1)}$ (the same reasonning is valid for $z^{(2)}$).

Starting from:

$$\begin{aligned}
x_{t+1}^{(1,1)} - x^{(1,1)} &= \eta y^{(1)}\left(1+\hat{K}_1 x^{(1,1)}\right) - \left(1-\rho+\rho\left(\left(1-\eta\right)x^{(1,1)}+\eta y^{(1)}\left(1+\hat{K}_1 x^{(1,1)}\right)\right)\right)\frac{z^{(1)}+1}{2\left(1+\rho\right)\left(1-z^{(1)}\right)}\\
&\quad -\eta x^{(1,1)}\\
x_{t+1}^{(1,2)} - x^{(1,2)} &= -\left(1-\rho+\rho\left(\left(1-\eta\right)x^{(1,1)}+\eta y^{(1)}\left(1+\hat{K}_1 x^{(1,1)}\right)\right)\right)\frac{z^{(1)}}{2\left(1+\rho\right)\left(1-z^{(1)}\right)} - \eta x^{(1,2)}
\end{aligned}$$

and expressing $y^{(1)}$ as a function of $x^{(1,1)}$ and $x^{(1,2)}$ one has:

$$\begin{aligned}
x_{t+1}^{(1,1)} - x^{(1,1)} &= \eta\left(\left(1-2z^{(1)}\right)x^{(1,1)}+\sqrt{1-\left(1-2z^{(1)}\right)^2}x^{(1,2)}\right)\left(1-\rho\frac{z^{(1)}+1}{2\left(1+\rho\right)\left(1-z^{(1)}\right)}\right)\left(1+\hat{K}_1 x^{(1,1)}\right)\\
&\quad -\left(1-\rho+\rho\left(\left(1-\eta\right)x^{(1,1)}\right)\right)\frac{z^{(1)}+1}{2\left(1+\rho\right)\left(1-z^{(1)}\right)} - \eta x^{(1,1)}\\
x_{t+1}^{(1,2)} - x^{(1,2)} &= -\left(1-\rho+\rho\left(\left(1-\eta\right)x^{(1,1)}+\eta y^{(1)}\left(1+\hat{K}_1 x^{(1,1)}\right)\right)\right)\frac{z^{(1)}}{2\left(1+\rho\right)\left(1-z^{(1)}\right)} - \eta x^{(1,2)}
\end{aligned}$$

given that:

$$\begin{aligned}
\frac{\partial}{\partial z^{(1)}}\left(1-\rho\frac{z^{(1)}+1}{2\left(1+\rho\right)\left(1-z^{(1)}\right)}\right) &< 0\\
-\frac{\partial}{\partial z^{(1)}}\frac{z^{(1)}+1}{2\left(1+\rho\right)\left(1-z^{(1)}\right)} &< 0
\end{aligned}$$

one obtains

$$\begin{aligned} \delta x_{t+1}^{(1,1)} &> 0 \\ \delta x_{t+1}^{(1,2)} &> 0 \end{aligned}$$

for

$$\delta z^{(1)} < 0$$

Now, let

$$\begin{aligned} \cos u &= \left(1 - 2z^{(1)}\right) \\ \delta u &< 0 \end{aligned}$$

and start from $\cos u = \frac{1}{\sqrt{2}}$ to compute $\delta y_{t+1}^{(1)}$

$$\begin{aligned} \delta y_{t+1}^{(1)} &= (\cos u)\, \delta x_{t+1}^{(1,1)} + (\sin u)\, \delta x_{t+1}^{(1,2)} + \left((\cos u)\, x_{t+1}^{(1,2)} - (\sin u)\, x_{t+1}^{(1,1)}\right) \delta u \\ &= \frac{1}{\sqrt{2}} \left(\delta x_{t+1}^{(1,1)} + \delta x_{t+1}^{(1,2)} + \left(x_{t+1}^{(1,2)} - x_{t+1}^{(1,1)}\right) \delta u\right) \end{aligned}$$

Note that $\left(x_{t+1}^{(1,2)} - x_{t+1}^{(1,1)}\right) < 0$, so that

$$\left(x_{t+1}^{(1,2)} - x_{t+1}^{(1,1)}\right) \delta u > 0$$

As a consequence, if $z^{(1)}$ decreases, $x_{t+1}^{(1,1)}$and $x_{t+1}^{(1,2)}$ and then $y_{t+1}^{(1)}$ increases. Symetrically $y_{t+1}^{(2)}$ decreases and from (1-2z) we deduce that $\left(1 - 2z_{t+1}^{(1)}\right) - \left(1 - 2z_t^{(1)}\right) > 0$ and then $z_{t+1}^{(1)} - z_t^{(1)} < 0$.

This is the same reasoning for an arbitrary $1 - 2z^{(1)}$, so that departing from $1 - 2z_0^{(1)} = \frac{1}{\sqrt{2}}$ with $\delta z^{(1)} < 0$ leads to $z_t^{(1)} \to 0$.

b) This is the consequence of a). ■