Financial Interactions and Capital Accumulation

Pierre Gosselin* Aïleen Lotz†

May 2024

Abstract

In a series of precedent papers, we have presented a comprehensive methodology, termed Field Economics, for translating a standard economic model into a statistical field-formalism framework. This formalism requires a large number of heterogeneous agents, possibly of different types. It reveals the emergence of collective states among these agents or type of agents while preserving the interactions and microeconomic features of the system at the individual level. In two prior papers, we applied this formalism to analyze the dynamics of capital allocation and accumulation in a simple microeconomic framework of investors and firms.

Building upon our prior work, the present paper refines the initial model by expanding its scope. Instead of considering financial firms investing solely in real sectors, we now suppose that financial agents may also invest in other financial firms. We also introduce banks in the system that act as investors with a credit multiplier. Two types of interaction are now considered within the financial sector: financial agents can lend capital to, or choose to buy shares of, other financial firms. Capital now flows between financial agents and is only partly invested in real sectors, depending on their relative returns. We translate this framework into our formalism and study the diffusion of capital and possible defaults in the system, both at the macro and micro level.

At the macro level, we find that several collective states may emerge, each characterized by a distinct level of average capital and investors per sector. These collective states depend on external parameters such as level of connections between investors or firms' productivity. The multiplicity of possible collective states is the consequence of the nature of the system composed of interconnected heterogeneous agents. Several equivalent patterns of returns and portfolio allocation may emerge. The multiple collective states induce the unstable nature of financial markets, and some of them include defaults may emerge. At the micro level, we study the propagation of returns and defaults within a given collective state. Our findings highlight the significant role of banks, which can either stabilize the system through lending activities or propagate instability through loans to investors.

Key words: Financial Markets, Real Economy, Capital Allocation, Statistical Field Theory, Background fields, Collective states, Multi-Agent Model, Interactions.

JEL Classification: B40, C02, C60, E00, E1, G10

1 Introduction

The financial market is often depicted as an optimal tool for resource allocation. According to this view, the market efficiently allocates capital to sectors in need and enables firms to raise capital at a lower cost. However, this perspective assumes a market consisting solely of real firms seeking financial investors.

*Pierre Gosselin : Institut Fourier, UMR 5582 CNRS-UGA, Université Grenoble Alpes, BP 74, 38402 St Martin d'Hères, France. E-Mail: Pierre.Gosselin@univ-grenoble-alpes.fr

†Aïleen Lotz: Cerca Trova, BP 114, 38001 Grenoble Cedex 1, France. E-mail: a.lotz@cercatrova.eu

In reality, the market serves not only as a nexus for the real and financial sectors but also as a platform for investors to raise capital for their own investments. Additionally, it enables investors to invest in the performance of other investors, particularly through indices and derivative products. Furthermore, investors themselves utilize the market to access capital, with their activities representing a potential source of returns that may attract other investors.

The fact that returns in the market can be obtained through direct investment in the real economy or indirectly by investing in the performance of other investors questions the optimality of financial markets. It underscores their dual nature, since capital may not always flow directly to the real economy but can be redirected to other financial actors for return-seeking purposes. The circulation of capital in pursuit of optimal returns, not only within the real sector but also among financial agents, complicates capital exchanges, potentially creates tensions between the financial and real sectors, and may divert capital away from the real economy.

Studying the optimality of indirect capital allocation and its effects, along with the diffusion and circulation of capital across different financial actors, requires moving beyond the representative agent framework. It involves considering a large number of interacting agents, among whom one can trace how capital circulates.

Field Economics allows this type of analysis. This formalism, based on Field Theory and developed in a series of papers, describes the interactions among a large number of economic agents and the resulting collective and individual states. What's more, it does so while keeping track of the microeconomic systems on which they are built and considering them as a whole.

In a previous paper, we studied the dynamics of capital allocation and accumulation for a system with a large number of agents, dividing between a real sector and a financial sector investing solely in the real sector. We observed the different dynamics of capital and how it was distributed among different sectors of the real economy.

The present paper builds upon and extends the previous model by considering that investors can now invest in the financial sector by taking stakes or lending to other investors. This allows us to examine how investment in the real economy is influenced when the investment process is delegated to other investors.

We address these questions by introducing interactions between two types of financial investors: banks and non-banks. Two forms of investment are considered: equity participation and loans. Introducing loans in the model enables us to study the strength of capital circulation and the potential for defaults within the setup.

At the macro level, we show that, due to interconnections between financial agents and leverage effects, multiple potential collective states exist. Each state is characterized by a total level of capital, the number of agents of each type, and their average return per sector. This multiplicity reflects the instability of the financial system and its sensitivity to slight changes in parameters. Additionally, certain collective states, including defaults in certain parts of the economy, also exist. Transitions between collective states of the economy are possible and may be induced by external conditions.

Banks impact the stability of the system in two contradictory ways. Acting as lenders, they stabilize the system by reducing the occurrence of collective states, but also contribute to increasing the leverage of investors. The balance between these effects depends on the relative size of the banking sectors.

Dynamically, we examine the mechanisms that realize these transitions at the individual level. We study the dynamics of different groups of agents and show how defaults may arise through indirect effects of interactions between various types of agents, propagating to other sectors and leading to the emergence of collective default states.

The paper is organized as follows. Section 2 provides a literature review. In the Preamble, we present a field formalism for a system with a large number of agents, possibly of different groups. Section 3 presents the general method for translating a microeconomic model into a field model. Sections 4 and 5 present applications of this translation, both at the collective level, through the computations of background fields and averages, and at the individual level, through the computation of the transition functions.

In Part 1, we apply this field formalism to a system of two types of agents: firms and investors. In section 6, we present the microeconomic model, and in section 7, its field translation. Sections 8 and 9 present the resolution for the firms' and investors' background fields, respectively. Sections 10 and 11 present the equations for investors' returns and capital levels per sector. The solution is presented in section 12, and section 13 computes the possible averages of the system. These two sections close the resolution for collective states in terms of average capital and returns per sector along with the global capital level. In section 14, we consider the inclusion of defaults and their impact on collective states. Section 15 inspects the dynamical aspect of excess returns and default propagation within a given collective state defined by a collection of interacting heterogeneous groups of agents. Some dynamical features of the default mechanism are studied. Part 2 of this work present a refined model with a third type of agent, banks. Sections 16 and 17 present the microeconomic framework and its translation. Sections 18 presents the minimization equations for the background field of firms, investors, and banks, respectively. Section 19 solves for the firm's background field and average level of capital per sector. In sections 20 and 21, we solve the investors' and banks' returns as functions of banks and investors' average capital. Sections 22, 23, and 24 close the resolution of the model by deriving the collective state in terms of total capital and returns per sector, and global average capital for investors and banks. In section 25, we revisit the dynamical aspect of excess returns and default propagation from section 15 in the presence of banks. We synthetize and discuss our results in section 26. Section 27 concludes

2 Literature review

Several branches of the economic literature seek to replace the representative agent with a collection of heterogeneous ones. Among other things, they differ in the way they model this collection of agents.

The first branch of the literature represents this collection of agents by probability densities. This is the approach followed by mean field theory, heterogeneous agents new Keynesian (HANK) models, and the information-theoretic approach to economics.

Mean field theory studies the evolution of agents' density in the state space of economic variables. It includes the interactions between agents and the population as a whole but does not consider the direct interactions between agents. This approach is thus at an intermediate scale between the macro and micro scale: it does not aggregate agents but replaces them with an overall probability distribution. Mean field theory has been applied to game theory (Bensoussan et al. 2018, Lasry et al. 2010a, b) and economics (Gomes et al. 2015). However, these mean fields are actually probability distributions. In our formalism, the notion of fields refers to some abstract complex functions defined on the state space and is similar to the second-quantized-wave functions of quantum theory. Interactions between agents are included at the individual level. Densities of agents are recovered from these fields and depend directly on interactions.

Heterogeneous agents' new Keynesian (HANK) models use a probabilistic treatment similar to mean fields theory. An equilibrium probability distribution is derived from a set of optimizing heterogeneous agents in a new Keynesian context (see Kaplan and Violante 2018 for an account). Our approach, on the contrary, focuses on the direct interactions between agents at the microeconomic level. We do not look for an equilibrium probability distribution for each agent, but rather directly build a probability density for the system of N agents seen as a whole, that includes interactions, and then translate this probability density in terms of fields. The states' space we consider is thus much larger than those considered in the above approaches. Because it is the space of all paths for a large number of agents, it allows studying the agents' economic structural relations and the emergence of the particular phases or collective states induced by these specific micro-relations, that will in turn impact each agent's stochastic dynamics at the microeconomic level. Other differences are worth mentioning. While HANK models stress the role of an infinite number of heterogeneously-behaved consumers, our formalism dwells on the relations between physical and financial capital[1]. Besides, our formalism does not rely on agents' rationality assumptions, since for a large number of agents, behaviours, be they fully or partly rational, can be modeled as random.

The information theoretic approach to economics (see Yang 2018) considers probabilistic states around the equilibrium. It is close to our methodological stance: it replaces the Walrasian equilibrium with a statistical equilibrium derived from an entropy maximisation program. Our statistical weight is similar to the one they use, but is directly built from microeconomic dynamic equations. The same difference stands for the rational inattention theory (Sims 2006) in which non-gaussian density laws are derived from limited information and constraints: our setting directly includes constraints in the random description of an agent (Gosselin, Lotz, Wambst 2020).

The differences highlighted above between these various approaches and our work also manifest at the micro-scale in the description of agents' dynamics. Actually, in the field framework, once the collective states have been found, we can recover both the types of individual dynamics depending on the initial conditions and the "effective" form of interactions between two or more agents: At the individual level, agents are distributed along some probability law. However, this probability law is directly conditioned by the collective state of the system and the effective interactions. Different collective states, given different parameters, yield different individual dynamics. This approach allows for coming back and forth between collective and individual aspects of the system. Different categories of agents appear in the emerging collective state. Dynamics may present very different patterns, given the collective state's form and the agents' initial conditions.

A second branch of the literature is closest to our approach since it considers the interacting system of agents in itself. It is the multi-agent systems literature, notably agent-based models (see Gaffard Napoletano 2012, Mandel et al. 2010 2012) and economic networks (Jackson 2010).

Agent-based models deal with the macroeconomic level, whereas network models lower-scale phenomena such as contract theory, behaviour diffusion, information sharing, or learning. In both settings, agents are typically defined by and follow various sets of rules, leading to the emergence of equilibria and dynamics otherwise inaccessible to the representative agent setup. Both approaches are however highly numerical and model-dependent and rely on microeconomic relations - such as ad-hoc reaction functions - that may be too simplistic. Statistical fields theory on the contrary accounts for transitions between scales. Macroeconomic patterns do not emerge from the sole dynamics of a large set of agents: they are grounded in behaviours and interaction structures. Describing these structures in terms of field theory allows for the emergence of phases at the macro scale, and the study of their impact at the individual level.

A third branch of the literature, Econophysics, is also related to ours since it often considers the set of agents as a statistical system (for a review, see Abergel et al. 2011a,b and references therein; or Lux 2008, 2016). But it tends to focus on empirical laws, rather than apply the full potential of field theory to economic systems. In the same vein, Kleinert (2009) uses path integrals to model stock prices' dynamics. Our approach, in contrast, keeps track of usual microeconomic concepts, such

[1] Note that our formalism could also include heterogeneous consumers (see Gosselin, Lotz, Wambst 2020).

as utility functions, expectations, and forward-looking behaviours, and includes these behaviours into the analytical treatment of multi-agent systems by translating the main characteristics of optimizing agents in terms of statistical systems. Closer to our approach, Bardoscia et al (2017) study a general equilibrium model for a large economy in the context of statistical mechanics, and show that phase transitions may occur in the system. Our problematic is similar, but our use of field theory deals with a large class of dynamic models.

The literature on interactions between finance and real economy or capital accumulation takes place mainly in the context of DGSE models. (for a review of the literature, see Cochrane 2006; for further developments see Grassetti et al. 2022, Grosshans and Zeisberger 2018, Böhm et al. 2008, Caggese and Orive, Bernanke e al. 1999, Campello et al. 2010, Holmstrom and Tirole 1997, Jermann, and Quadrini 2012, Khan Thomas 2013, Monacelli et al. 2011). Theoretical models include several types of agents at the aggregated level. They describe the interactions between a few representative agents such as producers for possibly several sectors, consumers, financial intermediaries, etc. to determine interest rates, levels of production, and asset pricing, in a context of ad-hoc anticipations.

Our formalism differs from this literature in three ways. First, we consider several groups of a large number of agents to describe the emergence of collective states and study the continuous space of sectors. Second, we consider expected returns and the longer-term horizon as somewhat exogenous or structural. Expected returns are a combination of elements, such as technology, returns, productivity, sectoral capital stock, expectations, and beliefs. These returns are also a function defined over the sectors' space: the system's background fields are functionals of these expected returns. Taken together, the background fields of a field model describe an economic configuration for a given environment of expected returns. As such, expected returns are at first seen as exogenous functions. It is only in the second step, when we consider the dynamics between capital accumulation and expectations, that expectations may themselves be seen as endogenous. Even then, the form of relations between actual and expected variables specified are general enough to derive some types of possible dynamics.

Last but not least, we do not seek individual or even aggregated dynamics, but rather background fields that describe potential long-term equilibria and may evolve with the structural parameters. For such a background, agents' individual typical dynamics may nevertheless be retrieved through Green functions (see GLW). These functions compute the transition probabilities from one capital-sector point to another. But backgrounds themselves may be considered as dynamical quantities. Structural or long-term variations in the returns' landscape may modify the background and in turn the individual dynamics. Expected returns themselves depend on and interact with, capital accumulation.

Ultimately, there is a vast literature concerning default risk and contagion of default in a financial system (see for example, Reinhart and Rogoff 2009, Gennaioli et al. 2012, Acharya et al 2017, Adrian and Brunnermeier 2016, Allen and Gale 2000). In terms of modeling, some studies focus on the structures of connection between agents and their impact on default contagion (Gai and Kapadia 2010, Battiston et al. 2012, Battiston et al. 2020, Langfield et al. 2020), while other works develop microeconomic models and look for the possibilities of equilibrium and the risk of default depending on the level of connections between agents (Acemoglu et al. 2015, Bardoscia et al. 2019, Cifuentes et al. 2005, Elliott et al. 2014, Haldane and May 2011) These models consider networks of agents linked by mutual participations measured by leverage matrices.

Our work has a similar starting point, except that we include firms and banks, account for disparities in firms' returns, and include loans between agents. Moreover, our diffusion matrix takes into account the characteristics of the sectors impacted during the diffusion of returns and sector-dependent feedback loops.

More importantly, our use of Field Theory accounts for the possible emergence of multiple collective states, and allows their description in terms of average capital per sector, number of agents and average return per sectors. This description goes beyond those of the aforementioned models, since our formalism goes back and forth between the micro and macro levels.

Preamble. Field formalism for a model of firms and investors

In the first part of this work, we describe the field formalism for an economic system, its application to derive the potential collective states of the system and the individual dynamics within such collective states. Ultimately we apply this formalism to translate a model with large number of interacting investors and firms.

3 General method of translation

The formalism we propose transforms an economic model of dynamic agents into a statistical field model. In classical models, each agent's dynamics is described by an optimal path for some vector variable, say $A_i(t)$, from an initial to a final point, up to some fluctuations.

But this system of agents could also be seen as probabilistic: each agent could be described by a *probability density* centered around the classical optimal path, up to some idiosyncratic uncertainties[2] [3]. In this probabilistic approach, each possible trajectory of the whole set of N agents has a specific probability. The classical model is therefore described by the set of trajectories of the group of N agents, each one being endowed with its own probability, its statistical weight. The statistical weight is therefore a function that associates a probability with each trajectory of the group.

This probabilistic approach can be translated into a more compact *field formalism*[4] that preserves the essential information encoded in the model but implements a change in perspective. A field model is a structure governed by its own intrinsic rules that encapsulate the economic model chosen. This field model contains all possible realizations that could arise from the initial economic model, i.e. all the possible global outcomes, or collective state, permitted by the economic model. So that, once constructed, the field model provides a unique advantage over the standard economic model: it allows to compute the probabilities of each of the possible outcomes for each collective state of the economic model. These probabilities are computed indirectly through the *action functional* of the model, a function that assigns a specific value to each realization of the field. Technically, the random N agents' trajectories $\{\mathbf{A}_i(t)\}$ are replaced by a field, a random variable whose realizations are complex-valued functions Ψ of the variables $\mathbf{A}$, and the statistical weight of the N agents' trajectories $\{\mathbf{A}_i(t)\}$ in the probabilistic approach is translated into a statistical weight for each realization Ψ. They encapsulate the collective states of the system.

Once the probabilities of each collective state computed, the most probable collective state among all other collective states, can be found. In other words, a field model allows to consider the true global outcome induced by any standard economic model. This is what we will call the

[2] Because the number of possible paths is infinite, the probability of each individual path is null. We, therefore, use the word "probability density" rather than "probability".

[3] See Gosselin, Lotz and Wambst (2017, 2020, 2021).

[4] Ibid.

expression of the field model, more usually called the *background field* of the model.

This most probable realization of the field, the expression or background field of the model, should not be seen as a final outcome resulting from a trajectory, but rather as its most recurring realization. Actually, the probability of the realizations of the model is peaked around the expression of the field. This expression, which is characteristic of the system, will determine the nature of individual trajectories within the structure, in the same way as a biased dice would increase the probability of one event. The field in itself is therefore static, insofar as each realization of the system of agents only contributes to the emergence of the proper expression of the field. However, studying variations in the parameters of the system indirectly induce a time parameter at the field or macro level.

3.1 Statistical weight and minimization functions for a classical system

In an economic framework with a large number of agents, each agent is characterized by one or more stochastic dynamic equations. Some of these equations result from the optimization of one or several objective functions. Deriving the statistical weight from these equations is straightforward: it associates, to each trajectory of the group of agents $\{T_i\}$, a probability that is peaked around the set of optimal trajectories of the system, of the form:

$$W\left(s\left(\{T_i\}\right)\right)=\exp\left(-s\left(\{T_i\}\right)\right) \tag{1}$$

where $s\left(\{T_i\}\right)$ measures the distance between the trajectories $\{T_i\}$ and the optimal ones.

This paper considers two types of agents characterized by vector-variables $\{\mathbf{A}_i(t)\}_{i=1,...N}$, and $\left\{\hat{\mathbf{A}}_l(t)\right\}_{i=1,...\hat{N}}$ respectively, where N and $\hat{N}$ are the number of agents of each type, with vectors $\mathbf{A}_i(t)$ and $\hat{\mathbf{A}}_l(t)$ of arbitrary dimension. For such a system, the statistical weight writes:

$$W\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)=\exp\left(-s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)\right) \tag{2}$$

The optimal paths for the system are assumed to be described by the sets of equations:

$$\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l...}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)=\epsilon_i,\ i=1...N \tag{3}$$

$$\frac{d\hat{\mathbf{A}}_l(t)}{dt}-\sum_{i,j,k...}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)=\hat{\epsilon}_l,\ i=1...\hat{N} \tag{4}$$

where the ϵ_i and $\hat{\epsilon}_i$ are idiosynchratic random shocks. These equations describe the general dynamics of the two types agents, including their interactions with other agents. They may encompass the dynamics of optimizing agents where interactions act as externalities so that this set of equations is the full description of a system of interacting agents[5][6].

For equations (3) and (4), the quadratic deviation at time t of any trajectory with respect to the optimal one for each type of agent are:

$$\left(\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l...}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \tag{5}$$

[5] Expectations of agents could be included by replacing $\frac{d\mathbf{A}_i(t)}{dt}$ with $E\frac{d\mathbf{A}_i(t)}{dt}$, where E is the expectation operator. This would amount to double some variables by distinguishing "real variables" and expectations. However, for our purpose, in the context of a large number of agents, at least in this work, we discard as much as possible this possibility.

[6] A generalisation of equations (3) and (4), in which agents interact at different times, and its translation in term of field is presented in appendix 1.

and:

$$\left(\frac{d\hat{\mathbf{A}}_l(t)}{dt}-\sum_{i,j,k...}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \tag{6}$$

Since the function (2) involves the deviations for all agents over all trajectories, the function $s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)$ is obtained by summing (5) and (6) over all agents, and integrate over t. We thus find:

$$\begin{aligned}s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right) &= \int dt\sum_i\left(\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l...}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \quad (7)\\ &+\int dt\sum_l\left(\frac{d\hat{\mathbf{A}}_l(t)}{dt}-\sum_{i,j,k...}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2\end{aligned}$$

There is an alternate, more general, form to (7). We can assume that the dynamical system is originally defined by some equations of type (3) and (4), plus some objective functions for agents i and l, and that these agents aim at minimizing respectively:

$$\sum_{j,k,l...}g\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right) \tag{8}$$

and:

$$\sum_{i,j,k..}\hat{g}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right) \tag{9}$$

In the above equations, the objective functions depend on other agents' actions seen as externalities[7]. The functions (8) and (9) could themselves be considered as a measure of the deviation of a trajectory from the optimum. Actually, the higher the distance, the higher (8) and (9).

Thus, rather than describing the systm by a full system of dynamic equations, we can consider some ad-hoc equations of type (3) and (4) and some objective functions (8) and (9) to write the alternate form of $s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)$ as:

$$\begin{aligned}&s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right) \quad (10)\\ =&\int dt\sum_i\left(\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l...}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2\\ &+\int dt\sum_l\left(\frac{d\hat{\mathbf{A}}_l(t)}{dt}-\sum_{i,j,k...}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2\\ &+\int dt\sum_{i,j,k,l...}\left(g\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)+\hat{g}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)\end{aligned}$$

In the sequel, we will refer to the various terms arising in equation (10) as the "minimization functions", i.e. the functions whose minimization yield the dynamics equations of the system[8].

[7] We may also assume intertemporal objectives, see (GLW).

[8] A generalisation of equation (10), in which agents interact at different times, and its translation in term of field is presented in appendix 1.

3.2 Translation techniques

The statistical weight $W(s(\{T_i\}))$ defined in (1) once computed, it can be translated in terms of field. To do so, and for each type α of agent, the sets of trajectories $\{\mathbf{A}_{\alpha i}(t)\}$ are replaced by a field $\Psi_\alpha(\mathbf{A}_\alpha)$, a random variable whose realizations are complex-valued functions Ψ of the variables $\mathbf{A}_\alpha$[9]. The statistical weight for the whole set of fields $\{\Psi_\alpha\}$ has the form $\exp(-S(\{\Psi_\alpha\}))$. The function $S(\{\Psi_\alpha\})$ is called the *fields action functional*. It represents the interactions among different types of agents. Ultimately, the expression $\exp(-S(\{\Psi_\alpha\}))$ is the statistical weight for the field[10] that computes the probability of any realization $\{\Psi_\alpha\}$ of the field.

The form of $S(\{\Psi_\alpha\})$ is obtained directly from the classical description of our model. For two types of agents, we start with expression (10). The various minimizations functions involved in the definition of $s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)$ will be translated in terms of field and the sum of these translations will produce finally the action functional $S(\{\Psi_\alpha\})$. The translation method can itself be divided into two relatively simple processes, but varies slightly depending on the type of terms that appear in the various minimization functions.

3.2.1 Terms without temporal derivative

In equation (10), the terms that involve indexed variables but no temporal derivative terms are the easiest to translate. They are of the form:

$$\sum_i \sum_{j,k,l,m...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right)$$

These terms describe the whole set of interactions both among and between two groups of agents. Here, agents are characterized by their variables $\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t)$... and $\hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)$... respectively, for instance in our model firms and investors.

In the field translation, agents of type $\mathbf{A}_i(t)$ and $\hat{\mathbf{A}}_l(t)$ are described by a field $\Psi(\mathbf{A})$ and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$, respectively.

In a first step, the variables indexed i such as $\mathbf{A}_i(t)$ are replaced by variables $\mathbf{A}$ in the expression of g. The variables indexed j, k, l, m..., such as $\mathbf{A}_j(t)$, $\mathbf{A}_k(t)$, $\hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)$... are replaced by $\mathbf{A}', \mathbf{A}''$, $\hat{\mathbf{A}}$, $\hat{\mathbf{A}}'$, and so on for all the indices in the function. This yields the expression:

$$\sum_i \sum_{j,k,l,m...} g\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right)$$

In a second step, each sum is replaced by a weighted integration symbol:

$$\sum_i \rightarrow \int |\Psi(\mathbf{A})|^2 d\mathbf{A},\ \sum_j \rightarrow \int |\Psi(\mathbf{A}')|^2 d\mathbf{A}',\ \sum_k \rightarrow \int |\Psi(\mathbf{A}'')|^2 d\mathbf{A}''$$
$$\sum_l \rightarrow \int \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 d\hat{\mathbf{A}},\ \sum_m \rightarrow \int \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}}'$$

[9]In the following, we will use indifferently the term "field" and the notation Ψ for the random variable or any of its realization Ψ.

[10]In general, one must consider the integral of $\exp(-S(\{\Psi_\alpha\}))$ over the configurations $\{\Psi_\alpha\}$. This integral is the partition function of the system.

which leads to the translation:

$$\begin{aligned}&\sum_i\sum_j\sum_{j,k...} g\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\\ \rightarrow\ &\int g\left(\mathbf{A},\mathbf{A}',\mathbf{A}'',\hat{\mathbf{A}},\hat{\mathbf{A}}'...\right)|\Psi(\mathbf{A})|^2|\Psi(\mathbf{A}')|^2|\Psi(\mathbf{A}'')|^2\times...d\mathbf{A}d\mathbf{A}'d\mathbf{A}''...\\ &\times\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2\left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2\times...d\hat{\mathbf{A}}d\hat{\mathbf{A}}'...\end{aligned}\tag{11}$$

where the dots stand for the products of square fields and integration symbols needed.

3.2.2 Terms with temporal derivative

In equation (10), the terms that involve a variable temporal derivative are of the form:

$$\sum_i\left(\frac{d\mathbf{A}_i^{(\alpha)}(t)}{dt}-\sum_{j,k,l,m...}f^{(\alpha)}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2\tag{12}$$

This particular form represents the dynamics of the α-th coordinate of a variable $\mathbf{A}_i(t)$ as a function of the other agents.

The method of translation is similar to the above, but the time derivative adds an additional operation.

In a first step, we translate the terms without derivative inside the parenthesis:

$$\sum_{j,k,l,m...}f^{(\alpha)}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\tag{13}$$

This type of term has already been translated in the previous paragraph, but since there is no sum over i in equation (13), there should be no integral over $\mathbf{A}$, nor factor $|\Psi(\mathbf{A})|^2$.

The translation of equation (13) is therefore, as before:

$$\int f^{(\alpha)}\left(\mathbf{A},\mathbf{A}',\mathbf{A}'',\hat{\mathbf{A}},\hat{\mathbf{A}}'...\right)|\Psi(\mathbf{A}')|^2|\Psi(\mathbf{A}'')|^2d\mathbf{A}'d\mathbf{A}''\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2\left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2d\hat{\mathbf{A}}d\hat{\mathbf{A}}'\tag{14}$$

A free variable $\mathbf{A}$ remains, which will be integrated later, when we account for the external sum $\sum_i$. We will call $\Lambda(\mathbf{A})$ the expression obtained:

$$\Lambda(\mathbf{A})=\int f^{(\alpha)}\left(\mathbf{A},\mathbf{A}',\mathbf{A}'',\hat{\mathbf{A}},\hat{\mathbf{A}}'...\right)|\Psi(\mathbf{A}')|^2|\Psi(\mathbf{A}'')|^2d\mathbf{A}'d\mathbf{A}''\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2\left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2d\hat{\mathbf{A}}d\hat{\mathbf{A}}'\tag{15}$$

In a second step, we account for the derivative in time by using field gradients. To do so, and as a rule, we replace :

$$\sum_i\left(\frac{d\mathbf{A}_i^{(\alpha)}(t)}{dt}-\sum_j\sum_{j,k...}f^{(\alpha)}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2\tag{16}$$

by:

$$\int\Psi^{\dagger}(\mathbf{A})\left(-\nabla_{\mathbf{A}^{(\alpha)}}\left(\frac{\sigma^2_{\mathbf{A}^{(\alpha)}}}{2}\nabla_{\mathbf{A}^{(\alpha)}}-\Lambda(\mathbf{A})\right)\right)\Psi(\mathbf{A})\,d\mathbf{A}\tag{17}$$

The variance $\sigma^2_{\mathbf{A}^{(\alpha)}}$ reflects the probabilistic nature of the model which is hidden behind the field formalism. This variance represents the characteristic level of uncertainty of the system's dynamics. It is a parameter of the model. Note also that in (17), the integral over $\mathbf{A}$ reappears at the end, along with the square of the field $|\Psi(\mathbf{A})|^2$. This square is split into two terms, $\Psi^{\dagger}(\mathbf{A})$ and $\Psi(\mathbf{A})$, with a gradient operator inserted in between.

3.3 Action functional

The field description is ultimately obtained by summing all the terms translated above and introducing a time dependency. This sum is called the action functional. It is the sum of terms of the form (11) and (17), and is denoted $S\left(\Psi,\Psi^{\dagger}\right)$.

For example, in a system with two types of agents described by two fields $\Psi\left(\mathbf{A}\right)$and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$, the action functional has the form:

$$\begin{aligned} S\left(\Psi,\Psi^{\dagger}\right) &= \int\Psi^{\dagger}\left(\mathbf{A}\right)\left(-\nabla_{\mathbf{A}^{(\alpha)}}\left(\frac{\sigma_{\mathbf{A}^{(\alpha)}}^{2}}{2}\nabla_{\mathbf{A}^{(\alpha)}}-\Lambda_{1}(\mathbf{A})\right)\right)\Psi\left(\mathbf{A}\right)d\mathbf{A} \\ &+\int\hat{\Psi}^{\dagger}\left(\hat{\mathbf{A}}\right)\left(-\nabla_{\hat{\mathbf{A}}^{(\alpha)}}\left(\frac{\sigma_{\hat{\mathbf{A}}^{(\alpha)}}^{2}}{2}\nabla_{\hat{\mathbf{A}}^{(\alpha)}}-\Lambda_{2}(\hat{\mathbf{A}})\right)\right)\hat{\Psi}\left(\hat{\mathbf{A}}\right)d\hat{\mathbf{A}} \\ &+\sum_{m}\int g_{m}\left(\mathbf{A},\mathbf{A}',\mathbf{A}'',\hat{\mathbf{A}},\hat{\mathbf{A}}'...\right)\left|\Psi\left(\mathbf{A}\right)\right|^{2}\left|\Psi\left(\mathbf{A}'\right)\right|^{2}\left|\Psi\left(\mathbf{A}''\right)\right|^{2}\times...d\mathbf{A}d\mathbf{A}'d\mathbf{A}''... \\ &\times\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^{2}\left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^{2}\times...d\hat{\mathbf{A}}d\hat{\mathbf{A}}'... \end{aligned} \tag{18}$$

where the sequence of functions g_{m} describes the various types of interactions in the system.

Note that the collective states described by the fields are structural states of the system. The fields have their own dynamics at the macro-scale, which will be discussed later in the paper. This is why the usual microeconomic time variable used in standard models has disappeared in formula (18). However, time dependency may at times be required in fields, so that a time variable, written θ could be introduced by replacing:

$$\begin{aligned} \Psi\left(\mathbf{A}\right) &\rightarrow \Psi\left(\mathbf{A},\theta\right) \\ \hat{\Psi}\left(\hat{\mathbf{A}}\right) &\rightarrow \hat{\Psi}\left(\hat{\mathbf{A}},\theta\right) \end{aligned}$$

More about this point can be found in appendix 1.

4 Use of the field model

Once the field action functional S is found, we can use field theory to study the system of agents. This can be done at two levels: the collective and the individual level. At the collective level, the system is described by the background fields of the system that condition average quantities of economic variables of the system.

At the individual level, the field formalism allows to compute agents' individual dynamics in the state defined by the background fields, through the transition functions of the system.

4.1 Collective level: background fields and averages

At the collective level, the background fields of the system can be computed. These background fields are the particular functions, $\Psi\left(\mathbf{A}\right)$ and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$, and their adjoints fields $\Psi^{\dagger}\left(\mathbf{A}\right)$ and $\hat{\Psi}^{\dagger}\left(\hat{\mathbf{A}}\right)$, that minimize the action functional S. Once the background field(s) obtained, the associated density of agents defined by a given A and a given $\hat{A}$ are:

$$\left|\Psi\left(\mathbf{A}\right)\right|^{2}=\Psi^{\dagger}\left(\mathbf{A}\right)\Psi\left(\mathbf{A}\right) \tag{19}$$

and:

$$\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^{2}=\hat{\Psi}^{\dagger}\left(\hat{\mathbf{A}}\right)\hat{\Psi}\left(\hat{\mathbf{A}}\right) \tag{20}$$

respectively. With these density functions at hand, we can compute various average quantities in the collective state. Actually, the averages for the system in the state defined by $\Psi(\mathbf{A})$ and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$ of components $(\mathbf{A})_k$ or $\left(\hat{\mathbf{A}}\right)_l$ are:

$$\langle(\mathbf{A})_k\rangle = \frac{\int (\mathbf{A})_k |\Psi(\mathbf{A})|^2 d\mathbf{A}}{\int |\Psi(\mathbf{A})|^2 d\mathbf{A}}$$

$$\left\langle\left(\hat{\mathbf{A}}\right)_l\right\rangle = \frac{\int \left(\hat{\mathbf{A}}\right) \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 d\hat{\mathbf{A}}}{\int \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 d\hat{\mathbf{A}}}$$

respectively. We can also define both partial densities and averages by integrating some components and fixing the values of others, as will be detailled in the particular model considered in the next sections.

4.2 Individual level: agents transition functions and their field expression

4.2.1 Transition functions in a classical framework

In a classical perspective, the statistical weight (2) can be used to compute the transition probabilities of the system, i.e. the probabilities for any number of agents of both types to evolve from an initial state $\{\mathbf{A}_l\}_{l=1,\dots}, \left\{\hat{\mathbf{A}}_l\right\}_{l=1,\dots}$ to a final state in a given timespan. These transition functions describe the dynamic of the agents of the system.

To do so, we first compute the integral of equation (2) over all paths between the initial and the final points considered. Defining $\{\mathbf{A}_l(s)\}_{l=1,\dots,N}$ and $\left\{\hat{\mathbf{A}}_l(s)\right\}_{l=1,\dots,\hat{N}}$ the sets of paths for agents of each type, where N and $\hat{N}$ are the numbers of agents of each type, we consider the set of $N+\hat{N}$ independent paths written:

$$\mathbf{Z}(s) = \left(\{\mathbf{A}_l(s)\}_{l=1,\dots,N}, \left\{\hat{\mathbf{A}}_l(s)\right\}_{l=1,\dots,\hat{N}}\right)$$

The weight (2) can now be written $\exp(-W(\mathbf{Z}(s)))$.

The transition functions $T_t\left(\underline{(\mathbf{Z})}, \overline{(\mathbf{Z})}\right)$ compute the probability for the $(N, \hat{N})$ agents to evolve from the initial points $Z(0) \equiv \underline{\mathbf{Z}}$ to the final points $Z(t) \equiv \overline{(\mathbf{Z})}$ during a time span t. This probability is defined by:

$$T_t\left(\underline{\mathbf{Z}}, \overline{(\mathbf{Z})}\right) = \frac{1}{\mathcal{N}} \int_{\substack{\mathbf{Z}(0)\equiv\underline{\mathbf{Z}} \\ \mathbf{Z}(t)\equiv\overline{(\mathbf{Z})}}} \exp(-W(\mathbf{Z}(s))) \mathcal{D}(\mathbf{Z}(s)) \tag{21}$$

The integration symbol $D\mathbf{Z}(s)$ covers all sets of $N \times \hat{N}$ paths constrained by $\mathbf{Z}(0) \equiv \underline{\mathbf{Z}}$ and $\mathbf{Z}(t) \equiv \overline{(\mathbf{Z})}$. The normalisation factor sets the total probability defined by the weight (2) to 1 and is equal to:

$$\mathcal{N} = \int \exp(-W(\mathbf{Z}(s))) \mathcal{D}\mathbf{Z}(s)$$

The interpretation of (21) is straightforward. Instead of studying the full trajectory of one or several agents, we compute their probability to evolve from one configuration to another, and in average, the usual trajectory approach remains valid.

Equation (21) can be generalized to define the transition functions for $k \leqslant N$ and $\hat{k} \leqslant \hat{N}$ agents of each type. The initial and final points respectively for this set of $k+\hat{k}$ agents are written:

$$\mathbf{Z}(0)^{[k,\hat{k}]} \equiv \underline{\mathbf{Z}}^{[k,\hat{k}]}$$

and:

$$\mathbf{Z}(t)^{[k,\hat{k}]} \equiv \overline{(\mathbf{Z})}^{[k,\hat{k}]}$$

The transition function for these agents is written:

$$T_t\left(\underline{(\mathbf{Z})}^{[k,\hat{k}]}, \overline{(\mathbf{Z})}^{[k,\hat{k}]}\right)$$

and the generalization of equation (21) is:

$$T_t\left(\underline{(\mathbf{Z})}^{[k,\hat{k}]}, \overline{(\mathbf{Z})}^{[k,\hat{k}]}\right) = \frac{1}{\mathcal{N}} \int_{\substack{\mathbf{Z}(0)^{[k,\hat{k}]} = \underline{(\mathbf{Z})}^{[k,\hat{k}]} \\ \mathbf{Z}(t)^{[k,\hat{k}]} = \overline{(\mathbf{Z})}^{[k,\hat{k}]}}} \exp\left(-W\left(\left(\mathbf{Z}(s)\right)\right)\right) \mathcal{D}\left(\left(\mathbf{Z}(s)\right)\right) \tag{22}$$

The difference with (21) is that only k paths are constrained by their initial and final points.

Ultimately, the Laplace transform of $T_t\left(\underline{(Z)}^{[k,\hat{k}]}, \overline{(Z)}^{[k,\hat{k}]}\right)$ computes the - time averaged - transition function for agents with random lifespan of mean $\frac{1}{\alpha}$, up to a factor $\frac{1}{\alpha}$, and is given by:

$$G_\alpha\left(\underline{(\mathbf{Z})}^{[k,\hat{k}]}, \overline{(\mathbf{Z})}^{[k,\hat{k}]}\right) = \int_0^\infty \exp(-\alpha t)\, T_t\left(\underline{(\mathbf{Z})}^{[k,\hat{k}]}, \overline{(\mathbf{Z})}^{[k,\hat{k}]}\right) dt \tag{23}$$

This formulation of the transition functions is relatively intractable. Therefore, we will now propose an alternative method based on the field model.

4.2.2 Field-theoretic expression

The transition functions (22) and (23) can be retrieved using the field theory transition functions - or Green functions, which compute the probability for a variable number $\left(k, \hat{k}\right)$ of agents to transition from an initial state $\underline{(\mathbf{Z}, \boldsymbol{\theta})}^{[k,\hat{k}]}$ to a final state $\overline{(\mathbf{Z}, \boldsymbol{\theta})}^{[k,\hat{k}]}$, where $\underline{(\boldsymbol{\theta})}^{[k,\hat{k}]}$ and $\overline{(\boldsymbol{\theta})}^{[k,\hat{k}]}$ are vectors of initial and final times for $k + \hat{k}$ agents respectively.

We will write:

$$T_t\left(\underline{(\mathbf{Z}, \boldsymbol{\theta})}^{[k,\hat{k}]}, \overline{(\mathbf{Z}, \boldsymbol{\theta})}^{[k,\hat{k}]}\right)$$

the transition function between $\underline{(\mathbf{Z}, \boldsymbol{\theta})}^{[k,\hat{k}]}$ and $\overline{(\mathbf{Z}, \boldsymbol{\theta})}^{[k,\hat{k}]}$ with $\overline{(\boldsymbol{\theta})}_i < t$, $\forall i$, and:

$$G_\alpha\left(\underline{(\mathbf{Z}, \boldsymbol{\theta})}^{[k,\hat{k}]}, \overline{(\mathbf{Z}, \boldsymbol{\theta})}^{[k,\hat{k}]}\right)$$

its Laplace transform. Setting $\underline{(\boldsymbol{\theta})}_i = 0$ and $\overline{(\boldsymbol{\theta})}_i = t$ for $i = 1, ..., k + \hat{k}$, these functions reduce to (22) or (23): the probabilistic formalism of the transition functions is thus a particular case of the field formalism definition. In the sequel we therefore will use the term transition function indiscriminately.

The computation of the transition functions relies on the fact that $\exp(-S(\Psi))$ itself represents a statistical weight for the system. Gosselin, Lotz, Wambst (2020) showed that $S(\Psi)$ can be modified in a straightforward manner to include source terms:

$$S(\Psi, J) = S(\Psi) + \int \left(J(Z,\theta)\,\Psi^\dagger(Z,\theta) + J^\dagger(Z,\theta)\,\Psi(Z,\theta)\right) d(Z,\theta) \tag{24}$$

where $J(Z,\theta)$ is an arbitrary complex function, or auxiliary field.

Introducing $J(Z,\theta)$ in $S(\Psi,J)$ allows to compute the transition functions by successive derivatives. Actually, we can show that:

$$G_{\alpha}\left(\underline{(\mathbf{Z},\theta)}^{[k,\hat{k}]},\overline{(\mathbf{Z},\theta)}^{[k,\hat{k}]}\right)=\left[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{(\mathbf{Z},\theta)}_{i_l}\right)}\frac{\delta}{\delta J^{\dagger}\left(\overline{(\mathbf{Z},\theta)}_{i_l}\right)}\right)\int\exp\left(-S\left(\Psi,J\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}\right]_{J=J^{\dagger}=0} \tag{25}$$

where the notation $\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}$ denotes an integration over the space of functions $\Psi(Z,\theta)$ and $\Psi^{\dagger}(Z,\theta)$, i.e. an integral in an infinite dimensional space. Even though these integrals can only be computed in simple cases, a series expansion of $G_{\alpha}\left(\underline{(\mathbf{Z},\theta)}^{[k,\hat{k}]},\overline{(\mathbf{Z},\theta)}^{[k,\hat{k}]}\right)$ can be found using Feynman graphs techniques.

Once $G_{\alpha}\left(\underline{(\mathbf{Z},\theta)}^{[k,\hat{k}]},\overline{(\mathbf{Z},\theta)}^{[k,\hat{k}]}\right)$ is computed, the expression of $T_t\left(\underline{(\mathbf{Z},\theta)}^{[k,\hat{k}]},\overline{(\mathbf{Z},\theta)}^{[k,\hat{k}]}\right)$ can be retrieved in principle by an inverse Laplace transform. In field theory, formula (25) shows that the transition functions (23) are correlation functions of the field theory with action $S(\Psi)$.

5 Computing Transition Functions using Field Theory

The formula (25) provides a precise and compact definition of the transition functions for multiple agents in the system. However, in practice, this formula is not directly applicable and does not shed much light on the connection between the collective and microeconomic aspects of the considered system. To calculate the dynamics of the agents, we will proceed in three steps.

Firstly, we will minimize the system's action functional and determine the background field, which represents the collective state of the system. Once the background field is found, we will perform a series expansion of the action functional around this background field, referred to as the effective action of the system. It is with this effective action that we can compute the transition functions for the state defined by the background field. We will discover that each term in this expansion has an interpretation in terms of a transition function.

Instead of directly computing the transition functions, we can consider a series expansion of the action functional around a specific background field of the system.

5.1 Step 1: finding the background field

For a particular type of agent, background fields are defined as the fields $\Psi_0(Z,\theta)$ that maximize the statistical weight $\exp(-S(\Psi))$ or, alternatively, minimize $S(\Psi)$:

$$\frac{\delta S(\Psi)}{\delta\Psi}\mid_{\Psi_0(Z,\theta)}=0,\ \frac{\delta S\left(\Psi^{\dagger}\right)}{\delta\Psi^{\dagger}}\mid_{\Psi_0^{\dagger}(Z,\theta)}=0$$

The field $\Psi_0(Z,\theta)$ represents the most probable configuration, a specfic state of the entire system that influences the dynamics of agents. It serves as the background state from which probability transitions and average values can be computed. As we will see, the agents' transitions explicitely depend on the chosen background field $\Psi_0(Z,\theta)$, which represents the macroeconomic state in which the agents evolve.

When considering two or more types of agents, the background field satisfies the following condition:

$$\frac{\delta S\left(\Psi,\hat{\Psi}\right)}{\delta\Psi}\mid_{\Psi_0(Z,\theta)}=0,\ \frac{\delta S\left(\Psi,\hat{\Psi}\right)}{\delta\Psi^{\dagger}}\mid_{\Psi_0^{\dagger}(Z,\theta)}=0$$
$$\frac{\delta S\left(\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}}\mid_{\hat{\Psi}_0(Z,\theta)}=0,\ \frac{\delta S\left(\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}^{\dagger}}\mid_{\hat{\Psi}_0^{\dagger}(Z,\theta)}=0$$

5.2 Step 2: Series expansion around the background field

In a given background state, the *effective action*[11] is the series expansion of the field functional $S(\Psi)$ around $\Psi_0(Z,\theta)$. We will present the expansion for one type of agent, but generalizing it to two or several agents is straightforward.

The series expansion around the background field simplifies the computations of transition functions and provides an interpretation of these functions in terms of individual interactions within the collective state. To perform this series expansion, we decompose Ψ as:

$$\Psi=\Psi_0+\Delta\Psi$$

and write the series expansion of the action functional:

$$\begin{aligned}S(\Psi) &= S(\Psi_0)+\int\Delta\Psi^{\dagger}(Z,\theta)\,O(\Psi_0(Z,\theta))\,\Delta\Psi(Z,\theta) \\ &\quad+\sum_{k>2}\int\prod_{i=1}^{k}\Delta\Psi^{\dagger}(Z_i,\theta)\,O_k(\Psi_0(Z,\theta),(Z_i))\prod_{i=1}^{k}\Delta\Psi(Z_i,\theta)\end{aligned}\tag{26}$$

The series expansion can be interpreted economically as follows. The first term, $S(\Psi_0)$, describes the system of all agents in a given macroeconomic state, Ψ_0. The other terms potentially describe all the fluctuations or movements of the agents around this macroeconomic state considered as given. Therefore, the expansion around the background field represents the microeconomic reality of a historical macroeconomic state. More precisely, it describes the range of microeconomic possibilities allowed by a macroeconomic state.

The quadratic approximation is the first term of the expansion and can be written as:

$$S(\Psi)=S(\Psi_0)+\int\Delta\Psi^{\dagger}(Z,\theta)\,O(\Psi_0(Z,\theta))\,\Delta\Psi(Z,\theta)\tag{27}$$

It will allow us to find the transition functions of agents in the historical macro state, where all interactions are averaged. The other terms of the expansion allow us to detail the interactions within the nebula, and are written as follows:

$$\sum_{k>2}\int\prod_{i=1}^{k}\Delta\Psi^{\dagger}(Z_i,\theta)\,O_k(\Psi_0(Z,\theta),(Z_i))\prod_{i=1}^{k}\Delta\Psi(Z_i,\theta)$$

They detail, given the historical macroeconomic state, how the interactions of two or more agents can impact the dynamics of these agents. Mathematically, this corresponds to correcting the transition probabilities calculated in the quadratic approximation.

Here, we provide an interpretation of the third and fourth-order terms.

[11] Actually, this paper focuses on the *classical effective action*, which is an approximation sufficient for the computations at hand.

The third order introduces possibilities for an agent, during its trajectory, to split into two, or conversely, for two agents to merge into one. In other words, the third-order terms take into account or reveal, in the historical macroeconomic environment, the possibilities for any agent to undergo modifications along its trajectory. However, this assumption has been excluded from our model.

The fourth order reveals that in the macroeconomic environment, due to the presence of other agents and their tendency to occupy the same space, all points in space will no longer have the same probabilities for an agent. In fact, the fourth-order terms reveal the notion of geographical or sectoral competition and potentially intertemporal competition. These terms describe the interaction between two agents crossing paths, which leads to a deviation of their trajectories due to the interaction.

We do not interpret higher-order terms, but the idea is similar. The even-order terms (2n) describe interactions among n agents that modify their trajectories. The odd-order terms modify the trajectories but also include the possibility of agents reuniting or splitting into multiple agents. We will see in more detail how these terms come into play in the transition functions.

5.3 Step 3: Computation of the transition functions

5.3.1 Quadratic approximation

In the first approximation, transition functions in a given background field $\Psi_0(Z,\theta)$ can be computed by replacing $S(\Psi)$ in (25), with its quadratic approximation (27). In formula (27), $O(\Psi_0(Z,\theta))$ is a differential operator of second order. This operator depends explicitly on $\Psi_0(Z,\theta)$. As a consequence, transition functions and agent dynamics explicitly depend on the collective state of the system. In this approximation, the formula for the transition functions (25) becomes:

$$\begin{aligned} G_{\alpha}\left(\underline{(Z,\theta)}^{[k]},\overline{(Z,\theta)}^{[k]}\right) &= \left[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{(Z,\theta)}_{i_l}\right)}\frac{\delta}{\delta J^{\dagger}\left(\overline{(Z,\theta)}_{i_l}\right)}\right)\right. \\ &\left.\times\int\exp\left(-\int\Delta\Psi^{\dagger}(Z,\theta)\,O\left(\Psi_0(Z,\theta)\right)\Delta\Psi(Z,\theta)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}\right]_{J=J^{\dagger}=0} \end{aligned} \tag{28}$$

Using this formula, we can show that the one-agent transition function is given by:

$$G_{\alpha}\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right)=O^{-1}\left(\Psi_0(Z,\theta)\right)\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right) \tag{29}$$

where:

$$O^{-1}\left(\Psi_0(Z,\theta)\right)\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right)$$

is the kernel of the inverse operator $O^{-1}\left(\Psi_0(Z,\theta)\right)$. It can be seen as the $\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right)$ matrix element of $O^{-1}\left(\Psi_0(Z,\theta)\right)$[12].

More generally, the k-agents transition functions are the product of individual transition functions:

$$G_{\alpha}\left(\underline{(Z,\theta)}^{[k]},\overline{(Z,\theta)}^{[k]}\right)=\prod_{i=1}^{k}G_{\alpha}\left(\underline{(Z,\theta)_i}^{[1]},\overline{(Z,\theta)_i}^{[1]}\right) \tag{30}$$

[12] The differential operator $O\left(\Psi_0(Z,\theta)\right)$ can be seen as an infinite dimensional matrix indexed by the double (infinite) entries $\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right)$. With this description, the kernel $O^{-1}\left(\Psi_0(Z,\theta)\right)\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right)$ is the $\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right)$ element of the inverse matrix.

The above formula shows that in the quadratic approximation, the transition probability from one state to another for a group is the product of individual transition probabilities. In this approximation, the trajectories of these agents are therefore independent. The agents do not interact with each other and only interact with the environment described by the background field.

The quadratic approximation must be corrected to account for individual interactions within the group, by including higher-order terms in the expansion of the action.

5.3.2 Higher-order corrections

To compute the effects of interactions between agents of a given group, we consider terms of order greater than 2 in the effective action. These terms modify the transition functions. Writing the expansion:

$$\exp\left(-S\left(\Psi\right)\right)=\exp\left(-\left(S\left(\Psi_0\right)+\int\Delta\Psi^{\dagger}\left(Z,\theta\right)O\left(\Psi_0\left(Z,\theta\right)\right)\right)\right)\left(1+\sum_{n\geqslant 1}\frac{A^n}{n!}\right)$$

where:

$$A=\sum_{k>2}\int\prod_{i=1}^{k}\Delta\Psi^{\dagger}\left(Z_i,\theta\right)O\left(\Psi_0\left(Z,\theta\right),\left(Z_i\right)\right)\prod_{i=1}^{k}\Delta\Psi\left(Z_i,\theta\right)$$

is the sum of all possible interaction terms, leads to the series expansion of (25):

$$\begin{aligned}G_{\alpha}\left(\underline{\left(Z,\theta\right)}^{[k]},\overline{\left(Z,\theta\right)}^{[k]}\right) &= \left[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{\left(Z,\theta\right)}_{i_l}\right)}\frac{\delta}{\delta J^{\dagger}\left(\overline{\left(Z,\theta\right)}_{i_l}\right)}\right)\right. \\ &\left.\int\exp\left(-\int\Delta\Psi^{\dagger}\left(Z,\theta\right)O\left(\Psi_0\left(Z,\theta\right)\right)\Delta\Psi\left(Z,\theta\right)\right)\left(1+\sum_{n\geqslant 1}\frac{A^n}{n!}\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}\right]_{J=J^{\dagger}=0}\end{aligned} \tag{31}$$

These corrections can be computed using graphs' expansion.

More precisely, the first term of the series:

$$\left[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{\left(Z,\theta\right)}_{i_l}\right)}\frac{\delta}{\delta J^{\dagger}\left(\overline{\left(Z,\theta\right)}_{i_l}\right)}\right)\int\exp\left(-\int\Delta\Psi^{\dagger}\left(Z,\theta\right)O\left(\Psi_0\left(Z,\theta\right)\right)\Delta\Psi\left(Z,\theta\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}\right]_{J=J^{\dagger}=0} \tag{32}$$

is a transition function in the quadratic approximation. The other contributions of the series expansion correct the approximated n agents transtns functions (30).

Typically a contribution:

$$\begin{aligned}G_{\alpha}\left(\underline{\left(Z,\theta\right)}^{[k]},\overline{\left(Z,\theta\right)}^{[k]}\right) &= \left[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{\left(Z,\theta\right)}_{i_l}\right)}\frac{\delta}{\delta J^{\dagger}\left(\overline{\left(Z,\theta\right)}_{i_l}\right)}\right)\right. \\ &\left.\int\exp\left(-\int\Delta\Psi^{\dagger}\left(Z,\theta\right)O\left(\Psi_0\left(Z,\theta\right)\right)\Delta\Psi\left(Z,\theta\right)\right)\frac{A^n}{n!}\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}\right]_{J=J^{\dagger}=0}\end{aligned} \tag{33}$$

can be depicted by a graph. The power $\frac{A^n}{n!}$ translates that agents interact n times along their path. The trajectories of each agent of the group is broken n times between its initial and final points. At each time of interaction the trajectories of agents are deviated. To such a graph is associated a probility that modifies the quadratic approximation transition functions.

In the sequel we will only focus on the first order corrections to the two-agents transition functions.

Part 1. System of investors and firms

6 Microeconomic framework

Let us now present the microeconomic framework that will be turned into a field model using our general method. The interactions between the real and the financial economy are pictured by two groups of agents, producers, and investors. In the following, we will refer to producers or firms i indistinctively, and use the upper script $\hat{}$ for variables describing investors.

6.1 Investors' allocation of capital

Each investor, denoted by j, is characterized by their level of disposable capital, $\hat{K}_j$, and their position, $\hat{X}_j$, in the sector space. The disposable capital for each investor at time t is the sum of their private capital plus the total amount of participations and loans entrusted to them by other investors.

Investors have the flexibility to invest this disposable capital across the entire sector space, but they typically prefer sectors that are close to their position. At the end of each period, any lent capital is repaid with interest, if feasible.

Investors tend to diversify their capital in four ways: each investor j chooses to allocate his entire capital $\hat{K}_j$ among various firms i or among other investors. In each scenario, this allocation can manifest either as a participation in the ownership of the firm or as a credit extended to other investors at a specified interest rate.

6.1.1 Allocation to firms

The capital allocated by investor j to firm i, denoted as $\hat{K}_j^{(i)}$, is given by:

$$\hat{K}_j^{(i)}(t) = \left(\hat{F}_2\left(R_i, \hat{X}_j\right)\hat{K}_j\right)(t) \tag{34}$$

where:

$$\hat{F}_2\left(R_i, \hat{X}_j\right) = \frac{F_2(R_i)\,G\left(X_i - \hat{X}_j\right)}{\sum_l F_2(R_l)\,G\left(X_l - \hat{X}_j\right)} \tag{35}$$

where F_2 is an arbitrary function that depends on the expected return of firm i and the distance between sectors X_i and $\hat{X}_j$. The function $\hat{F}_2\left(R(K_i, X_i), \hat{X}_j\right)$ is thus the function F_2, expressed as share of invested capital in a specific firm. It captures the dependency of investments on firms' relative attractivity. The equation (34) is thus a general form of risk-averse portfolio allocation[13].

[13] Actually, an investor allocating capital exclusively in a sector X_i and optimizing the function:

$$\frac{R_i}{\sum_l R_l}s_j - s_j^2 Var\left(\frac{R_i}{\sum_l R_l}\right) \tag{36}$$

where the share s_j satisfies $\sum s_j = 1$, will set $s_j = \frac{R_i}{\sum_l R_l}$. If we were to introduce the possibility of investing in multiple sectors and consider more general preferences than this simple quadratic function, we should introduce the functions $G\left(X_i - \hat{X}_j\right)$ and $F_2(R_i)$ in the solutions of (36), leading to (34).

Each agent allocated capital is divided between loans and equity participation, expressed as:

$$\left(\hat{F}_2\left(R_i, \hat{X}_j\right)\hat{K}_j\right)(t) = k_1\left(X_i, \hat{X}_j\right)\left(\hat{F}_2\left(R_i, \hat{X}_j\right)\hat{K}_j\right)(t) + k_2\left(X_i, \hat{X}_j\right)\left(\hat{F}_2\left(R_i, \hat{X}_j\right)\hat{K}_j\right)(t)$$

where k_{1ij} represents the proportion allocated to loans and k_{2ij} represents the proportion allocated to equity participation, such that $k_1\left(X_i, \hat{X}_j\right)+k_2\left(X_i, \hat{X}_j\right) = 1$. The allocated capital (34) constitutes only a portion of investor j's total capital. Below, we will express it as:

$$k_i\left(X_i, \hat{X}_j\right)\left(\hat{F}_2\left(R_i, \hat{X}_j\right)\hat{K}_j\right) \rightarrow k_i\left(X_i, \hat{X}_j\right)$$

6.1.2 Allocation to other investors

The rest of capital is invested between various other investors. Investor j acquires equity stakes in other investors, up to the amount:

$$\sum_l \hat{k}_1\left(\hat{K}_l(t), \hat{K}_j(t)\right)\hat{K}_j(t)$$

where $\hat{k}_1\left(\hat{K}_l(t), \hat{K}_j(t)\right)\hat{K}_j(t)$ represents the capital invested by investor j in the investor l. Investor j also lends a portion of its capital to other investors, up to an amount:

$$\sum_l \hat{k}_2\left(\hat{K}_l(t), \hat{K}_j(t)\right)\hat{K}_j(t)$$

where $\hat{k}_2\left(\hat{K}_l(t), \hat{K}_j(t)\right)\hat{K}_j(t)$ represents the capital lent from investor j to investor l, and $\hat{k}_i\left(\hat{X}_l(t), \hat{X}_j(t)\right)$ depends on the return R_l provided by investor l:

$$\hat{k}_i\left(\hat{X}_l(t), \hat{X}_j(t), R_l\right)$$

Thus $\hat{K}_j(t)$ decomposes as:

$$\begin{aligned}
\hat{K}_j(t) &= \sum_l \hat{k}_1\left(\hat{K}_l(t), \hat{K}_j(t)\right)\hat{K}_j(t) + \sum_l \hat{k}_2\left(\hat{K}_l(t), \hat{K}_j(t)\right)\hat{K}_j(t) + \sum_i\left(\hat{F}_{2,1}\left(R_i, \hat{X}_j\right)\hat{K}_j\right) \\
&= \sum_l \hat{k}_1\left(\hat{K}_l(t), \hat{K}_j(t)\right)\hat{K}_j(t) + \sum_l \hat{k}_2\left(\hat{K}_l(t), \hat{K}_j(t)\right)\hat{K}_j(t) \\
&\quad + \sum_i\left(k_1\left(X_i, K_i, \hat{X}_j\right) + k_2\left(X_i, K_i, \hat{X}_j\right)\right)\hat{K}_j(t)
\end{aligned}$$

or equivalently:

$$\sum_l \hat{k}\left(\hat{X}_l(t), \hat{X}_j(t)\right) + \sum_i \hat{F}_{2,1}\left(R_i, \hat{X}_j\right) = 1$$

with:

$$\hat{k}\left(\hat{X}_l(t), \hat{X}_j(t)\right) = \hat{k}_1\left(\hat{K}_l(t), \hat{K}_j(t)\right) + \hat{k}_2\left(\hat{K}_l(t), \hat{K}_j(t)\right)$$

Investor j's private capital, denoted as $\hat{K}_{jp}(t)$, can be expressed as a function of disposable capital, $\hat{K}_j(t)$, which includes both investor j's private capital, $\hat{K}_{jp}(t)$, and the capital entrusted to him by other investors. We can decompose the latter into shares of other investors in investor j, and lent capital. Disposable capital $\hat{K}_j(t)$ of investor j thus writes:

$$\hat{K}_j(t) = \hat{K}_{jp}(t) + \sum_l\left(\hat{k}_1\left(\hat{K}_{jp}(t), \hat{X}_j(t), \hat{X}_l(t)\right) + \hat{k}_2\left(\hat{K}_{jp}(t), \hat{X}_j(t), \hat{X}_l(t)\right)\right)\hat{K}_l(t)$$

Inversely, private capital $\hat{K}_{jp}$ can be expressed, in the linear approximation[14], as :

$$\hat{K}_{jp}(t) = \frac{\hat{K}_j(t)}{1 + \sum_l \left(\hat{k}_{1jl} + \hat{k}_{2jl}\right) \hat{K}_l(t)} \tag{37}$$

where:

$$\begin{aligned} \hat{k}_{ajl} &= \hat{k}_a\left(\hat{X}_j(t), \hat{X}_l(t)\right) \\ k_{ajl} &= k_a\left(X_j(t), \hat{X}_l(t)\right) \end{aligned}$$

and:

$$\begin{aligned} \hat{k}_{jl} &= \hat{k}_{1jl} + \hat{k}_{2jl} \\ k_{jl} &= k_{1jl} + k_{2jl} \end{aligned}$$

6.1.3 Investors' disposable capital

The investors' disposable capital can be either decomposed according to its source or to its allocation.

Disposable capital comes from the investor private capital plus the capital entrusted to him by other investors. As such, total disposable capital $\hat{K}_j(t)$ can be expressed as:

$$\hat{K}_j(t) = \frac{\hat{K}_j(t)}{1 + \sum_v \hat{k}_{jv}\hat{K}_v(t)} + \frac{\sum_v \hat{k}_{2jv}\hat{K}_j(t)\hat{K}_v(t)}{1 + \sum_v \hat{k}_{jv}\hat{K}_v(t)} + \frac{\sum_v \hat{k}_{1jv}\hat{K}_j(t)\hat{K}_v(t)}{1 + \sum_v \hat{k}_{jv}\hat{K}_v(t)}$$

that are private capital, loans, and participations from other investors respectively.

Disposable capital is decomposed into several investments. It writes:

$$\begin{aligned} \hat{K}_j(t) &= \sum_i k_{1ij} K_{ip} \hat{K}_j(t) + \sum_i k_{2ij} K_{ip} \hat{K}_j(t) + \sum_l \hat{k}_{1lj} \hat{K}_{lp}(t) \hat{K}_j(t) + \sum_l \hat{k}_{2lj} \hat{K}_{lp}(t) \hat{K}_j(t) \\ &= \sum_i \frac{k_{1ij} K_i}{1 + \sum_v k_{iv} \hat{K}_v(t)} \hat{K}_j(t) + \sum_i \frac{k_{2ij} K_i}{1 + \sum_v k_{iv} \hat{K}_v(t)} \hat{K}_j(t) \\ &\quad + \sum_l \frac{\hat{k}_{1lj} \hat{K}_l(t)}{1 + \sum_v \hat{k}_{jv} \hat{K}_v(t)} \hat{K}_j(t) + \sum_l \frac{\hat{k}_{2lj} \hat{K}_l(t)}{1 + \sum_v \hat{k}_{jv} \hat{K}_v(t)} \hat{K}_j(t) \end{aligned}$$

and implies the constraint, for all j:

$$\sum_i \frac{k_{ij} K_i}{1 + \sum_v k_{iv} \hat{K}_v(t)} + \sum_l \frac{\hat{k}_{lj} \hat{K}_l(t)}{1 + \sum_v \hat{k}_{jv} \hat{K}_v(t)} = 1$$

6.1.4 Firms' disposable capital

Firms' disposable capital is defined by the sum of their private capital, investor participations, and loans. In the linear approximation and assuming investments are proportional to $K_{ip}(t)$, disposable capital for firm i is defined by:

$$K_i(t) = K_{ip}(t) + \left(\sum_v (k_{1iv} + k_{2iv}) \hat{K}_v(t)\right) K_{ip}(t) \tag{38}$$

[14] See Appendix 1.

where K_{ip} is the private capital, which also writes:

$$\begin{aligned} K_{ip}(t) &= \frac{K_i(t)}{1+\sum_v (k_{1iv}+k_{2iv})\hat{K}_v(t)} \\ &= \frac{K_i(t)}{1+\sum_v k_{iv}\hat{K}_v(t)} \end{aligned}$$

6.2 Capital accumulation under a no-default scenario

6.2.1 Firms' private returns

Producers are a set of firms, operating each in a single sector, so that a single firm with subsidiaries in different countries and/or offering differentiated products can be modeled as a set of independent firms. Similarly, a sector refers to a set of firms with similar productions, so that sectors can be decomposed into sectors per country to account for local specificities, or in several sectors.

Firms move across a vector space of sectors, which is of arbitrary dimension. Firms are defined by their respective sector X_i and physical capital K_i, two variables subject to dynamic changes. They may change their capital stocks over time or altogether shift sectors.

Each firm produces a single differentiated good. However, rather than dealing with a full system of producers-consumers, we will merely consider the return each producer may provide to its investors. This is sufficient for our goal to study financial capital circulation and diffusion.

The return of producer i at time t, denoted r_i, depends on K_i, X_i and on the level of competition in the sector. It is written:

$$r_i = r(K_i, X_i) - \gamma \sum_j \delta(X_i - X_j)\frac{K_j}{K_i} \tag{39}$$

where $\delta(X_i - X_j)$ is the Dirac δ function which is equal to 0 for $X_i \neq X_j$. The first term in formula (39) is an arbitrary function that depends on the sector and the level of capital per firm in this sector. It represents the return of capital in a specific sector X_i under no competition. We deliberately keep the form of $r(K_i, X_i)$ unspecified, since most of the results of the model rely on the general properties of the functions involved. When needed, we will give a standard Cobb-Douglas form to the returns $r(K_i, X_i)$. The second term in (39) is the decreasing return of capital. In any given sector, it is proportional to both the number of competitors and the specific level of capital per firm used.

We also assume that, for all i, firm i has a market valuation defined by both its price, P_i, and the variation of this price on financial markets, $\dot{P}_i$. This variation is itself assumed to be a function of an expected long-term return denoted R_i, or more precisely the relative return $\bar{R}_i$ of firm i against the whole set of firms:

$$\frac{\dot{P}_i}{P_i} = F_1\left(\bar{R}_i, \frac{\dot{K}_i}{K_i} - \left\langle \frac{\dot{K}_i}{K_i} \right\rangle\right) \tag{40}$$

where $\frac{\dot{K}_i}{K_i} - \left\langle \frac{\dot{K}_i}{K_i} \right\rangle$ computes the relative variation of capital with respect to the average.

Formula (40) includes the main features of models of price dynamics. In this equation, the time dependency of variables is implicit.

Formula (40) reflects the impact of capital and location on the price of the firm through its expected returns. It also reflects how variations in capital impact its growth prospects, through competition and dividends (see (39)). Actually, the higher the capital of the firm, the lower impact of competition and the higher the dividends.

We assume R_i to have the general form:

$$R_i = R(K_i, X_i)$$

Expected long-term returns depend on the capital and sector in which the firm operates, but also on external parameters, such as technology, ... which are encoded in the shape of $R(K_i, X_i)$.

The relative return $\bar{R}_i$ arising in (40) is defined by:

$$\bar{R}_i = \bar{R}(K_i, X_i) = \frac{R_i}{\sum_l R_l} \tag{41}$$

The function F_1 in (40) is arbitrary and reflects the preferences of the market relatively to the firm's relative returns. We will allow the number of firms per sector to vary around some sector-dependent exogeneous density[15].

6.2.2 Firms' returns to investors

The total return of the firm to the investor is given by the sum:

$$r_i'\left(K_i(t), \frac{\dot{K}_i}{K_i}\right) = r_i + F_1\left(\bar{R}_i, \frac{\dot{K}_i}{K_i}\right)$$

where r_i represents dividends, and $F_1\left(\bar{R}_i, \frac{\dot{K}_i}{K_i}\right)$ denotes the variation in stock price.

We rewrite $r_i'\left(K_i(t), \frac{\dot{K}_i}{K_i}\right)$ by specifying[16] the form of $F_1\left(\bar{R}_i, \frac{\dot{K}_i}{K_i}\right)$, and find:

$$r_i'\left(K_i(t), \frac{\dot{K}_i}{K_i}\right) = r_i + F_1\left(\bar{R}(K_i, X_i)\right) + \tau\left(\bar{R}(K_i, X_i)\right)\Delta f_1'(K_i(t))$$

with:

$$\Delta f_1'(K_i(t)) = f_1'(K_i(t)) - \langle f_1'(K_i(t))\rangle$$

Here, the average $\langle f_1'(K_i(t))\rangle$ is calculated over all firms. The expression for $r_i'\left(K_i(t), \frac{\dot{K}_i}{K_i}\right)$ reflects that the return on the variation of stock prices depends on expectations of returns $F_1\left(\bar{R}(K_i, X_i)\right)$ and firm growth.

6.2.3 Firms' private capital accumulation

The gain realized by the firm corresponds to its private capital plus its loans muliplied by the return, from which we substract the amount paid back for loans, where $\bar{r}$ is the loan interest rate. The increase in capital is thus:

$$K_{ip}(t+1) - K_{ip}(t) = \left(1 + \sum_{\nu} k_{2j\nu}\hat{K}_{\nu}(t)\right) r_i K_{ip}(t) - \bar{r}\sum_{v} k_{2lv}\hat{K}_v(t) K_{ip}(t)$$

or, in a continuous approximation, as:

$$\frac{d}{dt}K_{ip}(t) = f_1'(K_i(t)) K_{ip}(t) \tag{42}$$

with:

$$f_1'(K_i(t)) = \left(1 + \sum_{\nu} k_{2j\nu}\hat{K}_{\nu}(t)\right) r_i - \bar{r}\sum_{v} k_{2lv}\hat{K}_v(t) \tag{43}$$

[15] Here, we depart from our previous paper, as we do not assume that firms relocate in the sector space according to returns. Instead, we are interested in financial capital allocation, which operates on a shorter time scale compared to firm dynamics.

[16] See Appendix 1.

6.2.4 Investors' returns

For investors, we focus on the dynamics of disposable capital. The investor borrows and manages the stakes entrusted to them, so their investment decisions commit sums that far exceed their private capital. Unlike firms, which produce, the investor redistributes this available capital, which in turn contributes to available capital. The return for $\hat{K}_j(t)$, denoted R_j decomposes into two parts.

Direct returns to investor j First, returns directly obtained by agent j:

$$\left(1+\sum_v \hat{k}_{2jv}\hat{K}_v(t)\right)\hat{K}_{jp}(t)\,R'_j$$

Here R'_j is the return obtained by investing in firms, funds, or lending capital, written in terms of disposable capital, as:

$$\frac{1+\sum_v \hat{k}_{2jv}\hat{K}_v(t)}{1+\sum_v\left(\hat{k}_{1jv}+\hat{k}_{2jv}\right)\hat{K}_v(t)}R'_j\hat{K}_j(t)$$

and per unit of capital, as :

$$\frac{1+\sum_v \hat{k}_{2jv}\hat{K}_v(t)}{1+\sum_v\left(\hat{k}_{1jv}+\hat{k}_{2jv}\right)\hat{K}_v(t)}R'_j$$

Indirect returns due to other investors The other component of the return comes from investing in other investors, yielding a return per unit of disposable capital:

$$\sum_l \frac{\hat{k}_{1lj}\hat{K}_l(t)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v(t)}R_l$$

where R_l is the return of investors l in which agent j has invested.

The total return therefore satisfies:

$$R_j=R'_j+\sum_l \frac{\hat{k}_{1lj}\hat{K}_l(t)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v(t)}R_l \tag{44}$$

The direct return R'_j decomposes into loans to firms, loans to investors, and participations in firms and other investors:

$$\begin{aligned}R'_j &= \bar{r}\sum_l \frac{\hat{k}_{2lj}\hat{K}_l(t)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v(t)}+\bar{r}\sum_i \frac{k_{2ij}K_i(t)}{1+\sum_v\left(k_{1iv}+k_{2iv}\right)\hat{K}_v(t)}\\ &\quad+\sum_i\left(r_i+F_1\left(\bar{R}_i,\frac{\dot{K}_i(t)}{K_i(t)}\right)+\tau\left(\bar{R}(K_i,X_i)\right)\Delta f'_1(K_i(t))\right)\frac{k_{1ij}K_i(t)}{1+\sum_v\left(k_{1iv}+k_{2iv}\right)\hat{K}_v(t)}\end{aligned} \tag{45}$$

6.2.5 Investors' capital accumulation

Appendix 1.4 shows that, in the continuous approximation, the dynamics for an investor i disposable capital is :

$$\frac{d}{dt}\hat{K}_j(t)=\sum_l\left(1-\frac{\hat{k}_{1jl}\hat{K}_j(t)}{1+\sum_v \hat{k}_{jv}\hat{K}_v(t)}\right)^{-1}\hat{f}_l\hat{K}_l(t)=\sum_l(1-M)^{-1}_{jl}\,\hat{f}_l\hat{K}_l(t) \tag{46}$$

where $\hat{f}_j$ represents the total return of investor j, calculated as the sum of all capital committed by the investor multiplied by their return R_j, minus the interest payments on loans:

$$\hat{f}_j = \left(1 + \sum_v \hat{k}_{2jv} \hat{K}_v(t)\right) R_j - \bar{r} \sum_v \hat{k}_{2jv} \hat{K}_v(t) \tag{47}$$

and matrix M corresponds to the shares taken by investor j in other investors. This is why the return of investor j depends on other investors in the equation (46). The expression for M is:

$$M_{jm} = \frac{\hat{k}_{1jm} \hat{K}_j(t)}{1 + \sum_\nu \hat{k}_{j\nu} \hat{K}_\nu(t)} \tag{48}$$

which indeed corresponds to the share of participation of investor j in investor m.

We also show that the return equation writes:

$$\begin{aligned}\frac{\hat{f}_j - \bar{r}}{1 + \sum_v \hat{k}_{2jv} \hat{K}_v(t)} &= \sum_l \left(\frac{\hat{k}_{1lj} \hat{K}_l(t)}{1 + \sum_v \left(\hat{k}_{1lv} + \hat{k}_{2lv}\right) \hat{K}_v(t)}\right) \left(\frac{\hat{f}_l - \bar{r}}{1 + \sum_v \hat{k}_{2lv} \hat{K}_v(t)}\right) \\ &+ \sum_i \frac{\left(r_i + F_1\left(\bar{R}_i, \frac{\dot{K}_i(t)}{K_i(t)}\right) + \tau\left(\bar{R}(K_i, X_i)\right) \Delta f_1'(K_i(t)) - \bar{r}\right) k_{1ij} K_i(t)}{1 + \sum_v (k_{1iv} + k_{2iv}) \hat{K}_v(t)}\end{aligned} \tag{49}$$

This equation decomposes the investor's excess return over the interest rate into two components: the returns to the investor from the returns of other investors in which it has taken a stake, plus the direct returns from the firms in which it has taken a stake.

6.3 Capital accumulation under a default scenario

Until now, we had derived the equations for capital accumulation under the assumption of zero default. Loans were repaid, and there was no loss of capital. We will now introduce the possibility of defaults occuring in our setup.

6.3.1 Defaults of firms

Firms' capital accumulation (42) is only valid when private capital at the end of each period exceeds the loans to be repaid. When this is not the case, the firm defaults. The default condition is that the overall return R_j, on private capital, private loans, and bank loans, does not generate sufficient funds to repay with interest the amount borrowed at rate $\bar{r}$, which is expressed as:

$$K_{ip} \left(1 + \sum_m k_{2im} \hat{K}_m(t)\right) (1 + f_1(K_i)) < K_{ip} (1 + \bar{r}) \sum_m k_{2im} \hat{K}_m(t)$$

This can also be expressed as:

$$\frac{1 + f_1(K_i)}{\sum_m k_{2im} \hat{K}_m(t)} + f_1(K_i) < \bar{r} \tag{50}$$

This inequality is the condition for the firm to repay its loans and avoid default. Given (50), the default condition writes in a compact form:

$$1 + f_1'(K_i(t)) < 0$$

that is when returns and private capital cannot cover loans anymore. When this inequality is satisfied, the lenders suffer losses. The difference between final capital and loaned capital writes:

$$\begin{aligned} & K_{ip}\left(1+\sum_m k_{2im}\hat{K}_m(t)\right)(1+f_1(K_i))-K_{ip}(1+\bar{r})\sum_m k_{2im}\hat{K}_m(t) \\ = & (1+f_1'(K_i(t)))K_{ip} \end{aligned}$$

The last expression represents the amount of capital that cannot be repaid to the investors. Each of them will therefore incur a loss proportional to the loan they granted, relative to the total loans. Investor j thus suffers an overall loss of:

$$k_{2ij}K_{ip}(1+f_1'(K_i(t)))\hat{K}_j(t)=(1+f_1'(K_i(t)))\frac{k_{2ij}K_i(t)}{1+\sum_v k_{iv}\hat{K}_v(t)}\hat{K}_j(t) \tag{51}$$

corresponding to a loss per unit of investment.of:

$$(1+f_1'(K_i(t)))\frac{k_{2ij}K_i(t)}{1+\sum_v k_{iv}\hat{K}_v(t)}$$

The loss (51) will be accounted for in the investors return by adding to investor j's return equation a contribution:

$$\sum_i\left(\bar{r}-\frac{(1+f_1'(K_i(t)))}{\sum_v k_{iv}\hat{K}_v(t)}\right)H\left(-\left(1+\hat{f}\left(\hat{K}_{vp}(t)\right)\right)\right)\frac{k_{2ij}K_i(t)}{1+\sum_v k_{iv}\hat{K}_v(t)}\hat{K}_j(t) \tag{52}$$

where H is the Heaviside function.

6.3.2 Defaults of investors

An investor defaults when the total private capital is insufficient to repay the loans.

$$\left(1+\sum_m \hat{k}_{2vm}\hat{K}_\nu\right)(1+R_\nu)\hat{K}_{vp}(t)<(1+\bar{r})\sum_m \hat{k}_{2vm}\hat{K}_{vp}(t)$$

where the LHS is the total level of invested capital including returns, and the RHS represents the amount of loans plus interest. The previous equation is compactly rewritten as:

$$\left(1+\hat{f}\left(\hat{K}_v(t)\right)\right)\hat{K}_{vp}(t)<0$$

with:

$$\hat{f}\left(\hat{K}_v(t)\right)=\left(1+\sum_m \hat{k}_{2vm}\hat{K}_\nu\right)(1+R_\nu)$$

If this situation arises, the default of investor modifies other investors' returns $\bar{R}_j'$ by a term:

$$\sum_l\left(\bar{r}-\frac{\left(1+\hat{f}\left(\hat{K}_{vp}(t)\right)\right)}{\sum_m \hat{k}_{2vm}\hat{K}_m}\right)H\left(-\left(1+\hat{f}\left(\hat{K}_{vp}(t)\right)\right)\right)\frac{\hat{k}_{2lj}\hat{K}_l(t)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v(t)} \tag{53}$$

where H is the Heaviside function. This term only appears when $1+\bar{f}_\nu$ is negative, meaning when the return $\bar{f}_\nu<-1$, indicating that the return is so negative that it even destroys the investor's own capital. When this term appears, the loss incurred by investor l is:

$$\left(\bar{r}-\frac{\left(1+\hat{f}\left(\hat{K}_{vp}(t)\right)\right)}{\sum_m \hat{k}_{2vm}\hat{K}_m}\right)\frac{\hat{k}_{2lj}\hat{K}_l(t)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v(t)}$$

The equations for returns accounting for defaults are provided in the following section. Note that the full loss of the investor is written:

$$\left(1+\hat{f}\left(\hat{K}_{vp}\left(t\right)\right)\right)\hat{K}_{vp}\left(t\right)$$

where:

$$\hat{f}\left(\hat{K}_{v}\left(t\right)\right)=\left(1+\sum_{m}\hat{k}_{2vm}\hat{K}_{\nu}\right)R_{\nu}-\bar{r}\sum_{m}\hat{k}_{2vm}$$

The rate of loss for investor v is given by:

$$\frac{\left(1+\hat{f}\left(\hat{K}_{vp}\left(t\right)\right)\right)\hat{K}_{vp}\left(t\right)}{\sum_{m}\hat{k}_{2vm}\hat{K}_{vp}\left(t\right)\hat{K}_{m}}=\frac{\left(1+\hat{f}\left(\hat{K}_{vp}\left(t\right)\right)\right)}{\sum_{m}\hat{k}_{2vm}\hat{K}_{m}}$$

6.3.3 Investors' returns under a default scenario

The possibility of default modifies the return equation (49):

$$\begin{aligned}
&\sum_{l}\left(\delta_{jl}-\frac{\hat{k}_{1lj}\hat{K}_{l}\left(t\right)}{1+\sum_{v}\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_{v}\left(t\right)}\right)\left(\frac{\hat{f}_{l}-\bar{r}}{1+\sum_{v}\hat{k}_{2jv}\hat{K}_{v}\left(t\right)}\right) \qquad (54)\\
&+\sum_{l}\left(\bar{r}-\frac{\left(1+\hat{f}\left(\hat{K}_{vp}\left(t\right)\right)\right)}{\sum_{m}\hat{k}_{2vm}\hat{K}_{m}}\right)H\left(-\left(1+\hat{f}\left(\hat{K}_{vp}\left(t\right)\right)\right)\right)\frac{\hat{k}_{2lj}\hat{K}_{l}\left(t\right)}{1+\sum_{v}\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_{v}\left(t\right)}\\
&+\sum_{i}\left(\bar{r}-\frac{\left(1+f_{1}^{\prime}\left(K_{i}\left(t\right)\right)\right)}{\sum_{m}k_{2im}\hat{K}_{m}}\right)H\left(-\left(1+f_{1}^{\prime}\left(K_{i}\left(t\right)\right)\right)\right)\frac{k_{2ij}K_{i}\left(t\right)}{1+\sum_{v}\left(k_{1iv}+k_{2iv}\right)\hat{K}_{v}\left(t\right)}\\
=\;&\sum_{i}\left(\frac{f_{1}^{\prime}\left(K_{i}\left(t\right)\right)-\bar{r}}{1+\sum_{\nu}k_{2j\nu}\hat{K}_{\nu}\left(t\right)}+\Delta F_{\tau}\left(\bar{R}\left(K_{i},X_{i}\right)\right)\right)\frac{k_{1ij}K_{i}\left(t\right)}{1+\sum_{v}\left(k_{1iv}+k_{2iv}\right)\hat{K}_{v}\left(t\right)}
\end{aligned}$$

where:

$$f_{1}^{\prime}\left(K_{i}\left(t\right)\right)=\left(1+\sum_{\nu}k_{2j\nu}\hat{K}_{\nu}\left(t\right)\right)r_{i}-\bar{r}\sum_{v}k_{2lv}\hat{K}_{v}\left(t\right)$$

and H is the Heaviside function.

Equation (54) is the same as the default-free return equation (49), with the addition of the losses (53) incurred by the investor if an investor in which they have invested defaults, and their loss (52) if a firm in which they have invested defaults.

7 Field translation

Actually, equations (3) and (4) correspond to the minimization functions (5) and (6), which are translated into fields. In our context, what corresponds to (3) and (4) are the two capital accumulation equations (46) and (42) for investors and firms respectively. What corresponds here to (5) and (6) are the following two minimization functions. We can write the minimization functions for the dynamics of $\hat{K}_{k}$ and K_{ip}, which yields:

$$\left\{\frac{d}{dt}\hat{K}_{k}\left(t\right)-\sum_{l}\left(1-M\right)_{jl}^{-1}\hat{f}_{l}\hat{K}_{l}\left(t\right)\right\}^{2} \qquad (55)$$

$$\left\{\frac{d}{dt}K_{ip}(t)-f_1'(K_i(t))K_{ip}(t)\right\}^2 \tag{56}$$

leading to an action functional for the field version of the system. The field version of these dynamic equations is presented in Appendix 1. Moreover, in addition to these two equations, we have implemented the field version of (54). This is done in Appendix 1 by including a potential in the investor's action functional. We present the results of the translation.

7.1 Investors' action functional

We denote $\hat{\Psi}$ the field describing the investors. It depends on the two variables $\hat{K}$ and $\hat{X}$, and its action functional is obtained by applying the translation formula (16) and (17) to the minimization function (55):

$$-\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\left(\nabla_{\hat{K}}\left(\sigma_{\hat{K}}^2\nabla_{\hat{K}}-\hat{g}\left(\hat{K},\hat{X}\right)\hat{K}\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right)+\frac{1}{2\hat{\mu}}\left(\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2-\left|\hat{\Psi}_0\left(\hat{X}\right)\right|^2\right)^2$$

where:

$$\hat{g}\left(\hat{K},\hat{X}\right)=\left(1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\right)^{-1}\hat{f}\left(\hat{K}',\hat{X}'\right)$$

and

$$\begin{aligned}\hat{f}\left(\hat{X}\right)&=\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\left(1+R_\nu\left(\hat{X}\right)\right)\\&=\left(1+\int\hat{k}_2\left(\hat{X},\hat{X}_1\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2d\hat{K}_1d\hat{X}_1\right)\left(1+R_\nu\left(\hat{X}\right)\right)\end{aligned}$$

7.2 Investors' field return equations

The translation of the function (54) is obtained by applying the translation provided by (11)

Appendix 2.3.4 translates the return equation (54) including defaults in terms of field:

$$\begin{aligned}&\frac{f\left(\hat{X}\right)-\bar{r}}{1+\underline{\hat{k}}_2\left(\hat{X}\right)}-\int\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\frac{f\left(\hat{X}'\right)-\bar{r}}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)}d\hat{K}'d\hat{X}' \\ =&\int\left(\bar{r}+\frac{1+f\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\hat{X}'\right)}H\left(-\frac{1+f\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\hat{X}'\right)}\right)\right)\frac{\hat{k}_2\left(\hat{X}',\hat{X}\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)}d\hat{K}'d\hat{X}'\\&+\int\left(\bar{r}+\frac{1+f_1'(X')}{\underline{k}_2(X')}H\left(-\frac{1+f_1'(X')}{\underline{k}_2(X')}\right)\right)\frac{k_2\left(X',\hat{X}\right)\left|\Psi(K',X')\right|^2K'}{1+\underline{k}(X')}dK'dX'\\&+\int\frac{\left|\Psi(K',X')\right|^2k_1\left(X',\hat{X}\right)K'}{1+\underline{k}(X')}\left(\frac{f_1'(K,X)-\bar{r}k_2\left(\hat{X}\right)}{1+k_2\left(\hat{X}\right)}+\Delta F_\tau\left(\bar{R}(K,X)\right)\right)dK'dX'\end{aligned} \tag{57}$$

The first term $\frac{f(\hat{X})-\bar{r}}{1+\underline{\hat{k}}_2(\hat{X})}$ represents the return of an investor in sector X, and all the other terms describe how it decomposes. The second term is the portion of return from other investors retrieved by the investor in X.On the right side of the equation, the first term calculates the loss due to the default of other investors, when it occurs. The second term calculates the loss due to the default of firms in which the investor had invested, and the last term is the return generated by the firms in which the investor had invested.

7.3 Firms' action functional

The translation of the function (56) is obtained by applying the translation provided by (16) and (17). The action functional for the field of firms is:

$$-\Psi^{\dagger}(K,X)\left(\nabla_{K_p}\left(\sigma_K^2\nabla_{K_p}-f_1'(K,X)K_p\right)\right)\Psi(K,X)+\frac{1}{2\epsilon}\left(|\Psi(K,X)|^2-|\Psi_0(X)|^2\right)^2$$

where:

$$\begin{aligned}
f_1'(X) &= \left(1+\underline{k}_2(X)\right)f_1(X)-\bar{r}\underline{k}_2\left(\hat{X}\right) \\
&= f_1(X)+\left(f_1(X)-\bar{r}\right)\int k_2\left(X,\hat{X}_1\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1
\end{aligned}$$

7.4 Use of the field model and computation of averages per sector

Section 4 provides us with the interpretation of the different fields in the model. Once the field action functional S is found, we can use field theory to study the system of agents, both at the collective and individual levels. At the collective level, we can compute the averages of the system in a given background field. The individual level, described by the transition functions, will be studied in the third part.

Recall that the background fields emerging at the collective level are particular functions, $\Psi(K,X)$ and $\hat{\Psi}\left(\hat{K},\hat{X}\right)$, and their adjoints fields $\Psi^{\dagger}(K,X)$ and $\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)$, that minimize the action functional of the system.

Once the background fields are obtained, the associated number of firms and investors per sector for a given average capital K can be computed. They are given by:

$$|\Psi(K,X)|^2=\Psi^{\dagger}(K,X)\Psi(K,X) \tag{58}$$

and:

$$\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2=\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\hat{\Psi}\left(\hat{K},\hat{X}\right) \tag{59}$$

With these two density functions at hand, various average quantities in the collective state can be computed.

The number of producers $\|\Psi(X)\|^2$ and investors $\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2$ in sectors are computed using the formula:

$$\|\Psi(X)\|^2 \equiv \int |\Psi(K,X)|^2 dK \tag{60}$$

$$\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2 \equiv \int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\hat{K} \tag{61}$$

The total invested capital $\hat{K}_X$ in sector X is defined by a partial average:

$$\hat{K}_{\hat{X}}=\int\hat{K}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 d\hat{K}=\int\hat{K}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2 d\hat{K} \tag{62}$$

and the average invested capital per firm in sector X is defined by:

$$K_X=\frac{\int\hat{K}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 d\hat{K}}{\|\Psi(X)\|^2} \tag{63}$$

Ultimately, the distributions of invested capital per investor and of capital per firm, given a collective state and a sector X, are $\frac{\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2}{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}$ and $\frac{|\Psi(K,X)|^2}{\|\Psi(X)\|^2}$, respectively.

Gathering equations (60), (61) and (63), each collective state is singularly determined by the collection of data that characterizes each sector: the number of firms, investors, the average capital, and the distribution of capital. All these quantities allow the study of capital allocation among sectors and its dependency in the parameters of the system, such as expected long-term and short-term returns, and any other parameter. This "static" point of view, will be extended by introducing some fluctuations in the expectations, leading to a dynamic of the average capital at the macro-level. In the following, we solve the system for the background fields and compute the average associated quantities.

8 Resolution for firms

8.1 Minimization equation and background field

We consider the field action for the firms:

$$S\left(\Psi\right)=-\Psi^{\dagger}\left(K,X\right)\left(\nabla_{K_p}\left(\sigma_{\hat{K}}^2\nabla_{K_p}-f_1'\left(K_p,X\right)K_p\right)\right)\Psi\left(K,X\right)+\frac{1}{\epsilon}\left(\left|\Psi\left(K,X\right)\right|^2-\left|\Psi_0\left(X\right)\right|^2\right)$$

We show in appendix that a change of variables leads to the modified action:

$$\begin{aligned}S\left(\Psi\right) &= -\Psi^{\dagger}\left(K,X\right)\sigma_{\hat{K}}^2\nabla_{K_p}^2\Psi\left(K,X\right) \\ &\quad +\Psi^{\dagger}\left(K,X\right)\left(\frac{\left(f_1'\left(K_p,X\right)K_p\right)^2}{2\sigma_{\hat{K}}^2}+\frac{1}{2}f_1'\left(K_p,X\right)\right)\Psi\left(K,X\right)+\frac{1}{\epsilon}\left(\left|\Psi\left(K,X\right)\right|^2-\left|\Psi_0\left(X\right)\right|^2\right)\end{aligned} \tag{64}$$

which is easier to deal with.

The background field for the system is the solution of the first-order minimization equations:

$$\frac{\delta S\left(\Psi\right)}{\delta\Psi^{\dagger}\left(K,X\right)}=0$$

$$\frac{\delta S\left(\Psi\right)}{\delta\Psi\left(K,X\right)}=0$$

These equations, also known as the saddle point equations, for (64) reduce to:

$$0=\left(\frac{f_1'^2\left(K,X\right)}{\sigma_{\hat{K}}^2}+\frac{\frac{d}{dK}f_1'\left(K,X\right)}{2}\right)+\frac{1}{\epsilon}\left(\left|\Psi\left(K,X\right)\right|^2-\left|\Psi_0\left(X\right)\right|^2\right)$$

whose solution is given by:

$$\left|\Psi\left(K,X\right)\right|^2=\left|\Psi_0\left(X\right)\right|^2-\epsilon\left(\frac{\left(f_1^{(e)}\left(X\right)K_p-\bar{C}\left(X\right)\right)^2}{\sigma_{\hat{K}}^2}+\frac{f_1^{(e)}\left(X\right)}{2}\right) \tag{65}$$

where $f_1^{(e)}\left(X\right)$ is the return of the firm, corresponding to the net return of production, from which the paiements of loans are substracted:

$$\begin{aligned}f_1^{(e)}\left(X\right) &= \left(1+\underline{k}_2\left(X\right)\right)f_1\left(X\right)-\underline{k}_2\left(X\right)\bar{r} \\ \bar{C}^{(e)}\left(X\right) &= \left(1+\underline{k}_2\left(X\right)\right)\bar{C}\left(X\right)\end{aligned}$$

$$|\Psi_0(X)|^2 - \epsilon \frac{f_1^{(e)}(X)}{2} - \epsilon \frac{\left(f_1^{(e)}(X) K_p - \bar{C}^{(e)}(X)\right)^2}{\sigma_{\hat{K}}^2}$$

In the following, we will replace:

$$\begin{aligned} f_1^{(e)}(X) &\rightarrow f_1(X) \\ \bar{C}^{(e)}(X) &\rightarrow \bar{C}(X) \end{aligned}$$

To compute returns and average capital for firms per sector, some assumptions about productivities $f_1(X)$ must be made. For the sake of simplicity, we will consider constant returns to scale. Non-constant returns will be studied in Appendix 12.

8.2 Average capital per sector for constant return to scale

Several cases can occur. Depending on the productivity $f_1(X)$, some agents may have either very low or even no capital. Alternately, all agents may have a minimal level of capital in a given sector.

8.2.1 Case 1 : No minimum capital

In this case, the following condition holds:

$$|\Psi_0(X)|^2 - \epsilon \frac{f_1(X)}{2} - \epsilon \frac{\left(\bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2} > 0$$

which, given (65), translates to $|\Psi(0,X)|^2 > 0$, indicating that some firms have either no or very low private capital.

Appendix 10.1.1 demonstrates that there exists a maximal level of capital per sector. Additionally, it also shows that the background field per sector is given by:

$$\begin{aligned} |\Psi(X)|^2 &= \left(\frac{2}{3}\sigma_{\hat{K}}^2 \left(\frac{|\Psi_0(X)|^2}{\epsilon} - \frac{f_1(X)}{2}\right) + \frac{1}{3}\left(\sqrt{\sigma_{\hat{K}}^2 \left(\frac{|\Psi_0(X)|^2}{\epsilon} - \frac{f_1(X)}{2}\right)} - \bar{C}(X)\right)\bar{C}(X)\right) \\ &\quad \times \epsilon \frac{K_0}{\sigma_{\hat{K}}^2} \end{aligned} \tag{66}$$

and that the amount of capital, $K_X |\Psi(X)|^2$ is:

$$K_X |\Psi(X)|^2 = \frac{\epsilon K_0}{\sigma_{\hat{K}}^2 f_1(X)} \left\{ \frac{1}{4}\left(2X^2(\bar{C}+X) - (X-\bar{C})\left(X^2 + (\bar{C})^2\right)\right) - \frac{\bar{C}}{3}\left(X(X-\bar{C}) + C^2\right) \right\} \tag{67}$$

where:

$$X = \sqrt{\sigma_{\hat{K}}^2 \left(\frac{|\Psi_0(X)|^2}{\epsilon} - \frac{f_1(X)}{2}\right)}$$

The appendix also derives the average capital in sector X:

$$\begin{aligned} K_X &= \frac{1}{f_1(X)} \frac{\frac{1}{4}\left(2X^2(C+X) - (X-C)\left(X^2+C^2\right)\right) - \frac{C}{3}\left(X(X-C)+C^2\right)}{\frac{2}{3}X^2 + \frac{1}{3}(X-C)C} \\ &= \frac{1}{4f_1(X)}(3X-C)\frac{(C+X)}{2X-C} \end{aligned}$$

as well as the overall return of capital in sector X:

$$\begin{aligned}
&\left|\Psi\left(X\right)\right|^{2}\left(\left(f_{1}\left(X\right)-\bar{r}\right)K_{X}-\bar{C}\left(X\right)\right) \qquad (68)\\
\rightarrow\ &\frac{\left(C^{(e)}+X^{(e)}\right)\left(\frac{2}{3}X^{2}+\frac{1}{3}\left(X-C^{(e)}\right)C^{(e)}\right)\epsilon}{\sigma_{\hat{K}}^{2}f_{1}^{(e)}\left(X\right)}\left(\frac{\left(f_{1}\left(X\right)-\bar{r}\right)}{4f_{1}^{(e)}\left(X\right)}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}-C\right)
\end{aligned}$$

This will be used later to compute investors' returns.

8.2.2 Case 2: Minimum capital

In this case, we assume that all firms have a minimum capital, thus:

$$\left|\Psi_{0}\left(X\right)\right|^{2}-\epsilon\frac{f_{1}\left(X\right)}{2}-\epsilon\frac{\left(\bar{C}\left(X\right)\right)^{2}}{\sigma_{\hat{K}}^{2}}<0$$

In this case the capital is confined within two bounds. We will write K_{0-} the minimal level of firms capital in the sector, and K, the maximal level. Both are defined by setting $\left|\Psi\left(K_{0-},X\right)\right|^{2}=\left|\Psi\left(K_{0+},X\right)\right|^{2}=0$, that is these level are such that there is no firms below or above these levels respectively. We find:

$$K_{0\pm}=\frac{\bar{C}\left(X\right)\pm\sqrt{\sigma_{\hat{K}}^{2}\left(\frac{\left|\Psi_{0}(X)\right|^{2}}{\epsilon}-\frac{f_{1}(X)}{2}\right)}}{f_{1}\left(X\right)}$$

and we define the range of possible capital ΔK_{0} within the sector by:

$$\Delta K_{0}=K_{0+}-K_{0-}$$

Appendix 7.1 computes the field $\left|\Psi\left(X\right)\right|^{2}$:

$$\left|\Psi\left(X\right)\right|^{2}=\frac{2}{3}\left(\frac{\left|\Psi_{0}\left(X\right)\right|^{2}}{\epsilon}-\frac{f_{1}\left(X\right)}{2}\right)\epsilon\Delta K_{0}$$

and the amount of capital $K_{X}\left|\Psi\left(X\right)\right|^{2}$ in a sector:

$$K_{X}\left|\Psi\left(X\right)\right|^{2}=\frac{2\epsilon\Delta K_{0}\bar{C}\left(X\right)}{3f_{1}\left(X\right)\sigma_{\hat{K}}^{2}}X^{2}$$

with:

$$X=\sqrt{\sigma_{\hat{K}}^{2}\left(\frac{\left|\Psi_{0}\left(X\right)\right|^{2}}{\epsilon}-\frac{f_{1}\left(X\right)}{2}\right)}$$

Ultimately, the average capital is given by:

$$K_{X}=\frac{\bar{C}\left(X\right)}{f_{1}\left(X\right)}$$

while the return of the whole sector is:

$$\left|\Psi\left(X\right)\right|^{2}K_{X}\frac{f_{1}'\left(X\right)-\bar{r}}{1+\underline{k}_{2}\left(X\right)}=\left|\Psi\left(X\right)\right|^{2}\left(\left(f_{1}\left(X\right)-\bar{r}\right)K_{X}-\bar{C}\left(X\right)\right)=\frac{4\epsilon}{3\sigma_{\hat{K}}^{2}f_{1}^{(e)}}X^{3}\bar{C}\left(X\right)\left(-\frac{\bar{r}}{f_{1}^{(e)}}\right)$$

9 Resolution for investors

9.1 Field action and change of variables

Starting from the investors field action functional:

$$S\left(\hat{\Psi}\right)=-\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\left(\nabla_{\hat{K}}\left(\sigma_{\hat{K}}^{2}\nabla_{\hat{K}}-\hat{g}\left(\hat{K},\hat{X}\right)\hat{K}\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right)+\frac{1}{2\hat{\mu}}\left(\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}-\left|\hat{\Psi}_{0}\left(\hat{X}\right)\right|^{2}\right)^{2}$$

a change of variable:

$$\begin{aligned}\hat{\Psi}\left(\hat{K},\hat{X}\right) &\rightarrow \exp\left(-\int\hat{f}\left(\hat{K},\hat{X}\right)\hat{K}d\hat{K}\right)\hat{\Psi}\\ \hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right) &\rightarrow \exp\left(\int\hat{f}\left(\hat{K},\hat{X}\right)\hat{K}d\hat{K}\right)\hat{\Psi}^{\dagger}\end{aligned}$$

leads to minimize the action:

$$\begin{aligned}S\left(\hat{\Psi}\right) &= -\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\sigma_{\hat{K}}^{2}\nabla_{\hat{K}}^{2}\hat{\Psi}\left(\hat{K},\hat{X}\right)\\ &\quad+\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\left(\frac{\left(\hat{g}\left(\hat{K},\hat{X}\right)\hat{K}\right)^{2}}{2\sigma_{\hat{K}}^{2}}+\frac{\hat{g}\left(\hat{K},\hat{X}\right)}{2}\right)\hat{\Psi}\left(\hat{K},\hat{X}\right)+\frac{1}{2\hat{\mu}}\left(\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}-\left|\hat{\Psi}_{0}\left(\hat{X}\right)\right|^{2}\right)^{2}\end{aligned}\tag{69}$$

where:

$$\hat{g}\left(\hat{K},\hat{X}\right)=\left(1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}\right)^{-1}\hat{f}\left(\hat{K}',\hat{X}'\right)$$

and the elements of the matrix are given by:

$$M\left(\hat{K},\hat{X},\hat{K}',\hat{X}'\right)=\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}}{1+\underline{\hat{k}}\left(\hat{X}\right)}$$

9.2 Minimization equation for the background field

If we neglect the gradient terms, the saddle point equation for the background field becomes:

$$\begin{aligned}0 &= \left(\frac{\hat{g}^{2}\left(\hat{K}_{1},\hat{X}_{1}\right)}{2\sigma_{\hat{K}}^{2}}+\frac{\hat{g}\left(\hat{K}_{1},\hat{X}_{1}\right)}{2\hat{K}_{1}}\right)+\int\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K}_{1},\hat{X}_{1}\right)\right|^{2}}\left(\frac{\hat{g}^{2}\left(\hat{K},\hat{X}\right)}{2\sigma_{\hat{K}}^{2}}+\frac{\hat{g}\left(\hat{K},\hat{X}\right)}{2\hat{K}}\right)\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}\\ &\quad+\frac{1}{\hat{\mu}}\left(\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}-\left|\hat{\Psi}_{0}\left(\hat{X}\right)\right|^{2}\right)\end{aligned}$$

Appendix 8 estimates the derivative $\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K}_{1},\hat{X}_{1}\right)\right|^{2}}\hat{g}\left(\hat{K},\hat{X}\right)$ and we find:

$$\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_{1},\hat{X}_{1}\right)\right|^{2}}\hat{g}\left(\hat{K},\hat{X}\right)\simeq\frac{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_{1}\right)}{\left\|\hat{\Psi}\right\|^{2}}\hat{g}\left(\hat{X}_{1}\right)-\frac{\hat{K}_{1}}{\left\langle\hat{K}\right\rangle}\frac{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)}{\left\|\hat{\Psi}\right\|^{2}}\left\langle\hat{g}\right\rangle\tag{70}$$

and the minimization equation writes:

$$0 = \frac{\hat{K}_1^2 \hat{g}^2\left(\hat{K}_1, \hat{X}_1\right)}{2\sigma_{\hat{K}}^2} + \frac{\hat{g}\left(\hat{K}_1, \hat{X}_1\right)}{2} \tag{71}$$
$$+\int \left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2 \left(\frac{\hat{K}^2 \hat{g}^2\left(\hat{K}, \hat{X}, \Psi, \hat{\Psi}\right)}{\sigma_{\hat{K}}^2} + \frac{1}{2}\right)\left(\frac{\underline{\hat{k}}\left(\left\langle \hat{X}\right\rangle, \hat{X}_1\right)}{\left\|\hat{\Psi}\right\|^2}\hat{g}\left(\hat{X}_1\right) - \frac{\hat{K}_1}{\left\langle \hat{K}\right\rangle}\frac{\underline{\hat{k}}\left(\left\langle \hat{X}\right\rangle, \left\langle \hat{X}\right\rangle\right)}{\left\|\hat{\Psi}\right\|^2}\left\langle \hat{g}\right\rangle\right)$$
$$+\frac{1}{\hat{\mu}}\left(\left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2 - \left|\hat{\Psi}_0\left(\hat{X}\right)\right|^2\right)$$

This equation must be taken into account along with the returns equation (57) derived previously.

9.3 Solving the minimization equation and average capital per sector

Solving for $\left|\hat{\Psi}\left(\hat{K}_1, \hat{X}_1\right)\right|^2$ yields:

$$\left|\hat{\Psi}\left(\hat{K}_1, \hat{X}_1\right)\right|^2 = \left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}\left\{\left(\frac{\hat{K}_1^2 \hat{g}^2\left(\hat{X}_1\right)}{2\sigma_{\hat{K}}^2} + \frac{\hat{g}\left(\hat{X}_1\right)}{2}\right)\right. \tag{72}$$
$$\left. + \left(\frac{\left\langle \hat{K}\right\rangle^2 \left\langle \hat{g}\right\rangle}{\sigma_{\hat{K}}^2} + \frac{1}{2}\right)\left(\underline{\hat{k}}\left(\left\langle \hat{X}\right\rangle, \hat{X}_1\right)\hat{g}\left(\hat{X}_1\right) - \frac{\hat{K}_1}{\left\langle \hat{K}\right\rangle}\underline{\hat{k}}\left(\left\langle \hat{X}\right\rangle, \left\langle \hat{X}\right\rangle\right)\left\langle \hat{g}\right\rangle\right)\right\}$$

We also study the stability of this solution in Appendix 9.

9.3.1 Maximal level of capital per sector

In Appendix 10.1, we compute the maximal value of capital $\hat{K}_0$ for sector $\hat{X}$. This is found by setting:

$$\left|\hat{\Psi}\left(\hat{K}_1, \hat{X}_1\right)\right|^2 = 0$$

which yields:

$$\hat{K}_0^2 \simeq 2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}} + \left(\frac{\left\langle \hat{K}\right\rangle^2 \left\langle \hat{g}\right\rangle^2}{\sigma_{\hat{K}}^2} + \frac{\left\langle \hat{g}\right\rangle}{2}\right)\left(\left(\frac{\left\langle \hat{K}_0\right\rangle}{\left\langle \hat{K}\right\rangle} - \frac{\underline{\hat{k}}\left(\left\langle \hat{X}\right\rangle, \hat{X}_1\right)}{\underline{\hat{k}}\left(\left\langle \hat{X}\right\rangle, \left\langle \hat{X}\right\rangle\right)}\right)\underline{\hat{k}}\left(\left\langle \hat{X}\right\rangle, \left\langle \hat{X}\right\rangle\right)\right)\right) \tag{73}$$

9.3.2 Global averages for field and capital

We then compute the averages over all sectors $\left\langle \hat{K}_0\right\rangle^2$, $\left\|\hat{\Psi}\right\|^2$ and $\left\langle \hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2$ required to solve for the average maximum level of capital. We obtain:

$$\left\langle \hat{K}_0\right\rangle^2 = 2\frac{\sigma_{\hat{K}}^2\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}} + \frac{\left\langle \hat{g}\right\rangle}{2}\left(\frac{1-r}{r}\right)\hat{k}\right)}{\left\langle \hat{g}\right\rangle^2\left(1 - 2r\left(1-r\right)\hat{k}\right)}$$

$$\left\|\hat{\Psi}\right\|^2 \simeq \frac{\hat{\mu}V}{3\sigma_{\hat{K}}^2}\left(2\frac{\sigma_{\hat{K}}^2\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}} + \frac{\left\langle \hat{g}\right\rangle}{2}\left(\frac{1-r}{r}\right)\hat{k}\right)}{\left\langle \hat{g}\right\rangle^2\left(1 - 2r\left(1-r\right)\hat{k}\right)}\right)^{\frac{3}{2}}\left(1 - \frac{2 + \hat{k} - \sqrt{\left(2+\hat{k}\right)^2 - \hat{k}}}{4\hat{k}}\hat{k}\right)\left\langle \hat{g}\right\rangle^2 \tag{74}$$

$$\left\langle \hat{K} \right\rangle \left\| \hat{\Psi} \right\|^2 = \frac{\hat{\mu} V \sigma_{\hat{K}}^2}{2 \langle \hat{g} \rangle^2} \left(\frac{\frac{\|\hat{\Psi}_0\|^2}{\hat{\mu}} + \frac{\langle \hat{g} \rangle}{2} \left(\frac{1-r}{r} \right) \hat{k}}{1 - 2r(1-r)\hat{k}} \right)^2 \left(1 - 2 \frac{2 + \hat{k} - \sqrt{\left(2 + \hat{k}\right)^2 - \hat{k}}}{9} \right) \tag{75}$$

The quantity (75) computes the average overall financial capital in the system. The average capital per investor is:

$$\left\langle \hat{K} \right\rangle = \frac{3}{4 \langle \hat{g} \rangle} \sqrt{\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}} \frac{1 - 2\frac{2+\hat{k}-\sqrt{\left(2+\hat{k}\right)^2 - \hat{k}}}{9}}{1 - \frac{2+\hat{k}-\sqrt{\left(2+\hat{k}\right)^2 - \hat{k}}}{4\hat{k}}} \sqrt{\frac{\left\| \hat{\Psi}_0 \right\|^2 + \hat{\mu} \frac{\langle \hat{g} \rangle}{2} \left(\frac{1-r}{r} \right) \hat{k}}{1 - 2r(1-r)\hat{k}}} \tag{76}$$

9.3.3 Average field and capital per sector

We then find the field $\left\| \hat{\Psi}\left(\hat{X}_1\right) \right\|^2$ and the amount of capital $\hat{K}_{\hat{X}_1} \left\| \hat{\Psi}\left(\hat{X}_1\right) \right\|^2$ and the average capital $\hat{K}_{\hat{X}_1}$ in sector X as functions of $\hat{K}_0^2$ and averages $\left\langle \hat{K} \right\rangle$ and $\left\langle \hat{K}_0 \right\rangle$:

$$\begin{aligned} \left\| \hat{\Psi}\left(\hat{X}_1\right) \right\|^2 &\simeq \hat{\mu} \frac{\hat{K}_0^3}{\sigma_{\hat{K}}^2} \left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{3} - \frac{\left\langle \hat{K} \right\rangle \langle \hat{g} \rangle^2}{2\left\langle \hat{K}_0 \right\rangle} \underline{\hat{k}}\left(\left\langle \hat{X} \right\rangle, \left\langle \hat{X} \right\rangle \right) \right) \\ &= \frac{\hat{\mu}}{\sigma_{\hat{K}}^2} \left(2 \frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)} \left(\frac{\left\| \hat{\Psi}_0\left(\hat{X}_1\right) \right\|^2}{\hat{\mu}} - D\left(\hat{X}_1\right) \right) \right)^{\frac{3}{2}} \left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{3} - \frac{\left\langle \hat{K} \right\rangle \langle \hat{g} \rangle^2}{2\left\langle \hat{K}_0 \right\rangle} \hat{k} \right) \end{aligned} \tag{77}$$

$$\begin{aligned} \hat{K}\left[\hat{X}_1\right] &= \hat{K}_{\hat{X}} \left\| \hat{\Psi}\left(\hat{X}_1\right) \right\|^2 \simeq \hat{\mu} \frac{\hat{K}_0^4}{2\sigma_{\hat{K}}^2} \left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4} - \frac{\left\langle \hat{K} \right\rangle \langle \hat{g} \rangle^2}{3\left\langle \hat{K}_0 \right\rangle} \hat{k}\left(\left\langle \hat{X} \right\rangle, \left\langle \hat{X} \right\rangle \right) \right) \\ &= \frac{\hat{\mu}}{2\sigma_{\hat{K}}^2} \left(2 \frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)} \left(\frac{\left\| \hat{\Psi}_0\left(\hat{X}_1\right) \right\|^2}{\hat{\mu}} - D\left(\hat{X}_1\right) \right) \right)^2 \left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4} - \frac{r \langle \hat{g} \rangle^2}{3} \underline{\hat{k}} \right) \end{aligned} \tag{78}$$

and the average capital per investor in sector $\hat{X}_1$ is:

$$\hat{K}_{\hat{X}_1} = \frac{\hat{K}\left[\hat{X}_1\right]}{\left\| \hat{\Psi}\left(\hat{X}_1\right) \right\|^2} = \frac{\sqrt{2 \frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)} \left(\frac{\left\| \hat{\Psi}_0\left(\hat{X}_1\right) \right\|^2}{\hat{\mu}} - D\left(\hat{X}_1\right) \right)} \left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4} - \frac{r \langle \hat{g} \rangle^2}{3} \hat{k} \right)}{2 \left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{3} - \frac{\left\langle \hat{K} \right\rangle \langle \hat{g} \rangle^2}{2\left\langle \hat{K}_0 \right\rangle} \hat{k} \right)} \tag{79}$$

10 Investors' returns under a no-default scenario

So far, we have found the level of capital per sector and the return firms provide to investors. By reintroducing these results into the equation for investor returns, we will be able to express investors' excess returns in terms of capital per sector, returns provided by firms, and the number of agents per sector, which we will then interpret.

10.1 Derivation of the investors' returns equation

The link between returns $g\left(\hat{X}_1\right)$ and f, given by:

$$g\left(\hat{X}\right)=\int(1-M)^{-1}\left(\hat{X},\hat{X}'\right)f\left(\hat{X}'\right)d\hat{X}'$$

and the return equation (57) without default:

$$\begin{aligned}&\frac{f\left(\hat{X}\right)-\bar{r}}{1+\underline{\hat{k}}_2\left(\hat{X}\right)}-\int\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\frac{f\left(\hat{X}'\right)-\bar{r}}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)}d\hat{K}'d\hat{X}'\\ =&\int\frac{\left|\Psi\left(K',X'\right)\right|^2k_1\left(X',\hat{X}\right)K'}{1+\underline{k}\left(X'\right)}\left(\frac{f_1'\left(K,X\right)-\bar{r}k_2\left(\hat{X}\right)}{1+k_2\left(\hat{X}\right)}+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)dK'dX'\end{aligned}$$

can be rewritten as an equation for $g\left(\hat{X}\right)$ as a function of firms returns:

$$\int\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\hat{X}\right)}-\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)}\right)(1-M)\left(g-\bar{r}'\right)d\hat{X}'=\frac{k_1\left(X\right)}{1+\underline{k}\left(X\right)}f_1' \tag{80}$$

where:

$$\bar{r}'=(1-M)^{-1}\bar{r}$$

$$\frac{k_1\left(X\right)}{1+\underline{k}\left(X\right)}=\int\frac{\left|\Psi\left(K',X'\right)\right|^2k_1\left(X',\hat{X}\right)K'}{1+\underline{k}\left(X'\right)}dK'dX'$$

and f_1' stands for the returns investors make from their participation in a firm located in X:

$$\frac{f_1'\left(K,X\right)-\bar{r}k_2\left(\hat{X}\right)}{1+k_2\left(\hat{X}\right)}+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)$$

The expression for f_1' has been determined for constant returns (see (68)). We derive the expressions for both sides of (80) in Appendices 11.1 and 11.2.

We define:

$$\underline{k}\left(X\right)=k\left(X\right)\frac{\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\langle K\rangle\left|\Psi_0\left(X\right)\right|^2}$$

which implies that the capital invested in firms in sector X is a fraction of the investors' disposable capital in X divided by the number of firms in the sector. We show in Appendix 11.5 that for $\underline{k}\left(X\right)>>1$ and $\epsilon<\sigma_{\hat{K}}^2f_1\left(X\right)$, (80) becomes in terms of excess return $\hat{g}\left(\hat{X}_1\right)-\bar{r}'$:

$$\begin{aligned}&\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\hat{X}\right)}-\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)\left(\hat{g}\left(\hat{X}_1\right)-\bar{r}'\right)\\ =&\frac{1}{3}\frac{(1-\beta)\left(X+C\beta\right)^2\epsilon}{\sigma_{\hat{K}}^2\left(f_1\left(X\right)+\beta\underline{k}\left(X\right)R\right)}\left(\frac{R\left(3X-\beta C\right)\left(\beta C+X\right)}{4\left(f_1\left(X\right)+\beta\underline{k}\left(X\right)R\right)}-\frac{C\left(2X-C\beta\right)}{1+\underline{k}\left(X\right)}\right)\end{aligned} \tag{81}$$

10.2 Solution of the investors' returns equation

We show in appendix 11.5 that equation (81) becomes after resolution for $\underline{k}(X)$ and a first order expansion in R:

$$\begin{aligned}\hat{g}\left(\hat{X}_1\right)-\bar{r}' &= \int\left(1-\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1} \\ &\quad\times\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\left(\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)\right)\end{aligned} \tag{82}$$

with:

$$\begin{aligned}A\left(\hat{X}'\right) &= \frac{\epsilon\left(1-\beta\right)\left(X+C\beta\right)^2}{3\sigma_{\hat{K}}^2}\frac{\left(3X-\beta C\right)\left(\beta C+X\right)}{4C\left(2X-C\beta\right)} \\ B\left(\hat{X}'\right) &= \frac{\epsilon\left(1-\beta\right)\left(X+C\beta\right)^2}{3\sigma_{\hat{K}}^2}\beta RC\left(2X-C\beta\right)\end{aligned}$$

with::

$$X=\sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0\left(X\right)\right|^2}{\epsilon}-\frac{f_1\left(X\right)}{2}\right)}\rightarrow\sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0\left(X\right)\right|^2}{\epsilon}-\left(\frac{f_1\left(X\right)}{2\left(\left(1+\underline{k}\left(X\right)\frac{\hat{K}\left[\hat{X}\right]}{K\left[X\right]}\right)\langle K\rangle\right)^r}-\frac{C_0}{2}\right)\right)}$$

An estimation for $\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$ is given in Appendix 11.3:

$$\begin{aligned}&\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right) \\ \simeq &\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\left\langle\hat{k}_1\right\rangle\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)}{1+\hat{k}\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{\left(1+\hat{k}\left(\hat{X}\right)\right)\left(1+\frac{\hat{k}_2\left(\hat{X},\left\langle\hat{X}\right\rangle\right)}{1-\left\langle\hat{k}\right\rangle}\right)}\right) \\ &\times\frac{\sqrt{\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2-\hat{\mu}D\left(\hat{X}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)^{\frac{3}{2}}}{\left(\left\|\hat{\Psi}_0\right\|^2-\hat{\mu}\left\langle D\right\rangle\right)^2}\end{aligned}$$

10.3 Interpretation

The equation for $\hat{g}\left(\hat{X}_1\right)$ represents the returns of the financial sector $\hat{X}_1$, which can be a geographical sector, a type of activity, etc. It depends on the direct returns of the firms to which it lends or in which it takes stakes, and the returns of other firms, indirectly, through equity stakes and loans to other investors. When the expression $\left(1-\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1}$ is expanded, we formally have:

$$1+\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)+\left[\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right]^2+\left[\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right]^3+... \tag{83}$$

When this series (83) is applied to the returns of firms:

$$\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\left(\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)\right) \tag{84}$$

it yields an infinite number of contributions. In the series (83), the first constant, 1, directly reflects the return (84) of the firm, in which the investor has directly invested. The second term in the series, $\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$, applied to (84), yields the indirect return gained by investors on other investors who got returns from firms. The third term amplifies the phenomenon: it is the return obtained on the investment in investors who themselves have invested in investors who have obtained a return. It's a kind of domino effect. The magnitude of these compounded effects is measured by the magnitude of $\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$. The magnitude increases with the borrowing rate for investments, measured by $\underline{\hat{k}}_2\left(\hat{X}_1\right)$. The higher the borrowing rates, the more significant the domino effect. The second important element is given by the connectivity between agents in the system, measured by $\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$. It's a characteristic property of the system that measures the connections between different actors. This matrix is sector-dependent. Some returns will diffuse from one place to another, but not to others, etc. These characteristics are measured by the coefficients in $\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$, which are interpreted just below.

10.4 The diffusion matrix

The various components of the diffusion matrix $\hat{S}_1^E\left(\hat{X},\hat{X}'\right)$ illustrate how returns disseminate from one sector $\hat{X}'$, to another, $\hat{X}$. The coefficients $\hat{k}$ represent the proportion of investments made by $\hat{X}'$in $\hat{X}$. There are three terms in this matrix:

$$\frac{\hat{k}\left(\hat{X},\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\hat{X}\right)}$$

signifies the proportion of investments made by $\hat{X}'$ in $\hat{X}$. Fluctuations in $\hat{X}'$ returns can prompt $\hat{X}'$ to adjust its investments in $\hat{X}$, thereby influencing $\hat{X}$'s returns as well. This term also reflects the proportion of investments made by other investors in $\hat{X}$. Any changes in their returns lead to corresponding variations in the invested portion, thereby impacting the returns of $\hat{X}$.

The second term describes the reciprocal scenario:

$$\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)}$$

$\hat{X}$ has invested in $\hat{X}'$, entitling it to a share of $\hat{X}'$'s return. This represents the direct involvement of investors in each other's sectors, facilitating the direct transmission of returns from firms in sector $\hat{X}'$.

The two terms described above directly depict how the returns of one influence the returns of the other, illustrating their direct connection.

The third term moderates this influence by acknowledging that both investors are also linked to the broader pool of investors:

$$-\frac{\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}$$

Investor $\hat{X}'$ not only invests in $\hat{X}$, but also in other sectors represented by the average $\left\langle\hat{X}\right\rangle$. Similarly, investor $\hat{X}$ invests in the overall pool, not exclusively in $\hat{X}'$. Consequently, this third term mitigates the impact of the direct return from $\hat{X}'$ on $\hat{X}$. Although $\hat{X}$ may receive returns from $\hat{X}'$, a portion of these returns originates not from $\hat{X}'$ itself, but from investments made by $\hat{X}'$ with other investors. Whereas the two first terms of the diffusion matrix suggest that returns originate directly from the sector where the investment was made, the third term underscores that these returns are, at least partially, derived from investments made by both parts on other investors. This dampens the direct reciprocal transmission between $\hat{X}'$ and $\hat{X}$.

The final term in the diffusion process is a weighting factor determined by the number of investors per sector. A higher concentration of investors in sector $\hat{X}'$ increases its impact on sector $\hat{X}$. Conversely, as the number of investors in sector $\hat{X}$ increases, the impact on each individual agent in that sector decreases, as they have all invested proportionally in $\hat{X}'$ and inversely in $\hat{X}$.

$$\frac{\sqrt{\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2-\hat{\mu}D\left(\hat{X}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)^{\frac{3}{2}}}{\left(\left\|\hat{\Psi}_0\right\|^2-\hat{\mu}\langle D\rangle\right)^2}$$

Remark We are working with $\hat{g}\left(\hat{X}_1\right)$, which is not exactly the return of the investors but is related to it, denoted by $\hat{f}\left(\hat{X}_1\right)$. The term $\hat{g}\left(\hat{X}_1\right)$ also includes returns coming from investors who have provided capital to the investor we are studying. The relationship between $\hat{f}\left(\hat{X}_1\right)$ and $\hat{g}\left(\hat{X}_1\right)$ is given by the following transformation:

$$\hat{g}\left(\hat{X}_1\right)-\bar{r}'=(1-M)^{-1}\left(\hat{f}\left(\hat{X}_1\right)-\bar{r}\right)$$

This ensures that the results regarding $\hat{f}\left(\hat{X}_1\right)$ or $\hat{g}\left(\hat{X}_1\right)$ are interpreted similarly, and it is more convenient to work with $\hat{g}\left(\hat{X}_1\right)$, especially when dealing with capital by sector. However, note that the equation (82) is also an equation for the return $\hat{f}$:

$$\begin{aligned}\left(\hat{f}\left(\hat{X}_1\right)-\bar{r}\right) &= (1-M)\int\left(1-\left(1+\hat{\underline{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1} \\ &\quad\times\left(1+\hat{\underline{k}}_2\left(\hat{X}_1\right)\right)\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)\end{aligned} \tag{85}$$

11 Equation for total investors' capital per sector

We obtained the equation for investor returns by sector. We can also rephrase this equation as an equation concerning the total capital by sector.

11.1 Derivation of the equation

Given the formula (344) for the total capital amount $\hat{K}\left[\hat{X}_1\right]$ in sector $\hat{X}_1$:

$$\begin{aligned}\hat{K}\left[\hat{X}_1\right] &= \frac{2\sigma_{\hat{K}}^2}{\hat{\mu}}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)}{\hat{g}^2\left(\hat{X}_1\right)}\right)^2\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{r\left\langle\hat{g}\right\rangle^2}{3}\hat{\underline{k}}\right) \\ &\simeq \frac{2\sigma_{\hat{K}}^2}{\hat{\mu}\hat{g}^2\left(\hat{X}_1\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)^2\left(\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}\right)\end{aligned} \tag{86}$$

we can write the return equation as an equation for $\hat{K}\left[\hat{X}_1\right]$, the total capital in sector $\hat{X}_1$. The derivation is presented in the appendix 11.4 and we have:

$$\hat{g}\left(\hat{X}_1\right)\simeq\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}}}{\sqrt{\frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{2\sigma_{\hat{K}}^2}}}$$

where:

$$D\left(\hat{X}_1\right)=\left(\frac{\left\langle\hat{K}\right\rangle^2\left\langle\hat{g}\right\rangle^2}{\sigma_{\hat{K}}^2}+\frac{\left\langle\hat{g}\right\rangle}{2}\right)\left(\frac{\hat{\underline{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)}{\hat{\underline{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)}-\frac{6\hat{\underline{k}}}{2+\hat{\underline{k}}-\sqrt{\left(2+\hat{\underline{k}}\right)^2-\hat{\underline{k}}}}\right)\hat{\underline{k}}$$

This enables to rewrite (81) as an equation for average capital per sector as:

$$\begin{aligned}&\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}-\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)\left(\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}}}{\sqrt{\frac{\hat{\mu}\hat{K}\left[\hat{X}'\right]}{2\sigma_{\hat{K}}^2}}}-\bar{r}'\right) \\ =&\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)\end{aligned} \tag{87}$$

This is the same equation as for returns but described in terms of overall sectoral capital. The two variables are interconnected, and solving for one will consequently solve for the other.

11.2 Interpretation

The interpretation is similar to that of returns. It's worth noting that the same diffusion phenomenon exists, and the level of capital will be contingent on the firm returns. Capital levels between sectors will be interconnected, resulting in a global resolution. This leads to both collective states and the multiplicity of these collective states. There can be multiple collective states, such as capital levels by sector adjusting to the return of all firms. Different distributions can satisfy this equation because, in a way, it's the overall firm return that determines the possible capital distributions. Our focus will be on solving for capital and examining its dependency on parameters to study capital accumulation.

12 Total investors' capital per sector

In this section, we study the equation for the average invested capital per sector, (87) This equation aggregates capital from all sectors, defining a collective state in which average capitals across sectors are interconnected. Initially, we will examine the straightforward scenario of constant returns to scale, where invested capital yields uniform returns regardless of the firm or sector. Subsequently, we will introduce the impact of decreasing returns to scale, incorporating first-order adjustments to accommodate non-uniform returns. A comprehensive treatment of decreasing returns to scale returns is provided in Appendix 12.

12.1 Total investors' capital per sector under constant return to scale

The sectors of firms and the firms themselves are characterized by the potential returns on investment they could generate for investors. This implies that the return on investment, expressed as a percentage, is assumed to be independent of the level of invested capital. Although this assumption is obviously unrealistic, it serves as an initial benchmark for our model, allowing us to refine it subsequently.

The resolution proceeds in two steps. Firstly, we analyze the capital within a financial sector, considered in isolation from others, and examine its dependence on firm productivity. Subsequently, we incorporate the effects of interconnected investors across different sectors.

12.1.1 Isolated investors

Since we study a group of interconnected investors, the return equation cannot be solved analytically. Therefore, we will begin with a basic case and expand our analysis from there.

We first establish a benchmark with a sector that is not interconnected with others.

Here, we assume temporarily that the sectors operate independently; there are no interactions between investors. Each investor focuses solely on their specific firm sector and does not invest in other investors. This implies that the diffusion matrix is null, and the return equation is:

$$\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}}}{\sqrt{\frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{2\sigma_{\hat{K}}^2}}}-\bar{r}'=\left(\left(\frac{A\left(\hat{X}_1\right)}{f_1^2\left(\hat{X}_1\right)}+\frac{B\left(\hat{X}_1\right)}{f_1^3\left(\hat{X}_1\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,\hat{X}_1\right)\right)\right)\right)$$

We saw that:

$$\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\simeq\tau\left(f_1\left(X\right)-\left\langle f_1\left(X\right)\right\rangle\right)$$

We can replace in first approximation the parameters $A\left(\hat{X}_1\right)$ and $B\left(\hat{X}_1\right)$ by their averages:

$$\begin{aligned}A\left(\hat{X}_1\right)&\rightarrow A\\B\left(\hat{X}_1\right)&\rightarrow B\end{aligned}$$

We also define:

$$N\left(\hat{X}_1\right)=\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}}\sqrt{\frac{2\sigma_{\hat{K}}^2}{\hat{\mu}}}$$

Here, $N\left(\hat{X}_1\right)$ represents a specific parameter for each sector, which depends on the number of investors in the sector and the magnitude of investments made within it. The capital equation is written as:

$$\pm\frac{N\left(\hat{X}_1\right)}{\sqrt{\hat{K}\left[\hat{X}_1\right]}}-\bar{r}'\simeq\left(\frac{A}{\left(f_1\left(X\right)\right)^2}+\frac{B}{\left(f_1\left(X\right)\right)^3}\right)\left(\left(f_1\left(X\right)-\bar{r}'\right)+\tau F\left(X\right)\left(f_1\left(X\right)-\left\langle f_1\left(X\right)\right\rangle\right)\right)$$

Dependency in productivity Even though the solution is only partial because we have isolated the agents, we can still calculate the dependency of $\hat{K}\left[\hat{X}_1\right]$ in $f_1(X)$ as a benchmark. It is given by:

$$\begin{aligned}&\frac{\left(f_1\left(X\right)\right)^4}{\left(\hat{K}\left[\hat{X}_1\right]\right)^{\frac{3}{2}}}\frac{d\hat{K}\left[\hat{X}_1\right]}{df_1\left(X_1\right)} \\ =\ &\pm 2\left(B\left(2f_1\left(X\right)-3r\right)+Af_1\left(X\right)\left(f_1\left(X\right)-2r\right)+F\tau\left(B\left(2f_1\left(X\right)-3\left\langle f_1\left(X\right)\right\rangle\right)+Ax\left(f_1\left(X\right)-2\left\langle f_1\left(X\right)\right\rangle\right)\right)\right)\end{aligned} \tag{88}$$

The sign of the right-hand side depends on whether we are seeking a positive or negative return solution. For a positive solution, it acts as an increasing function if it surpasses a certain threshold. For a negative solution, it increases as long as it stays below the threshold. In the case of negative returns, the level of capital diminishes quickly with returns. Even when returns are negative, the level of capital in the sector remains very low but does not reach zero. Due to the interia in firms displacement, collective states with negative return and low capital persist. This is the consequence of our field model, where the average capital maintains a persistent value, corresponding to the capital level for firms in this sector. Dynamically, agents vanish and are replaced by others experiencing the same loss.

The formula (88) indicates that the capital level in the sector generally increases as the average productivity of firms in the sector, $f_1(X)$, increases. Typically, for investments to be profitable, the firm's productivity must exceed double the interest rate, r, which is usually the case. More is said about that while introducing the decreasing returns.

The variation $\frac{d\hat{K}\left[\hat{X}_1\right]}{df_1(X_1)}$ depends on two terms. One term is related to the dividend, expressed as:

$$B\left(2f_1\left(X\right)-3r\right)+Af_1\left(X\right)\left(f_1\left(X\right)-2r\right)$$

The other term:

$$F\tau\left(B\left(2f_1\left(X\right)-3\left\langle f_1\left(X\right)\right\rangle\right)+Ax\left(f_1\left(X\right)-2\left\langle f_1\left(X\right)\right\rangle\right)\right)$$

represents the price dividend, which quantifies the increase in capital when productivity exceeds the average productivity across all sectors.

12.1.2 Interconnected investors

Stating that investors are interconnected implies that the diffusion matrix $\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$ alters previous outcomes. However, when returns to scale are constant, this diffusion matrix remains unaffected by the levels of capital invested in the sectors. Consequently, the diffusion of returns becomes independent of the capitals invested in various sectors. As a result, variations in the level of invested capital do not lead to changes in productivity and, therefore, returns.

As a consequence, in the scenario of constant returns to scale and interconnected investors, we can derive the level of capital invested in a sector as a function of firm returns, yielding:

$$\pm\frac{N\left(\hat{X}_1\right)}{\sqrt{\hat{K}\left[\hat{X}_1\right]}}-\bar{r}' \simeq \int\left(1-\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1}\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)$$
$$\times\left(\frac{A}{\left(f_1\left(X'\right)\right)^2}+\frac{B}{\left(f_1\left(X'\right)\right)^3}\right)\left(\left(f_1\left(X'\right)-\bar{r}'\right)+\tau F\left(X\right)\left(f_1\left(X'\right)-\left\langle f_1\left(X'\right)\right\rangle\right)\right)$$

12.1.3 Interpretation

The equation reveals that the average capital level in a sector is contingent on the global returns of firms and the investors' interconnections. Firm returns diffuse to financial investors based on the diffusion matrix, representing the structure of these connections. Simplifying to constant returns to scale (CRS) has enabled us to derive a single collective state for a set of firm returns. Indeed, under CRS, the level of capital between sectors does not affect investment returns, since the latter are constant. However, this also demonstrates that when returns are not constant, the capital level of a sector influences the diffusion and thus the other capital levels in other sectors, leading to a possible multiplicity of solutions to the capital equation, and therefore collective states.

Dependency in the parameters The dependency of the capital level in one sector on the returns of a firm from another sector is derived directly and expressed as:

$$\frac{\frac{d\hat{K}\left[\hat{X}_1\right]}{df_1\left(X'\right)}}{\left(\hat{K}\left[\hat{X}_1\right]\right)^{\frac{3}{2}}} = \pm\left(1-\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1}\frac{\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)}{\left(f_1\left(X'\right)\right)^4}$$
$$\times 2\left(B\left(2f_1\left(X'\right)-3r\right)+Af_1\left(X\right)\left(f_1\left(X'\right)-2r\right)\right.$$
$$\left.+F\tau\left(B\left(2f_1\left(X'\right)-3\left\langle f_1\left(X\right)\right\rangle\right)+A\left(f_1\left(X'\right)-2\left\langle f_1\left(X\right)\right\rangle\right)\right)\right)$$

The relative change in capital in the sector, expressed as $\frac{d\hat{K}\left[\hat{X}_1\right]}{df_1\left(X'\right)}$is determined by the variation in the return of a specific firm:

$$2\left(B\left(2f_1\left(X'\right)-3r\right)+Af_1\left(X\right)\left(f_1\left(X'\right)-2r\right)+F\tau\left(B\left(2f_1\left(X'\right)-3\left\langle f_1\left(X\right)\right\rangle\right)+A\left(f_1\left(X'\right)-2\left\langle f_1\left(X\right)\right\rangle\right)\right)\right)$$

multiplied by a diffusion coefficient between the two sectors.

$$\left(1-\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1}\frac{\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)}{\left(f_1\left(X'\right)\right)^4}$$

12.2 Total investors' capital per sector under decreasing returns to scale

Now that we have established the baseline case of constant returns to scale, we can introduce first-order corrections to account for slowly decreasing returns to scale. As previously, we will begin by examining the case where investors are not interconnected, and do not invest among themselves. Subsequently, we will incorporate interactions between investors. A more detailed treatment is presented in Appendices 12.3 and 12.4.

12.2.1 Isolated investors

For weakly decreasing returns, we simply replace the productivities $f_1(X)$ in the formulas with their equivalent under decreasing returns, denoted as $f_1\left(X, \hat{K}\left[\hat{X}\right]\right)$:

$$f_1\left(X, \hat{K}\left[\hat{X}\right]\right) \equiv \frac{f_1(X)}{\left(\left(1+\frac{k(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_X\right)^r} - C_0 \simeq \frac{f_1(X)}{\left(\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)\right)^r} - C_0$$

The term in the denominator decreases the return when the total invested capital increases. The term C_0 stands for a fixed cost. In the case of constant returns, this term did not appear, as it is directly included in the definition of $f_1(X)$The return equation for investors solely connected to their sector becomes:

$$\begin{aligned} &\pm\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}} - \bar{r}' \qquad (89)\\ \simeq \quad &\left(\frac{A}{\left(f_1\left(X', \hat{K}\left[\hat{X}'\right]\right)\right)^2} + \frac{B}{\left(f_1\left(X', \hat{K}\left[\hat{X}'\right]\right)\right)^3}\right) \\ &\times\left[\left(f_1\left(X', \hat{K}\left[\hat{X}'\right]\right) - \bar{r}'\right) + \tau F(X)\left(f_1\left(X', \hat{K}\left[\hat{X}'\right]\right) - \left\langle f_1\left(X, \hat{K}\left[\hat{X}\right]\right)\right\rangle\right)\right]\end{aligned}$$

The dependency of $\hat{K}\left[\hat{X}\right]$ in $f_1(X)$ is calculated in Appendix 11.6 and is generally increasing, as expected.

12.2.2 Interconnected investors

To further understand the possibility of multiple states, we search for solutions of (89). The equation is more complex than the one studied under the CRS assumption because now decreasing returns reveal the capital of investors in the returns provided by the firms. It cannot be solved exactly, so we will look for solutions that will be seen as corrections to a state where investors are isolated, which therefore serves as a benchmark. Let us call $\Delta\left(\frac{N(\hat{X})}{\sqrt{\hat{K}[\hat{X}]}}\right)$ the correction due to interactions between investors, compared to a state without interactions between sectors. Besides, we define $\hat{K}_1\left[\hat{X}\right]$ as the solution without interactions between sectors. Thus, we can write:

$$\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}} = \frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}_1\left[\hat{X}\right]}} + \Delta\left(\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\right)$$

The sought-after solution is therefore the sum of the benchmark plus the correction to this benchmark due to interactions.

Appendix 11.8 derives an approximate equation for $\Delta\left(\frac{N(\hat{X})}{\sqrt{\hat{K}[\hat{X}]}}\right)$:

$$\left(\delta\left(\hat{X}-\hat{X}'\right)\pm_{\hat{X}}\frac{2H_2\left(\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)}{\left((1-H_1)^2-2H_2\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\frac{N(\hat{X}')}{\sqrt{\hat{K}_1[\hat{X}']}}\right)^{\frac{3}{2}}}\right)\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}\left[\hat{X}'\right]}}\right)$$

$$=\frac{(1-H_1)\pm_{\hat{X}}\sqrt{(1-H_1)^2-\left(2H_2\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\frac{N(\hat{X}')}{\sqrt{\hat{K}_1[\hat{X}']}}\right)}}{H_2}$$

where the sign $\pm_{\hat{X}}$ means that the choice of sign depends on the sector considered. There is indeed a double possibility per sector. Furthermore, the equation is not local; the first line shows that there is diffusion (presence of $\hat{S}_1^E\left(\hat{X}',\hat{X}\right)$), and thus the choice of sign in one sector impacts the other sectors. The withdrawal of capital invested in one sector will impact the other sectors connected to it, via investment decisions among investors. Consequently, entire blocks can end up in a low-capital or high-capital state. The diffusion and amplification effect depend on the level of indebtedness $1+\underline{\hat{k}}_2\left(\hat{X}\right)$. It is indeed a diffusion amplified by the level of lending among investors.

These multiple values are only constrained by the overall average of the system. The consistency of the equations imposes that in averages:

$$\int\hat{K}_1\left[\hat{X}\right]+\Delta\hat{K}\left[\hat{X}\right]d\hat{X}=\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2$$

so that for a given global state many different collective may arise, with our without disparity. On the other the overall multiple state may be seen as the consequence of the local multiple equilibria.

12.2.3 Interpretation

When investors are interconnected, and assuming decreasing returns to scale, the level of capital invested in a sector becomes relevant in the diffusion of capital, as it impacts the returns of firms. This leads to the emergence of multiple collective states due to circular effects. Even slight modifications in the capital level of a sector or firm can trigger changes in returns, impacting other investors' returns and altering their own capital levels. Consequently, investments and returns of adjacent firms are modified, initiating a chain reaction. In such scenarios, the emergence of a single collective state becomes the exception, as indicated by approximate calculations.

The degree of correlation among investors' states increases with the level of interconnection. Therefore, the different possible collective states do not represent independent investors but rather groups of investors, potentially characterized by varying capital levels, such as one group with high capital and another with low capital.

Dependency in other sectors' parameters Given that here, investors are connected, we can calculate the impact of a variation in firm returns in one sector on the overall capital level of the investors. If we expand the equation for capital (89) in detail:

$$\pm\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}-\bar{r}'\simeq\int\left(1-\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)^{-1}\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right) \tag{90}$$

$$\times\left(\frac{A}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)\right)^2}+\frac{B}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)\right)^3}\right)$$

$$\times\left[\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-\bar{r}'\right)+\tau F\left(X'\right)\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-\left\langle f_1\left(X,\hat{K}\left[\hat{X}\right]\right)\right\rangle\right)\right]$$

we see a relationship between the capital levels of all sectors. A collective state consists of a large number of related values of capital. The interdependence of capital in one sector $\hat{X}_1$ with respect to another sector $\hat{X}'$ is computed in Appendix 11.6, and yields:

$$\frac{\frac{d\hat{K}\left[\hat{X}_1\right]}{df_1(X')}}{\left(\hat{K}\left[\hat{X}_1\right]\right)^{\frac{3}{2}}} = \pm\left(1-\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1}$$

$$\times\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\frac{\left(1+\underline{k}\left(X'\right)\hat{K}\left[\hat{X}'\right]\right)\left(AF_1\left(X'\right)+BF_2\left(X'\right)\right)}{f_1\left(X\right)\left(AF_1\left(X'\right)+BF_2\left(X'\right)\right)-\frac{1}{2}F_3\left(X'\right)}$$

with:

$$F_1\left(X'\right) = \frac{f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-2r\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-\bar{r}'\right)}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)\right)^3}$$

$$F_2\left(X'\right) = \frac{f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-3r\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-\bar{r}'\right)}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)\right)^4}$$

$$F_3\left(X'\right) = \frac{\left(\left(1+\underline{k}\left(X'\right)\hat{K}\left[\hat{X}\right]\right)^{1+r}\right)H\left(\hat{X}_1\right)}{\left(\hat{K}\left[\hat{X}_1\right]\right)^{\frac{3}{2}}}$$

In general, this relationship is positive, showing that returns from distant firms can diffuse to the capital level of a sector through successive and complex investment chains. Similar to the case of constant returns to scale (CRS), productivity must exceed a certain threshold depending on the interest rate, to have a positive impact on the level of capital, measured by the term:

$$\frac{f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-2r\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-\bar{r}'\right)}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)\right)^3}$$

which is positive in general since $2r << 1$.

13 Global average capital and return

In the entire system, the parameters depend on the overall average quantities. Now that we have found solutions in terms of average quantities, we can compute them. Since there are various collective states, the averages themselves may be multiple, characterizing the collective states.

We compute the various averages arising in the model in Appendix 11.7. These quantities arise in the previous equations and solutions to compute the various parameters on which the solutions depend. These averages thus complete the solutions of the model.

13.1 Firms global average capital and return

13.1.1 Firms global average capital

The private average capital per firm, $\langle K\rangle$, is:

$$\langle K\rangle \simeq \langle K\rangle_1 \left(1-\frac{\left(3\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)\left(\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}+\beta C\right)}{4\langle f_1\rangle\langle K\rangle_1\left(2\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)}\right)^{\frac{1}{r}}$$

with:

$$\langle K\rangle_1=\frac{\langle f_1\rangle^{\frac{1+r^2}{r}}}{C_0^{\frac{1}{r}}\left(\langle f_1\rangle^r+\langle \underline{k}\rangle\left\langle \hat{K}\right\rangle C_0^r\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\right)}$$

This average capital per firm is increasing with average productivity $\langle f_1\rangle$, decreases with the cost of capital C_0, and decreases with the number of firms in the sector, since the firms share capital invested in the sector:

$$\langle \underline{k}\rangle\left\langle \hat{K}\right\rangle\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}$$

We see that $\langle K\rangle$ is inversely proportional to the invested capital. This doesn't mean that the more we invest, the less the firm has capital, but rather that the share of private capital decreases. The private capital $\langle K\rangle$ is displaced by the investment $\left\langle \hat{K}\right\rangle$, augmented by the effect of investor diffusion. Indeed, the average global available capital level for firms is given by:

$$\langle K\rangle\left(1+\langle \underline{k}\rangle\left\langle \hat{K}\right\rangle\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\right)$$

And this quantity, in turn, increases with the invested capital, provided that $\langle f_1\rangle > C_0$, which is generally the case since it is the condition for production. The term $\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}$ represents the ratio of the number of investors to the number of firms. The more investors there are, the more capital comes to invest, and the more the share of private capital decreases.

13.1.2 Firms' global average return

We can also compute the average return of firms, $\left\langle f_1^{(e)}\right\rangle$:

$$\begin{aligned}\left\langle f_1^{(e)}\right\rangle \simeq\; & \langle f_1\rangle+\beta\langle \underline{k}\rangle\frac{\left(\frac{1}{4}-\frac{r}{3}\hat{k}\right)}{\left(\frac{1}{3}-\frac{r}{2}\hat{k}\right)\bar{r}'}\\ & \times\frac{4\left\langle f_1^{(e)}\right\rangle\left(2\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)\sqrt{\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(\left\|\hat{\Psi}_0\right\|^2-\hat{\mu}\langle D\rangle\right)}}{\left(3\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)\left(\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}+\beta C\right)}\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\left(\langle f_1\rangle-\bar{r}\right)\end{aligned}$$

The actual return of firms $\left\langle f_1^{(e)} \right\rangle$ sees its deviation from the average increase with the share of loans from investors, rather than equity stakes, and with the ratio of the number of investors to the number of firms in the sector.

Indeed, if firms have a return higher than the interest rate, they benefit from having made a loan rather than issuing equity stakes.

13.2 Investors' global average capital and return

13.2.1 Investors' global average return

The average return for investor, $\langle \hat{g} \rangle$, is:

$$\langle \hat{g} \rangle = \langle \Delta \rangle + \bar{r}' \tag{91}$$

where $\langle \Delta \rangle$ represents the average excess return for the overall set of investors. It satisfies the average of equation (82), given by:

$$\langle \Delta \rangle = \left\langle \left(\frac{A(X)}{\left(f_1^{(r)}(X)\right)^2} + \frac{B(X)}{\left(f_1^{(r)}(X)\right)^3} \right) \right\rangle \left\langle R + \Delta F_\tau \left(\bar{R}(K,X) \right) \right\rangle \tag{92}$$

with:

$$\begin{aligned} f_1^{(r)}(X) &= \frac{f_1(X)}{\left(1 + \langle \underline{k} \rangle \frac{\langle \hat{K} \rangle \|\hat{\Psi}_0\|^2}{\|\Psi_0\|^2}\right)^r} - C_0 \\ R &= \frac{f_1(X)}{\left(1 + \langle \underline{k} \rangle \frac{\langle \hat{K} \rangle \|\hat{\Psi}_0\|^2}{\|\Psi_0\|^2}\right)^r} - C_0 - \bar{r} \end{aligned}$$

13.2.2 Investors' global average capital

Using equations (253) and (254), we find the average number of investors, $\left\|\hat{\Psi}\right\|^2$:

$$\left\|\hat{\Psi}\right\|^2 \simeq \frac{\hat{\mu} V}{3\sigma_{\hat{K}}^2 \langle \hat{g} \rangle} \left(2 \frac{\sigma_{\hat{K}}^2 \left(\frac{\|\hat{\Psi}_0\|^2}{\hat{\mu}} + \frac{\langle \hat{g} \rangle}{2} \left(\frac{1-r}{r}\right) \hat{\underline{k}} \right)}{\left(1 - 2r(1-r)\hat{\underline{k}}\right)} \right)^{\frac{3}{2}} \left(1 - \frac{2 + \hat{\underline{k}} - \sqrt{\left(2 + \hat{\underline{k}}\right)^2 - \hat{\underline{k}}}}{4\hat{\underline{k}}} \hat{\underline{k}} \right) \tag{93}$$

and the total available capital, $\left\langle \hat{K} \right\rangle \left\|\hat{\Psi}\right\|^2$:

$$\left\langle \hat{K} \right\rangle \left\|\hat{\Psi}\right\|^2 = \frac{\hat{\mu} V \sigma_{\hat{K}}^2}{2 \langle \hat{g} \rangle^2} \left(\frac{\left(\frac{\|\hat{\Psi}_0\|^2}{\hat{\mu}} + \frac{\langle \hat{g} \rangle}{2} \left(\frac{1-r}{r}\right) \hat{\underline{k}} \right)}{\left(1 - 2r(1-r)\hat{\underline{k}}\right)} \right)^2 \left(1 - 2 \frac{2 + \hat{\underline{k}} - \sqrt{\left(2 + \hat{\underline{k}}\right)^2 - \hat{\underline{k}}}}{9} \right) \tag{94}$$

where $\langle \hat{g} \rangle$ is given by (91), and with:

$$r = \frac{2 + \hat{\underline{k}} - \sqrt{\left(2 + \hat{\underline{k}}\right)^2 - \hat{\underline{k}}}}{6\hat{\underline{k}}}$$

Appendix 11.7 shows that $\left\Vert \hat{\Psi}\right\Vert ^{2}$ and $\left\langle \hat{K}\right\rangle \left\Vert \hat{\Psi}\right\Vert ^{2}$ increase with average productivity $\langle f_1(X)\rangle$. Formula (93) and (94) show that both $\left\Vert \hat{\Psi}\right\Vert ^{2}$ and $\left\langle \hat{K}\right\rangle \left\Vert \hat{\Psi}\right\Vert ^{2}$ increase with $\hat{\underline{k}}$, the average interconnectivity of investors.

13.2.3 Interpretation

The excess return compared to the interest rate is proportional to the excess productivity compared to the interest rate and relative to the average productivity, $R+\Delta F_{\tau}\left(\bar{R}(K,X)\right)$, as expected. It is also proportional to the percentage of equity stake $(1-\beta)$ in investments. This outcome was also expected, as it contrasts with the situation for firms. When returns are significant, firms benefit from loans, while investors benefit from having taken equity stakes. Factors like $(X+C\beta)$ essentially depend on the number of firms in the economy. The more firms there are, the higher the return. Additionally, the return is proportional to the ratio $\frac{\epsilon}{3\sigma_{\hat{K}}^{2}}$, which computes the variability of the number of firms in a sector. This ratio reflects the ease with which firms can position themselves across sectors. The higher it is, the more firms can move across sectors; conversely, the lower it is, the more static the firms will be.

The gains of firms, on average, propagate to investors. This propagation is also proportional to the number of firms $\Vert\Psi_0\Vert^2$: the more firms there are, the greater the returns. Conversely, it is inversely proportional to the number of investors, $\left\Vert \hat{\Psi}_{0}\right\Vert ^{2}$, who must share the returns.

13.3 Multiple averages

Combining (91) nd (94) leads to rewrite the link between average amount of capital and average return as an equation for average capital. Equation (94) leads to :

$$\langle\hat{g}\rangle=\frac{\sqrt{\frac{V\sigma_{\hat{K}}^{2}}{2\hat{\mu}}}\left\Vert \hat{\Psi}_{0}\right\Vert ^{2}}{\left(1-2r\left(1-r\right)\hat{\underline{k}}\right)\sqrt{\frac{\langle\hat{K}\rangle\Vert\hat{\Psi}\Vert^{2}}{1-2\frac{2+\hat{\underline{k}}-\sqrt{\left(2+\hat{\underline{k}}\right)^{2}-\hat{\underline{k}}}}{9}}}-\frac{1}{2}\left(\frac{1-r}{r}\right)\hat{\underline{k}}}$$

and comparing to (92) leads to:

$$\begin{aligned}
&\frac{\sqrt{\frac{V\sigma_{\hat{K}}^{2}}{2\hat{\mu}}}\left\Vert \hat{\Psi}_{0}\right\Vert ^{2}}{\left(1-2r\left(1-r\right)\hat{\underline{k}}\right)\sqrt{\frac{\langle\hat{K}\rangle\Vert\hat{\Psi}\Vert^{2}}{1-2\frac{2+\hat{\underline{k}}-\sqrt{\left(2+\hat{\underline{k}}\right)^{2}-\hat{\underline{k}}}}{9}}}-\frac{1}{2}\left(\frac{1-r}{r}\right)\hat{\underline{k}}}\\
=\;&\bar{r}'+\left(\frac{A}{\left(\left\langle f_{1}^{(r)}(X)\right\rangle\right)^{2}}+\frac{B}{\left(\left\langle f_{1}^{(r)}(X)\right\rangle\right)^{3}}\right)\left\langle\left(f_{1}^{(r)}(X)+\Delta F_{\tau}\left(\bar{R}(K,X)\right)\right)\right\rangle
\end{aligned}$$

with:

$$f_{1}^{(r)}(X)=\frac{f_{1}(X)}{\left(1+\langle\hat{\underline{k}}\rangle\frac{\langle\hat{K}\rangle\Vert\hat{\Psi}\Vert^{2}}{\Vert\Psi_{0}\Vert^{2}}\right)^{r}}-C_{0}$$

For:

$$\left\langle\left(R+\Delta F_{\tau}\left(\bar{R}(K,X)\right)\right)\right\rangle>0$$

this equation yields multiple solutions across a broad range of parameters. It is worth noting that when $\left\|\hat{\Psi}_0\right\|^2$ is low, the solution is unique. In general three solutions for capital arise. Furthermore, as the total amount of capital increases, the average return tends to decrease.

For:

$$\left\langle \left(R + \Delta F_{\tau}\left(\bar{R}\left(K, X\right)\right)\right)\right\rangle < 0$$

only one solution exists when capital is low. This capital is not equal to 0, as even with a small number of agents, it does not vanish completely.

13.4 Synthesis on collective states under a non-default scenario

Our results in sections 12 and 13, obtained by sector and on average, show that, in general, there is a multiplicity of collective states. They may arise either at the level of global averages, or in the distribution of these global averages.

At the level of global averages, several possibilities arise, spanning from a high capital level, a relatively low excess return and a large number of agents, to that of a lower capital with a smaller number of agents and higher returns.

Furthermore, for a given average capital level, multiple collective states corresponding to different capital distributions per sector may emerge, from very homogeneous to very heterogeneous distributions.

These situations result from the interconnection of sectors and firm returns, which depend on the capital invested. Firms could lack invested capital due to a highly concentrated capital in other sectors. This lack of capital can arise from a vicious circle, where the lack of return implies the lack of investment, itself implying a lack of return... With these results at hand, we can now turn to the study of investor returns under a default scenario.

14 Investors' returns under a default scenario

To account for defaults in the resolution of the system, we must write the return equation recursively: we start with a minimal set of sectors experiencing a default, then study their recursive propagation. The collective state with default is reached when the recursive process converges toward an entire sector space divided into two stable subsets, each corresponding to a default or non-default sets, respectively. We present below the recursive resolution.

14.1 Resolution

Let us consider a minimal domain D, such that the return of the sector $\hat{f}\left(\hat{X}_1\right)$, given by (85), satisfies:

$$\hat{f}\left(\hat{X}_1\right) - \bar{r} < -1$$

This corresponds to the set of sectors such defaults occur. The return is so low that the firm's or investor's equity alone is insufficient to repay the loans.

The return equation, in terms of average capital per sector, is modified by this default set, denoted as DF_0.

$$\left(\frac{A\left(\hat{X}_1\right)}{f_1^2\left(\hat{X}_1\right)}+\frac{B\left(\hat{X}_1\right)}{f_1^3\left(\hat{X}_1\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,\hat{X}_1\right)\right)\right)=\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}}}{\sqrt{\frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{2\sigma_{\hat{K}}^2}}}-\bar{r}'$$

$$-\int_{H/DF_0}\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\left(\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}}}{\sqrt{\frac{\hat{\mu}\hat{K}\left[\hat{X}'\right]}{2\sigma_{\hat{K}}^2}}}-\bar{r}'\right)-\int_{DF_0}\hat{k}_2\left(\hat{X}_1,\hat{X}'\right)\frac{1+\hat{f}\left(\hat{X}'\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)}$$

The modification arises in the second line. Now, the diffusion of returns is constrained to the sectors with no default, denoted as H/DF_0. Conversely, the loss stems from defaulting sectors located in the subspace DF_0:

$$\int_{DF_0}\hat{k}_2\left(\hat{X}_1,\hat{X}'\right)\frac{1+\bar{g}\left(\hat{X}'\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)}$$

and propagates to the whole sector space.

As a consequence for $\hat{X}_1$ not belonging to D, that is for $\hat{X}_1$ a sector with no defaults, the return equation, expressed in terms of $\hat{f}\left(\hat{X}_1\right)$ becomes:

$$\begin{aligned}\left(\hat{f}\left(\hat{X}_1\right)-\bar{r}\right) &= (1-M)\int\left(1-\frac{3\sigma_{\hat{K}}^2 f_1(X)}{C(2X-C\beta)(1-\beta)(X+C\beta)^2\epsilon}\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)-1_{DF_0}\frac{\hat{k}_2\left(\hat{X}',\hat{X}_1\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\right)^{-1}\\ &\times\left(\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}(K,X)\right)\right)-\frac{1_{DF_0}\hat{k}_2\left(\hat{X}',\hat{X}_1\right)(1+\bar{r}')}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\right)\end{aligned}$$

This equation shows that the sector $\hat{X}_1$, initially considered as non-defaulting, may become part of the default set if the the loss associated with the default of other sectors exceeds the threshold for default to occur.

This outcome suggests the resolution method to identify the stable default set of some collective state. Beginning with an initial default set, we recursively define the n-th default set:

$$DF_n=\left\{\hat{X}_1,\hat{f}_{n-1}\left(\hat{X}_1\right)<-1\right\}$$

where $\hat{f}_n\left(\hat{X}_1\right)$ is defined by:

$$\begin{aligned}\left(\hat{f}_{n+1}\left(\hat{X}_1\right)-\bar{r}\right) &= (1-M)\int\left(1-\frac{3\sigma_{\hat{K}}^2 f_1(X)}{C(2X-C\beta)(1-\beta)(X+C\beta)^2\epsilon}\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)-1_{DF_n}\frac{\hat{k}_2\left(\hat{X}',\hat{X}_1\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\right)^{-1}\\ &\times\left(\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}(K,X)\right)\right)-\frac{1_{DF_n}\hat{k}_2\left(\hat{X}',\hat{X}_1\right)(1+\bar{r}')}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\right)\end{aligned}$$

Ultimately, the return for a collective state with default is given by the limit:

$$\hat{f}_n\left(\hat{X}_1\right)\rightarrow\hat{f}\left(\hat{X}_1\right)$$

The resolution resembles to a transmission mechanism. Beginning with a default-free state, we assume that some sectors experience default. This is feasible for real sectors with low returns relative to the average and collective state characterized by low capital. In such scenarios, the collective state undergoes a transition towards a state with negative returns for some sectors. Describing this new collective state with defaulting sectors may imply that other sectors are affected, transmitting defaults to other sectors, ultimately leading to a collective default state. However, it is important to note that this dynamic description should not obscure the fact that the resulting default state is not solely the consequence of adverse shocks but rather a intrinsic possibility of the system.

This aspect can be studied more precisely by dividing the sector space into several interacting groups. Introducing some dynamic aspects in default transmission can be achieved by considering these groups as independent and weakly interacting.

15 Dynamics Interactions of agents inside or across groups of investors

The averages were computed across the entire space. However, we can consider that sector spaces, investors, and firms are organized into heterogeneous groups or sub-markets, which are relatively weakly interconnected.

In this section, rather than studying the system by sectors and/or globally, we will consider subgroups and investigate the dynamics of investors within these subgroups, interacting with agents from other groups. This approach allows for a dynamic examination of the possibility of transitions between collective states.

As a benchmark and to introduce some notations, we first reconsider the case of a homogeneous group: a homogeneous group of investors and firms, with approximately homogeneous returns, connections, capital, and productions. We describe the system's averages and then consider the interactions among agents from multiple groups.

15.1 Benchmark: dynamics of agents inside an homogeneous group of agents

We describe a homogeneous group using averages, which simplifies the notations. We then consider the effective action of agents within this group, describing their interactions and transition functions. This describes the dynamic aspect within a given collective state. The aim is to understand the dynamic and micro aspects that can lead to transitions between collective states.

15.1.1 Averages for an homogeneous group

When we consider the system as a group of homogeneous agents with identical average returns, Appendix 13.1 shows that the return equation, including defaults is:

$$\left(\frac{f\left(\hat{X}\right)}{1+\underline{\hat{k}}_2\left(\hat{X}\right)}+\bar{r}\frac{\underline{\hat{k}}_2\left(\hat{X}\right)}{1+\underline{\hat{k}}_2\left(\hat{X}\right)}\right)-\frac{\hat{K}\hat{k}_{1E}\left(\hat{X}\right)}{1+\underline{\hat{k}}}\left(\frac{f}{1+\underline{\hat{k}}_2}+\bar{r}\frac{\underline{\hat{k}}_2}{1+\underline{\hat{k}}_2}\right) \tag{95}$$

$$= \left\{\left(\bar{r}+\frac{1+f}{\underline{\hat{k}}_2}H\left(-\frac{1+f}{\underline{\hat{k}}_2}\right)\right)\frac{\hat{k}_{2E}\left(\hat{X}\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\right.$$

$$\left.+\left(\bar{r}+\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)}H\left(-\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)}\right)\right)\frac{k_{2E}\left(\hat{X}\right)}{1+\underline{k}\left(X'\right)}+\frac{k_{1E}\left(\hat{X}\right)}{1+\underline{k}\left(X'\right)}f_1\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)\right\}$$

where $\hat{\underline{k}}_2$ and $\hat{\underline{k}}_1$ are the averages of $\hat{\underline{k}}_2\left(\hat{X}\right)$ and $\hat{\underline{k}}_2\left(\hat{X}\right)$, representing the average share of investment in sector $\hat{X}$ by other sectors. Recall that $\hat{k}_{1E}\left(\hat{X}\right)$ denotes the average share of outgoing investment from sector $\hat{X}$. We define f as the average return for investors, and f_1 as the average return for firms. In Appendix 13.1, we show that considering the averages, the return equation for the group, including defaults, writes:

$$\frac{f}{1+\hat{\underline{k}}}=\left(\frac{1+f}{\hat{\underline{k}}_2}\right)H\left(-\left(1+f\right)\right)\frac{\hat{\underline{k}}_2}{1+\hat{\underline{k}}}+\left(\frac{1+f_1'}{\underline{k}_2}\right)H\left(-\left(1+f_1'\right)\right)\frac{k_2}{1+\underline{k}}+\bar{r}\frac{k_2}{1+\underline{k}}+\frac{k_1}{1+\underline{k}}f_1$$

note that without default, this reduces to:

$$\frac{f}{1+\hat{\underline{k}}}=\bar{r}\frac{k_2}{1+\underline{k}}+\frac{k_1}{1+\underline{k}}f_1 \tag{96}$$

depicting that returns are divided into returns from loans and firms, along with:

$$\frac{k_2}{1+\underline{k}}+\frac{k_1}{1+\underline{k}}=\frac{\underline{k}}{1+\underline{k}}=\frac{1}{1+\hat{\underline{k}}}$$

15.1.2 Dynamics in one homogeneous group

We analyze the dynamics within a group of roughly homogeneous agents; they fluctuate around the same average. Let's begin with a group without defaults. Recall that the return equation:

$$\int\left(\delta\left(\hat{X}'-\hat{X}\right)-\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)}\right)\frac{f\left(\hat{X}'\right)-\bar{r}}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)}d\hat{K}'d\hat{X}'$$

$$=\int\frac{\left|\Psi\left(K',X'\right)\right|^2k_1\left(X',\hat{X}\right)K'}{1+\underline{k}\left(X'\right)}\left(\frac{f_1'\left(K,X\right)-\bar{r}k_2\left(\hat{X}\right)}{1+k_2\left(\hat{X}\right)}+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)dK'dX'$$

is included in the field action through a potential term, and that the diffusion between sectors $\hat{X}'$ and $\hat{X}$ writes:

$$\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)}$$

We apply the method outlined in section 5 to compute the transition functions using Field Theory. The dynamics are studied by calculating the effective action (26) of the system around the background field. From this effective action, we can compute the individual transition functions using formulas (27) and (29). The transition functions for a group of agents are given by formula (30), which, without interactions, corresponds to a product of individual transition functions. Finally, we include in these transition functions, based on the interactions of agents, the possible transmission of defaults using formula (31).

15.1.3 Effective action

The dynamics are studied by calculating the effective action (26) of the system around the background field. In principle, a series expansion around the background field is computed, which also implies a series expansion in returns. To simplify these calculations and allow for the emergence of dynamic fluctuations in returns, we consider the effective action obtained by keeping the return at its level given by the background field, and then extend the effective action by introducing a field describing the excess returns relative to the background value. This is a simplification of the effective action, but it maintains the idea that agents have dynamics around the background that will be impacted by changes in returns.

This is done by introducing a field $\Xi\left(\hat{X},\delta f_1\right)$ measuring the modifications in returns. The value of δf_1 measures the excess return compared to the average of the group. The dynamics for investors inside the group is given by the effective action:

$$\begin{aligned}
S_0 &= -\int \hat{\Psi}^{\dagger}\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}-\hat{K}\left(f\left(\hat{X},K_{\hat{X}}\right)+\int \delta f_1\left|\Xi\left(\hat{X},\delta f_1\right)\right|^2 d\left(\delta f_1\right)\right)\right)\right)\hat{\Psi} \\
&\quad +\hat{\mu}\left(\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2-\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2\right)^2 \\
&\quad -\int \Xi^{\dagger}\left(\hat{X},\delta f_1\right)\sigma_{\delta f_1}^2\nabla_{\delta f_1}^2\Xi\left(\hat{X},\delta f_1\right)d\left(\delta f_1\right)+\Xi^{\dagger}\left(\hat{X},\delta f_1\right)J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)+J^{\dagger}\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\Xi\left(\hat{X},\delta f_1\right)
\end{aligned}$$

where the terme :

$$-\Xi^{\dagger}\left(\hat{X},\delta f_1\right)\sigma_{\delta f_1}^2\nabla_{\delta f_1}^2\Xi\left(\hat{X},\delta f_1\right)$$

represents the dynamics of the excess return δf_1, and its fluctuations, and the term:

$$\Xi^{\dagger}\left(\hat{X},\delta f_1\right)J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)+J^{\dagger}\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\Xi\left(\hat{X},\delta f_1\right)$$

represents the interaction of δf_1 with a vector of external perturbations $J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)$, where $\mathbf{E}$ denotes the external parameters.

Around the background field, we can consider that $\hat{\mu}\left(\left\|\hat{\Psi}\right\|^2-\left\|\hat{\Psi}_0\right\|^2\right)^2 << 1$, and that the group has a fixed number of investors.

15.1.4 Transition function for agents without interactions

the transition function for agents inside the group is first computed without interactions. This is done by inverting the operator, as in (29), for the dynamics of capital:

$$-\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}-\hat{K}f\left(\hat{X},K_{\hat{X}}\right)+\int \delta f_1\left|\Xi\left(\hat{X},\delta f_1\right)\right|^2 d\left(\delta f_1\right)\right)$$

Appendix 14 shows that this yields the partial Green function conditioned to the initial state and final state for returns as:

$$\begin{aligned}
&\sqrt{\left|\frac{f\left(\hat{X},K_{\hat{X}}\right)+\left(\frac{\delta f_1+\delta f_1'}{2}\right)}{\sigma_{\hat{K}}^2\left(1-\exp\left(2\left(f\left(\hat{X},K_{\hat{X}}\right)+\left(\frac{\delta f_1+\delta f_1'}{2}\right)\right)\Delta t\right)\right)}\right|} \qquad (97)\\
&\times\exp\left(f\left(\hat{X},K_{\hat{X}}\right)\frac{\left(\hat{K}-\exp\left(\left(f\left(\hat{X},K_{\hat{X}}\right)+\frac{\delta f_1+\delta f_1'}{2}\right)\Delta t\right)\hat{K}'\right)^2}{\sigma_{\hat{K}}^2\left(1-\exp\left(2\left(f\left(\hat{X},K_{\hat{X}}\right)+\frac{\delta f_1+\delta f_1'}{2}\right)\Delta t\right)\right)}\right)\\
&\times\exp\left(\frac{\left(f\left(\hat{X},K_{\hat{X}}\right)+\left(\frac{\delta f_1'-\delta f_1}{\Delta t}\right)\right)}{\sigma_{\hat{K}}^2\left(1-\exp\left(2\left(f\left(\hat{X},K_{\hat{X}}\right)+\left(\frac{\delta f_1'-\delta f_1}{\Delta t}\right)\right)\Delta t\right)\right)}\left(\hat{K}-\exp\left(\left(f\left(\hat{X},K_{\hat{X}}\right)+\left(\frac{\delta f_1'-\delta f_1}{\Delta t}\right)\right)\Delta t\right)\hat{K}'\right)^2\right)
\end{aligned}$$

We proceed in the same manner for the excess returns, and we invert the operator:

$$-\Xi^{\dagger}\left(\hat{X},\delta f_1\right)\sigma_{\delta f_1}^2\nabla_{\delta f_1}^2\Xi\left(\hat{X},\delta f_1\right)+\Xi^{\dagger}\left(\hat{X},\delta f_1\right)J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)+J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\Xi\left(\hat{X},\delta f_1\right)$$

and we obtain the partial transition function:

$$\sqrt{\frac{1}{\sigma_{\delta f_1}^2}}\exp\left(-\frac{\left(\delta f_1-\delta f_1'\right)^2}{\sigma_{\delta f_1}^2}+J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\delta f_1-J\left(\hat{X}',K_{\hat{X}'},\mathbf{E}'\right)\delta f_1'\right) \tag{98}$$

So that the transition function for an agent between a state with capital K and return δf_1, to a state with capital K' and return $\delta f_1'$ is:

$$\begin{aligned}
&\left\langle \hat{K}',\delta f_1'\ \hat{K},\delta f_1\right\rangle\\
= &\sqrt{\left|\frac{f\left(\hat{X},K_{\hat{X}}\right)+\left(\frac{\delta f_1+\delta f_1'}{2}\right)}{\sigma_{\hat{K}}^2\left(1-\exp\left(2\left(f\left(\hat{X},K_{\hat{X}}\right)+\left(\frac{\delta f_1+\delta f_1'}{2}\right)\right)\Delta t\right)\right)}\right|}\\
&\times\exp\left(\frac{\left(f\left(\hat{X},K_{\hat{X}}\right)+\left(\frac{\delta f_1+\delta f_1'}{2}\right)\right)}{\sigma_{\hat{K}}^2\left(1-\exp\left(2\left(f\left(\hat{X},K_{\hat{X}}\right)+\left(\frac{\delta f_1+\delta f_1'}{2}\right)\right)\Delta t\right)\right)}\left(\hat{K}-\exp\left(\left(f\left(\hat{X},K_{\hat{X}}\right)+\left(\frac{\delta f_1+\delta f_1'}{2}\right)\right)\Delta t\right)\hat{K}'\right)^2\right)\\
&\times\exp\left(-\frac{\left(\delta f_1-\delta f_1'\right)^2}{\sigma_{\delta f_1}^2}+J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\delta f_1-J^{\dagger}\left(\hat{X}',K_{\hat{X}'},\mathbf{E}'\right)\delta f_1'\right)\\
\equiv\ &G\left(\Delta t,f_1,\hat{K}_1,\hat{K}_1'\right)G_{\delta f_1}\left(\Delta t,\delta f_1,\delta f_1'\right)
\end{aligned}$$

This transition function calculates, the probability for an agent to transition from a state with capital and excess return, $\hat{K},\delta f_1$, to a state $\hat{K}',\delta f_1'$. The term under the square root is a normalization factor. The first exponential term calculates the transition probability for the agent's capital, and roughly follows the group's trend, since $\hat{K}'$ is centered on the expression $\exp\left(f\left(\hat{X},K_{\hat{X}}\right)\Delta t\right)\hat{K}$, which corresponds to accumulation when the return is constant and equal to the sector's average $f\left(\hat{X},K_{\hat{X}}\right)$.

The second exponential term represents the dynamics of excess returns relative to a group's average. This dynamic includes perturbation due to external sources. The term $\frac{\left(\delta f_1-\delta f_1'\right)^2}{\sigma_{\delta f_1}^2}$ describes internal fluctuations with amplitude $\sigma_{\delta f_1}$, and the following term $J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\delta f_1-J^{\dagger}\left(\hat{X}',K_{\hat{X}'},\mathbf{E}'\right)\delta f_1'$ deviates the trajectory of the agent's excess return from the average of their sector and the group. These two terms combine: fluctuations move the agent away from the average, which is amplified by the perturbation. Rather than simply fluctuating around an average, these perturbations can deviate returns.

15.1.5 Effective action including interactions

Finally, we include in these transition functions, based on the interactions of agents, the possible transmission of defaults using formula (31). We incorporate into the effective action the interactions provided by the diffusion of returns among themselves.

The constraint for return is included through a potential inside the effective action:

$$
\begin{aligned}
S = & -\int \hat{\Psi}^{\dagger}\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^{2}}{2}\nabla_{\hat{K}}-\hat{K}\left(f\left(\hat{X},K_{\hat{X}}\right)+\int \delta f_{1}\left|\Xi\left(\hat{X},\delta f_{1}\right)\right|^{2} d\left(\delta f_{1}\right)\right)\right)+\right)\hat{\Psi} \\
& +\hat{\mu}\left(\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^{2}-\left\|\hat{\Psi}_{0}\left(\hat{X}\right)\right\|^{2}\right)^{2} \\
& -\Xi^{\dagger}\left(\hat{X},\delta f_{1}\right)\nabla_{\delta f_{1}}^{2}\Xi\left(\hat{X},\delta f_{1}\right)+\Xi^{\dagger}\left(\hat{X},\delta f_{1}\right)J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)+J^{\dagger}\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\Xi\left(\hat{X},\delta f_{1}\right) \\
& +\int\left|\hat{\Psi}\left(\hat{X},K_{\hat{X}}\right)\right|^{2} V\left(\hat{\Psi},\hat{X},K,\delta f_{1},\delta f_{1}^{\prime}\right)\left|\Xi\left(\hat{X},\delta f_{1}\right)\right|^{2}\left|\Xi\left(\hat{X},\delta f_{1}^{\prime}\right)\right|^{2}
\end{aligned}
$$

where $V\left(\hat{\Psi},\hat{X},K,\delta f_{1}\right)$ is a Dirac delta function type potential imposing ex-post the return equation (95) on agents experiencing independent returns. Assuming a return modification $\delta f_{1}^{\prime}$ in some sector $\hat{X}^{\prime}$, this impacts sectors $\hat{X}$ by a variation δf_{1} obtained by the first-order expansion of (95) around its average:

$$
\delta f_{1}^{\prime}=\frac{\hat{k}_{1}\left(\hat{X}^{\prime},\hat{X}\right)\left\langle\hat{K}^{\prime}\right\rangle\left|\hat{\Psi}\left(\hat{K}^{\prime},\hat{X}^{\prime}\right)\right|^{2}}{1+\underline{\hat{k}}\left(\hat{X}^{\prime}\right)}\delta f_{1} \tag{99}
$$

where the right-hand side represents the share of investment of agents of sector $\hat{X}$ in sector $\hat{X}^{\prime}$.

To consider that such constraint has to be imposed for all agents after interaction, we include in the effective potential:

$$
\begin{aligned}
& V\left(\hat{\Psi},\hat{X},K,\delta f_{1}\right) \\
= & \ \delta\left[\int\left(\delta\left(\hat{X}^{\prime}-\hat{X}\right)-\frac{\hat{k}_{1}\left(\hat{X}^{\prime},\hat{X}\right)\hat{K}^{\prime}\left|\hat{\Psi}\left(\hat{K}^{\prime},\hat{X}^{\prime}\right)\right|^{2}}{1+\underline{\hat{k}}\left(\hat{X}^{\prime}\right)}\right)\frac{\int\left|\Xi\left(\hat{X},\delta f_{1}\right)\right|^{2}\left|\Xi\left(\hat{X},\delta f_{1}^{\prime}\right)\right|^{2} d\left(\delta f_{1}\right)d\left(\delta f_{1}^{\prime}\right)}{1+\underline{\hat{k}}_{2}\left(\hat{X}^{\prime}\right)}d\hat{K}^{\prime}d\hat{X}^{\prime}\right]
\end{aligned} \tag{100}
$$

with $\delta[X]$ being the Dirac-delta function. δf_{1} and $\delta f_{1}^{\prime}$ are deviations from the average return of the sector experienced by two investors located in $\hat{X}$ and $\hat{X}^{\prime}$ respectvly. The term in brackets inside the δ function is the translation of (99) in terms of field.

In Appendix 14, we show that expanding a function in series as $\int\left|\Xi\left(\hat{X},\delta f_{1}\right)\right|^{2} d\left(\delta f_{1}\right)$ amounts to impose the constraint:

$$
\delta\left(\delta f_{1i}^{\prime}\left(\hat{X}_{i}\right)-\sum_{j}\frac{\hat{k}_{1}\left(\hat{X}_{i},\hat{X}_{j}\right)\left\langle\hat{K}_{j}\right\rangle\left|\hat{\Psi}\left(\hat{K}_{j},\hat{X}_{j}\right)\right|^{2}}{1+\underline{\hat{k}}\left(\hat{X}_{j}\right)}\delta f_{1j}\left(\hat{X}_{j}\right)\right) \tag{101}
$$

between the interacting agents.

The Dirac delta function type potential means that we impose the return equation ex post after modification of these returns for different agents. The potential $V\left(\hat{\Psi},\hat{X},K,\delta f_{1}\right)$ only shows successive deviations $\delta f_{1}^{\prime}$ and δf_{1} because the return equation is already satisfied for the averages and the returns of the firms assumed to be non-fluctuating. The chosen interaction thus propagates excess returns among investors. A variation in excess return by one investor is propagated to those interacting with them.

15.1.6 Transition functions within the group without default

The transition functions are calculated using the formulas (30) and (31). Without the interaction V defined in (100), the transition of the group would be given by:

$$\begin{aligned}
& \left\langle \left(\hat{K}_i', \delta f_{1i}'\right)_i T \left(\hat{K}_i, \delta f_{1i}\right)_i \right\rangle \\
= & \prod_i \left\langle \hat{K}_i', \delta f_{1i}' \ \hat{K}_i, \delta f_{1i} \right\rangle \\
= & G\left(\Delta t, f_{1,1}, \hat{K}_1, \hat{K}_1'\right) ... G\left(\Delta t, f_{1,1}, \hat{K}_n, \hat{K}_n'\right) \times G_{\delta f_1}\left(\Delta t, \delta f_1, \delta f_1'\right) ... G_{\delta f_1}\left(\Delta t, \delta f_p, \delta f_p'\right)
\end{aligned}$$

reflecting that the transition probabilities of investors are independent. Taking interactions into account, this result is corrected using (31).

The contribution associated to one interaction are given fr n agents by the potential:

$$\begin{aligned}
& G\left(\Delta t, f_{1,1}, \hat{K}_1, \hat{K}_1'\right) ... G\left(\Delta t, f_{1,1}, \hat{K}_n, \hat{K}_n'\right) \times G_{\delta f_1}\left(\Delta t, \delta f_1, \delta f_1'\right) ... G_{\delta f_1}\left(\Delta t, \delta f_p, \delta f_p'\right) \\
& \times V_n\left(\hat{X}_1, \hat{K}_1' ... \hat{X}_n, \hat{K}_n'\right) \\
& \times G\left(\Delta t, f_{1,1}', \hat{K}_1', \hat{K}_1''\right) ... G\left(\Delta t, f_{1,n}', \hat{K}_n', \hat{K}_n''\right) \times G_{\delta f_1}\left(\Delta t, \delta f_1', \delta f_1''\right) ... G_{\delta f_1}\left(\Delta t, \delta f_p', \delta f_p''\right)
\end{aligned}$$

The potential is seen as an interaction term modifying the trajectories of agents. For these n agents, the integral form is replaced by a specific sum involving these agents.

$$\begin{aligned}
& V_n\left(\hat{X}_1, \hat{K}_1' ... \hat{X}_n, \hat{K}_n'\right) \\
= & \prod_i \delta \left\{ \sum_j \left(\delta\left(\hat{X}_i - \hat{X}_j\right) - \frac{\hat{k}_1\left(\hat{X}_j, \hat{X}_i\right) \hat{K}_i' \left|\hat{\Psi}\left(\hat{K}_j', \hat{X}_j\right)\right|^2}{1 + \underline{\hat{k}}\left(\hat{X}'\right)} \right) \frac{\delta f_1'\left(\hat{X}_j\right)}{1 + \underline{\hat{k}}_2\left(\hat{X}'\right)} \right\}
\end{aligned} \tag{102}$$

It is this interaction term that will determine the transitions of the interacting agents. It imposes the constraint of correlated returns among investors, where an investor's return is given by $f\left(\hat{X}_j\right) + \delta f_1'\left(\hat{X}_j\right)$, that is, the sector average plus an excess. The dynamics of agents within the group are not independent; they all influence each other through this constraint (101).

To compute the full effect of interactions we have to sum over a series of successive interactins induced by V_n. La transition est modifiée par la formule (31), et est donnée par:the transition inside the group is given by series:

$$\begin{aligned}
& \left\langle \left(\hat{K}_i', \delta f_{1i}'\right)_i T \left(\hat{K}_i, \delta f_{1i}\right)_i \right\rangle \\
= & \prod_i \left\langle \hat{K}_i', \delta f_{1i}' \ \hat{K}_i, \delta f_{1i} \right\rangle \\
& + G\left(\Delta t, f_{1,1}, \hat{K}_1, \hat{K}_1'\right) ... G\left(\Delta t, f_{1,1}, \hat{K}_n, \hat{K}_n'\right) G_{\delta f_1}\left(\Delta t, \delta f_1, \delta f_1'\right) ... G_{\delta f_1}\left(\Delta t, \delta f_p, \delta f_p'\right) \\
& \times V_n\left(\hat{X}_1, \hat{K}_1' ... \hat{X}_n, \hat{K}_n'\right) \times G\left(\Delta t, f_{1,1}', \hat{K}_1', \hat{K}_1''\right) ... G\left(\Delta t, f_{1,n}', \hat{K}_n', \hat{K}_n''\right) \times G_{\delta f_1}\left(\Delta t, \delta f_1', \delta f_1''\right) ... G_{\delta f_1}\left(\Delta t, \delta f_p', \delta f_p''\right) \\
& + [G...G] V_n [G...G] + ... [G...G] V_n [G...G] V_n ... [G...G] V_n [G...G]
\end{aligned}$$

where $[G...G] V_n [G...G]$ denotes blocks of convolutions between Green functions and potential.

If one or more agents experience an excess return at a given moment, it alters the returns of those who have interacted with them through the term $V_n\left(\hat{X}_1, \hat{K}_1' ... \hat{X}_n, \hat{K}_n'\right)$. This, in turn, modifies their Green function. If an agent was close to default, an adverse shock experienced by an interacting agent can push it into default, subsequently impacting the initial agent and causing it to default as well. However, to model this phenomenon accurately, we need to consider multiple groups. In the homogeneous case, where agents are all relatively similar, all agents should be close to default for such a situation to arise. Considering multiple groups allows us to differentiate the situation of agents based on their group.

15.1.7 Transition functions within the group with default

Assume that at some point some of the agents in the system reach a default state. We denote DS as the set of agents with default at some points This may correspond to an adverse δf_1 for these agents, that modifies the potential:

$$\begin{aligned}
& V_n^D\left(\hat{X}_1,\hat{K}_1'...\hat{X}_n,\hat{K}_n'\right) \\
= & \prod_i \frac{1}{\rho}\left\{\sum_j\left(\delta\left(\hat{X}_i-\hat{X}_j\right)-\frac{\hat{k}_1\left(\hat{X}_j,\hat{X}_i\right)\hat{K}_i'\left|\hat{\Psi}\left(\hat{K}_j',\hat{X}_j\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\right)\frac{\delta f_1'\left(\hat{X}_j\right)}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)}\right. \\
& +\sum_{j\in DS}\left(\bar{r}+\frac{1+f\left(\hat{X}_j\right)}{\underline{\hat{k}}_2\left(\hat{X}_j\right)}\right)\frac{\hat{k}_2\left(\hat{X}_j,\hat{X}_i\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}_j',\hat{X}_j\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}_j\right)} \\
& \left.-\sum_j\frac{\left|\Psi\left(\hat{K}_j',\hat{X}_j\right)\right|^2 k_1\left(\hat{X}_j,\hat{X}_i\right)K'}{1+\underline{k}\left(\hat{X}_j\right)}\left(\frac{f_1'\left(K_j,\hat{X}_j\right)-\bar{r}k_2\left(\hat{X}_j\right)}{1+k_2\left(\hat{X}\right)}+\Delta F_\tau\left(\bar{R}\left(K_i,\hat{X}_j\right)\right)\right)\right\}^2
\end{aligned}$$

and the transitions of the system after default decompose as:

$$\begin{aligned}
& \left\langle\left(\hat{K}_i',\delta f_{1i}'\right)_i T\left(\hat{K}_i,\delta f_{1i}\right)_i\right\rangle \\
= & \prod_i\left\langle\hat{K}_i',\delta f_{1i}'\ \hat{K}_i,\delta f_{1i}\right\rangle \\
& +G\left(\Delta t,f_{1,1},\hat{K}_1,\hat{K}_1'\right)...G\left(\Delta t,f_{1,1},\hat{K}_n,\hat{K}_n'\right)G_{\delta f_1}\left(\Delta t,\delta f_1,\delta f_1'\right)...G_{\delta f_1}\left(\Delta t,\delta f_p,\delta f_p'\right) \\
& \times V_n\left(\hat{X}_1,\hat{K}_1'...\hat{X}_n,\hat{K}_n'\right)\times G\left(\Delta t,f_{1,1}',\hat{K}_1',\hat{K}_1''\right)...G\left(\Delta t,f_{1,n}',\hat{K}_n',\hat{K}_n''\right)\times G_{\delta f_1}\left(\Delta t,\delta f_1',\delta f_1''\right)...G_{\delta f_1}\left(\Delta t,\delta f_p',\delta f_p''\right) \\
& +\left[G...G\right]V_n^D\left[G...G\right]+.....\left[G...G\right]V_n^D\left[G...G\right]V_n^D...\left[G...G\right]V_n^D\left[G...G\right]
\end{aligned}$$

the potential V_n^D drives returns towards lower values, increasing the probability that some investors enter the default zone, thereby inducing amplified default rates.

15.2 Interactions of agents inside or across groups of investors

The approach focusing on a single group is constrained by the fact that dynamics and fluctuations are relative to the group's mean. Consequently, in a homogeneous group without defaults, it becomes difficult to identify a single sector that would deviate significantly from this mean. Essentially, the collective state is always governed by an average, pulling agents towards it.

To investigate the transmission of defaults, we must therefore divide the system into multiple groups, distinguishing between those with defaults or at risk, and separately analyze the dynamics of agents in groups without defaults versus those in at-risk or default groups. This allows us to understand how the dynamics of one group impact those of another.

Our challenge remains consistent: in collective states, there is no singular event or dynamic that precipitates significant change. Instead, it is necessary to isolate groups of collective states, and it is the dynamics within these distinct groups that can have a meaningful impact.

In a homogeneous group, our aim is to study the dynamics of agents, enabling us to subsequently explore how agents in one group dynamically influence those in another.

15.2.1 Average for several groups

Let us now define $\underline{\hat{k}}_{\eta}^{[ii]}$, $\underline{k}_{\eta}^{[ii]}$, the average coefficients within the group, and $\underline{\hat{k}}_{\eta}^{[ji]}$ and $\underline{k}_{\eta}^{[ji]}$, the average connections from i t j. We define the total share of capital invested in sector i, comprising intra-sectoral investments (from i to i), and inter-sectoral investments (from j to i) as:

$$\underline{\hat{k}}_{\eta}^{[i]} = \underline{\hat{k}}_{\eta}^{[ii]} + \underline{\hat{k}}_{\eta}^{[ij]}$$

where the sum over the indices is implied. The return equation involving several groups becomes:

$$\begin{aligned}
&\left(\frac{f^{[i]}}{1+\underline{\hat{k}}_{2}^{[i]}} + \bar{r}\frac{\underline{\hat{k}}_{2}^{[i]}}{1+\underline{\hat{k}}_{2}^{[i]}}\right) - \frac{\hat{k}_{1}^{[ji]}}{1+\underline{\hat{k}}^{[j]}}\left(\frac{f^{[j]}}{1+\underline{\hat{k}}_{2}^{[j]}} + \bar{r}\frac{\underline{\hat{k}}_{2}^{[j]}}{1+\underline{\hat{k}}_{2}^{[j]}}\right) \qquad (103)\\
=\;& \left(\bar{r} + \frac{1+f^{[i]}}{\underline{\hat{k}}_{2}^{[i]}} H\left(-\left(1+f^{[i]}\right)\right)\right)\frac{\underline{\hat{k}}_{2}^{[ii]}}{1+\underline{\hat{k}}^{[i]}} + \left(\bar{r} + \frac{1+f^{[j]}}{\underline{\hat{k}}_{2}^{[j]}} H\left(-\left(1+f^{[j]}\right)\right)\right)\frac{\underline{\hat{k}}_{2}^{[ji]}}{1+\underline{\hat{k}}^{[j]}}\\
&+\left(\bar{r} + \frac{1+f_{1}'^{[i]}}{\underline{k}_{2}^{[i]}} H\left(-\frac{1+f_{1}'^{[i]}}{\underline{k}_{2}^{[i]}}\right)\right)\frac{\underline{k}_{2}^{[ii]}}{1+\underline{k}^{i}} + \frac{\underline{k}_{1}^{[ii]}}{1+\underline{k}^{i}} f_{1}^{[i]} + \left(\bar{r} + \frac{1+f_{1}'^{[j]}}{\underline{k}_{2}^{[j]}} H\left(-\frac{1+f_{1}'^{[j]}}{\underline{k}_{2}^{[j]}}\right)\right)\frac{\underline{k}_{2}^{[ji]}}{1+\underline{k}^{j}} + \frac{\underline{k}_{1}^{[ij]}}{1+\underline{k}^{j}} f_{1}^{[j]}
\end{aligned}$$

with several constraints between the coefficients detailed in Appendix 13.2.

This can be conveniently reformulated in matricial form using the alternate description presented in Appendix 4.This alternate description involves relative shares of investmnt $\underline{\hat{S}}_{1}^{[ii]}$, $\underline{\hat{S}}_{1}^{[jj]}$. We find in Appendix 13.2 that:

$$\begin{aligned}
0 =\;& \begin{pmatrix} 1-\underline{\hat{S}}_{1}^{[ii]} & -\underline{\hat{S}}_{1}^{[ji]} \\ -\underline{\hat{S}}_{1}^{[ij]} & 1-\underline{\hat{S}}_{1}^{[jj]} \end{pmatrix}\begin{pmatrix} \left(f^{[i]}-\bar{r}\right)\underline{\hat{s}}_{1}^{[i]} \\ \left(f^{[j]}-\bar{r}\right)\underline{\hat{s}}_{1}^{[j]} \end{pmatrix} \qquad (104)\\
&-\begin{pmatrix} \underline{\hat{S}}_{2}^{[ii]} & \underline{\hat{S}}_{2}^{[ji]} \\ \underline{\hat{S}}_{2}^{[ij]} & \underline{\hat{S}}_{2}^{[jj]} \end{pmatrix}\begin{pmatrix} \left(1+f^{[i]}\right)\underline{\hat{s}}_{2}^{[i]} H\left(-\left(1+f^{[i]}\right)\right) \\ \left(1+f^{[j]}\right)\underline{\hat{s}}_{2}^{[j]} H\left(-\left(1+f^{[j]}\right)\right) \end{pmatrix}\\
&-\begin{pmatrix} \underline{S}_{2}^{[ii]} & \underline{S}_{2}^{[ji]} \\ \underline{S}_{2}^{[ij]} & \underline{S}_{2}^{[jj]} \end{pmatrix}\begin{pmatrix} \left(1+f_{1}'^{[i]}\right)\underline{s}_{2}^{[i]} H\left(-\left(1+f_{1}'^{[i]}\right)\right) \\ \left(1+f_{1}'^{[j]}\right)\underline{s}_{2}^{[j]} H\left(-\left(1+f_{1}'^{[j]}\right)\right) \end{pmatrix} - \begin{pmatrix} \underline{S}_{1}^{[ii]} & \underline{S}_{1}^{[ji]} \\ \underline{S}_{1}^{[ij]} & \underline{S}_{1}^{[jj]} \end{pmatrix}\begin{pmatrix} \left(f_{1}'^{[i]}-\bar{r}\right)\underline{s}_{1}^{[i]} \\ \left(f_{1}'^{[j]}-\bar{r}\right)\underline{s}_{1}^{[j]} \end{pmatrix}
\end{aligned}$$

with:

$$\begin{aligned}
\underline{\hat{s}}_1^{[i]} &= \frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}\right)}, \ \underline{\hat{s}}_1^{[j]} = \frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[jj]}+\underline{\hat{S}}_1^{[ji]}\right)} \\
\underline{\hat{s}}_2^{[i]} &= \frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{\underline{\hat{S}}_2^{[ii]}+\underline{\hat{S}}_2^{[ij]}}, \ \underline{\hat{s}}_2^{[j]} = \frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{\underline{\hat{S}}_2^{[jj]}+\underline{\hat{S}}_2^{[ji]}} \\
\underline{s}_1^{[i]} &= \frac{1-\left(\underline{S}^{[ii]}+\underline{S}^{[ij]}\right)}{1-\left(\underline{S}_1^{[ii]}+\underline{S}_1^{[ij]}\right)}, \ \underline{s}_1^{[j]} = \frac{1-\left(\underline{S}^{[jj]}+\underline{S}^{[ij]}\right)}{1-\left(\underline{S}_1^{[jj]}+\underline{S}_1^{[ji]}\right)} \\
\underline{s}_2^{[i]} &= \frac{1-\left(\underline{S}^{[ii]}+\underline{S}^{[ij]}\right)}{\underline{S}_2^{[ii]}+\underline{S}_2^{[ij]}}, \ \underline{s}_2^{[j]} = \frac{1-\left(\underline{S}^{[jj]}+\underline{S}^{[ji]}\right)}{\underline{S}_2^{[jj]}+\underline{S}_2^{[ji]}}
\end{aligned}$$

This formulation allows for the study of transitions between independent collective states towards entangled states. The diagonal elements of the matrices represent intra-sectoral investments, while the non-diagonal elements represent inter-sectoral investments. The independent collective states are thus defined by setting the non-diagonal matrices $\underline{\hat{S}}_\eta^{[ji]} = 0$ and $\underline{S}_\eta^{[ji]} = 0$. Corrections due to interactions between groups are obtained by considering the non-diagonal elements. This enables the study of default transmission between different groups, introducing a dynamic element in the occurrence of defaults. The addition of interactions allows defaults to propagate.

Note that we could have addressed the same point in the previous section by examining the propagation of default from a single agent within a sector. However, it is more realistic to consider that the economy can withstand the default of a single agent if it does not spread. It is more realistic to consider that an entire group can default, and that this default propagates. The transmission of default from investors in sector j to sector i is expressed by the term:

$$\left(1+f^{[j]}\right)\frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{\underline{\hat{S}}_2^{[jj]}+\underline{\hat{S}}_2^{[ji]}} H\left(-\left(1+f^{[j]}\right)\right) \tag{105}$$

and for the transmission of default from firms:

$$\left(1+f_1^{\prime[j]}\right)\frac{1-\left(\underline{S}^{[jj]}+\underline{S}^{[ji]}\right)}{\underline{S}_2^{[jj]}+\underline{S}_2^{[ji]}} \tag{106}$$

This transmission depends on the non-diagonal coefficients in equation (104). These coefficients represent the diffusion of returns from j to i , and thus the loss due to default (105) or (106) will decrease the returns of sector i, possibly leading to the creation of another default.

Furthermore, dividing the system into multiple groups allows us to study, based on the connections between groups of sectors, which groups will be impacted by defaults from other groups and which will remain unaffected.

15.3 Dynamics for agents in several groups

In the case of multiple groups, we can consider that the field for investors decomposes into several components, each defined over a certain region of the sector space. These fields interact but are considered independent in a first approximation. We thus replace the field describing the system by several fields, one for each group and define:

$$\left[\hat{\Psi}\right]\left(\left\{\hat{K}_{r_i}, \hat{X}_{r_i}\right\}_{G_i}\right)$$

which stands for a set of field, one defined for each group $\left\{\hat{K}_{r_i},\hat{X}_{r_i}\right\}_{G_i}$.

15.3.1 Effective action

We revisit the functional action of each group and add a term for interaction between the different groups.

Each group has its own sector-spatial extension, and the effective action is a generalization of the single-group action.

$$\begin{aligned}&\sum\left[\hat{\Psi}\right]\left(\left\{\hat{K}_{r_i},\hat{X}_{r_i}\right\}_{G_i}\right)\nabla_{\hat{K}_{r_i}}\left(\nabla_{\hat{K}_{r_i}}-\hat{K}_{r_i}f^{[i]}\right)\left[\hat{\Psi}\right]^{\dagger}\left(\left\{\hat{K}_{r_i},\hat{X}_{r_i}\right\}_{G_i}\right)\\&+\sum\prod\left|\left[\hat{\Psi}\right]\left(\left\{\hat{K}_{r_i},\hat{X}_{r_i}\right\}_{G_i}\right)\right|^2\delta\left(V\left(\left(\left[\hat{\Psi}\right]\left(\left\{\hat{K}_{r_i},\hat{X}_{r_i}\right\}_{G_i}\right)\right)\right)\right)\\&-\Xi^{\dagger}\left(\hat{X},\delta f_1\right)\sigma^2_{\delta f_1}\nabla^2_{\delta f_1}\Xi\left(\hat{X},\delta f_1\right)+\Xi^{\dagger}\left(\hat{X},\delta f_1\right)J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)+J^{\dagger}\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\Xi\left(\hat{X},\delta f_1\right)\end{aligned}$$

15.3.2 Return equations and potential

We will write the excess return equations for each group by including their interactions. These interactions will be considered as perturbations that modify the transition functions. To do so, we consider that agents belonging to different groups are linked by the return equation (104) and that groups are weakly connected. This implies considering Dirac-type potentials, one describing intra-group interactions, and one describing inter-group interactions. Excluding defaults, these potentials, are obtained through a first order expansion in $\delta f^{[i]}$ and $\delta f^{[j]}$ of (104) which writes:

$$0=\begin{pmatrix}\delta f^{[i]\prime}\\\delta f^{[j]\prime}\end{pmatrix}-\begin{pmatrix}\underline{\hat{S}}_1^{[ii]}&\underline{\hat{S}}_1^{[ji]}\\\underline{\hat{S}}_1^{[ij]}&\underline{\hat{S}}_1^{[jj]}\end{pmatrix}\begin{pmatrix}\delta f^{[i]}\frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}\right)}\\\delta f^{[j]}\frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[jj]}+\underline{\hat{S}}_1^{[ji]}\right)}\end{pmatrix}\tag{107}$$

we can write the intra-group interaction potentials by considering the diagonal terms in (107). It leads to the potential:

$$\sum_i V_i\left(\hat{\Psi},\hat{X},K,\delta f_1\right)=\sum_i\delta\left[\delta f_1^{[i]}\left(\hat{X}\right)-\underline{\hat{S}}_1^{[ii]}\frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}\right)}\frac{\delta f_1^{[i]\prime}\left(\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)}dX'\right]\tag{108}$$

while the inter-group interaction potentials is obtained by considering the off-diagonal terms in (107). We find:

$$\begin{aligned}&\sum_i W_i\left(\hat{\Psi},\hat{X},K,\delta f_{1i}\right)\\=&\sum_i\delta\left(\delta f^{[i]\prime}\left(\hat{X}\right)-\underline{\hat{S}}_1^{[ji]}\frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[jj]}+\underline{\hat{S}}_1^{[ji]}\right)}\frac{\delta f^{[j]\prime}\left(\hat{X}'\right)}{\left(1+\underline{\hat{k}}_2\left(\hat{X}'\right)\right)}\right)\end{aligned}\tag{109}$$

The transitions are then driven both by intra-group (172) and inter-group (174) interactions. The potential (172) is similar to the homogeneous group and illustrates how, within the same group, the excess returns of one agent impact those of others. Even an agent that has not initially experienced a return variation will experience it subsequently due to the interaction term. The potential (174), on the other hand, demonstrates the transmission of a return variation from one block to another.

15.3.3 Transitions functions without default

As for one homogeneous group, the transitions are computed by series of terms of the type:

$$\begin{aligned}
&G\left(\Delta t, f_{1,1}, \left\{\hat{K}_{r_1}, \hat{X}_{r_1}\right\}_{G_1}, \left\{\hat{K}'_{r_1}, \hat{X}'_{r_1}\right\}_{G_1}\right) ...G\left(\Delta t, f_{1,n}, \left\{\hat{K}_{r_n}, \hat{X}_{r_n}\right\}_{G_n}, \left\{\hat{K}'_{r_n}, \hat{X}'_{r_n}\right\}_{G_n}\right) \\
&\times G_{\delta f_1}\left(\Delta t, \delta f_1, \delta f'_1\right) ...G_{\delta f_1}\left(\Delta t, \delta f_p, \delta f'_p\right) \\
&\times \left(V\left(\left(\left[\hat{\Psi}\right]\left(\left\{\hat{K}_{r_i}, \hat{X}_{r_i}\right\}_{G_i}\right)\right)\right)\right) \\
&\times G\left(\Delta t, f'_{1,1}, \left\{\hat{K}'_{r_1}, \hat{X}'_{r_1}\right\}_{G_1}, \left\{\hat{K}''_{r_1}, \hat{X}''_{r_1}\right\}_{G_1}\right) ...G\left(\Delta t, f'_{1,n}, \left\{\hat{K}'_{r_n}, \hat{X}'_{r_n}\right\}_{G_n}, \left\{\hat{K}''_{r_n}, \hat{X}''_{r_n}\right\}_{G_n}\right) \\
&\times G_{\delta f_1}\left(\Delta t, \delta f'_1, \delta f''_1\right) ...G_{\delta f_1}\left(\Delta t, \delta f'_p, \delta f''_p\right)
\end{aligned}$$

where:

$$V\left(\left(\left[\hat{\Psi}\right]\left(\left\{\hat{K}_{r_i}, \hat{X}_{r_i}\right\}_{G_i}\right)\right)\right)$$

stands for the intra- and inter-group interactions.

15.3.4 Transitions functions including default

As before, some default may arise in the process of transition. This is modeled by including the defaults investors term in (172) and (174). Starting with the constraint in the default case:

$$\begin{aligned}
0 &= \begin{pmatrix} \delta \hat{f}^{[i]} \\ \delta \hat{f}^{[j]} \end{pmatrix} - \begin{pmatrix} \underline{\hat{S}}_1^{[ii]} & \underline{\hat{S}}_1^{[ji]} \\ \underline{\hat{S}}_1^{[ij]} & \underline{\hat{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \delta \hat{f}^{[i]\prime} \frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}\right)} \\ \delta \hat{f}^{[j]\prime} \frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[jj]}+\underline{\hat{S}}_1^{[ji]}\right)} \end{pmatrix} \\
&\quad - \begin{pmatrix} \underline{\hat{S}}_2^{[ii]} & \underline{\hat{S}}_2^{[ji]} \\ \underline{\hat{S}}_2^{[ij]} & \underline{\hat{S}}_2^{[jj]} \end{pmatrix} \begin{pmatrix} \left(1+\hat{f}^{[i]\prime}+\delta \hat{f}^{[i]\prime}\right) \underline{\hat{s}}_2^{[i]} H\left(-\left(1+\hat{f}^{[i]\prime}+\delta \hat{f}^{[i]\prime}\right)\right) \\ \left(1+\hat{f}^{[j]\prime}+\delta \hat{f}^{[j]\prime}\right) \underline{\hat{s}}_2^{[i]} H\left(-\left(1+\hat{f}^{[j]\prime}+\delta \hat{f}^{[j]\prime}\right)\right) \end{pmatrix}
\end{aligned} \tag{110}$$

the default potentials $V_i^D\left(\hat{\Psi}, \hat{X}, K, \delta f_1\right)$ and $W_i^D\left(\hat{\Psi}, \hat{X}, K, \delta f_{1i}\right)$ are derived straightforwardly:

$$\begin{aligned}
\sum_i V_i^D\left(\hat{\Psi}, \hat{X}, K, \delta f_1\right) &= \sum_i \delta\left[\delta f_1^{[i]}\left(\hat{X}\right) - \underline{\hat{S}}_1^{[ii]} \frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}\right)} \frac{\delta f_1^{[i]}\left(\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)} dX' \right.\\
&\quad \left. -\underline{\hat{S}}_2^{[ii]}\left(1+\hat{f}^{[i]\prime}+\delta \hat{f}^{[i]\prime}\right) \underline{\hat{s}}_2^{[i]} H\left(-\left(1+\hat{f}^{[i]\prime}+\delta \hat{f}^{[i]\prime}\right)\right)\right]
\end{aligned}$$

and:

$$\begin{aligned}
&\sum_i W_i^D\left(\hat{\Psi}, \hat{X}, K, \delta f_{1i}\right) \\
&= \sum_i \delta\left[\delta f^{[i]\prime}\left(\hat{X}\right) - \underline{\hat{S}}_1^{[ji]} \frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[jj]}+\underline{\hat{S}}_1^{[ji]}\right)} \frac{\delta f^{[j]\prime}\left(\hat{X}'\right)}{\left(1+\underline{\hat{k}}_2\left(\hat{X}'\right)\right)} \right.\\
&\quad \left. -\underline{\hat{S}}_2^{[ji]}\left(1+\hat{f}^{[j]\prime}+\delta \hat{f}^{[j]\prime}\right) \underline{\hat{s}}_2^{[i]} H\left(-\left(1+\hat{f}^{[j]\prime}+\delta \hat{f}^{[j]\prime}\right)\right)\right]
\end{aligned} \tag{111}$$

The transitions thus include two possible effects.

First, a sequence of interactions:

$$[G...G]\,V_n\,[G...G] + ...\,[G...G]\,V_n\,[G...G]\,V_n...\,[G...G]\,V_n\,[G...G] \tag{112}$$

where V_n describes transitions between agents of several groups, which may drive some agents to default. The following transitions are thus driven by terms like:

$$[G...G]\,V_n^D\,[G...G] + ...\,[G...G]\,V_n^D\,[G...G]\,V_n^D...\,[G...G]\,V_n\,[G...G] \tag{113}$$

which increases the number of default in group.

Then through inter-groups potential the default propagates to agents of different groups.

The terms (187) and (188) describe the sequential dynamics of interactions among multiple investors within different groups. Initially, investors evolve independently, and their transition function is simply the product of their individual transition functions $[G...G]$. The interaction between different investors is measured by the term V_n, which diffuses variations in returns among investors. Once the returns are modified by this initial interaction, investors continue their accumulation dynamics with returns modified by the interaction and a transition function $[G...G]$, then interact again, and so on.

If some investors default, particularly due to their interactions with other investors in a group structurally close to default, the subsequent interaction between investors will be described by the default interaction V_n^D, which takes defaults into account. This interaction will reduce the returns of other investors, whose accumulation dynamics will be impacted. Their probability of default increases, and the risk of default increases with each interaction.

Between groups of investors, the diffusion of defaults can take more complex and indirect patterns. Some investors in group A, without defaulting themselves, may experience a negative shock that propagates to group B, causing their returns to decrease, thereby impacting group A in return, potentially leading it to default... This spiral effect dynamically reflects the transition from one collective state to another. From a collective state characterized by a limited number of defaults, there can be a sudden transition to another collective state characterized by a massive number of defaults. The dynamics are relatively rapid, and all defaults in the collective state will materialize.

Part 2. System including banks

We have introduced a simple system where investors could choose to invest in firms or in other investors. Now, we will further sophisticate this initial benchmark by adding another type of investors to the system: banks, which can lend more than their private or available capital allows.

16 Microeconomic framework for banks, investors and firms

The system follows the initial setup, with the addition of banks, which modifies the system while keeping the principles intact. In the subsequent discussion, we will denote variables related to firms without superscript, variables related to investors with a hat symbol, and variables related to banks with a bar symbol.

16.1 Disposable capital

16.1.1 Banks

Banks are defined by their private capital $\bar{K}_{jp}(t)$, their diposable capital $\bar{K}_{j0}(t)$ and the capital $\bar{K}_j(t)$ lent to other agents, which is proportional to $\bar{K}_{jp}(t)$ with a ratio κ, giving $\bar{K}_j(t) = \kappa\bar{K}_{jp}(t)$. Banks engage in lending and taking participation with each other, with shares $\bar{k}_2$ and $\bar{k}_1$ respectively.

The disposable capital for banks will be invested in loans and participations in other banks on one hand, and in participations in firms on the other hand:

$$\begin{aligned}
\bar{K}_{j0}(t) &= \sum_l \bar{k}_1\left(\bar{K}_{lp}(t), \bar{X}_l(t), \bar{X}_j(t)\right)\bar{K}_{j0}(t) + \sum_l \bar{k}_2\left(\bar{K}_{lp}(t), \bar{X}_l(t), \bar{X}_j(t)\right)\bar{K}_{j0}(t) \\
&\quad + \sum_l \bar{k}_1\left(\hat{K}_{lp}(t), \hat{X}_l(t), \bar{X}_j(t)\right)\bar{K}_{j0}(t) + \sum_i \left(\hat{F}_{2,1}\left(R_i, \hat{X}_j\right)\bar{K}_{j0}(t)\right)
\end{aligned}$$

Conversely, the available capital of a bank is the sum of its private capital, participations from other banks, and loans from other banks. To simplify, investors do not lend to or take participation in banks.

The link between private capital and disposable capital is as follows:

$$\bar{K}_{j0}(t) = \bar{K}_{jp}(t) + \sum_l \left(\bar{k}_1\left(\bar{K}_{jp}(t), \bar{X}_j(t), \bar{X}_l(t)\right) + \bar{k}_2\left(\bar{K}_{jp}(t), \bar{X}_j(t), \bar{X}_l(t)\right)\right)\bar{K}_{l0}(t)$$

which writes, in the linear approximation:

$$\bar{K}_{j0}(t) = \bar{K}_{jp}(t) + \sum_l \left(\bar{k}_1\left(\bar{X}_j(t), \bar{X}_l(t)\right) + \bar{k}_2\left(\bar{X}_j(t), \bar{X}_l(t)\right)\right)\bar{K}_{jp}(t)\bar{K}_{l0}(t)$$

We can express the private capital as a function of the disposable capital:

$$\bar{K}_{jp}(t) = \frac{\bar{K}_{j0}(t)}{1 + \sum_l \left(\bar{k}_1\left(\bar{X}_j(t), \bar{X}_l(t)\right) + \bar{k}_2\left(\bar{X}_j(t), \bar{X}_l(t)\right)\right)\bar{K}_{l0}(t)} \tag{114}$$

As in Part 1, we will use the notations:

$$\bar{k}_\varepsilon\left(\bar{X}_j(t), \bar{X}_l(t)\right) \to \bar{k}_{\varepsilon lj}$$

and:

$$\bar{k}_{1lj} + \bar{k}_{1lj} \to \bar{k}_{lj}$$

The lent capital is proportional to the private capital of the bank. The factor denoted here as κ is the credit multiplier. Equation (114) implies that lent capital can also be expressed as a function of the disposable capital:

$$\bar{K}_j(t) = \kappa\bar{K}_{jp}(t) = \frac{\kappa\bar{K}_{j0}(t)}{1 + \sum_l \left(\bar{k}_1\left(\bar{X}_j(t), \bar{X}_l(t)\right) + \bar{k}_2\left(\bar{X}_j(t), \bar{X}_l(t)\right)\right)\bar{K}_{l0}(t)}$$

16.1.2 Investors

The setup for investors remains the same as before. However, there is a difference in Part 1 due to the disposable capital of investors. As banks may also take participations or lend to any investor, the investor's disposable capital is now a combination of private capital, participations from investors and banks, and loans from investors and banks.

The disposable capital for investor j can be expressed as follows:

$$\begin{aligned}\hat{K}_j(t) &= \hat{K}_{jp}(t)+\sum_l\left(\hat{k}_1\left(\hat{K}_{jp}(t),\hat{X}_j(t),\hat{X}_l(t)\right)+\hat{k}_2\left(\hat{K}_{jp}(t),\hat{X}_j(t),\hat{X}_l(t)\right)\right)\hat{K}_l(t)\\&+\sum_l\left(\hat{k}_1^B\left(\hat{K}_{jp}(t),\hat{X}_j(t),\bar{X}_l(t)\right)\bar{K}_{l0}(t)+\hat{k}_2^B\left(\hat{K}_{jp}(t),\hat{X}_j(t),\bar{X}_l(t)\right)\bar{K}_l(t)\right)\end{aligned}$$

which writes in the linear approximation:

$$\hat{k}_1\left(\hat{K}_{jp}(t),\hat{X}_j(t),\hat{X}_l(t)\right)+\hat{k}_2\left(\hat{K}_{jp}(t),\hat{X}_j(t),\hat{X}_l(t)\right)=\left(\hat{k}_{1jl}+\hat{k}_{2jl}\right)\hat{K}_{jp}(t)$$

and:

$$\left(\hat{k}_1^B\left(\hat{K}_{jp}(t),\hat{X}_j(t),\bar{X}_l(t)\right)\bar{K}_{l0}(t)+\hat{k}_2^B\left(\hat{K}_{jp}(t),\hat{X}_j(t),\bar{X}_l(t)\right)\bar{K}_l(t)\right)=\left(\hat{k}_{1jl}^B\bar{K}_{l0}(t)+\hat{k}_{2jl}^B\bar{K}_l(t)\right)\hat{K}_{jp}(t)$$

We define the participation share of investor l in investor j as $\hat{k}_{1jl}$ and the loan share of investor l in investor j as $\hat{k}_{2jl}$:

$$\begin{aligned}\hat{k}_{1jl} &= \hat{k}_1\left(\hat{X}_j(t),\hat{X}_l(t)\right)\\\hat{k}_{2jl} &= \hat{k}_2\left(\hat{X}_j(t),\hat{X}_l(t)\right)\end{aligned}$$

And the total sum of the two shares as $\hat{k}_{jl}$, i.e.:

$$\hat{k}_{jl}=\hat{k}_{1jl}+\hat{k}_{2jl}$$

Similarly, in the linear approximation, the investment of a bank in an investor, with participation $\hat{k}_1^B$ and loan $\hat{k}_2^B$, is expressed as follows:

$$\begin{aligned}\hat{k}_1^B\left(\hat{K}_{jp}(t),\hat{X}_j(t),\hat{X}_l(t)\right) &= \hat{k}_1^B\left(\hat{X}_j(t),\hat{X}_l(t)\right)\hat{K}_{jp}(t)\\\hat{k}_2^B\left(\hat{K}_{jp}(t),\hat{X}_j(t),\hat{X}_l(t)\right) &= \hat{k}_2^B\left(\hat{X}_j(t),\hat{X}_l(t)\right)\hat{K}_{jp}(t)\end{aligned}$$

and we can express the disposable capital of investor j, $\hat{K}_j(t)$, as :

$$\hat{K}_j(t)=\hat{K}_{jp}(t)+\sum_l\left(\hat{k}_1+\hat{k}_2\right)\hat{K}_l(t)+\sum_l\left(\hat{k}_1^B\bar{K}_{l0}(t)+\hat{k}_2^B\bar{K}_l(t)\right)$$

so that private capital can be expressed as a function of the disposable capital:

$$\begin{aligned}\hat{K}_{jp}(t) &= \frac{\hat{K}_j(t)}{1+\sum_l\hat{k}_{jl}\hat{K}_l(t)+\sum_l\left(\hat{k}_{1jl}^B\bar{K}_{l0}(t)+\hat{k}_{2jl}^B\bar{K}_l(t)\right)}\\&= \frac{\hat{K}_j(t)}{1+\sum_l\hat{k}_{jl}\hat{K}_l(t)+\sum_l\hat{k}_{1jl}^B\bar{K}_{l0}(t)+\kappa\sum_l\hat{k}_{2jl}^B\frac{\bar{K}_{j0}(t)}{1+\sum_m\bar{k}_{vm}\bar{K}_{m0}(t)}}\end{aligned}\tag{115}$$

Note that the ratio:

$$\kappa\frac{\bar{K}_{j0}(t)}{1+\sum_m\bar{k}_{vm}\bar{K}_{m0}(t)}$$

in the bank loan models that bank loans are proportional to the bank's private capital, rather than their disposable capital.

16.1.3 Firms

The description of firms' private capital and disposable capital is similar to Part 1, except that now firms can borrow from both banks and investors. Their disposable capital now decomposes in the following way:

$$K_i(t) = K_{ip}(t) + \left(\sum_v k_{1iv}^{(B)} \bar{K}_{v0}(t) + k_{2iv}^{(B)} \bar{K}_v(t)\right) K_{ip}(t) + \left(\sum_v (k_{1iv} + k_{2iv}) \hat{K}_v(t)\right) K_{ip}(t)$$

where $k_{1iv}^{(B)}$ and $k_{2iv}^{(B)}$ are the shares invested through participation and loans respectively by bank ν to firm i. These participations are proportional to the bank's disposable capital, as for investors, but loans are proportional to the level of possible lent captal, i.e. the private capital of the bank times the multiplier κ. We will consider $\kappa >> 1$.

The firm private capital thus writes:

$$\begin{aligned}
K_{ip}(t) &= \frac{K_i(t)}{1 + \left(\sum_v k_{1iv}^{(B)} \bar{K}_{v0}(t) + k_{2iv}^{(B)} \bar{K}_v(t)\right) + \left(\sum_v k_{iv} \hat{K}_v(t)\right)} \\
&= \frac{K_i(t)}{1 + \left(\sum_v k_{1iv}^{(B)} \bar{K}_{v0}(t) + k_{2iv}^{(B)} \kappa \frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm} \bar{K}_{m0}(t)}\right) + \left(\sum_v k_{iv} \hat{K}_v(t)\right)}
\end{aligned}$$

This formula is the similar to Part 1.

16.2 Capital allocation

16.2.1 Banks

Banks' disposable capital is decomposed into several investments and can be expressed as:

$$\begin{aligned}
\bar{K}_{j0}(t) &= \sum_i \frac{k_{1il}^{(B)} K_i(t)}{1 + \sum_v k_{1iv}^{(B)} \bar{K}_{v0}(t) + k_{2iv}^{(B)} \kappa \frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm} \bar{K}_{m0}(t)} + \sum_v k_{iv} \hat{K}_v(t)} \bar{K}_{j0}(t) \\
&+ \sum_l \frac{\hat{k}_{1lj}^B \hat{K}_l(t)}{1 + \sum_\nu \hat{k}_{l\nu} \hat{K}_l(t) + \sum_\nu \hat{k}_{1l\nu}^B \bar{K}_{l0}(t) + \kappa \sum_\nu \hat{k}_{2l\nu}^B \frac{\bar{K}_{j0}(t)}{1+\sum_m \bar{k}_{vm} \bar{K}_{m0}(t)}} \bar{K}_{j0}(t) \\
&+ \sum_l \frac{\left(\bar{k}_{1lj} + \bar{k}_{2lj}\right) \bar{K}_{l0}(t)}{1 + \sum_\nu \left(\bar{k}_{1l\nu} + \bar{k}_{2l\nu}\right) \bar{K}_{l0}(t)} \bar{K}_{j0}(t)
\end{aligned} \tag{116}$$

where $\bar{K}_{j0}(t)$ is the disposable capital of the bank, and the other elements are proportionality factors in which the terms $k_{1il}^{(B)}$, $\hat{k}_{1lj}^B$, and $\left(\bar{k}_{1lj} + \bar{k}_{2lj}\right)$represent the leverage effects that the investor or the bank, as an investor, chooses to apply to their investment. Each time, the available capital of the investment $K_i(t)$, $\hat{K}_l(t)$, $\bar{K}_{l0}(t)$, divided by a denominator, represents the private capital of the agent in which one invests. Equation (116) implies the constraint:

$$\begin{aligned}
1 &= \sum_i \frac{k_{1il}^{(B)} K_i(t)}{1 + \sum_v k_{1iv}^{(B)} \bar{K}_{v0}(t) + k_{2iv}^{(B)} \kappa \frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm} \bar{K}_{m0}(t)} + \sum_v k_{iv} \hat{K}_v(t)} \\
&+ \sum_l \frac{\hat{k}_{1lj}^B \hat{K}_l(t)}{1 + \sum_\nu \hat{k}_{l\nu} \hat{K}_l(t) + \sum_\nu \hat{k}_{1l\nu}^B \bar{K}_{l0}(t) + \kappa \sum_\nu \hat{k}_{2l\nu}^B \frac{\bar{K}_{j0}(t)}{1+\sum_m \bar{k}_{vm} \bar{K}_{m0}(t)}} \\
&+ \sum_l \frac{\left(\bar{k}_{1lj} + \bar{k}_{2lj}\right) \bar{K}_{l0}(t)}{1 + \sum_\nu \left(\bar{k}_{1l\nu} + \bar{k}_{2l\nu}\right) \bar{K}_{l0}(t)}
\end{aligned}$$

The first sum describes participations in firms, the second sum represents participations in investors. Loans to firms and investors are not included because they are accounted for separately as monetary creation. The third sum consists of participation $\bar{k}_{1lj}$ and loans $\bar{k}_{2l\nu}$ in other banks.

16.2.2 Investors

For investors, the principle remains the same. Capital is invested in firms and other investors, and the coefficients represent the leverage effects that multiply the private capital of the agent in which one invests. Disposable capital is decomposed into several investments and can be expressed as:

$$\begin{aligned}\hat{K}_j(t) &= \sum_i \frac{(k_{1ij}+k_{2ij})K_i(t)}{1+\sum_v k_{1iv}^{(B)}\bar{K}_{v0}(t)+k_{2iv}^{(B)}\kappa\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}+\sum_v k_{iv}\hat{K}_v(t)}\hat{K}_j(t)\\ &+\sum_l \frac{\left(\hat{k}_{1lj}+\hat{k}_{2lj}\right)\hat{K}_l(t)}{1+\sum_\nu \hat{k}_{l\nu}\hat{K}_l(t)+\sum_\nu \hat{k}_{1l\nu}^B\bar{K}_{l0}(t)+\kappa\sum_\nu \hat{k}_{2l\nu}^B\frac{\bar{K}_{j0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}}\hat{K}_j(t)\end{aligned}$$

and this identity implies, for all j, the constraint:

$$\begin{aligned}1 &= \sum_i \frac{(k_{1ij}+k_{2ij})K_i(t)}{1+\sum_v k_{1iv}^{(B)}\bar{K}_{v0}(t)+k_{2iv}^{(B)}\kappa\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}+\sum_v k_{iv}\hat{K}_v(t)}\\ &+\sum_l \frac{\left(\hat{k}_{1lj}+\hat{k}_{2lj}\right)\hat{K}_l(t)}{1+\sum_\nu \hat{k}_{l\nu}\hat{K}_l(t)+\sum_\nu \hat{k}_{1l\nu}^B\bar{K}_{l0}(t)+\kappa\sum_\nu \hat{k}_{2l\nu}^B\frac{\bar{K}_{j0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}}\end{aligned}$$

with:

$$k_{ij}=k_{1ij}+k_{2ij}$$

and:

$$\hat{k}_{ij}=\hat{k}_{1ij}+\hat{k}_{2ij}$$

the overall leverage effects of the investment from an investor to a firm or another investor, respectively.

16.3 Returns and capital accumulation under no-default scenario

16.3.1 Firms

Returns Returns for firms come from production and price increases, thus they are the same as in Part 1. For a given return on invested capital R_j, firm j has the following return on its private capital:

$$\begin{aligned}f_j &= \sum_l \left(1+\sum_v \hat{k}_{2lv}\hat{K}_v(t)+\kappa\sum_v \hat{k}_{2jv}^B\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}\right)R_j\\ &-\bar{r}\left(\sum_v \hat{k}_{2jv}\hat{K}_v(t)+\kappa\sum_v \hat{k}_{2jv}^B\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}\right)\end{aligned} \quad (117)$$

The formula is the same as in Part 1, except that banks have been introduced. The return on private capital that the firm generates for itself is the return on investment rate R_j, multiplied by all the capital it has engaged, minus the interest on the loans it must repay. The difference from Part 1 is that the amount of capital raised by the firm itself through borrowing includes loans from banks. For the subsequent analysis, especially for the dynamics of firm accumulation, it will be useful to re-express R_j as a function of f_j:

$$R_j = \frac{f_j + \bar{r}\left(\sum_v \hat{k}_{2jv}\hat{K}_v(t) + \kappa \sum_v \hat{k}^B_{2jv}\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{\bar{k}}_{vm}\bar{K}_{m0}(t)}\right)}{1 + \sum_v \hat{k}_{2lv}\hat{K}_v(t) + \kappa \sum_v \hat{k}^B_{2jv}\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{\bar{k}}_{vm}\bar{K}_{m0}(t)}} \tag{118}$$

Firms' accumulation of capital Similar to Part 1, the accumulation of a firm occurs at a rate determined by $f_i(K_i(t))$, and the dynamics for firms' capital accumulation can be expressed as:

$$\frac{d}{dt}K_{ip}(t) = f_i(K_i(t))\, K_{ip}(t) \tag{119}$$

16.3.2 Investors

Investors' returns The investors' returns are similar to those in Part 1, except that bank loans and participation arise in the disposable capital. The investors' return $\hat{R}_j$ determined by equations (251) and (45):

$$\hat{R}_j = \sum_l \frac{\hat{k}_{1lj}\hat{K}_l(t)}{1 + \sum_v \hat{k}_{lv}\hat{K}_v(t) + \sum_v \hat{k}^B_{1lv}\bar{K}_{v0}(t) + \kappa \sum_v \hat{k}^B_{2lv}\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}}\hat{R}_l + \hat{R}'_j \tag{120}$$

Formula (120) shows that the returns of investors can be decomposed into two parts. The first part corresponds to the return from other investors, $\hat{R}_l$, weighted by the size of investor j's participations in investor l. These participations are directly proportional to the private capital of investor l. The second part, $\hat{R}'_j$, represents the direct return obtained by investor j, who grants loans to firms and investors and takes participations in firms. The direct return $\hat{R}'_j$ decomposes as:

$$\begin{aligned}
\hat{R}'_j &= \bar{r}\sum_l \frac{\hat{k}_{2lj}\hat{K}_l(t)}{1 + \sum_l \hat{k}_{jl}\hat{K}_l(t) + \sum_l \hat{k}^B_{1jl}\bar{K}_{l0}(t) + \kappa \sum_l \hat{k}^B_{2jl}\frac{\bar{K}_{j0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}} \\
&+\bar{r}\sum_i \frac{k_{2ij}K_i(t)}{1 + \sum_v k_{iv}\hat{K}_v(t) + \sum_v k^{(B)}_{1iv}\bar{K}_{v0}(t) + k^{(B)}_{2iv}\kappa\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}} \\
&+\sum_i \frac{\left(r_i + F_1\left(\bar{R}_i, \frac{\dot{K}_i(t)}{K_i(t)}\right) + \tau\left(\bar{R}(K_i, X_i)\right)\Delta f'_1(K_i(t))\right)k_{1ij}K_i(t)}{1 + \sum_v k_{iv}\hat{K}_v(t) + \sum_v k^{(B)}_{1iv}\bar{K}_{v0}(t) + k^{(B)}_{2iv}\kappa\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}}
\end{aligned}$$

As in Part 1, the return equation can be reformulated in terms of excess return relative to the interest rate:

$$\begin{aligned}
&\sum_l \left(\delta_{jl} - \frac{\hat{k}_{1lj}\hat{K}_l(t)}{1 + \sum_v \hat{k}_{jv}\hat{K}_l(t) + \sum_v \hat{k}^B_{1jv}\bar{K}_{l0}(t) + \kappa \sum_v \hat{k}^B_{2jv}\frac{\bar{K}_{j0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}}\right) \\
&\times\frac{\hat{f}_l - \bar{r}}{1 + \sum_v \hat{k}_{2lv}\hat{K}_v(t) + \kappa \sum_v \hat{k}^B_{2lv}\frac{\bar{K}_{\nu 0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}} \\
= &\sum_i \frac{\left(r_i + F_1\left(\bar{R}_i, \frac{\dot{K}_i(t)}{K_i(t)}\right) + \tau\left(\bar{R}(K_i, X_i)\right)\Delta f'_1(K_i(t)) - \bar{r}\right)k_{1ij}K_i(t)}{1 + \sum_v \hat{k}_{jv}\hat{K}_v(t) + \sum_v \hat{k}^B_{1jv}\bar{K}_{l0}(t) + \kappa \sum_v \hat{k}^B_{2jv}\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}}
\end{aligned} \tag{121}$$

with:

$$\begin{aligned}\hat{f}_j &= \left(1+\sum_v \hat{k}_{2jv}\hat{K}_v(t)+\kappa\sum_l \hat{k}^B_{2jl}\frac{\bar{K}_{j0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}\right)R_j \\ &\quad -\left(\sum_v \hat{k}_{2jv}\hat{K}_v(t)+\kappa\sum_l \hat{k}^B_{2jl}\frac{\bar{K}_{j0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}\right)\bar{r}\end{aligned}$$

Investors' capital accumulation Appendix 17.1 computes the dynamics for $\hat{K}_j(t)$:

$$\frac{d}{dt}\hat{K}_j(t)=\hat{R}_j+\sum_l \hat{M}_{jl}\frac{d}{dt}\hat{K}_l(t)-\sum_l \bar{N}_{jl}\frac{d}{dt}\bar{K}_{0l}(t)$$

The right-hand side of equation (122) indicates that the variation in investor j's disposable capital depends not only on its return $\hat{R}_j$, but also on the capital provided by other investors through the matrix $\hat{M}$, as well as that provided by banks through the matrix $\bar{N}$. This equation can be reformulated as a dynamic process involving the variation of all the capital at the investor's disposal:

$$\sum_l \left(\delta_{jl}-\hat{M}\right)_{jl}\frac{d}{dt}\hat{K}_l(t)+\sum_l \bar{N}_{jl}\frac{d}{dt}\bar{K}_{0l}(t)=\hat{R}_j \tag{122}$$

where:

$$\hat{M}_{jm}=\frac{\hat{k}_{jm}\hat{K}_j(t)}{1+\sum_l \hat{k}_{jl}\hat{K}_l(t)+\sum_l \hat{k}^B_{1jl}\bar{K}_{l0}(t)+\kappa\sum_l \hat{k}^B_{2jl}\frac{\bar{K}_{l0}(t)}{1+\sum_m \bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}} \tag{123}$$

and:

$$\bar{N}_{jl}=\frac{\left(\hat{k}^B_{1jl}+\kappa\hat{k}^B_{2jl}\frac{1}{1+\sum_m \bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}-\kappa\frac{\sum_m \hat{k}^B_{2jm}\bar{K}_{m0}(t)\bar{\bar{k}}_{ml}}{\left(1+\sum_n \bar{\bar{k}}_{mn}\bar{K}_{n0}(t)\right)^2}\right)\hat{K}_j}{1+\sum_l \hat{k}_{jl}\hat{K}_l(t)+\sum_l \hat{k}^B_{1jl}\bar{K}_{l0}(t)+\kappa\sum_l \hat{k}^B_{2jl}\frac{\bar{K}_{j0}(t)}{1+\sum_m \bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}}$$

16.3.3 Banks

Banks' returns Similar to investors, the return of bank j, denoted as $\bar{R}_j$, decomposes into a sum of direct returns $\bar{R}'_j$ and returns from bank j's participations in investors and other banks:

$$\begin{aligned}\bar{R}_j &= \bar{R}'_j+\sum_l \frac{\bar{k}_{1lj}\bar{K}_{l0}(t)}{1+\sum_v \bar{k}_{lv}\bar{K}_{v0}(t)}\bar{R}_l \\ &\quad +\sum_l \frac{\bar{k}_{1lj}\bar{K}_{j0}(t)}{1+\sum_v \hat{k}_{lv}\hat{K}_v(t)+\sum_v \bar{k}_{1lv}\bar{K}_{v0}(t)+\kappa\sum_v \bar{k}_{2lv}\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}}R_l\end{aligned} \tag{124}$$

The direct return of bank j, denoted as $\bar{R}'_j$, originates from loans granted to other banks, loans granted to investors, loans granted to firms, and direct participation in firms, which can be expressed as follows:

$$\begin{aligned}
\bar{R}'_j &= \bar{r}\sum_l \frac{\bar{k}_{2lj}\bar{K}_{l0}(t)}{1+\sum_v \bar{k}_{lv}\bar{K}_{v0}(t)} \\
&+\bar{r}\sum_l \frac{\kappa\hat{k}_{2lj}\hat{K}_l(t)}{\left(1+\sum_v \hat{k}_{lv}\hat{K}_v(t)+\sum_{lv}\hat{k}^B_{1lv}\bar{K}_{v0}(t)+\kappa\sum_s \hat{k}^B_{2ls}\frac{\bar{K}_{s0}(t)}{1+\sum_m \bar{k}_{sm}\bar{K}_{m0}(t)}\right)\left(1+\sum_v \bar{k}_{jv}\bar{K}_{v0}(t)\right)} \\
&+\bar{r}\sum_i \frac{\kappa k^{(B)}_{2ij}K_i(t)}{\left(1+\sum_v k_{iv}\hat{K}_v(t)+\left(\sum_v k^{(B)}_{1iv}\bar{K}_{v0}(t)+k^{(B)}_{2iv}\kappa\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}\right)\right)\left(1+\sum_v \bar{k}_{jv}\bar{K}_{v0}(t)\right)} \\
&+\sum_i \frac{k^{(B)}_{1ij}K_i(t)\left(r_i+F_1\left(\bar{R}_i,\frac{\dot{K}_i(t)}{K_i(t)}\right)+\tau\left(\bar{R}(K_i,X_i)\right)\Delta f'_1(K_i(t))\right)}{\left(1+\sum_v k_{iv}\hat{K}_v(t)+\left(\sum_v k^{(B)}_{1iv}\bar{K}_{v0}(t)+k^{(B)}_{2iv}\kappa\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}\right)\right)\left(1+\sum_v \bar{k}_{jv}\bar{K}_{v0}(t)\right)}
\end{aligned}$$

The remaining disposable capital of the bank is invested in taking participations in other banks and investors, which will provide returns $\bar{R}_l$ and R_l.

By combining these two sources of returns, the overall return equation for bank j can be rewritten as:

$$\begin{aligned}
&\sum_l \left(\delta_{jl}-\frac{\hat{k}_{1lj}\bar{K}_{l0}(t)}{1+\sum_v \bar{k}_{lv}\bar{K}_{v0}(t)}\right)\left(\frac{\bar{f}_l-\bar{r}}{1+\sum_v \bar{k}_{2lv}\bar{K}_{v0}(t)}\right) \\
&-\sum_l \frac{\hat{k}^B_{1lj}\hat{K}_l(t)}{1+\sum_v \hat{k}_{lv}\hat{K}_l(t)+\sum_v \hat{k}^B_{1lv}\bar{K}_{l0}(t)+\kappa\sum_v \hat{k}^B_{2lv}\frac{\bar{K}_{l0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}} \\
&\times\left(\frac{\hat{f}_l-\bar{r}}{1+\sum_v \hat{k}_{2lv}\hat{K}_v(t)+\kappa\sum_v \hat{k}^B_{2lv}\frac{\bar{K}_{\nu0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}}\right) \\
= &\sum_i \frac{\left(r_i+F_1\left(\bar{R}_i,\frac{\dot{K}_i(t)}{K_i(t)}\right)+\tau\left(\bar{R}(K_i,X_i)\right)\Delta f'_1(K_i(t))-\bar{r}\right)k^B_{1ij}K_i(t)}{1+\sum_v \hat{k}_{jv}\hat{K}_v(t)+\sum_v \hat{k}^B_{1jv}\bar{K}_{l0}(t)+\kappa\sum_v \hat{k}^B_{2jv}\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}}
\end{aligned} \tag{125}$$

Banks' capital accumulation We show in appendix 17.2 that in the continuous approximation, the capital accumulation for banks writes:

$$\frac{d}{dt}\bar{K}_{0j}(t)=\bar{R}_j+\sum_l M_{jl}\frac{d}{dt}\bar{K}_{0l}(t)$$

Similar to investors, the variation in available capital for bank j is the sum of its total return, denoted as $\bar{R}_j$, plus the variation in participations and loans from banks investing in bank j, through the matrix $\bar{M}$. As before, this equation can be reformulated as a dynamic process involving the variation of all the capital at bank j's disposal.

$$\sum_l \left(\delta_{jl}-M_{jl}\frac{d}{dt}\right)\bar{K}_{0l}(t)=\bar{R}'_j \tag{126}$$

with:

$$\bar{M}_{jm}=\frac{\bar{k}_{jm}\bar{K}_{j0}(t)}{1+\sum_\nu \bar{k}_{j\nu}\hat{K}_\nu(t)}$$

Here, the overall leverage effect $\bar{k}_{jm}$ provided by bank m to bank j decomposes into the leverage effect associated with participations and loans, denoted as $\bar{k}_{1jm}$ and $\bar{k}_{2jm}$ respectively:

$$\bar{k}_{1jm}+\bar{k}_{2jm}=\bar{k}_{jm}$$

16.3.4 Coupling investors' and banks' capital accumulation

The dynamics of accumulation for both banks and investors are interrelated. This is evident in equation (122), which is expected, as in our model, the disposable capital of investors directly depends on banks. The dependency of banks on investors is indirect, through the returns. Accumulations of investors and banks are interdependent and must be considered together. They can be expressed in matrix form:

$$\begin{pmatrix} 1-\hat{M} & -\bar{N} \\ 0 & 1-\bar{M} \end{pmatrix} \begin{pmatrix} \frac{d}{dt}\hat{K}(t) \\ \frac{d}{dt}\bar{K}_0(t) \end{pmatrix} = \begin{pmatrix} \hat{R}' \\ \bar{R}' \end{pmatrix} \tag{127}$$

whose solution is:

$$\begin{bmatrix} \frac{d}{dt}\hat{K}(t) \\ \frac{d}{dt}\bar{K}_0(t) \end{bmatrix} = \begin{pmatrix} \left(1-\hat{M}\right)^{-1} & \left(1-\hat{M}\right)^{-1}\bar{N}\left(1-\bar{M}\right)^{-1} \\ 0 & \left(1-\bar{M}\right)^{-1} \end{pmatrix} \begin{bmatrix} \hat{R}' \\ \bar{R}' \end{bmatrix} \tag{128}$$

We can see from equations (127) and (128) that the dynamics of capital accumulation for investors and banks are linked through participations and loans. For this reason, both types of agents will generally be solved simultaneously.

16.4 Returns and capital accumulation under default scenario

16.4.1 Default of banks

As in Part 1, let's assume that bank v defaults if its total private capital cannot cover its loans:

$$\left(1+\sum_m \bar{k}_{2vm}\bar{K}_{m0}(t)\right)\left(1+\bar{R}_v\right) < (1+\bar{r})\sum_m \bar{k}_{2vm}\bar{K}_{m0}(t)$$

If this occurs, the default of bank ν modifies other banks returns $\bar{R}'_j$ by a term:

$$\sum_l \left(\bar{r} - \frac{\left(1+\bar{f}_\nu\right)}{\sum_m \bar{k}_{2vm}\bar{K}_m}\right) \frac{H\left(-\left(1+\bar{f}_\nu\right)\right)\bar{k}_{2lj}\bar{K}_l(t)}{1+\sum_v\left(\bar{k}_{1lv}+\bar{k}_{2lv}\right)\bar{K}_v(t)}$$

where H is the Heaviside step function. This term only appears when $1+\bar{f}_\nu$ is negative, that is when the return $\bar{f}_\nu < -1$, indicating that the return is so negative that it even erodes the bank's private capital. When this term appears, the loss incurred by bank l, which has lent to bank v, is:

$$\left(\bar{r} - \frac{\left(1+\bar{f}_\nu\right)}{\sum_m \bar{k}_{2vm}\bar{K}_m}\right) \frac{\bar{k}_{2lj}\bar{K}_l(t)}{1+\sum_v\left(\bar{k}_{1lv}+\bar{k}_{2lv}\right)\bar{K}_v(t)}$$

The equations for returns accounting for default are provided in the appendix.

16.4.2 Default of investors

The loans received by investors are divided into loans from other investors and loans from banks, where κ is the banking multiplier.

$$\sum_v \hat{k}_{2jv}\hat{K}_v(t) + \kappa\sum_v \hat{k}^B_{2jv}\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{\bar{k}}_{vm}\bar{K}_{m0}(t)}$$

The default condition on this loan is therefore that the return on private capital, private loans, and bank loans do not generate enough returns to repay the borrowed sum with interest, such that:

$$\sum_{l}\left(1+\sum_{v}\hat{k}_{2lv}\hat{K}_{v}\left(t\right)+\kappa\sum_{v}\hat{k}_{2jv}^{B}\frac{\bar{K}_{v0}\left(t\right)}{1+\sum_{m}\bar{\bar{k}}_{vm}\bar{K}_{m0}\left(t\right)}\right)\left(1+\hat{R}_{j}\right)$$
$$<\quad\left(1+\bar{r}\right)\left(\sum_{v}\hat{k}_{2jv}\hat{K}_{v}\left(t\right)+\kappa\sum_{v}\hat{k}_{2jv}^{B}\frac{\bar{K}_{v0}\left(t\right)}{1+\sum_{m}\bar{\bar{k}}_{vm}\bar{K}_{m0}\left(t\right)}\right)$$

In this case, the default of an investor ν affects both the return $\bar{R}_{j}'$ of the banks that lent to them, by an amount:

$$\sum_{l}\left(\bar{r}-\frac{\left(1+\hat{f}_{\nu}\right)}{\sum_{m}\hat{k}_{2vm}\hat{K}_{m}+\kappa\sum_{m}\hat{k}_{2vm}^{B}\frac{\bar{K}_{m0}(t)}{1+\sum_{s}\bar{k}_{ms}\bar{K}_{s0}(t)}}\right)$$
$$\times\frac{H\left(-\left(1+\hat{f}_{\nu}\right)\right)\hat{k}_{2lj}^{B}\hat{K}_{l}\left(t\right)}{1+\sum_{\nu}\hat{k}_{l\nu}\hat{K}_{l}\left(t\right)+\sum_{\nu}\hat{k}_{1l\nu}^{B}\bar{K}_{l0}\left(t\right)+\kappa\sum_{\nu}\hat{k}_{2l\nu}^{B}\frac{\bar{K}_{j0}(t)}{1+\sum_{m}\bar{k}_{vm}\bar{K}_{m0}(t)}}$$

and the return $\hat{R}_{j}'$ of an investor, by an amount of:

$$\sum_{l}\left(\bar{r}-\frac{\left(1+\hat{f}_{\nu}\right)}{\sum_{m}\hat{k}_{2vm}\hat{K}_{m}+\kappa\sum_{m}\hat{k}_{2vm}^{B}\frac{\bar{K}_{m0}(t)}{1+\sum_{s}\bar{k}_{ms}\bar{K}_{s0}(t)}}\right)$$
$$\times\frac{H\left(-\left(1+\hat{f}_{\nu}\right)\right)\hat{k}_{2lj}\hat{K}_{l}\left(t\right)}{1+\sum_{\nu}\hat{k}_{l\nu}\hat{K}_{l}\left(t\right)+\sum_{\nu}\hat{k}_{1l\nu}^{B}\bar{K}_{l0}\left(t\right)+\kappa\sum_{\nu}\hat{k}_{2l\nu}^{B}\frac{\bar{K}_{j0}(t)}{1+\sum_{m}\bar{k}_{vm}\bar{K}_{m0}(t)}}$$

The equations for returns accounting for default are provided in the appendix.

16.4.3 Default of firms

We now examine the defaults of firms. The default condition is that the overall return R_j on private capital, private loans, and bank loans does not generate enough returns to repay the borrowed sum at the interest rate $\bar{r}$, such that:

$$\left(1+R_{j}\right)\left(1+\sum_{v}\hat{k}_{2jv}\hat{K}_{v}\left(t\right)+\kappa\sum_{v}\hat{k}_{2jv}^{B}\frac{\bar{K}_{v0}\left(t\right)}{1+\sum_{m}\bar{\bar{k}}_{vm}\bar{K}_{m0}\left(t\right)}\right)$$
$$<\quad\left(1+\bar{r}\right)\left(\sum_{v}\hat{k}_{2jv}\hat{K}_{v}\left(t\right)+\kappa\sum_{v}\hat{k}_{2jv}^{B}\frac{\bar{K}_{v0}\left(t\right)}{1+\sum_{m}\bar{\bar{k}}_{vm}\bar{K}_{m0}\left(t\right)}\right)$$

Default for firms i modifies the return for banks $\bar{R}_{j}'$ by:

$$\sum_{i}\left(\bar{r}-\frac{\left(1+f_{1}'\left(K_{i}\left(t\right)\right)\right)}{\left(\sum_{v}k_{2iv}^{(B)}\kappa\frac{\bar{K}_{v0}(t)}{1+\sum_{m}\bar{k}_{vm}\bar{K}_{m0}(t)}\right)+\left(\sum_{v}k_{2iv}\hat{K}_{v}\left(t\right)\right)}\right)$$
$$\times\frac{H\left(-\left(1+f_{1}'\left(K_{i}\left(t\right)\right)\right)\right)k_{2ij}^{(B)}K_{i}\left(t\right)}{1+\left(\sum_{v}k_{1iv}^{(B)}\bar{K}_{v0}\left(t\right)+k_{2iv}^{(B)}\kappa\frac{\bar{K}_{v0}(t)}{1+\sum_{m}\bar{k}_{vm}\bar{K}_{m0}(t)}\right)+\left(\sum_{v}k_{iv}\hat{K}_{v}\left(t\right)\right)}$$

and the investor j's return, denoted as $\hat{R}'_j$, by:

$$\sum_i \left(\bar{r} - \frac{(1+f_1'(K_i(t)))}{\left(\sum_v k_{2iv}^{(B)} \kappa \frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}\right) + \left(\sum_v k_{2iv}\hat{K}_v(t)\right)}\right)$$
$$\times \frac{H(-(1+f_1'(K_i(t))))\,k_{2ij}K_i(t)}{1+\left(\sum_v k_{1iv}^{(B)}\bar{K}_{v0}(t) + k_{2iv}^{(B)}\kappa \frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}\right) + \left(\sum_v k_{iv}\hat{K}_v(t)\right)}$$

The formulas for the return including default are provided in the appendix.

17 Field Translation

Similar to the simple model in Part 1, investors are described by $\hat{\Psi}\left(K, \hat{X}\right)$ and firms by $\Psi(K, X)$. Banks are described by the field $\bar{\Psi}\left(\hat{K}', \hat{X}'\right)$. The translation and definitions of coefficients and matrices are given in Appendices 18.1 and 18.2. The resolution method is the same as in Part 1; we simply present the results. The major difference is the introduction of the additional field for banks. It is worth noting that, for simplicity, and as in Part 1, we perform a change of variable on the fields, and all functional actions are rewritten with the new variables.

17.1 Banks' action functional

The field $\bar{\Psi}$ describes the banking system. It depends on the two variables $\bar{K}$ and $\bar{X}$, and its action functional is given by:

$$-\bar{\Psi}^{\dagger}\left(\bar{K}, \bar{X}\right) \nabla^2 \bar{\Psi}\left(\bar{K}, \bar{X}\right) + \left(\frac{\bar{g}^2\left(\bar{K}, \bar{X}\right)}{2\sigma_{\bar{K}}^2} + \frac{\bar{g}\left(\bar{K}, \bar{X}\right)}{2\bar{K}}\right) \left|\bar{\Psi}\left(\bar{K}, \bar{X}\right)\right|^2 \tag{129}$$

where:

$$\bar{g}\left(\hat{K}, \hat{X}\right) = \left(1 - \bar{M}\left|\bar{\Psi}\left(\bar{K}, \bar{X}\right)\right|^2\right)^{-1} \bar{f}\left(\bar{K}, \bar{X}\right)$$

and $\bar{M}$ is the matrix translating its microeconomic equivalent (123).

17.2 Investors' action functional

As in part 1, the field $\hat{\Psi}$ describing the investors depends on the two variables $\hat{K}$ and $\hat{X}$, and its action functional is given by:

$$-\hat{\Psi}^{\dagger}\left(\hat{K}, \hat{X}\right) \nabla^2 \hat{\Psi}\left(\hat{K}, \hat{X}\right) + \left(\frac{\hat{g}^2\left(\hat{K}, \hat{X}\right)}{2\sigma_{\hat{K}}^2} + \frac{\hat{g}\left(\hat{K}, \hat{X}\right)}{2\hat{K}}\right) \left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2 \tag{130}$$

where we define:

$$\hat{g}\left(\hat{K}, \hat{X}\right) = \left(1 - \hat{M}\left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2\right)^{-1} \hat{f}\left(\hat{K}, \hat{X}\right) + \left(1 - \hat{M}\right)^{-1} N\left(1 - \bar{M}\right)^{-1} \bar{f}\left(\bar{K}, \bar{X}\right)$$

The field definition of the matrices $\bar{M}$, $\hat{M}$ and N are given in Appendix 18.2.

17.3 Firms' action functional

The translation of the function (56) is obtained by applying the mapping provided by (16) and (17).

The action functional for the field of firms is:

$$-\Psi^{\dagger}(K,X)\left(\nabla_{K_p}\left(\sigma_K^2\nabla_{K_p}-f_1'(K,X)K_p\right)\right)\Psi(K,X)+\frac{1}{2\epsilon}\left(\left|\Psi(K,X)\right|^2-\left|\Psi_0(X)\right|^2\right)^2$$

where:

$$f_1'(K,X)=\left(1+\underline{k}_2(X)\right)f_1(X)-\bar{r}\underline{k}_2\left(\hat{X}\right)$$

where $\underline{k}_2\left(\hat{X}\right)$ is defined in appendix 18.1.

17.4 Return equations under a no-default scenario

For later purpose, we write the translation of return equations (120) and (124) in terms of $\hat{g}\left(\hat{K}_1,\hat{X}_1\right)$ and $\bar{g}\left(\bar{K}_1,\bar{X}_1\right)$. The equations with defaults are presented in Appendix 18.4.

17.4.1 Bank returns' equation

The translation of (124) is:

$$\begin{aligned}&\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\bar{\underline{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\bar{\underline{k}}_2\left(\bar{X}'\right)} \\ &-\int\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\left(\bar{N}\bar{g}\right)\left(\bar{K}',\bar{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\frac{\underline{\hat{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}\left(\bar{X}\right)}}d\bar{X}' \\ =\;&\frac{\underline{k}_1^{(B)}\left(X',\bar{X}\right)}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}\frac{\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}\end{aligned}\tag{131}$$

The left-hand side represents the returns for banks located in $\bar{X}$, including participations in other banks located in $\bar{X}'$, which provide a return $\bar{g}$ as well as participations in investors located in $\hat{X}'$, which yield a return $\hat{g}\left(\hat{K}',\hat{X}'\right)$. The returns of the banks depend indirectly the returns of other banks through the term $\left(\bar{N}\bar{g}\right)$: the return of investors in which a bank invests depends directly on the investments of other banks and investors. The right-hand side describes the returns provided by firms in which the bank invests.

17.4.2 Investors returns' equation

The translation of (120) is:

$$\begin{aligned}&\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)}\right)\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)} \\ =\;&\frac{k_1\left(X',X'\right)\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)}{\left(1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)\left(1+\underline{k}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)}\end{aligned}\tag{132}$$

The left-hand side represents the returns for investors located in $\hat{X}$, including participations in other investors located in $\hat{X}'$ providing return $\hat{g}\left(\hat{K}', \hat{X}'\right)$. Similarly, the returns of the banks indirectly depend on the returns of other banks through the term $(\bar{N}\bar{g})$. The right-hand side describes the returns provided by firms in which the bank invests.

17.4.3 Firms' returns

The derivation of firm return, average capital, and background field is similar to Part 1. Taking into account the banking sector, the firm's disposable capital is given by:

$$K = \left(1 + \underline{k}\left(\hat{X}'\right) + \underline{k}_1^{(B)}\left(\bar{X}'\right) + \kappa\left[\frac{\underline{k}_2^B}{1+\bar{\underline{k}}}\right](X')\right) K_p$$

We consider constant returns to scale and include small corrections to account for decreasing returns to scale.

Constant returns to scale Using the results of Part 1 and including banks, the firm's return on its private capital is calculated by multiplying the firm's available capital, consisting of K_p, by various leverage effects:

$$K_p\left(1 + k\left(X, , \hat{X}'\right)\hat{K}'_{\hat{X}'} + \underline{k}_1^{(B)}\left(X, \bar{X}'\right)\bar{K}_{\bar{X}'} + \kappa\frac{\underline{k}_2^{(B)}\left(X, \bar{X}'\right)}{1+\bar{\underline{k}}\left(\bar{X}\right)}\bar{K}_{\bar{X}'}\right)$$

by the productivity $f_1(X)$, subtracting the fixed production cost C, and then dividing by the total available capital, to obtain a return percentage. Multiplying this return by K_p, we get the overall return of the firm in value, which is:

$$K_p\left(\frac{f_1(X)\left(1 + k\left(X, , \hat{X}'\right)\hat{K}'_{\hat{X}'} + \underline{k}_1^{(B)}\left(X, \bar{X}'\right)\bar{K}_{\bar{X}'} + \kappa\frac{\underline{k}_2^{(B)}(X, \bar{X}')}{1+\bar{\underline{k}}(\bar{X})}\bar{K}_{\bar{X}'}\right)K_p - C}{\left(1 + \underline{k}\left(\hat{X}'\right) + \underline{k}_1^{(B)}\left(\bar{X}'\right) + \kappa\left[\frac{\underline{k}_2^B}{1+\bar{\underline{k}}}\right](X')\right)K_p}\right) = f_1(X) K_p - \bar{C}(X) \tag{133}$$

with:

$$\bar{C}(X) = \frac{C}{1 + \int \underline{k}\left(\hat{X}\right)\hat{K}_{\hat{X}'}\frac{\left|\hat{\Psi}(\hat{X}')\right|^2}{|\Psi_0(X)|^2} + \int\left(\underline{k}_1^{(B)}\left(\bar{X}'\right) + \kappa\left[\frac{\underline{k}_2^B}{1+\bar{\underline{k}}}\right](X')\right)\bar{K}_{\bar{X}'}\frac{\left|\bar{\Psi}(\hat{X}')\right|^2}{|\Psi_0(X)|^2}} \tag{134}$$

$$\frac{C(X)}{1 + \underline{k}(X) + \underline{k}_1^B(X) + \kappa\left[\frac{\underline{k}_2^B}{1+\bar{\underline{k}}}\right](X)}$$

Note that $\bar{C}(X)$ is an effective cost, reduced by investments. The more investments in the firm, the more the fixed cost is diluted in terms of return.

Decreasing returns to scale Similar to Part 1, to incorporate corrections accounting for slightly decreasing returns, we will replace $f_1(X)$ in the formulas above with a productivity that decreases with invested capital:

$$f_1(X) \to \frac{f_1(X)}{\left(1 + \frac{k(X)}{K[X]}\hat{K}[X] + \frac{k_1^B(X)}{K[X]}\bar{K}[X] + \frac{\kappa\left[\frac{k_2^B}{1+\bar{k}}\right]}{K[X]}\bar{K}[X]\right)^r K_X^r} - C_0$$

The formula clearly demonstrates that the more capital invested, the lower the marginal productivity of capital.

18 Minimization equations

Now that we have provided the functional actions of the various agents in the system, we can determine the background fields that define the system's states. The methods of solving and interpreting background fields and capital per sector are similar to Part 1. We will begin with firms.

18.1 Minimization equation for firms

As before, the minimization of firms' action functional leads to the equation:

$$0=\left(\frac{\left(f_1^{(e)}(X)\right)^2}{\sigma_{\hat{K}}^2}+\frac{f_1^{(e)}(X)}{2}\right)+\frac{1}{\epsilon}\left(|\Psi(K,X)|^2-|\Psi_0(X)|^2\right) \tag{135}$$

where $f_1^{(e)}(X)$ is the return of the firm once loans are repaid:

$$f_1^{(e)}(X)=\left(1+\underline{k}_2(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right](X)\right)f_1'(X)-\left(\underline{k}_2(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right](X)\right)\bar{r} \tag{136}$$

and $f_1'(X)$ represents the return of the firm from its production, as given by (133):

$$f_1'(X)=f_1(X)K_p-\bar{C}(X)$$

18.2 Minimization equation for investors

Appendix 20.1 derives the minimization equation for (130):

$$\begin{aligned}0=&\frac{\hat{K}_1^2\hat{g}^2\left(\hat{K}_1,\hat{X}_1\right)}{2\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(\hat{K}_1,\hat{X}_1\right)}{2}\\&+\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2\left(\frac{\hat{K}^2\hat{g}\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\frac{\delta\hat{g}\left(\hat{K},\hat{X}\right)}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}+\frac{1}{\hat{\mu}}\left(\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2-\left|\hat{\Psi}_0\left(\hat{X}\right)\right|^2\right)\end{aligned} \tag{137}$$

The formula for $\frac{\delta\hat{g}\left(\hat{K},\hat{X}\right)}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}$ is given in Appendix 25.

18.3 Minimization equation for banks

Appendix 20.2 derives the minimization equation for (129):

$$\begin{aligned}0\;=\;&\left(\frac{\bar{K}_1^2\bar{g}^2\left(\bar{K}_1,\bar{X}_1\right)}{\sigma_{\hat{K}}^2}+\frac{\bar{g}\left(\bar{K}_1,\bar{X}_1\right)}{2}\right)\\&+\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2\left(\frac{\hat{K}^2\hat{g}\left(\hat{K},\hat{X}\right)}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\frac{\delta\hat{g}\left(\hat{K},\hat{X}\right)}{\delta\left|\bar{\Psi}\left(\bar{K}_1,\bar{X}_1\right)\right|^2}+\frac{1}{\hat{\mu}}\left(\left|\bar{\Psi}\left(\bar{K}_1,\bar{X}_1\right)\right|^2-\left|\bar{\Psi}_0\left(\bar{X}_1\right)\right|^2\right)\end{aligned} \tag{138}$$

and the formula for $\frac{\delta\hat{g}\left(\hat{K},\hat{X}\right)}{\delta\left|\bar{\Psi}\left(\bar{K}_1,\bar{X}_1\right)\right|^2}$ is derived in appendix 25. These equations will be used when computing average capital per sector.

19 Resolution for firms

19.1 Solution for the background field

The total return produced by a firm is given by (136):

$$f_1^{(e)}(X)K=\left(1+\underline{k}_2(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right](X)\right)\left(f_1'(X)K_p-\bar{C}(X)\right)-\left(\underline{k}_2(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right](X)\right)K_p\bar{r}$$

where the first term in the right-hand side is the return of the capital invested by the firm, including loans. with $\bar{C}(X)$ being the effective cost defined in (134).

As before, we assume constant returns to scale and include corrections later. The solution to the minimization equation for the firm's background field is:

$$|\Psi(X)|^2=|\Psi_0(X)|^2-\epsilon\frac{f_1^{(e)}(X)}{2}-\epsilon\frac{\left(f_1^{(e)}(X)K_p\right)^2}{\sigma_{\hat{K}}^2}$$

19.2 Solution for the average capital per sector

Appendix 20.4 computes the average capital as in Part 1:

$$K_X=\frac{1}{4f_1^{(e)}(X)}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}$$

19.3 Solution for the return to investors

Note that the return to investors will thus be:

$$\frac{f_1^{(e)}(X)-\bar{r}}{1+\underline{k}_2(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right](X)}=f_1(X)-\bar{r}-\frac{\bar{C}(X)}{K_p}$$

and that the amount of return involved in the banks and investors' return equations will be:

$$|\Psi(X)|^2K_X\frac{f_1'(X)-\bar{r}}{1+\underline{k}_2(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right](X)}=|\Psi(X)|^2\left(\left(f_1(X)-\bar{r}\right)K_X-\bar{C}(X)\right)$$

whose expanded expression is:

$$|\Psi(X)|^2K_X\frac{f_1'(X)-\bar{r}}{1+\underline{k}_2(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right](X)}$$
$$=\frac{\left(C^{(e)}+X^{(e)}\right)\left(\frac{2}{3}X^2+\frac{1}{3}\left(X-C^{(e)}\right)C^{(e)}\right)\epsilon\left\{3\left(X^{(e)}-C^{(e)}\right)^2-\frac{\bar{r}\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{f_1^{(e)}(X)}\right\}}{4\sigma_{\hat{K}}^2\left(1+\underline{k}_2(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right](X)\right)f_1^{(e)}(X)\left(2X^{(e)}-C^{(e)}\right)}$$

with:

$$X^{(e)}=\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}f_1^{(e)}(X)}$$

and:

$$\bar{C}^{(e)}(X)=\frac{\left(1+\underline{k}_2(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right](X)\right)C(X)}{1+\underline{k}(X)+\underline{k}_1^B(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right](X)}$$

20 Resolution for financial agents

Since investors and banks interact with each other, we will solve their minimization equations simultaneously, treating them as the financial agents of the system.

20.1 Compact form of the minimization equation

These minimization equations, denoted as (137) and (138), are rewritten using the expression for functional derivatives:

$$\frac{\delta \hat{g}\left(\hat{K},\hat{X}\right)}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}=-\frac{\left(\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\underline{\hat{k}}\right\rangle\right)+\left\langle\hat{k}_1^B\right\rangle\left\langle\hat{k}_2\right\rangle\right)\left(\langle\hat{g}\rangle+\frac{1}{1-\left\langle\underline{\hat{k}}\right\rangle}\bar{N}\left\langle\bar{g}\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle}}{\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left(1-\left(\left\langle\underline{\hat{k}}\right\rangle+\left(\left\langle\underline{\hat{k}}_1^B\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left\|\hat{\Psi}\right\|^2}$$

and:

$$\begin{aligned}\frac{\delta \hat{g}\left(\hat{K},\hat{X}\right)}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}&=-\left(\langle\hat{g}\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\right\rangle\right)\frac{\bar{K}}{\left\langle\hat{K}\right\rangle}\\&\times\frac{\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\hat{k}\right\rangle\right)+\left\langle\hat{k}_1^B\right\rangle\left\langle\hat{k}_2\right\rangle}{\left(1-\left(\left\langle\hat{k}\right\rangle+\left(\left\langle\hat{k}_1^B\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left\|\hat{\Psi}\right\|^2}\end{aligned}$$

with:

$$\bar{N}\rightarrow\left\langle\hat{k}_1^B\right\rangle+\kappa\frac{\left\langle\hat{k}_2^B\right\rangle}{1+\left\langle\bar{k}\right\rangle}\left(1-\frac{\left\langle\bar{k}\right\rangle}{\left(1+\left\langle\bar{k}\right\rangle\right)^2}\right)$$

These coefficients can be written in a more compact form as:

$$\frac{\delta \hat{g}\left(\hat{K},\hat{X}\right)}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}=-\left\langle\hat{g}^{ef}\right\rangle\frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}$$

and:

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\hat{g}\left(\hat{K},\hat{X}\right)=-\left\langle\hat{g}^{Bef}\right\rangle\frac{\bar{K}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}$$

Using these formula, the minimization equations for investors become:

$$\begin{aligned}0&=\frac{\hat{K}_1^2\hat{g}^2\left(\hat{K}_1,\hat{X}_1\right)}{2\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(\hat{K}_1,\hat{X}_1\right)}{2}\\&-\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2\left(\frac{\hat{K}^2\hat{g}\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{ef}\right\rangle\frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\\&+\frac{1}{\hat{\mu}}\left(\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2-\left|\hat{\Psi}_0\left(\hat{X}\right)\right|^2\right)\end{aligned}$$

or, which is equivalent:

$$0 = \frac{\hat{K}_1^2\hat{g}^2\left(\hat{K}_1,\hat{X}_1\right)}{2\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(\hat{K}_1,\hat{X}_1\right)}{2} - \left(\frac{\left\langle\hat{K}\right\rangle^2\left\langle\hat{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{ef}\right\rangle\frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle}+\frac{1}{\hat{\mu}}\left(\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2-\left|\hat{\Psi}_0\left(\hat{X}\right)\right|^2\right) \tag{139}$$

Similarly, the equation for the banks' field $\bar{\Psi}\left(\bar{K}_1,\bar{X}_1\right)$ becomes:

$$0 = \left(\frac{\bar{K}_1^2\bar{g}^2\left(\bar{K}_1,\bar{X}_1\right)}{\sigma_{\hat{K}}^2}+\frac{\bar{g}\left(\bar{K}_1,\bar{X}_1\right)}{2}\right) - \left(\frac{\left\langle\hat{K}\right\rangle^2\hat{g}\left(\hat{K},\hat{X}\right)}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{Bef}\right\rangle\frac{\bar{K}_1}{\left\langle\hat{K}\right\rangle}+\frac{1}{\hat{\mu}}\left(\left|\bar{\Psi}\left(\bar{K}_1,\bar{X}_1\right)\right|^2-\left|\bar{\Psi}_0\left(\bar{X}_1\right)\right|^2\right)$$

20.2 Solving the minimization equation for the background fields

The solutions for the background fields are:

$$\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 = \left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}\left\{\left(\frac{\hat{K}_1^2\hat{g}^2\left(\hat{X}_1\right)}{2\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(\hat{X}_1\right)}{2}\right) + \left(\frac{\left\langle\bar{K}\right\rangle^2\left\langle\bar{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)\frac{\left\|\bar{\Psi}\right\|^2\hat{K}_1}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}-\left(\frac{\left\langle\hat{K}\right\rangle^2\left\langle\hat{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{ef}\right\rangle\frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle}\right\}$$

$$\left|\bar{\Psi}\left(\bar{K}_1,\bar{X}_1\right)\right|^2 = \left|\bar{\Psi}_0\left(\bar{X}_1\right)\right|^2-\hat{\mu}\left\{\left(\frac{\bar{K}_1^2\bar{g}^2\left(\bar{X}_1\right)}{\sigma_{\hat{K}}^2}+\frac{\bar{g}\left(\bar{X}_1\right)}{2}\right) + \left(\frac{\left\langle\bar{K}\right\rangle^2\left\langle\bar{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left(\left\langle\underline{\hat{k}}^B\right\rangle^{ef}\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)+\frac{\Delta\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}\right)}{\left(1-\left\langle\bar{k}_1\right\rangle\right)\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\left\langle\bar{g}\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\bar{K}_1}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle} - \left(\frac{\left\langle\hat{K}\right\rangle^2\left\langle\hat{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{Bef}\right\rangle\frac{\bar{K}_1}{\left\langle\hat{K}\right\rangle}\right\}$$

where, for any quantity Q, the notation $\Delta\left(Q\left(\hat{X}\right)\right)$ stands for its deviation from the average:

$$\Delta\left(Q\left(\hat{X}\right)\right)=\left(Q\left(\hat{X}\right)\right)-\left\langle\Delta\left(Q\left(\hat{X}\right)\right)\right\rangle$$

20.3 Global averages for field and capital

20.3.1 Investors

Appendix 20.5 computes the averages of equations (143) and (144), obtaining the average disposable capital per investor across the entire system, denoted as $\left\langle \hat{K} \right\rangle$:

$$\left\langle \hat{K} \right\rangle^2 \simeq \frac{1}{6}\frac{\sigma_{\hat{K}}^2}{\hat{\mu}} \frac{\left(\left(2+\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1-\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)\left(4-\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)}\right)\right)^2 \left\| \hat{\Psi}_0 \right\|^2}{\left\langle \hat{g}^{ef} \right\rangle^2 \left(5+\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1-\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)\left(4-\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)}\right)} \tag{140}$$

and the total disposable capital for investors in the system, denoted as $\left\langle \hat{K} \right\rangle \left\| \hat{\Psi} \right\|^2$:

$$\begin{aligned}\left\langle \hat{K} \right\rangle \left\| \hat{\Psi} \right\|^2 &= \frac{2\sigma_{\hat{K}}^2}{\hat{\mu}} V \left(\frac{1}{4} - \frac{1}{18}\left(2+\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1-\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)\left(4-\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)}\right)\right) \langle \hat{g} \rangle^2 \\ &\quad \times \left(\frac{\left\| \bar{\Psi}_0 \right\|^2 - \frac{\hat{\mu}}{6}\left(\frac{\left\langle \hat{K} \right\rangle^2 \langle \hat{g} \rangle}{\sigma_{\hat{K}}^2}\right) \left\langle \hat{g}^{Bef} \right\rangle \left(2+\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1-\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)\left(4-\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)}\right)}{\langle \bar{g} \rangle^2}\right)^2\end{aligned} \tag{141}$$

The above formulas do not close the model, as the returns $\langle \hat{g} \rangle$ and $\left\langle \hat{g}^{ef} \right\rangle$ are themselves endogenous, and depend on the average capital. Below, we will derive the equation governing these returns, which will enable us to solve for both capital and returns.

20.3.2 Banks

Computing the average of equation (145a) yields the average disposable capital per bank in the entire system, denoted as $\langle \bar{K} \rangle$:

$$\langle \bar{K} \rangle \simeq 18\frac{\sigma_{\hat{K}}^2}{\hat{\mu}} \left(\frac{\sqrt{\langle \hat{g} \rangle^2 \left(1+\frac{3}{4}\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)} \left\| \bar{\Psi}_0 \right\| - \frac{3}{8}\langle \hat{g} \rangle \frac{\hat{g}^{Bef}(\bar{X}_1)}{\bar{g}(\bar{X}_1)} \left\| \hat{\Psi}_0 \right\|}{\sqrt{5+\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1-\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)\left(4-\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)}}}\right)^4 \left(\frac{1}{4\langle \bar{g} \rangle^2} - \frac{\langle \hat{g} \rangle \left\langle \hat{g}^{Bef} \right\rangle}{3\langle \bar{g} \rangle^4}\right)$$

and its associated field:

$$\left\| \bar{\Psi} \right\|^2 \simeq 18\frac{\sigma_{\hat{K}}^2}{\hat{\mu}} \left(\frac{\sqrt{\langle \hat{g} \rangle^2 \left(1+\frac{3}{4}\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)} \left\| \bar{\Psi}_0 \right\| - \frac{3}{8}\langle \hat{g} \rangle \frac{\hat{g}^{Bef}(\bar{X}_1)}{\bar{g}(\bar{X}_1)} \left\| \hat{\Psi}_0 \right\|}{\sqrt{5+\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1-\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)\left(4-\frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)}}}\right)^4 \left(\frac{1}{3\langle \bar{g} \rangle^2} - \frac{\langle \hat{g} \rangle \left\langle \hat{g}^{Bef} \right\rangle}{2\langle \bar{g} \rangle^4}\right)$$

The average of (146) yields the total amount of disposable capital for banks in the system, denoted as $\langle \bar{K} \rangle \left\| \bar{\Psi} \right\|^2$:

$$\langle \bar{K} \rangle \left\| \bar{\Psi} \right\|^2 \simeq 18 \frac{\sigma_{\hat{K}}^2}{\langle \bar{g} \rangle^2 \hat{\mu}} \left(\frac{\sqrt{\langle \hat{g} \rangle^2 \left(1 + \frac{3}{4} \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right)} \left\| \bar{\Psi}_0 \right\| - \frac{3}{8} \langle \hat{g} \rangle \frac{\langle \hat{g}^{Bef} \rangle}{\langle \bar{g} \rangle} \left\| \hat{\Psi}_0 \right\|}{\sqrt{5 + \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right) \left(4 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right)}}} \right)^4 \left(\frac{1}{4} - \frac{\langle \hat{g} \rangle \langle \hat{g}^{Bef} \rangle}{3 \langle \bar{g} \rangle^2} \right) \tag{142}$$

20.4 Average field and total capital per sector

20.4.1 Investors

The amount of capital for investors in sector $\hat{X}_1$ is:

$$\begin{aligned} \hat{K}\left[\hat{X}_1\right] &= \hat{K}_{\hat{X}} \left\| \hat{\Psi}\left(\hat{X}_1\right) \right\|^2 & (143) \\ &\simeq \frac{\hat{\mu}}{2\sigma_{\hat{K}}^2} \left(6 \frac{\sigma_{\hat{K}}^2}{\hat{\mu}} \frac{\left\| \hat{\Psi}_0\left(\hat{X}_1\right) \right\|^2}{\left(5 + \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right) \left(4 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right)} \right)} \right)^2 \left(\frac{1}{4\hat{g}^2\left(\bar{X}_1\right)} - \frac{\langle \hat{g} \rangle \hat{g}^{ef}\left(\hat{X}_1\right)}{3\hat{g}^4\left(\bar{X}_1\right)} \right) \end{aligned}$$

and its associated field is:

$$\left\| \hat{\Psi}\left(\hat{X}_1\right) \right\|^2 = \frac{\hat{\mu}}{2\sigma_{\hat{K}}^2} \left(6 \frac{\sigma_{\hat{K}}^2}{\hat{\mu}} \frac{\left\| \hat{\Psi}_0\left(\hat{X}_1\right) \right\|^2}{\left(5 + \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right) \left(4 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right)} \right)} \right)^{\frac{3}{2}} \left(\frac{1}{3\hat{g}^2\left(\bar{X}_1\right)} - \frac{\langle \hat{g} \rangle \hat{g}^{ef}\left(\hat{X}_1\right)}{2\hat{g}^4\left(\bar{X}_1\right)} \right) \tag{144}$$

20.4.2 Banks

The amount of capital for banks in sector $\bar{X}_1$ is given by:

$$\bar{K}\left[\bar{X}_1\right] \simeq 18 \frac{\sigma_{\hat{K}}^2}{\hat{\mu}} \left(\frac{\sqrt{\langle \hat{g} \rangle^2 \left(1 + \frac{3}{4} \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right)} \left\| \bar{\Psi}_0\left(\hat{X}_1\right) \right\| - \frac{3}{8} \langle \hat{g} \rangle \frac{\hat{g}^{Bef}(\bar{X}_1)}{\bar{g}(\bar{X}_1)} \left\| \hat{\Psi}_0\left(\bar{X}_1\right) \right\|}{\sqrt{5 + \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right) \left(4 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right)}}} \right)^4 \left(\frac{1}{4\bar{g}^2\left(\bar{X}_1\right)} - \frac{\langle \hat{g} \rangle \hat{g}^{Bef}\left(\bar{X}_1\right)}{3\bar{g}^4\left(\bar{X}_1\right)} \right) \tag{145a}$$

along with its associated field:

$$\left\| \bar{\Psi}\left(\bar{X}_1\right) \right\|^2 \simeq 18 \frac{\sigma_{\hat{K}}^2}{\hat{\mu}} \left(\frac{\sqrt{\langle \hat{g} \rangle^2 \left(1 + \frac{3}{4} \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right)} \left\| \bar{\Psi}_0\left(\hat{X}_1\right) \right\| - \frac{3}{8} \langle \hat{g} \rangle \frac{\hat{g}^{Bef}(\bar{X}_1)}{\bar{g}(\bar{X}_1)} \left\| \hat{\Psi}_0\left(\bar{X}_1\right) \right\|}{\sqrt{5 + \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right) \left(4 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right)}}} \right)^4 \left(\frac{1}{3\bar{g}^2\left(\bar{X}_1\right)} - \frac{\langle \hat{g} \rangle \hat{g}^{Bef}\left(\bar{X}_1\right)}{2\bar{g}^4\left(\bar{X}_1\right)} \right) \tag{146}$$

20.5 Investors' and banks' returns per sector

Using the formula for the total amount of capital per sector, as found in Appendix 21, we can find the investors and banks' returns per sector as functions of fields.

20.5.1 Computation of $\hat{g}$

The return per sector for investors is given by:

$$\hat{g}\left(\hat{X}_1\right) \simeq \frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\left(5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)\sqrt{\frac{2\hat{\mu}}{9\sigma_{\hat{K}}^2}\hat{K}\left[\hat{X}_1\right]+\frac{6\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle\hat{g}\rangle^3}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\left(5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)}\right)^2}}$$

This formula will be used alongside the return equation to derive an equation linking the average capital across different sectors.

20.5.2 Computation of $\bar{g}$

Similarly, we can express the return per sector for banks:

$$\bar{g}\left(\bar{X}_1\right) \simeq \frac{\left(\sqrt{\langle\hat{g}\rangle^2\left(1+\frac{3}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\left\|\bar{\Psi}_0\left(\hat{X}_1\right)\right\|-\frac{3}{8}\langle\hat{g}\rangle\frac{\hat{g}^{Bef}\left(\bar{X}_1\right)}{\bar{g}\left(\bar{X}_1\right)}\left\|\hat{\Psi}_0\left(\bar{X}_1\right)\right\|\right)^2}{\left(5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)^2\sqrt{\frac{2\hat{\mu}}{9\sigma_{\hat{K}}^2}\hat{K}\left[\hat{X}_1\right]+\frac{4\langle\hat{g}\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{3\langle\hat{g}\rangle^4}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\left(5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)}\right)^2}}$$

21 Investors' and banks' returns under a no-default scenario

21.1 Investors' returns

21.1.1 Derivation of the investor returns' equation under constant return to scale

The formula for investor returns equation in term of $\hat{g}\left(\hat{X}\right)$ is similar to that in Part 1:

$$\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)} \tag{147}$$

$$= \frac{k_1\left(\hat{X}',X\right)\hat{K}'}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}\frac{f_1'\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}$$

21.1.2 Solution of the investor returns' equation

Applying the resolution procedure outlined in Part 1, Appendix 21 provides the following solution:

$$\begin{aligned}\hat{g}\left(\hat{X}_1\right)\left(\hat{X}_1\right)-\bar{r}' &= \int\left(1-\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1} \\ &\quad\times\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(X'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(X'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)\end{aligned} \tag{148}$$

with:

$$\begin{aligned}A\left(\hat{X}'\right) &= \frac{\epsilon\left(1-\beta\right)\delta C\left(X+C\frac{\beta\delta+\beta^B}{1+\delta}\right)^2}{3\left(1+\delta\right)\sigma_{\hat{K}}^2}\frac{\left(3X-\frac{\beta\delta+\beta^B}{1+\delta}C\right)\left(\frac{\beta\delta+\beta^B}{1+\delta}C+X\right)}{4C\left(2X-C\frac{\beta\delta+\beta^B}{1+\delta}\right)} \\ B\left(\hat{X}'\right) &= \frac{\epsilon\left(1-\beta\right)\delta C\left(X+C\frac{\beta\delta+\beta^B}{1+\delta}\right)^2}{3\left(1+\delta\right)\sigma_{\hat{K}}^2}\frac{\left(3X-\frac{\beta\delta+\beta^B}{1+\delta}C\right)\left(\frac{\beta\delta+\beta^B}{1+\delta}C+X\right)}{4C\left(2X-C\frac{\beta\delta+\beta^B}{1+\delta}\right)}\frac{\left(\beta\delta+\beta^B\right)}{\left(1+\delta\right)f_1^3\left(X\right)}\end{aligned} \tag{149}$$

$$R=f_1\left(X'\right)-\bar{r}$$

and where the matrix $\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$ is estimated as:

$$\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)=\frac{\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)}{1+\left[\underline{\hat{k}}\left(\hat{X}_1\right)\right]_\kappa}\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\underline{\hat{k}}_1\left(\left\langle X\right\rangle,\hat{X}\right)k\left(\left\langle X\right\rangle,X'\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}'\right)\right]_\kappa}\right)\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)}{\left(\left\|\hat{\Psi}_0\right\|^2-\hat{\mu}\left\langle D\right\rangle\right)^2} \tag{150}$$

21.1.3 Interpretation and diffusion matrix

Equation (148) is similar to that derived in Part 1, and the interpretation is the same: under constant return to scale, the solution is unique. The only difference arises from.the fact that the coefficient $\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa$ defined by:

$$1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa=1+\frac{\hat{k}_2\left(\hat{X}',\left\langle\hat{X}\right\rangle\right)+\kappa\left[\frac{\hat{k}_2^B\left(\hat{X}',\left\langle\bar{X}\right\rangle\right)}{1+\bar{k}}\right]\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}}{1-\left\langle\hat{k}^\Sigma\right\rangle}$$

where:

$$\left\langle\hat{k}^\Sigma\right\rangle=\left(\left\langle\hat{k}\right\rangle+\left(\left\langle\hat{k}_1^B\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)$$

depends on the ratio of total disposable capital owned by banks to the total disposable capital owned by investors. The higher the ratio, the higher the expression:

$$\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$$

which indicates that diffusion is higher when the ratio is higher. Therefore, although the form of the diffusion matrix $\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$ defined in (150) is similar to Part 1, the coefficients are increased by banks that favour the leverage effect.

21.1.4 Corrections due to decreasing returns to scale

As explained before, for slowly decreasing returns to scale, we can replace $f_1(X)$ in (148) by:

$$
\begin{aligned}
& f_1\left(X', \hat{K}\left[\hat{X}'\right], \bar{K}\left[\bar{X}'\right]\right) \\
\equiv\ & \frac{f_1(X)}{\left(1+\frac{k(X)}{K[X]}\hat{K}[X]+\frac{k_1^B(X)}{K[X]}\bar{K}\left[\bar{X}\right]+\frac{\kappa\left[\frac{k_2^B}{1+\bar{k}}\right]}{K[X]}\bar{K}\left[\bar{X}\right]\right)^r K_X^r}-C_0 \\
\simeq\ & \frac{f_1(X)}{\left(\left(1+k(X)\hat{K}\left[\hat{X}\right]\right)+\left(k_1^B\left(\bar{X}\right)+\kappa\left[\frac{k_2^B\left(\bar{X}\right)}{1+\bar{k}}\right]\right)\bar{K}\left[\bar{X}\right]\right)^r}-C_0
\end{aligned}
$$

As in Part 1, accounting for decreasing returns to scale results in a multiplicity of collective states. This will be discussed further when considering the equations for capital per sector.

21.2 Banks' returns

21.2.1 Derivation of bank returns' equation and solutions

Appendix 22 shows that the equation for bank returns is as follows:

$$
\begin{aligned}
\bar{g}\left(\bar{X}_1\right)-\bar{r}' =& \left(1-\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\bar{S}_1^E\left(\bar{X}',\hat{X}_1\right)+\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right)\right)^{-1} \\
&\times\left(\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\frac{1-\beta^B}{(1-\beta)\delta}\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}-\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)+\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)\right)\left(\bar{g}\left(\bar{X}'\right)-\bar{r}'\right)\right)
\end{aligned}
\tag{151}
$$

The coefficients are defined in the same appendix.

Using (148), we derive the equivalent formulation for the banks' return equation:

$$
\begin{aligned}
& \bar{g}\left(\bar{X}_1\right)-\bar{r}' \\
=\ & \left(1-\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\bar{S}_1^E\left(\bar{X}',\hat{X}_1\right)+\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right)\right)^{-1} \\
& \times\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\frac{1-\beta^B}{(1-\beta)\delta}\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}-\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)+\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)\right) \\
& \times\left(1-\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1}\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(X'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(X'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)
\end{aligned}
\tag{152}
$$

where $A\left(\hat{X}'\right)$ and $B\left(\hat{X}'\right)$ are defined in (149).

21.2.2 Interpretation and diffusion matrices

Equation (151) has a structure similar to (148) as it involves a diffusion matrix $\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)$ specific to the banks, whose interpretation is similar to the diffusion matrix $\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$ of investors. However (151) is more complex than (148) since it directly involves $\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$, but also two crossed matrices $\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)$ from banks to investors and $\bar{S}_1^E\left(\bar{X}',\hat{X}_1\right)$ from investors to banks. The precise formulation of these matrices are given in Appendix 22. The matrix $\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)$ measures the impact of investors' returns on banks having participations in these investors, while $\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)$ measures the indirect return of an investor I on a bank B through the banks in which B has invested, that themselve invested in I. The global factor $\frac{1-\beta^B}{(1-\beta)\delta}$ is directly proportional to the share that loans take in the bank activity. If this share is close to one, this global factor is negligible, and the bank is quite unaffected by diffusion. It is only when banks take participations that they are affected by diffusion.

22 Equation for total capital of financial agents per sector

22.1 Derivation of the equation

We show in appendices 20 and 21 that equations for returns (148) and (151) can also be expressed as equations for average capital per sector if we substitute in first approximation:

$$\hat{g}\left(\hat{X}_1\right)\simeq\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{1-\frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle\hat{g}\rangle}}}{\sqrt{\frac{2\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{\sigma_{\hat{K}}^2}}}\equiv\frac{\hat{N}\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\tag{153}$$

with:

$$D\left(\hat{X}_1\right)=\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle^2}{\sigma_{\hat{K}}^2}+\frac{\langle\hat{g}\rangle}{2}\right)\left(\frac{\underline{\hat{k}}^{ef}\left(\hat{X}_1\right)}{\underline{\hat{k}}_2^{Bef}\left(\left\langle\hat{X}\right\rangle\right)}-\frac{6\underline{\hat{k}}_2^{Bef}}{2+\underline{\hat{k}}_2^{Bef}-\sqrt{\left(2+\underline{\hat{k}}\right)^2-\underline{\hat{k}}_2^{Bef}}}\right)\underline{\hat{k}}_2^{Bef}$$

for investor returns, and:

$$\bar{g}\left(\bar{X}_1\right)=\frac{\left(\left\|\bar{\Psi}_0\left(\bar{X}'\right)\right\|^2-\hat{\mu}D\left(\bar{X}'\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{Bef}\left(\bar{X}'\right)}{\langle\bar{g}\rangle}\right)}}{\sqrt{\underline{k}\left(\bar{X}'\right)}}\equiv\frac{\bar{N}\left(\bar{X}_1\right)}{\sqrt{\bar{K}\left[\bar{X}_1\right]}}$$

with:

$$\begin{aligned}\bar{D}\left(\bar{X}_1\right)&=\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{Bef}\right\rangle\frac{\bar{K}_0}{\left\langle\hat{K}\right\rangle}\\&-\left(\frac{\left\langle\bar{K}\right\rangle^2\langle\bar{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left(\left\langle\underline{\hat{k}}^B\right\rangle^{ef}\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)+\frac{\Delta\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}_1\right)}{\left(1-\left\langle\bar{k}_1\right\rangle\right)\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\langle\bar{g}\rangle\right)Z\langle\bar{g}\rangle\end{aligned}$$

for bank returns.

22.1.1 Equation for investors' total capital per sector

Now, considering firms with slowly decreasing returns to scale directly, equations (148) and (151) writes:

$$\pm\frac{\hat{N}\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}-\bar{r}'\simeq\int\left(1-\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)^{-1}\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right) \tag{154}$$
$$\times\left(\frac{A}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)\right)^2}+\frac{B}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)\right)^3}\right)$$
$$\times\left[\left(f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)-\bar{r}'\right)+\tau F\left(X'\right)\left(f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)-\left\langle f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)\right\rangle\right)\right]$$

with:

$$f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right) \tag{155}$$
$$\equiv\frac{f_1\left(X\right)}{\left(1+\frac{k(X)}{K[X]}\hat{K}\left[X\right]+\frac{k_1^B(X)}{K[X]}\bar{K}\left[\bar{X}\right]+\frac{\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]}{K[X]}\bar{K}\left[\bar{X}\right]\right)^r K_X^r}-C_0$$
$$\simeq\frac{f_1\left(X\right)}{\left(\left(1+k\left(X\right)\hat{K}\left[\hat{X}\right]\right)+\left(k_1^B\left(\bar{X}\right)+\kappa\left[\frac{\underline{k}_2^B\left(\bar{X}\right)}{1+\bar{k}}\right]\right)\bar{K}\left[\bar{X}\right]\right)^r}-C_0$$

Equation (155) shows that both investors and banks' capital are involved in the return equation.

22.1.2 Equation for banks' total capital per sector

The solution must be sought along with the solutions for banks' capital per sector. This one is expressed as:

$$\pm\frac{\bar{N}\left(\bar{X}_1\right)}{\sqrt{\bar{K}\left[\bar{X}_1\right]}}-\bar{r}' \tag{156}$$
$$=\left(1-\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\bar{S}_1^E\left(\bar{X}',\hat{X}_1\right)+\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right)\right)^{-1}$$
$$\times\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\frac{1-\beta^B}{\left(1-\beta\right)\delta}\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}-\hat{S}_1^E\left(\hat{X}',\bar{X}_1\right)\right)+\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)\right)$$
$$\times\left(1-\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1}\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\left(\frac{A}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)\right)^2}+\frac{B}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)\right)^3}\right)$$
$$\times\left[\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-\bar{r}'\right)+\tau F\left(X'\right)\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-\left\langle f_1\left(X,\hat{K}\left[\hat{X}\right]\right)\right\rangle\right)\right]$$

with relation:

$$\left(1-\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\bar{S}_1^E\left(\bar{X}',\hat{X}_1\right)+\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right)\right)\left(\pm\frac{\bar{N}\left(\bar{X}_1\right)}{\sqrt{\bar{K}\left[\bar{X}_1\right]}}-\bar{r}'\right) \tag{157}$$
$$= \left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\frac{1-\beta^B}{\left(1-\beta\right)\delta}\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}-\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)+\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)\right)\left(\pm\frac{\hat{N}\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}-\bar{r}'\right)$$

As in Part 1, we search for approximate solutions as corrections to a state where investors are not connected, thus serving as a benchmark.

22.1.3 Interpretation

The interpretation remains the same as in Part 1. The interconnections between agents generally result in multiple solutions for total capital per sector. These solutions now involve both banks and investors. Depending on the share of loans in bank activity, the bank may remain relatively unaffected by diffusion. However, since the total bank capital is correlated with the investors' level of capital, these agents also experience multiple equilibria. We will now study approximate solutions to the capital equations.

23 Total capital of financial agents per sector

We will derive approximate solutions for the total capital of investors and banks, focusing directly on firms with decreasing returns to scale productivity.

23.1 Isolated investors and banks

The benchmark is directly obtained by solving:

$$\pm\frac{\hat{N}\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}-\bar{r}' \simeq \left(\frac{A}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)\right)^2}+\frac{B}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)\right)^3}\right) \tag{158}$$
$$\times\left[\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-\bar{r}'\right)+\tau F\left(X'\right)\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-\left\langle f_1\left(X,\hat{K}\left[\hat{X}\right]\right)\right\rangle\right)\right]$$

and:

$$\pm\frac{\bar{N}\left(\bar{X}_1\right)}{\sqrt{\bar{K}\left[\bar{X}_1\right]}}-\bar{r}' = \left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\frac{1-\beta^B}{\left(1-\beta\right)\delta}\right)\left(\frac{A}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)\right)^2}+\frac{B}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)\right)^3}\right)$$
$$\times\left[\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-\bar{r}'\right)+\tau F\left(X'\right)\left(f_1\left(X',\hat{K}\left[\hat{X}'\right]\right)-\left\langle f_1\left(X,\hat{K}\left[\hat{X}\right]\right)\right\rangle\right)\right]$$

and these solutions have the same form and interpretation as in Part 1.

Note that in this benchmark, a direct relationship has been established between the capital of the banking sector and the capital of investors of the same sector:

$$\frac{\bar{N}\left(\bar{X}_1\right)}{\sqrt{\bar{K}\left[\bar{X}_1\right]}}-\bar{r}' = \pm\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\frac{1-\beta^B}{\left(1-\beta\right)\delta}\right)\left(\frac{\hat{N}\left(\hat{X}_1\right)}{\sqrt{\hat{K}\left[\hat{X}_1\right]}}-\bar{r}'\right)$$

This relationship is proportional to $\frac{1-\beta^B}{(1-\beta)\delta}$, so that the constraint between banks and investors operating in the same isolated sector is only valid if the bank takes participation in the investors.

23.2 Interconnected financial agents

We consider (156) and (157) simultaneously, and find corrections to the solutions for isolated agents.

23.2.1 General approach

We denote $\Delta\left(\frac{N(\hat{X})}{\sqrt{\hat{K}[\hat{X}]}}\right)$ as the correction due to interactions between investors, compared to a state without interactions between sectors. Additionally, we define $\hat{K}_1\left[\hat{X}\right]$ as the solution without interactions between sectors. Hence, we can express:

$$\frac{\hat{N}\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}=\frac{\hat{N}\left(\hat{X}\right)}{\sqrt{\hat{K}_1\left[\hat{X}\right]}}+\Delta\left(\frac{\hat{N}\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\right)$$

$$\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}=\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}_1\left[\bar{X}\right]}}+\Delta\left(\frac{\bar{N}\left(\hat{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right)$$

The sought-after solution is therefore the sum of the benchmark plus the correction to this benchmark due to interactions.

A second-order expansion straightforwardly yields an approximate equation for $\Delta\left(\frac{\hat{N}(\hat{X})}{\sqrt{\hat{K}[\hat{X}]}}\right)$ and $\Delta\left(\frac{\bar{N}(\hat{X})}{\sqrt{\bar{K}[\bar{X}]}}\right)$:

$$\begin{aligned}
&\int\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\hat{X}\right)}-\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}}\right)-\int\left(\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}'\left(\hat{X}'\right)}}\right)\\
=\;& H_{\hat{1}}\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}}\right)+\frac{1}{2}H_{\hat{2}}\left(\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}}\right)\right)^2+H_{\bar{1}}\Delta\left(\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right)+\frac{1}{2}H_{\bar{2}}\left(\Delta\left(\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right)\right)^2\\
&+H_{\hat{1}\bar{1}}\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}}\right)\Delta\left(\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right)
\end{aligned}$$

where $H_{\hat{l}}$ are derivatives of:

$$\begin{aligned}
&\left(\frac{A}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)\right)^2}+\frac{B}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)\right)^3}\right)\\
&\times\left[\left(f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)-\bar{r}'\right)+\tau F\left(X'\right)\left(f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)-\left\langle f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)\right\rangle\right)\right]
\end{aligned}$$

with respect to $\hat{K}\left[\hat{X}\right]$, $H_{\bar{l}}$ represents the derivatives with respect to $\bar{K}\left[\bar{X}\right]$, and $H_{\hat{1}\bar{1}}$ is the cross derivative.

The constraint between the two variations is given by:

$$\left(1-\left(1+\bar{k}_{2}^{n}\left(\bar{X}_{1}\right)\right)\left(\bar{S}_{1}^{E}\left(\bar{X}',\hat{X}_{1}\right)+\bar{S}_{1}^{B}\left(\bar{X}',\bar{X}_{1}\right)\right)\right)\Delta\left(\frac{\bar{N}\left(\bar{X}_{1}\right)}{\sqrt{\bar{K}\left[\bar{X}_{1}\right]}}\right) \tag{159}$$

$$= \pm\left(1+\bar{k}_{2}^{n}\left(\bar{X}_{1}\right)\right)\left(\frac{1-\beta^{B}}{\left(1-\beta\right)\delta}\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\left[\hat{\underline{k}}_{2}^{n}\left(\hat{X}\right)\right]_{\kappa}}-\hat{S}_{1}^{E}\left(\hat{X}',\bar{X}_{1}\right)\right)+\hat{S}_{1}^{B}\left(\hat{X}',\bar{X}_{1}\right)\right)\Delta\left(\frac{\hat{N}\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\right)$$

and the solution for $\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}\left[\hat{X}'\right]}}\right)$ is, in first approximation:

$$\left(\delta\left(\hat{X}-\hat{X}'\right)\pm_{\hat{X}}\left[H\right]\left(\hat{X}',\hat{X}\right)\right)\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}\left[\hat{X}'\right]}}\right)=\frac{\left(1-H_{1}\right)\pm_{\hat{X}}\left[L\right]\left(\hat{X}\right)}{H_{2}} \tag{160}$$

with:

$$\left[H\right]\left(\hat{X}',\hat{X}\right)=\frac{2H_{2}\left(\left(1+\hat{\underline{k}}_{2}\left(\hat{X}\right)\right)\hat{S}_{1}^{E}\left(\hat{X}',\hat{X}\right)-H_{\hat{1}\bar{1}}\Delta\left(\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right)\right)}{\left(\left(1-H_{1}\right)^{2}-2H_{2}\left(1+\hat{\underline{k}}_{2}\left(\hat{X}\right)\right)\left(Q\left(\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right)+\int\hat{S}_{1}^{E}\left(\hat{X}',\hat{X}\right)\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}_{1}\left[\hat{X}'\right]}}\right)\right)^{\frac{3}{2}}}$$

$$\left[L\right]\left(\hat{X}\right)=\sqrt{\left(1-H_{1}\right)^{2}-\left(2H_{2}\left(1+\hat{\underline{k}}_{2}\left(\hat{X}\right)\right)\left(Q\left(\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right)+\int\hat{S}_{1}^{E}\left(\hat{X}',\hat{X}\right)\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}_{1}\left[\hat{X}'\right]}}\right)\right)}$$

and:

$$Q\left(\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right)=H_{\bar{1}}\Delta\left(\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right)+\frac{1}{2}H_{\bar{2}}\left(\Delta\left(\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right)\right)^{2}$$

where the sign $\pm_{\hat{X}}$ indicates that the choice of sign depends on the sector considered, with two possibilities per sector. Moreover, the equation is non-local; the first line indicates diffusion (presence of $\hat{S}_{1}^{E}\left(\hat{X}',\hat{X}\right)$), and thus the choice of sign in one sector impacts the other sectors. The withdrawal of capital invested in one sector will impact the other sectors connected to it, through investment decisions among investors. As a result, entire blocks can end up in a low-capital or high-capital state. The diffusion and amplification effect depend on the level of indebtedness $1+\hat{\underline{k}}_{2}\left(\hat{X}\right)$. It represents diffusion amplified by the level of lending among investors.

Using (159), $Q\left(\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right)$ can be replaced by its average in (160). Given (159), this can be estimated as:

$$\left\langle \Delta\left(\frac{\bar{N}\left(\bar{X}_1\right)}{\sqrt{\bar{K}\left[\bar{X}_1\right]}}\right)\right\rangle \tag{161}$$
$$= \pm\frac{\left\langle \left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\frac{1-\beta^B}{(1-\beta)\delta}\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}-\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)+\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)\right)\right\rangle}{\left\langle\left(1-\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\bar{S}_1^E\left(\bar{X}',\hat{X}_1\right)+\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right)\right)\right\rangle}\left\langle\Delta\left(\frac{\hat{N}\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\right)\right\rangle$$

The averages are computed via integration. As a consequence:

$$Q\left(\left\langle\Delta\left(\frac{\bar{N}\left(\bar{X}_1\right)}{\sqrt{\bar{K}\left[\bar{X}_1\right]}}\right)\right\rangle\right)\simeq H_{\bar{1}}\left\langle S\right\rangle\left\langle\Delta\left(\frac{\hat{N}\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\right)\right\rangle+\frac{1}{2}H_{\bar{2}}\left\langle S\right\rangle^2\left(\left\langle\Delta\left(\frac{\hat{N}\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\right)\right\rangle\right)^2$$

with:

$$\begin{aligned}\left\langle S\right\rangle &= \pm\frac{\left\langle \left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\frac{1-\beta^B}{(1-\beta)\delta}\left(\frac{1}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}-\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)+\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)\right)\right\rangle}{\left\langle\left(1-\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\bar{S}_1^E\left(\bar{X}',\hat{X}_1\right)+\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right)\right)\right\rangle}\\ &\rightarrow \frac{\frac{1-\beta^B}{(1-\beta)\delta}\left(1-\left\langle\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right\rangle\right)+\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)}{\left\langle\left(1-\left(\bar{S}_1^E\left(\bar{X}',\hat{X}_1\right)+\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right)\right)\right\rangle}\end{aligned}$$

23.2.2 First approximation

In first approximation, we can consider banks primarily as lenders. In this case:

$$\begin{aligned}1-\beta^B &< <1\\ \left\langle\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right\rangle &= \bar{r}'\\ \left\langle S\right\rangle &= 0\end{aligned}$$

Moreover, given that $\left\langle\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right\rangle$ is constant, we can deduce:

$$\Delta\left(\frac{\bar{N}\left(\bar{X}\right)}{\sqrt{\bar{K}\left[\bar{X}\right]}}\right)\simeq 0$$

The equation becomes:

$$\left(\delta\left(\hat{X}-\hat{X}'\right)\pm_{\hat{X}}\frac{2H_2\left(\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)}{\left(\left(1-H_1\right)^2-2H_2\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\int\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}_1\left[\hat{X}'\right]}}\right)^{\frac{3}{2}}}\right)\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}\left[\hat{X}'\right]}}\right)$$
$$=\frac{\left(1-H_1\right)\pm_{\hat{X}}\sqrt{\left(1-H_1\right)^2-\left(2H_2\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\int\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}_1\left[\hat{X}'\right]}}\right)}}{H_2}$$

In this simple case, the equation for the corrections $\Delta\left(\frac{N(\hat{X}')}{\sqrt{\hat{K}[\hat{X}']}}\right)$ is similar to Part 1. However, given the form of:

$$f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)\simeq\frac{f_1(X)}{\left(\left(1+k(X)\hat{K}\left[\hat{X}\right]\right)+\left(k_1^B(\bar{X})+\kappa\left[\frac{k_2^B(\bar{X})}{1+\bar{k}}\right]\right)\bar{K}\left[\bar{X}\right]\right)^r}-C_0$$

the derivatives of:

$$\left(\frac{A}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)\right)^2}+\frac{B}{\left(f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)\right)^3}\right)$$
$$\times\left[\left(f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)-\bar{r}'\right)+\tau F(X')\left(f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)-\left\langle f_1\left(X',\hat{K}\left[\hat{X}'\right],\bar{K}\left[\bar{X}'\right]\right)\right\rangle\right)\right]$$

decrease in amplitude with $\bar{K}\left[\bar{X}\right]$. In appendix 21.3 we show that $\hat{S}_1^E\left(\hat{X}',\hat{X}\right)$ has the same expression as in Part 1.

23.2.3 Interpretation of solutions

On average:

$$\left\langle\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right\rangle\simeq\left\langle\hat{k}\right\rangle\left(1-\left\langle\hat{k}_1\right\rangle\right)+\left\langle\hat{k}_1\right\rangle$$

but the normalization of coefficients in Part 1:

$$\hat{k}_\eta\left(\hat{X}',\hat{X}\right)\rightarrow\frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)}$$

and in Part 2:

$$\hat{k}_\eta\left(\hat{X}',\hat{X}\right)\rightarrow\frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle-\left(\left\langle\hat{k}_1^B\left(\hat{X}',\bar{X}\right)\right\rangle+\kappa\frac{\left\langle\hat{k}_2^B(\hat{X}',\bar{X})\right\rangle}{1+\left\langle\bar{k}\right\rangle\left\langle\bar{K}\right\rangle}\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)}$$

reveals that the diffusion matrix $\left\langle\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right\rangle$, including banks, has a higher amplitude than in Part 1. Thus, when comparing the amplitude of corrections in Part 1 and 2, two competing effects arise. Banks increase the disponable capital to firms, thereby somewhat reducing access to other investors. The amount of capital lent to firms reaches a level such that marginal returns stabilize around low value, stablizing the system. On the other hand, the banks also lend capital to investors, which may increase the leverage effect for their disposable capital. This increases instability and the discrepancies between the various collective states. However, the amplification due to the diffusion matrix remains constant, depending on the leverage effect of investors and banks, while the stabilization effect of loans to firms by banks may increase with the size of the banking system. Consequently, there should be a level of bank capital at which the stabilization effect outweighs the diffusion effect.

This result suggests that the multiplicity of global averages of capital should be reduced by introducing banks into the system. This point is studied in the next paragraph.

24 Global average returns and total capital for financial agents

24.1 Global average returns

The system is closed by computing the remaining average quantities: firms' capital and returns, investors' and banks' average returns. Appendices 23.2 and 23.3 shows that the investors' returns are:

$$\begin{aligned}\langle \hat{g} \rangle \quad &\simeq \quad \bar{r}' + \left(1 - \left(1 + \left\langle \hat{k}_2^n \right\rangle\right) \left(\left\langle \hat{k} \right\rangle \left(1 - \left\langle \hat{k}_1 \right\rangle\right) + \left\langle \hat{k}_1 \right\rangle\right)\right)^{-1} \\ &\times \left(1 + \left\langle \hat{k}_2^n \right\rangle\right) \left\langle \left(\frac{A}{\left(f_1\left(X, \hat{K}[X], \bar{K}[X]\right)\right)^2} + \frac{B}{\left(f_1\left(X, \hat{K}[X], \bar{K}[X]\right)\right)^3} \right) R \right\rangle\end{aligned} \tag{162}$$

and that the bank returns are:

$$\begin{aligned}\langle \bar{g} \rangle \quad &= \quad \bar{r}' + \frac{\left(1 - \beta^B\right)}{(1-\beta)\langle \delta \rangle} \left[\left(1 - \left(\left((1 - \langle \bar{k}_1 \rangle) \langle \bar{k} \rangle + \langle \bar{k}_1 \rangle \right) + \left\langle \underline{\hat{k}}_1^B \right\rangle \left(\left\langle \underline{\hat{k}}_1^B \right\rangle + \kappa \frac{\left\langle \underline{\hat{k}}_2^B \right\rangle}{1 + \underline{\bar{k}}} \left(1 - \frac{\kappa}{1 + \underline{\bar{k}}} \right) \right) \right) \right) \right]^{-1} \\ &\times \left(\frac{1}{\left\langle \hat{k}^B \right\rangle} - \left(\left(\left\langle \hat{k} \right\rangle \left(1 - \left\langle \hat{k}_1 \right\rangle \right) + \left\langle \hat{k}_1 \right\rangle \right) - \frac{(1-\beta)\langle \delta \rangle}{1 - \beta^B} \left(\left\langle \hat{k}_1^B \right\rangle \left(1 - \left\langle \hat{k} \right\rangle \right) \right) \right) \right) \left(\langle \hat{g} \rangle - \bar{r}' \right)\end{aligned} \tag{163a}$$

24.2 Global average total capital for financial agents

The equation for average capital is solved by combining (141), (142), (162), and (163a). To gain some insight into the solutions, we make several approximations in Appendix 23. Firstly, we consider that $1 - \beta^B << 1$, so that:

$$\langle \bar{g} \rangle = \bar{r}'$$

This corresponds to approximating the banks as lenders primarily. Then, we consider that $\delta >> 1$, which corresponds to considering that banks loans are dominant due to the credit multiplier. Ultimately, we loosely consider that interaction terms $\frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle}$ can be neglected. They can be reintroduced in a more refined approximation. Under these assumptions, the system reduces to the following three equations:

$$\begin{aligned}\left\langle \hat{K} \right\rangle \left\| \hat{\Psi} \right\|^2 &= \frac{9\sigma_{\hat{K}}^2}{2\hat{\mu} \langle \hat{g} \rangle^2} V \left\| \hat{\Psi}_0 \right\|^4 \\ \left\langle \bar{K} \right\rangle \left\| \bar{\Psi} \right\|^2 &\simeq \frac{9\sigma_{\hat{K}}^2 \langle \hat{g} \rangle^4}{2 (\bar{r}')^2 \hat{\mu}} \left\| \bar{\Psi}_0 \right\|^4\end{aligned} \tag{164}$$

$$\begin{aligned}\langle \hat{g} \rangle \quad &\simeq \quad \bar{r}' + \left(1 - \left(1 + \left\langle \hat{k}_2^n \right\rangle\right) \left(\left\langle \hat{k} \right\rangle \left(1 - \left\langle \hat{k}_1 \right\rangle\right) + \left\langle \hat{k}_1 \right\rangle\right)\right)^{-1} \\ &\times \left(1 + \left\langle \hat{k}_2^n \right\rangle\right) \frac{\epsilon (1-\beta) C (X + C\beta)^2}{3\sigma_{\hat{K}}^2} \left\langle \frac{R(3X - \beta C)(\beta C + X)}{4 f_1^2(X) C (2X - C\beta)} + \frac{\beta R}{f_1^3(X)} \right\rangle\end{aligned}$$

Appendix 23.4 shows that this reduces to an equation for $\left\langle \hat{K} \right\rangle \left\| \hat{\Psi} \right\|^2$:

$$\frac{\sqrt{\frac{9\sigma_{\hat{K}}^2}{2\hat{\mu}} V} \left\| \hat{\Psi}_0 \right\|^2}{\sqrt{\left\langle \hat{K} \right\rangle \left\| \hat{\Psi} \right\|^2}} - \bar{r}' \simeq \left(1 - \left(1 + \left\langle \hat{k}_2^n \right\rangle\right) \left(\left\langle \hat{k} \right\rangle \left(1 - \left\langle \hat{k}_1 \right\rangle\right) + \left\langle \hat{k}_1 \right\rangle\right)\right)^{-1} \left(1 + \left\langle \hat{k}_2^n \right\rangle\right) \times \left\langle \left(\frac{A}{\left(f_1\left(X, \hat{K}[X], \bar{K}[X]\right)\right)^2} + \frac{B}{\left(f_1\left(X, \hat{K}[X], \bar{K}[X]\right)\right)^3} \right) R \right\rangle \tag{165}$$

$$\left\langle f_1\left(X, \hat{K}[X], \bar{K}[X]\right) \right\rangle = \frac{\left\langle f_1(X) \right\rangle}{\left(1 + \langle k \rangle \left\langle \hat{K} \right\rangle \left\| \hat{\Psi} \right\|^2 + \left(\left\langle k_1^B \right\rangle + \kappa \left[\frac{k_2^B}{1+\bar{k}}\right]\right) \left\langle \bar{K} \right\rangle \left\| \bar{\Psi} \right\|^2\right)^r} - C_0$$

Given that the right-hand side of (165) depends on $\left\langle \hat{K} \right\rangle \left\| \hat{\Psi} \right\|^2$, we can expect that this equation has multiple solutions. However, as mentioned in the previous section, the correction to average capital $\Delta \left(\frac{N(\hat{X}')}{\sqrt{\hat{K}[\hat{X}']}} \right)$ depends negatively on $\bar{K}[\bar{X}]$, suggesting that banks should reduce the possibilities of multiple global averages. This point appears to be confirmed by some numerical studies.

25 Several groups of agents

We follow the same process as in Part 1. Instead of considering the system averages as a whole, we will consider subgroups and the interactions between these subgroups. We can consider that within sector spaces, investors are organized into heterogeneous groups or sub-markets, relatively weakly connected, as before. We directly consider several different groups. Each group is homogeneous and contains banks, investors, and firms, with approximately homogeneous returns, connections, capital, and production.

25.1 Investors: return equations for group averages

Let us now define $\underline{\hat{k}}_\eta^{[ii]}$, $\underline{k}_\eta^{[ii]}$, the average coefficients within the group, and $\underline{\hat{k}}_\eta^{[ji]}$ and $\underline{k}_\eta^{[ji]}$ the average connections from i to j. We define the total share of capital invested in sector i, comprising intra-sectoral investments (from i to i), and inter-sectoral investments (from j to i), as:

$$\underline{\hat{k}}_\eta^{[i]} = \underline{\hat{k}}_\eta^{[ii]} + \underline{\hat{k}}_\eta^{[ij]}$$

in which the sum over the indices is understood. The return equation involving several groups becomes in terms of matrices:

$$
\begin{aligned}
0 \quad = \quad & \begin{pmatrix} 1-\underline{\hat{S}}_1^{[ii]} & -\underline{\hat{S}}_1^{[ji]} \\ -\underline{\hat{S}}_1^{[ij]} & 1-\underline{\hat{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \frac{f^{[i]}-\bar{r}}{1+\underline{\hat{k}}_2^{[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]^{[i]}} \\ \frac{f^{[j]}-\bar{r}}{1+\underline{\hat{k}}_2^{[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]^{[j]}} \end{pmatrix} \\
& -\begin{pmatrix} \underline{\hat{S}}_2^{[ii]} & \underline{\hat{S}}_2^{[ji]} \\ \underline{\hat{S}}_2^{[ij]} & \underline{\hat{S}}_2^{[jj]} \end{pmatrix} \begin{pmatrix} \frac{1+f^{[i]}}{\underline{\hat{k}}_2^{[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]^{[i]}} H\left(-\left(1+f^{[i]}\right)\right) \\ \frac{1+f^{[j]}}{\underline{\hat{k}}_2^{[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]^{[j]}} H\left(-\left(1+f^{[j]}\right)\right) \end{pmatrix} \\
& -\begin{pmatrix} \underline{S}_2^{[ii]} & \underline{S}_2^{[ji]} \\ \underline{S}_2^{[ij]} & \underline{S}_2^{[jj]} \end{pmatrix} \begin{pmatrix} \frac{1+f_1'^{[i]}}{\underline{k}_2^{[i]}+\left[\frac{\underline{k}_2^B}{1+\underline{k}}\right]^{[i]}} H\left(-\left(1+f_1'^{[i]}\right)\right) \\ \frac{1+f_1'^{[j]}}{\underline{k}_2^{[j]}+\left[\frac{\underline{k}_2^B}{1+\underline{k}}\right]^{[j]}} H\left(-\left(1+f_1'^{[j]}\right)\right) \end{pmatrix} - \begin{pmatrix} \underline{S}_1^{[ii]} & \underline{S}_1^{[ji]} \\ \underline{S}_1^{[ij]} & \underline{S}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \frac{f_1'^{[i]}-\bar{r}}{1+\underline{k}_2^{[i]}+\left[\frac{\underline{k}_2^B}{1+\underline{k}}\right]^{[i]}} \\ \frac{f_1'^{[j]}-\bar{r}}{1+\underline{k}_2^{[j]}+\left[\frac{\underline{k}_2^B}{1+\underline{k}}\right]^{[j]}} \end{pmatrix}
\end{aligned} \tag{166}
$$

with several constraints between the coefficients detailed in Appendix 24.2.

As already stated in Part 1, the diagonal of the matrices represents intra-sectoral investments. The off-diagonal elements of the matrix represent inter-sectoral investments. The independent collective states are thus defined by setting the non-diagonal matrices $\underline{\hat{S}}_{\eta}^{[ji]}=0$ and $\underline{S}_{\eta}^{[ji]}=0$. The corrections due to interactions between groups are obtained by taking into account the off-diagonal elements. This enables the study of default transmission between different groups. The transmission of default from one group, j, to other groups varies depending on the type of agent that defaults. The transmission of default from investors in sector j to sector i, arises from the term:

$$\frac{1+f^{[j]}}{\underline{\hat{k}}_2^{[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]^{[j]}} H\left(-\left(1+f^{[j]}\right)\right) \tag{167}$$

while the transmission of firms' default in sector j is initiated by the term:

$$\frac{1+f_1'^{[j]}}{\underline{k}_2^{[j]}+\left[\frac{\underline{k}_2^B}{1+\underline{k}}\right]^{[j]}} H\left(-\left(1+f_1'^{[j]}\right)\right) \tag{168}$$

This transmission depends on the off-diagonal coefficients in equation (166). These coefficients represent the diffusion of returns from j to i, and thus the loss due to default (167) or (168) will modify the returns downward in sector i, potentially leading to the creation of another default. Moreover, splitting into groups allows us to see, based on the connections between sector groups, which groups will default and which will remain unaffected.

25.2 Bank returns' equation for group averages

For banks, the return equation with intra and inter-group interactions writes:

$$
\begin{aligned}
0 \quad = \quad & \begin{pmatrix} 1-\underline{\bar{S}}_1^{[ii]} & -\underline{\bar{S}}_1^{[ji]} \\ -\underline{\bar{S}}_1^{[ij]} & 1-\underline{\bar{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \frac{\bar{f}^{[i]}-\bar{r}}{1+\underline{\bar{k}}^{[i]}} \\ \frac{\bar{f}^{[j]}-\bar{r}}{1+\underline{\bar{k}}^{[j]}} \end{pmatrix} - \begin{pmatrix} \underline{\hat{S}}_1^{[ii]} & \underline{\hat{S}}_1^{[ij]} \\ \underline{\hat{S}}_1^{[ij]} & \underline{\hat{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \frac{\hat{f}^{[i]}-\bar{r}}{1+\underline{\hat{k}}_2^{[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}} \\ \frac{\hat{f}^{[j]}-\bar{r}}{1+\underline{\hat{k}}_2^{[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[j]}} \end{pmatrix} \\
& - \begin{pmatrix} \underline{\bar{S}}_2^{[ii]} & \underline{\bar{S}}_2^{[ji]} \\ \underline{\bar{S}}_2^{B[ij]} & \underline{\bar{S}}_2^{B[jj]} \end{pmatrix} \begin{pmatrix} \frac{1+\bar{f}^{[i]}}{\underline{\bar{\bar{k}}}_2^{[i]}} H\left(-\left(1+\bar{f}^{[i]}\right)\right) \\ \frac{1+\bar{f}^{[j]}}{\underline{\bar{\bar{k}}}_2^{[j]}} H\left(-\left(1+\bar{f}^{[j]}\right)\right) \end{pmatrix} - \begin{pmatrix} \underline{\hat{S}}_2^{B[ii]} & \underline{\hat{S}}_2^{B[ji]} \\ \underline{\hat{S}}_2^{B[ij]} & \underline{\hat{S}}_2^{B[jj]} \end{pmatrix} \begin{pmatrix} \frac{1+\hat{f}^{[i]}}{\underline{\hat{k}}_2^{[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}} H\left(-\left(1+\hat{f}^{[i]}\right)\right) \\ \frac{1+\hat{f}^{[j]}}{\underline{\hat{k}}_2^{[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[j]}} H\left(-\left(1+\hat{f}^{[j]}\right)\right) \end{pmatrix} \\
& - \begin{pmatrix} \underline{S}_2^{B[ii]} & \underline{S}_2^{B[ji]} \\ \underline{S}_2^{B[ij]} & \underline{S}_2^{B[jj]} \end{pmatrix} \begin{pmatrix} \frac{1+f_1'^{[i]}}{\underline{k}_2^{[i]}+\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]^{[i]}} H\left(-\left(1+f_1'^{[i]}\right)\right) \\ \frac{1+f_1'^{[j]}}{\underline{k}_2^{[j]}+\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]^{[j]}} H\left(-\left(1+f_1'^{[j]}\right)\right) \end{pmatrix} - \begin{pmatrix} \underline{S}_1^{B[ii]} & \underline{S}_1^{B[ji]} \\ \underline{S}_1^{B[ij]} & \underline{S}_1^{B[jj]} \end{pmatrix} \begin{pmatrix} f_1^{[i]}-\bar{r} \\ f_1^{[i]}-\bar{r} \end{pmatrix}
\end{aligned} \tag{169}
$$

25.3 Fields and effective action

In the case of multiple groups, we consider that the field for investors decomposes into several independent and interacting components. We can replicate what we just did in Part 1 and introduce:

$$\left[\hat{\Psi}\right]\left(\left\{\hat{K}_{r_i},\hat{X}_{r_i}\right\}_{G_i}\right)$$

which stands for a set of field of investors, one defined for each group $\left\{\hat{K}_{r_i},\hat{X}_{r_i}\right\}_{G_i}$. Similarly, we define a set of fields for banks:

$$[\bar{\Psi}]\left(\left\{\bar{K}_{r_i},\bar{X}_{r_i}\right\}_{G_i}\right)$$

We revisit the functional action of each group from Part 1, adding the banks and introducing an interaction term between the different groups. Each group has its own sector-spatial extension, and the effective action is a generalization of the one previously contained.

$$
\begin{aligned}
& -\sum\left[\hat{\Psi}\right]\left(\left\{\hat{K}_{r_i},\hat{X}_{r_i}\right\}_{G_i}\right)\nabla_{\hat{K}_{r_i}}\left(\nabla_{\hat{K}_{r_i}}-\hat{K}_{r_i}f^{[i]}\right)\left[\hat{\Psi}\right]^{\dagger}\left(\left\{\hat{K}_{r_i},\hat{X}_{r_i}\right\}_{G_i}\right) \\
& -\sum[\bar{\Psi}]\left(\left\{\bar{K}_{r_i},\bar{X}_{r_i}\right\}_{G_i}\right)\nabla_{\bar{K}_{r_i}}\left(\nabla_{\bar{K}_{r_i}}-\hat{K}_{r_i}f^{[i]}\right)[\bar{\Psi}]^{\dagger}\left(\left\{\bar{K}_{r_i},\bar{X}_{r_i}\right\}_{G_i}\right) \\
& +\sum\prod\left|\left[\hat{\Psi}\right]\left(\left\{\hat{K}_{r_i},\hat{X}_{r_i}\right\}_{G_i}\right)\right|^2\delta\left(\hat{V}\left(\left(\left[\hat{\Psi}\right]\left(\left\{\hat{K}_{r_i},\hat{X}_{r_i}\right\}_{G_i}\right)\right)\right)\right) \\
& +\sum\prod\left|\left[\hat{\Psi}\right]\left(\left\{\hat{K}_{r_i},\hat{X}_{r_i}\right\}_{G_i}\right)\right|^2\delta\left(\bar{V}\left([\bar{\Psi}]\left(\left\{\bar{K}_{r_i},\bar{X}_{r_i}\right\}_{G_i}\right)\right)\right) \\
& -\Xi^{\dagger}\left(\hat{X},\delta f_1\right)\sigma_{\delta f_1}^2\nabla_{\delta f_1}^2\Xi\left(\hat{X},\delta f_1\right)+\Xi^{\dagger}\left(\hat{X},\delta f_1\right)J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)+J^{\dagger}\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\Xi\left(\hat{X},\delta f_1\right) \\
& -\bar{\Xi}^{\dagger}\left(\hat{X},\delta\bar{f}_1\right)\sigma_{\delta f_1}^2\nabla_{\delta f_1}^2\bar{\Xi}\left(\hat{X},\delta\bar{f}_1\right)+\bar{\Xi}^{\dagger}\left(\hat{X},\delta\bar{f}_1\right)J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)+J^{\dagger}\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\bar{\Xi}\left(\hat{X},\delta\bar{f}_1\right)
\end{aligned}
$$

We have introduced a field $\bar{\Xi}\left(\hat{X},\delta\bar{f}_1\right)$ to account for excess returns in the banking sector, and a potential $\bar{V}\left([\bar{\Psi}]\left(\left\{\bar{K}_{r_i},\bar{X}_{r_i}\right\}_{G_i}\right)\right)$ to implement the return equation for banks.

25.3.1 Excess returns' equation and potential

We assume that agents belonging to different groups and linked by return equations (166) and (169) are weakly connected. As before, their interactions are modeled by intra-group potentials plus some inter-group potentials.

Discarding defaults, the intra-group interaction potentials are obtained through an expansion of (166) and (169) by replacing $f_a^{[i]} \rightarrow f_a^{[i]} + \delta f_a^{[i]}$ and $\bar{f}^{[i]} \rightarrow \bar{f}^{[i]} + \delta \bar{f}^{[i]}$ and keeping diagonal terms. The constraint (166) between excess returns of investors divided in several blocks is:

$$0 = \begin{pmatrix} \delta f^{[i]\prime} \\ \delta f^{[j]\prime} \end{pmatrix} - \begin{pmatrix} \hat{\underline{S}}_1^{[ii]} & \hat{\underline{S}}_1^{[ji]} \\ \hat{\underline{S}}_1^{[ij]} & \hat{\underline{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \delta f^{[i]} \frac{1-\left(\hat{\underline{S}}^{[ii]}+\hat{\underline{S}}^{[ij]}\right)}{1-\left(\hat{\underline{S}}_1^{[ii]}+\hat{\underline{S}}_1^{[ij]}\right)} \\ \delta f^{[j]} \frac{1-\left(\hat{\underline{S}}^{[jj]}+\hat{\underline{S}}^{[ij]}\right)}{1-\left(\hat{\underline{S}}_1^{[jj]}+\hat{\underline{S}}_1^{[ji]}\right)} \end{pmatrix} \tag{170}$$

and the constraint (169) between excess returns of banks divided in several blocks is:

$$0 = \begin{pmatrix} 1-\bar{\underline{S}}_1^{[ii]} & -\bar{\underline{S}}_1^{[ji]} \\ -\bar{\underline{S}}_1^{[ij]} & 1-\bar{\underline{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \frac{\delta \bar{f}^{[i]}}{1+\bar{\underline{k}}^{[i]}} \\ \frac{\delta \bar{f}^{[j]}}{1+\bar{\underline{k}}^{[j]}} \end{pmatrix} - \begin{pmatrix} \hat{\underline{S}}_2^{[ii]} & \hat{\underline{S}}_\eta^{[ij]} \\ \hat{\underline{S}}_\eta^{[ij]} & \hat{\underline{S}}_2^{[jj]} \end{pmatrix} \begin{pmatrix} \frac{\delta \hat{f}^{[i]}}{1+\hat{\underline{k}}_2^{[i]}+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\hat{k}}\right]^{[i]}} \\ \frac{\delta \hat{f}^{[j]}}{1+\hat{\underline{k}}_2^{[j]}+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\hat{k}}\right]^{[j]}} \end{pmatrix} \tag{171}$$

Once again, only excess returns are considered, since we assume that averages satisfy the constraint.

As in part 1 it leads to the intra-group potential for investors' returns:

$$\sum_i V_i\left(\hat{\Psi}, \hat{X}, K, \delta f_1\right) = \sum_i \delta\left[\delta f_{1i}\left(\hat{X}'\right) - \hat{\underline{S}}_1^{[ii]} \frac{1-\left(\hat{\underline{S}}^{[ii]}+\hat{\underline{S}}^{[ij]}\right)}{1-\left(\hat{\underline{S}}_1^{[ii]}+\hat{\underline{S}}_1^{[ij]}\right)} \frac{\delta f_{1j}'\left(\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)} dX'\right] \tag{172}$$

and for banks' returns:

$$\sum_i \bar{V}_i\left(\hat{\Psi}, \hat{X}, K, \delta \bar{f}^{[i]}\right) = \sum_i \delta\left(\delta \bar{f}^{[i]} - \bar{\underline{S}}_1^{[ii]} \frac{\delta \bar{f}^{[i]\prime}}{1+\bar{\underline{k}}^{[i]}} - \hat{\underline{S}}_1^{[ii]} \frac{\delta \hat{f}^{[i]\prime}}{1+\hat{\underline{k}}_2^{[i]}+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\hat{k}}\right]^{[i]}}\right) \tag{173}$$

These two potentials (172) and (173) impose a propagation of fluctuations $\delta \bar{f}^{[i]\prime}$ or $\delta \hat{f}^{[i]\prime}$ on the returns. When the interaction occurs, the fluctuations propagate and induce new excess returns through the matrices $\hat{\underline{S}}_1^{[ii]}$, $\bar{\underline{S}}_1^{[ii]}$, $\hat{\underline{S}}_2^{[ii]}$. Without default, the fluctuations propagate through the participations.

The inter-group interaction potentials, without default, are obtained by keeping the off-diagonal terms in (170) and (171), which translate into the following potentials, for investors:

$$\begin{aligned} &\sum_i W_i\left(\hat{\Psi}, \hat{X}, K, \delta f_{1i}\right) \\ = &\sum_i \delta\left(\delta f^{[i]} - \hat{\underline{S}}_1^{[ij]} \frac{1-\left(\hat{\underline{S}}^{[jj]}+\hat{\underline{S}}^{[ij]}\right)}{1-\left(\hat{\underline{S}}_1^{[jj]}+\hat{\underline{S}}_1^{[ji]}\right)} \delta f^{[j]\prime}\right) \end{aligned} \tag{174}$$

and for banks:

$$\sum_i \bar{W}\left(\hat{\Psi},\hat{X},K,\delta\bar{f}^{[i]}\right) \tag{175}$$
$$= \sum_i \delta\left(\delta\bar{f}^{[i]}-\underline{\bar{S}}_1^{[ji]}\frac{\delta\bar{f}^{[j]\prime}}{\left(1+\underline{\hat{k}}_2^{[j]}\right)}-\underline{\hat{S}}_1^{[ij]}\frac{\delta\hat{f}^{[j]\prime}}{\left(1+\underline{\hat{k}}_2^{[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}\right)}\right)$$

This is simlar to the inter-group constraints, with the exception that only different groups interact. The interaction between these groups can be considered relatively weak, so that the fluctuations coefficients $\underline{\bar{S}}_1^{[ji]}$ and $\underline{\hat{S}}_1^{[ij]}$ are relatively small.

25.4 Transition functions

25.4.1 Transition functions without interactions

The method is similar to Part 1. The transition functions without interactions for these agents are computed by inverting the operator, as in (29), for the dynamics of investor capital:

$$-\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}-\hat{K}f\left(\hat{X},K_{\hat{X}}\right)+\int\delta f_1\left|\Xi\left(\hat{X},\delta f_1\right)\right|^2 d\left(\delta f_1\right)\right)$$

and for the dynamics of banks:

$$-\nabla_{\bar{K}}\left(\frac{\sigma_{\bar{K}}^2}{2}\nabla_{\bar{K}}-\bar{K}f\left(\bar{X},K_{\bar{K}}\right)+\int\delta\bar{f}_1\left|\Xi\left(\bar{X},\delta\bar{f}_1\right)\right|^2 d\left(\delta\bar{f}_1\right)\right)$$

This yields the partial Green function conditioned to the initial state and final state for returns as:

$$\prod_{a=1,2}\sqrt{\left|\frac{f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}}{\sigma_{\hat{K}}^2\left(1-\exp\left(2\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)\Delta t\right)\right)}\right|} \tag{176}$$
$$\times\exp\left(\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)\frac{\left(K_a-\exp\left(\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)\Delta t\right)\hat{K}'\right)^2}{\sigma_{\hat{K}}^2\left(1-\exp\left(2\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)\Delta t\right)\right)}\right)$$
$$\times\exp\left(\frac{\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)}{\sigma_{\hat{K}}^2\left(1-\exp\left(2\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)\Delta t\right)\right)}\left(\hat{K}-\exp\left(\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)\Delta t\right)\hat{K}'\right)^2\right)$$

with:

$$K_1 = \hat{K}$$
$$K_2 = \bar{K}$$

We proceed in the same manner for the excess returns, and we invert the operator:

$$-\Xi^{\dagger}\left(\hat{X},\delta f_{a1}\right)\sigma_{\delta f_1}^2\nabla_{\delta f_1}^2\Xi\left(\hat{X},\delta f_{a1}\right)+\Xi^{\dagger}\left(\hat{X},\delta f_{a1}\right)J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)+J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\Xi\left(\hat{X},\delta f_{a1}\right)$$

and we obtain the partial transition function:

$$\prod_{a=1,2}\sqrt{\frac{1}{\sigma_{\delta f_1}^2}}\exp\left(-\frac{\left(\delta f_a^{[i]}-\delta f_a^{[i]\prime}\right)^2}{\sigma_{\delta f_1}^2}+J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\delta f_a^{[i]}-J\left(\hat{X}',K_{\hat{X}'},\mathbf{E}'\right)\delta f_a^{[i]\prime}\right) \tag{177}$$

So that the transition functions for an agent between a state with capital K and return δf_1 to a state with capital K' and return $\delta f_1'$ are:

$$\left\langle\left(K_a,\delta f_a^{[i]}\right)_a\left(K_a,\delta f_a^{[i]\prime}\right)_a\right\rangle \tag{178}$$

$$\begin{aligned}
&= \prod_{a=1,2}\sqrt{\left|\frac{f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}}{\sigma_{\hat{K}}^2\left(1-\exp\left(2\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)\Delta t\right)\right)}\right|}\\
&\times\exp\left(\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)\frac{\left(\hat{K}-\exp\left(\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)\Delta t\right)\hat{K}'\right)^2}{\sigma_{\hat{K}}^2\left(1-\exp\left(2\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)\Delta t\right)\right)}\right)\\
&\times\exp\left(\frac{\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)}{\sigma_{\hat{K}}^2\left(1-\exp\left(2\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)\Delta t\right)\right)}\left(\hat{K}-\exp\left(\left(f_a^{[i]}+\frac{\delta f_a^{[i]}+\delta f_a^{[i]\prime}}{2}\right)\Delta t\right)\hat{K}'\right)^2\right)\\
&\times\prod_{a=1,2}\sqrt{\frac{1}{\sigma_{\delta f_1}^2}}\exp\left(-\frac{\left(\delta f_a^{[i]}-\delta f_a^{[i]\prime}\right)^2}{\sigma_{\delta f_1}^2}+J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\delta f_a^{[i]}-J\left(\hat{X}',K_{\hat{X}'},\mathbf{E}'\right)\delta f_a^{[i]\prime}\right)\\
&\equiv G\left(\left(K_a,\delta f_a^{[i]}\right)_a,\left(K_a,\delta f_a^{[i]\prime}\right)_a\right)
\end{aligned} \tag{179}$$

As in Part 1, the transitions for a group of agents, investors, and banks, without interactions is the product of terms of the form (178):

$$\prod_l\left\langle\left(K_{al},\delta f_{al}^{[i]}\right)_{al}\left(K_{al},\delta f_{al}^{[i]\prime}\right)_{al}\right\rangle=\prod_l G\left(\left(K_{al},\delta f_{al}^{[i]}\right)_{al},\left(K_{al},\delta f_{al}^{[i]\prime}\right)_{al}\right)$$

25.4.2 Transition functions with interactions without default

The transitions are then driven both by intra-group (172) and (173), and inter-group (174) and (175) interactions. The potentials (172) and (173) are similar to those for a homogeneous group and illustrate how, within the same group, the excess returns of one agent impact those of others. Even an agent who has not initially undergone a return variation will experience one subsequently, due to the interaction term. The potentials (174) and (175) , on the other hand, show the transmission of a return variation from one block to another.

As for one homogeneous group, the transitions are computed by series of terms of the type:

$$G\left(\Delta t, f_{1,1}, \left\{\left(K_{a_1}, \delta f_{a_1}^{[i]}\right)_{a_1}\right\}_{G_1}, \left\{\left(K'_{a_1}, \delta f_{a_1}^{[i]\prime}\right)_{a_1}\right\}_{G_1}\right)... \tag{180}$$
$$\times G\left(\Delta t, f_{1,n}, \left\{\left(K_{a_n}, \delta f_{a_n}^{[i]}\right)_{a_n}\right\}_{G_n}, \left\{\left(K'_{a_n}, \delta f_{a_n}^{[i]\prime}\right)_{a_n}\right\}_{G_n}\right)$$
$$\times\left(V\left(\left(\left[\hat{\Psi}\right]\left(\left\{\hat{K}_{r_i}, \hat{X}_{r_i}\right\}_{G_i}\right)\right)\right)\right)$$
$$\times G\left(\Delta t, f_{1,1}, \left\{\left(K'_{a_1}, \delta f_{a_1}^{[i]\prime}\right)_{a_1}\right\}_{G_1}, \left\{\left(K''_{a_1}, \delta f_{a_1}^{[i]\prime\prime}\right)_{a_1}\right\}_{G_1}\right)...$$
$$\times G\left(\Delta t, f_{1,n}, \left\{\left(K''_{a_n}, \delta f_{a_n}^{[i]\prime\prime}\right)_{a_n}\right\}_{G_n}, \left\{\left(K''_{a_n}, \delta f_{a_n}^{[i]\prime\prime}\right)_{a_n}\right\}_{G_n}\right)$$

where:

$$V\left(\left(\left[\hat{\Psi}\right]\left(\left\{\hat{K}_{r_i}, \hat{X}_{r_i}\right\}_{G_i}\right)\right)\right)$$

stands for the intra- and inter-group interactions (172) and (173), and inter-group (174) and (175). The term (180) computes the diviation from the free transition function if we assume a punctual interaction.

The full transition function is thus the infinite series of products of the form (180), written compactly as:

$$\sum_n a_n \left[G...G\right] V_n \left[G...G\right] + ... \left[G...G\right] V_n \left[G...G\right] V_n ... \left[G...G\right] V_n \left[G...G\right]$$

where the terms a_n come from the expansion of the interaction potential in the effective action. Their specific form is irrelevant here. Again, in this compact formulation, V_n can take the form (172) and (173), and inter-group (174) and (175).

25.4.3 Interactions including defaults

As in Part 1, some defaults may arise in the process of transition. The default potentials:

$$V_i^D\left(\hat{\Psi}, \hat{X}, K, \delta f_1\right)$$

and:

$$V_{ij}^D\left(\hat{\Psi}, \hat{X}, K, \delta f_1\right)$$

are derived straightforwardly by including the terms describing the defaults of investors and banks in intra-group (172) and (173), and in inter-group (174) and (175) potentials. To do so, we rewrite the constraint for investors with default:

$$0 = \begin{pmatrix} \delta\hat{f}^{[i]} \\ \delta\hat{f}^{[j]} \end{pmatrix} - \begin{pmatrix} \underline{\hat{S}}_1^{[ii]} & \underline{\hat{S}}_1^{[ji]} \\ \underline{\hat{S}}_1^{[ij]} & \underline{\hat{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \delta\hat{f}^{[i]\prime}\frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}\right)} \\ \delta\hat{f}^{[j]\prime}\frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[jj]}+\underline{\hat{S}}_1^{[ji]}\right)} \end{pmatrix} \tag{181}$$
$$- \begin{pmatrix} \underline{\hat{S}}_2^{[ii]} & \underline{\hat{S}}_2^{[ji]} \\ \underline{\hat{S}}_2^{[ij]} & \underline{\hat{S}}_2^{[jj]} \end{pmatrix} \begin{pmatrix} \left(1+\hat{f}^{[i]\prime}+\delta\hat{f}^{[i]\prime}\right)\underline{\hat{s}}_2^{[i]} H\left(-\left(1+\hat{f}^{[i]\prime}+\delta\hat{f}^{[i]\prime}\right)\right) \\ \left(1+\hat{f}^{[j]\prime}+\delta\hat{f}^{[j]\prime}\right)\underline{\hat{s}}_2^{[i]} H\left(-\left(1+\hat{f}^{[j]\prime}+\delta\hat{f}^{[j]\prime}\right)\right) \end{pmatrix}$$

Similarly, we rewrite the constraint for banks (169) by including defaults:

$$0 = \begin{pmatrix} 1-\underline{\bar{S}}_1^{[ii]} & -\underline{\bar{S}}_1^{[ji]} \\ -\underline{\bar{S}}_1^{[ij]} & 1-\underline{\bar{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \frac{\delta \bar{f}^{[i]}}{1+\underline{\bar{k}}^{[i]}} \\ \frac{\delta \bar{f}^{[j]}}{1+\underline{\bar{k}}^{[j]}} \end{pmatrix} - \begin{pmatrix} \underline{\hat{S}}_2^{[ii]} & \underline{\hat{S}}_\eta^{[ij]} \\ \underline{\hat{S}}_\eta^{[ij]} & \underline{\hat{S}}_2^{[jj]} \end{pmatrix} \begin{pmatrix} \frac{\delta \hat{f}^{[i]}}{1+\underline{\hat{k}}_2^{[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}} \\ \frac{\delta \hat{f}^{[j]}}{1+\underline{\hat{k}}_2^{[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[j]}} \end{pmatrix} \tag{182}$$

$$- \begin{pmatrix} \underline{\bar{S}}_2^{[ii]} & \underline{\bar{S}}_2^{[ji]} \\ \underline{\bar{S}}_2^{[ij]} & \underline{\bar{S}}_2^{[jj]} \end{pmatrix} \begin{pmatrix} \left(1+\bar{f}^{[i]}+\delta\bar{f}^{[i]}\right)\underline{\hat{s}}_2^{[i]} H\left(-\left(1+\bar{f}^{[i]}+\delta\bar{f}^{[i]}\right)\right) \\ \left(1+\bar{f}^{[j]}+\delta\bar{f}^{[j]}\right)\underline{\hat{s}}_2^{[i]} H\left(-\left(1+\bar{f}^{[j]}+\delta\bar{f}^{[j]}\right)\right) \end{pmatrix}$$

$$- \begin{pmatrix} \underline{\hat{S}}_2^{B'[ii]} & \underline{\hat{S}}_2^{B'[ji]} \\ \underline{\hat{S}}_2^{B'[ij]} & \underline{\hat{S}}_2^{B'[jj]} \end{pmatrix} \begin{pmatrix} \left(1+\hat{f}^{[i]\prime}+\delta\hat{f}^{[i]\prime}\right)\underline{\hat{s}}_2^{[i]} H\left(-\left(1+\hat{f}^{[i]\prime}+\delta\hat{f}^{[i]\prime}\right)\right) \\ \left(1+\hat{f}^{[j]\prime}+\delta\hat{f}^{[j]\prime}\right)\underline{\hat{s}}_2^{[i]} H\left(-\left(1+\hat{f}^{[j]\prime}+\delta\hat{f}^{[j]\prime}\right)\right) \end{pmatrix}$$

And similarly to Part 1, we obtain the intra-group potentials with defaults for investors:

$$\sum_i V_i^D\left(\hat{\Psi},\hat{X},K,\delta f_1\right) = \sum_i \delta\left[\delta f_{1i}\left(\hat{X}'\right) - \underline{\hat{S}}_1^{[ii]}\frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}\right)}\frac{\delta f_{1i}'\left(\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)}dX' \right. \tag{183}$$
$$\left. -\underline{\hat{S}}_2^{[ii]}\left(\left(1+\hat{f}^{[i]\prime}\left(\hat{X}'\right)+\delta\hat{f}^{[i]\prime}\left(\hat{X}'\right)\right)\underline{\hat{s}}_2^{[i]}H\left(-\left(1+\hat{f}^{[i]\prime}\left(\hat{X}'\right)+\delta\hat{f}^{[i]\prime}\left(\hat{X}'\right)\right)\right)\right)dX'\right]$$

and for banks:

$$\sum_i \bar{V}_i^D\left(\hat{\Psi},\hat{X},K,\delta\bar{f}^{[i]}\right) = \sum_i \delta\left[\delta\bar{f}^{[i]\prime} - \underline{\bar{S}}_1^{[ii]}\frac{\delta\bar{f}^{[i]\prime}}{1+\underline{\bar{k}}^{[i]}} - \underline{\hat{S}}_i^{[ii]}\frac{\delta\hat{f}^{[i]\prime}}{1+\underline{\hat{k}}_2^{[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}}\right. \tag{184}$$
$$-\underline{\bar{S}}_2^{[ii]}\left(1+\bar{f}^{[i]}+\delta\bar{f}^{[i]}\right)\underline{\hat{s}}_2^{[i]}H\left(-\left(1+\bar{f}^{[i]}+\delta\bar{f}^{[i]}\right)\right)$$
$$\left.-\underline{\hat{S}}_2^{B'[ii]}\left(1+\hat{f}^{[i]\prime}+\delta\hat{f}^{[i]\prime}\right)\underline{\hat{s}}_2^{[i]}H\left(-\left(1+\hat{f}^{[i]\prime}+\delta\hat{f}^{[i]\prime}\right)\right)\right]$$

And the inter-group potentials for investors:

$$\sum_i W_i^D\left(\hat{\Psi},\hat{X},K,\delta f_{1i}\right) \tag{185}$$
$$= \sum_i \delta\left[\delta\bar{f}^{[i]} - \underline{\bar{S}}_1^{[ji]}\frac{\delta\bar{f}^{[j]\prime}}{\left(1+\underline{\hat{k}}_2^{[j]}\right)} - \underline{\hat{S}}_1^{[ij]}\frac{\delta\hat{f}^{[j]\prime}}{\left(1+\underline{\hat{k}}_2^{[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}\right)}\right.$$
$$\left.-\underline{\hat{S}}_2^{[ij]}\left(\left(1+\hat{f}^{[j]\prime}\left(\hat{X}'\right)+\delta\hat{f}^{[j]\prime}\left(\hat{X}'\right)\right)\underline{\hat{s}}_2^{[i]}H\left(-\left(1+\hat{f}^{[j]\prime}\left(\hat{X}'\right)+\delta\hat{f}^{[j]\prime}\left(\hat{X}'\right)\right)\right)\right)dX'\right]$$

and for banks:

$$\sum_i \bar{W}^D\left(\hat{\Psi},\hat{X},K,\delta\bar{f}^{[i]}\right) \tag{186}$$

$$= \sum_i \delta\left[\delta\bar{f}^{[i]} - \underline{\bar{S}}_1^{[ji]}\frac{\delta\bar{f}^{[j]\prime}}{2\left(1+\underline{\hat{k}}_2^{[j]}\right)} - \underline{\hat{S}}_1^{[ij]}\frac{\delta\hat{f}^{[j]\prime}}{\left(1+\underline{\hat{k}}_2^{[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[j]}\right)}\right.$$
$$-\underline{\bar{S}}_2^{[ii]}\left(1+\bar{f}^{[i]}+\delta\bar{f}^{[i]}\right)\underline{\hat{s}}_2^{[i]}H\left(-\left(1+\bar{f}^{[i]}+\delta\bar{f}^{[i]}\right)\right)$$
$$\left.-\underline{\hat{S}}_2^{B^{\prime}[ii]}\left(1+\hat{f}^{[i]\prime}+\delta\hat{f}^{[i]\prime}\right)\underline{\hat{s}}_2^{[i]}H\left(-\left(1+\hat{f}^{[i]\prime}+\delta\hat{f}^{[i]\prime}\right)\right)\right]$$

25.4.4 Transition functions with interactions including defaults

Considering the possibilities of defauls the transition may thus occur from two possible effects.

First a sequence of interaction:

$$[G...G]\,V_n\,[G...G]+.....\,[G...G]\,V_n\,[G...G]\,V_n...\,[G...G]\,V_n\,[G...G] \tag{187}$$

with V_n describes transitions between agents of several groups, may drive some agents to default.

The following transitions are thus driven by terms like:

$$[G...G]\,V_n^D\,[G...G]+.....\,[G...G]\,V_n^D\,[G...G]\,V_n^D...\,[G...G]\,V_n\,[G...G] \tag{188}$$

which increases the number of default in group.

Then through inter-groups potential the default propagates to agents of different groups.

25.4.5 Interpretation

While the mechanisms are similar to the system without banks, several differences arise.

Firstly, as quoted above in section 22, the diffusion coefficients are higher when banks are included, since bank loans increase the disposable capital of investors. However, when no defaults arise, the transmission coefficients for banks are low: if we consider that the main activity for banks is loans at fixed rates, the excess returns do not affect these agents, and they do not transmit fluctuations in returns. As a consequence, the presence of banks reduces the effect of transmission. As in the study of collective states, the larger the number of banks, the more we can expect the increase in diffusion to be offset. The situation is different if banks take large participations in investors and thus act as investors.

The situation is different if some defaults arise. As large lenders, we can expect the banks to amplify the loss of return when defaults occur, which increases the cascade of defaults. In this case, since banks are interconnected, depending on the level of interconnections, any default may propagate to banks. Thus, if banks seem to stabilize the system and reduce the risk of default when loans are their main activity, they may amplify the defaults when these occur.

26 Discussion

Field economics reveals the collective states that characterize an economic model. A collective state represents a set of significant data describing a model, and each model may give rise to one or several collective states. In our model, these significant are, for each and every sector, their average capital, number of agents and average return. This collection of data characterizes one collective state.

Each collective state is thus defined by a specific set of levels of capital, number of agents, and sector average returns. These data define an overall average capital, number of agents and return. However, the converse is not true: for any given average level of capital, agents, and returns in an economy, multiple collective states may exist.

These collective states serve as static descriptions of the system's condition. Within any given economic model, the number of collective states determines the system's stability or instability, as well as the transitions between stable and unstable states. Viewed in this context, collective states represent the intrinsic characteristics of the system—its various manifestations under specific conditions, whether temporary or structural.

They are not economic equilibria per se, rather potential states that the economy may experience. Any global state among the set of possible collective states may take place, but is bound to be replaced by another global state realizing another collective state. This change is discontinuous: shifts occur abruptly, and the economy switches from one collective state to another without transition. However this shift will not necessarily result in an immediate modification of individual dynamics.

Because of the nature of the collective state, the analysis can either focus on the full extension of the collective state, or concentrate on how a specific realization of a collective state impacts the individual dynamics of the agents. Therefore, we do not study a dynamic transition between collective states, but rather how the agents' individual dynamics materialize the changes between collective states.

In this paper, we conducted both studies, starting with collective states, as they influence the individual dynamics within them. We first assumed a baseline model consisting of two types of agents, investors and firms, before turning to a more complex analysis involving three types of agents, firms, investors, and banks.

In the two-type model of agents, investors accumulate and allocate capital through equity stakes and successive lending activities. The complex interconnections in loan activities highlight the critical role of leverage in capital allocation, and any change in returns or defaults within one sector can swiftly spread to other sectors. In this scenario, the system's instability arises from the multitude of collective states characterizing the system, and merely reflects the transitions between these multiple states.

The study of collective states reveals that several global averages can emerge, spanning from scenarios characterized by high total capital, a large number of investors, and relatively low average returns to situations with lower total capital, few investors, but significant returns. Each of these global averages can be associated with multiple collective states: for any given level of global capital, it can be allocated across sectors in numerous ways, each allocation defining a unique realization of the collective state. Consequently, a particular level of overall wealth can manifest in the form of multiple distinct collective states, each representing a different distribution of capital across sectors, ranging from more evenly distributed to more concentrated configurations.

Collective averages can be associated to multiple collective states, some of which may involve defaults. A default occurs when agents in a sector experience returns that are insufficient to cover their debts. This default state within a sector is structural in nature: it is bound to occur within the global collective state. Whatever the individual dynamics within the sector may be, the likelihood of realizing the structural default state will increase over time unless the collective state shifts to another collective state.

The collective state ought to be thought in its final and full extension, in which all possible defaults have already spread to other sectors. It is a static state describing the final and stabilized situation in its maximum extent, in which some sectors may see their level of capital affected, even if they do not necessarily default.

Collective states are therefore structural situations that condition the individual dynamics of agents, depending on their sector. However, individual dynamics exhibit persistence, so that a change of collective state may not immediately translate into the dynamics of agents within the sector. This accounts for the potential delays in the transition of agents from one collective state to the other.

Within a given collective state, it is possible to examine the dynamics of agents. In our model, since firms solely engage in production and their available capital is directly proportional to their private capital, we have focused on analyzing the dynamics of firms' private capital. However, we have opted to study the dynamics of investors' disposable capital as this variable plays a critical role in their investment decisions and in the propagation of leverage effects.

A firm can default either directly, following an adverse shock, or indirectly, due to the default of some of its investors. When capital is depleted, the firm becomes unable to meet its expenses.

Investors can also default, either directly through poor investments or indirectly through loans and stakes taken from other investors. These defaults can spread and transmit sometimes complex, indirect and even spiraling effects. Negative returns or returns lower than investors' expectations in a given sector can either prompt their partners to reduce their participation or induce an additional decrease that will indirectly impact investors' results through interconnectedness. Investors operating in a sector on the brink of default may default due to their ties with other investors, even if there initially appeared to be no imminent default risk. This indirect negative return has the potential to magnify the initial loss and eventually push investors over the edge, leading to a domino effect, boomerang effect, or liquidity crisis. This scenario underscores the multiplicity of collective states: the collective state represents the final stage where the chain of defaults has fully transpired, while the initial situation characterized by apparent fragility rather than actual defaults marks the beginning of a dynamic within a new collective state featuring defaults.

Our model thus allows us to study the default mechanisms of both types of agents: firms and investors. But it also facilitates the analysis of amplification effects and systemic risk within the system. In our model, all investments are underpinned by a level of private capital. Returns are amplified by leverage effects that are only available to investors. This structural discrepancy between firms and investors means that the latter have the potential to generate higher returns than the former. When an investor defaults, they not only lose their private capital but also forfeit all the stakes and loans extended to them. These stakes and loans, in turn, represent disposable capital, which is derived from equity. Therefore, disposable capital essentially condenses private capital. Consequently, when there is a default on disposable capital, which serves as a condensation of private capital, the loss impacts all those involved. Hence, there is a diffusion of the loss throughout the system.

It is when we delve into averages and the diffusion matrix that Field Economics truly reveals its value. Our findings unveil rather complex phenomena that can be understood both globally, with their macroeconomic implications, and at a microeconomic level, when examining individuals within the collective state. Our formulas for the diffusion matrix, capital levels, and the consequences of leverage in investor relations demonstrate that we can isolate successive effects on factors that might otherwise be overlooked, providing detailed insights into the mechanisms of financial risk transmission and contagion, as well as capital diffusion. For instance, the diffusion matrix is localized, from sector to sector, meaning that diffusion effects depend on the path taken. By considering varying connections in the sector space, one could study different transmission patterns.

Because Field Economics allows for the introduction of as many types of agents as desired, simply by adding fields to represent them, we were able to distinguish the role of banks from other investors, leading to the differentiation of diffusion matrices for investor-investor, bank-bank, investor-bank, and bank-investor interactions, each with its own characteristics. This enabled us, even at the collective level, to discern the role of banks in the diffusion between investors, and how the interrelations between groups of agents or certain agents can form.

To this preliminary study of a system with two types of agents, which serves as a benchmark, we add a third type of agent, banks, which act as investors, but have the unique ability to create money through the multiplier effect. Once again, the analysis is conducted at two levels: the level of collective states and the level of individual dynamics. To simplify the analysis, we assume that banks essentially act as lenders, and under the assumption of no default, their returns are reduced to the interest rate. In this scenario, if the average total capital across all sectors mirrors that of the benchmark, several distinctions become nonetheless apparent:

The first distinction lies in the diffusion matrix, which exhibits higher coefficients compared to the benchmark. The borrowing behavior of investors from banks results in increased leverage, thereby expanding their disposable capital. The second distinction arises when banks provide capital to firms, resulting in higher disposable capital for firms due to leverage and monetary creation. With increased capital at their disposal, firms experience a decrease in marginal returns. The number of potential averages for collective states decreases in comparison to the benchmark which contributes to the stabilization of the system. The third difference is that the total disposable capital of banks varies inversely with that of investors. Depending on the circumstances, capital may primarily be owned by banks or investors, leading to a majority of loans or stakes, depending on the interest rates applied. The fourth distinction is that, typically, when the average productivity of firms—and consequently, their returns—increases, the disposable capital of investors tends to increase, albeit at a slower rate compared to the benchmark. This contributes to a stabilizing effect. The fifth distinction, resulting from the previous two observations, is that in periods of lower returns, capital tends to be concentrated in banks. This is due to the lending-based nature of the system, which favors banks through monetary creation.

The introduction of banks into the model has two conflicting effects on collective states. On one hand, by increasing liquidity and providing loans, banks facilitate the diffusion of capital and augment the disposable capital of investors, thereby amplifying the occurrence and spread of defaults. However, on the other hand, by directly lending to firms, they constrain the proliferation of firm defaults. The size of the banking sector relative to that of investors exerts a stabilizing influence, leading to a reduced number of collective states.

Similar effects are observed in individual dynamics: in the presence of banks, losses incurred by investors are more likely to spread to other investors, due to the increase in the leverage effect. However, if banks confine themselves to their lending function, they are largely shielded from these losses and do not propagate them. Thus, banks play a stabilizing role in the system and mitigate the risks of default. Nevertheless, in the event of a default, the credit multiplier amplifies its effects.

However, it's important to note that these results are based on certain assumptions, particularly the assumption that banks limit themselves to their lending role. If this assumption is relaxed—for example, if banks take significant stakes—other effects may arise. Additionally, we have disregarded the influence of investors on banks, assuming a relatively robust and independent banking sector. Removing this assumption and reintroducing the impact of investors on banks could potentially reintroduce multiple equilibria. In conclusion, these observations suggest that the effects discussed emerge when banks exhibit behavior akin to that of investors.

27 Conclusion

Overall, our framework enables the analysis of financial speculation dynamics, the mutual impact of the financial and real sectors. Our exploration of collective states reveals how capital is allocated and how defaults can emerge structurally within these states. From a dynamic perspective, it delineates the flow of capital, the propagation of returns, the potential for defaults and their dynamic spread throughout part or all of the economy, either directly or indirectly, within a collective state. The introduction of banks into the system allows for an examination of how these specific investors can either stabilize or destabilize the system, depending on parameters such as the size of the banking sector.

More broadly, Field Economics should enable to analyse the emergence of clustered groups operating autonomously and observe how networks of agents and interests can form or disintegrate, and their impact on the system.

References

Abergel F, Chakraborti A, Muni Toke I and Patriarca M (2011a) Econophysics review: I. Empirical facts, Quantitative Finance, Vol. 11, No. 7, 991-1012

Abergel F, Chakraborti A, Muni Toke I and Patriarca M (2011b) Econophysics review: II. Agent-based models, Quantitative Finance, Vol. 11, No. 7, 1013-1041

Acemoglu, D., Ozdaglar, A. and Tahbaz-Salehi, A. (2015). Systemic risk and stability in financial networks. American Economic Review, 105(2), 564-608.

Acharya, V. V., Pedersen, L. H., Philippon, T., & Richardson, M. (2017). Measuring systemic risk. The Review of Financial Studies, 30(1), 2-47.

Adrian, T. and Brunnermeier, M. K. (2016). CoVaR. American Economic Review, 106(7), 1705-1741.

Allen, F. and Gale, D. (2000). Financial contagion. Journal of Political Economy, 108(1), 1-33.

Bardoscia M., Livan G., Marsili M. (2017), Statistical mechanics of complex economies, Journal of Statistical Mechanics: Theory and Experiment, Volume 2017

Bardoscia, M., Battiston, S., Caccioli, F., & Caldarelli, G. (2019). Pathways towards instability in financial networks. Nature communications, 10(1), 1-9.

Battiston, S., Puliga, M., Kaushik, R., Tasca, P., & Caldarelli, G. (2012). Debtrank: Too central to fail? Financial networks, the FED and systemic risk. Scientific reports, 2, 541.

Battiston, S., Caldarelli, G., May, R. M., Roukny, T., & Stiglitz, J. E. (2020). The price of complexity in financial networks. Proceedings of the National Academy of Sciences, 117(52), 32779-32786.

Bernanke B., Gertler, M. and S. Gilchrist (1999), The financial accelerator in a quantitative business cycle framework, Chapter 21 in Handbook of Macroeconomics, 1999, vol. 1, Part C, pp 1341-1393

Bensoussan A, Frehse J, Yam P (2018) Mean Field Games and Mean Field Type Control Theory. Springer, New York

Böhm, V., Kikuchi, T., Vachadze, G.: Asset pricing and productivity growth: the role of consumption scenarios. Comput. Econ. 32, 163–181 (2008)

Caggese A, Orive A P, The Interaction between Household and Firm Dynamics and the Amplification of Financial Shocks. Barcelona GSE Working Paper Series, Working Paper n^o 866, 2015

Campello, M., Graham, J. and Harvey, C.R. (2010). The Real Effects of Financial Constraints: Evidence from a Financial Crisis, Journal of Financial Economics, vol. 97(3), 470-487.

Cifuentes, R., Ferrucci, G. and Shin, H. S. (2005). Liquidity risk and contagion. Journal of the European Economic Association, 3(2-3), 556-566.

Elliott, M., Golub, B., and Jackson, M. O. (2014). Financial networks and contagion. American Economic Review, 104(10), 3115-3153.

Gaffard JL and Napoletano M Editors (2012): Agent-based models and economic policy. OFCE, Paris

Gai, P., & Kapadia, S. (2010). Contagion in financial networks. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 466(2120), 2401-2423.

Gomes DA, Nurbekyan L, Pimentel EA (2015) Economic Models and Mean-Field Games Theory, Publicações Matemáticas do IMPA, 30o Colóquio Brasileiro de Matemática, Rio de Janeiro

Gennaioli, N., Shleifer, A. and Vishny, R. W. (2012). Neglected risks, financial innovation, and financial fragility. Journal of Financial Economics, 104(3), 452-468.

Gosselin P, Lotz A and Wambst M (2017) A Path Integral Approach to Interacting Economic Systems with Multiple Heterogeneous Agents. IF_PREPUB. 2017. hal-01549586v2

Gosselin P, Lotz A and Wambst M (2020) A Path Integral Approach to Business Cycle Models with Large Number of Agents. Journal of Economic Interaction and Coordination volume 15, pages 899–942

Gosselin P, Lotz A and Wambst M (2021) A statistical field approach to capital accumulation. Journal of Economic Interaction and Coordination 16, pages 817–908 (2021)

Grassetti, F., Mammana, C. & Michetti, E. A dynamical model for real economy and finance. Math Finan Econ (2022). https://doi.org/10.1007/s11579-021-00311-3

Greenwood, R., & Hanson, S. G. (2013). Issuer quality and corporate bond returns. The Review of Financial Studies, 26(6), 1483-1525.

Grosshans, D., Zeisberger, S.: All's well that ends well? on the importance of how returns are achieved. J. Bank. Finance 87, 397–410 (2018)

Haldane, A. G., & May, R. M. (2011). Systemic risk in banking ecosystems. Nature, 469(7330), 351-355.

Holmstrom, B., and Tirole, J. (1997). Financial intermediation, loanable funds, and the

real sector. Quarterly Journal of Economics, 663-691.

Jackson M (2010) Social and Economic Networks. Princeton University Press, Princeton

Jermann, U.J. and Quadrini, V., (2012). "Macroeconomic Effects of Financial Shocks," American Economic Review, Vol. 102, No. 1.

Khan, A., and Thomas, J. K. (2013). "Credit Shocks and Aggregate Fluctuations in an Economy with Production Heterogeneity," Journal of Political Economy, 121(6), 1055-1107.

Kaplan G, Violante L (2018) Microeconomic Heterogeneity and Macroeconomic Shocks, Journal of Economic Perspectives, Vol. 32, No. 3, 167-194

Kleinert H (1989) Gauge fields in condensed matter Vol. I , Superflow and vortex lines, Disorder Fields, Phase Transitions, Vol. II, Stresses and defects, Differential Geometry, Crystal Melting. World Scientific, Singapore

Kleinert H (2009) Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets 5th edition. World Scientific, Singapore

Krugman P (1991) Increasing Returns and Economic Geography. Journal of Political Economy, 99(3), 483-499

Langfield, S., Liu, Z., & Ota, T. (2020). Mapping the network of financial linkages: an industry-level analysis of the UK. Journal of Banking & Finance, 118, 105946.

Lasry JM, Lions PL, Guéant O (2010a) Application of Mean Field Games to Growth Theory https://hal.archives-ouvertes.fr/hal-00348376/document

Lasry JM, Lions PL, Guéant O (2010b) Mean Field Games and Applications. Paris-Princeton lectures on Mathematical Finance, Springer. https://hal.archives-ouvertes.fr/hal-01393103

Lux T (2008) Applications of Statistical Physics in Finance and Economics. Kiel Institute for the World Economy (IfW), Kiel

Lux T (2016) Applications of Statistical Physics Methods in Economics: Current state and perspectives. Eur. Phys. J. Spec. Top. (2016) 225: 3255. https://doi.org/10.1140/epjst/e2016-60101-x

Mandel A, Jaeger C, Fürst S, Lass W, Lincke D, Meissner F, Pablo-Marti F, Wolf S (2010). Agent-based dynamics in disaggregated growth models. Documents de travail du Centre d'Economie de la Sorbonne. Centre d'Economie de la Sorbonne, Paris

Mandel A (2012) Agent-based dynamics in the general equilibrium model. Complexity Economics 1, 105–121

Monacelli, T., Quadrini, V. and A. Trigari (2011). "Financial Markets and Unemployment," NBER Working Papers 17389, National Bureau of Economic Research.

Reinhart, C. M., & Rogoff, K. S. (2009). This Time Is Different: Eight Centuries of Financial Folly. Princeton University Press.

Sims C A (2006) Rational inattention: Beyond the Linear Quadratic Case, American Economic Review, vol. 96, no. 2, 158-163

Yang J (2018) Information theoretic approaches to economics, Journal of Economic Survey, Vol. 32, No. 3, 940-960

Cochrane, J.H. (ed.): Financial Markets and the Real Economy, International Library of Critical Writings in Financial Economics, vol. 18. Edward Elgar (2006)

Part 1 Appendices

Appendix 1 Details of the micro setup

A1.1 Linear approximation of relation between disposable capital and private capital

In a linear approximation, we consider that invested shares and lent capital are proportional to the private capital, acting as collateral:

$$\hat{k}_1\left(\hat{K}_{jp}(t), \hat{X}_j(t), \hat{X}_l(t)\right) + \hat{k}_2\left(\hat{K}_{jp}(t), \hat{X}_j(t), \hat{X}_l(t)\right) = \left(\hat{k}_1\left(\hat{X}_j(t), \hat{X}_l(t)\right) + \hat{k}_2\left(\hat{X}_j(t), \hat{X}_l(t)\right)\right)\hat{K}_{jp}(t)$$

and:

$$k_1\left(X_i, K_i, \hat{X}_j\right) + k_2\left(X_i, K_i, \hat{X}_j\right) = \left(k_1\left(X_i, \hat{X}_j\right) + k_2\left(X_i, \hat{X}_j\right)\right)K_{ip}$$

Ultimately, we can express private capital as a function of disposable capital:

$$\hat{K}_j(t) = \left(1 + \sum_l \left(\hat{k}_1\left(\hat{X}_j(t), \hat{X}_l(t)\right) + \hat{k}_2\left(\hat{X}_j(t), \hat{X}_l(t)\right)\right)\hat{K}_l(t)\right)\hat{K}_{jp}(t) \tag{189}$$

that is:

$$\hat{K}_{jp}(t) = \frac{\hat{K}_j(t)}{1 + \sum_l \left(\hat{k}_1\left(\hat{X}_j(t), \hat{X}_l(t)\right) + \hat{k}_2\left(\hat{X}_j(t), \hat{X}_l(t)\right)\right)\hat{K}_l(t)} \tag{190}$$

Defining:

$$\begin{aligned}
\hat{k}_{ajl} &= \hat{k}_a\left(\hat{X}_j(t), \hat{X}_l(t)\right)\\
k_{ajl} &= k_a\left(X_j(t), \hat{X}_l(t)\right)
\end{aligned}$$

and:

$$\begin{aligned}
\hat{k}_{jl} &= \hat{k}_{1jl} + \hat{k}_{2jl}\\
k_{jl} &= k_{1jl} + k_{2jl}
\end{aligned}$$

this also rewrites:

$$\hat{K}_{jp}(t) = \frac{\hat{K}_j(t)}{1 + \sum_l \left(\hat{k}_{1jl} + \hat{k}_{2jl}\right)\hat{K}_l(t)} \tag{191}$$

A1.2 Normalization of coefficients

We assume that coefficients are normalized as follows. For investors' shares in firms, we assume that coefficients depend on the total capital involved for firms. Therefore, we replace:

$$k_{ajl} \rightarrow \frac{k_{ail}}{N\left\langle K_p(t)\right\rangle} \tag{192}$$

where N is the number of firms, and $\langle K_p(t)\rangle$ represents the average capital per firm. Similarly, we replace:

$$\hat{k}_{ajl} \rightarrow \frac{\hat{k}_{ajl}}{\hat{N}\left\langle \hat{K}_p(t)\right\rangle} \tag{193}$$

where $\hat{N}$ is the total number of investors, and $\left\langle \hat{K}_p(t) \right\rangle$ represents the average capital per investor.

Below, we will examine the dynamics for $K_p(t)$ and $\hat{K}(t)$. Equation (192) is sufficient for the analysis, but equation (193) needs to be rewritten as a function of $\hat{K}(t)$. For this purpose, we will rewrite (42) and (189):

$$\hat{K}_j(t) = \left(1 + \sum_l \frac{\left(\hat{k}_1\left(\hat{X}_j(t), \hat{X}_l(t)\right) + \hat{k}_2\left(\hat{X}_j(t), \hat{X}_l(t)\right)\right)}{\hat{N}\left\langle \hat{K}_p(t) \right\rangle} \hat{K}_l(t)\right) \hat{K}_{jp}(t) \tag{194}$$

The average over $\hat{K}_{jp}(t)$, written $\left\langle \hat{K}_p(t) \right\rangle$, is given by:

$$\left\langle \hat{K}_p(t) \right\rangle = \left\langle \hat{K}(t) \right\rangle - \sum_l \frac{\left\langle \left(\hat{k}_1\left(\hat{X}_j(t), \hat{X}_l(t)\right) + \hat{k}_2\left(\hat{X}_j(t), \hat{X}_l(t)\right)\right) \right\rangle}{N} \left\langle \hat{K}(t) \right\rangle \tag{195}$$

This leads us to:

$$\left\langle \hat{K}_p(t) \right\rangle = \left\langle \hat{K}(t) \right\rangle \left(1 - \left(\left\langle \hat{k}_1 \right\rangle + \left\langle \hat{k}_2 \right\rangle\right)\right)$$

where $\left\langle \hat{k}_a \right\rangle$ represents the average of $\hat{k}_a\left(\hat{X}_j(t), \hat{X}_l(t)\right)$, and the factor $\hat{N}$ represents the number of investors.

The normalization for $\hat{k}_{ajl}$ is thus:

$$\hat{k}_{ajl} \rightarrow \frac{\hat{k}_{ajl}}{\hat{N}\left\langle \hat{K}(t) \right\rangle \left(1 - \left(\left\langle \hat{k}_1 \right\rangle + \left\langle \hat{k}_2 \right\rangle\right)\right)} \tag{196}$$

This normalization will be kept implicit in the main part of the text for the sake of simplcity and reintroduced when needed.

A1.3 Returns from firms to investors

We derive $r_i'\left(K_i(t), \frac{\dot{K}_i}{K_i}\right)$ by specifying the form of $F_1\left(\bar{R}_i, \frac{\dot{K}_i}{K_i}\right)$. Given that:

$$K_i = K_{ip}(t)\left(1 + \sum_v k_{iv} \hat{K}_v(t)\right)$$

we have:

$$\frac{\dot{K}_{ip}(t)}{K_{ip}(t)} + \frac{\sum_v k_{iv} \frac{d}{dt} \hat{K}_v(t)}{1 + \sum_v k_{iv} \hat{K}_v(t)} + \frac{\sum_v \frac{d}{dt} k_{iv} \hat{K}_v(t)}{1 + \sum_v k_{iv} \hat{K}_v(t)} = \frac{\dot{K}_{ip}(t)}{K_{ip}(t)} + \frac{\sum_v k_{iv} \hat{f}_\nu \hat{K}_v(t)}{1 + \sum_v k_{iv} \hat{K}_v(t)}$$

Moreover, since the sum:

$$\frac{\sum_v k_{iv} \frac{d}{dt} \hat{K}_v(t)}{1 + \sum_v k_{iv} \hat{K}_v(t)}$$

is close to its average, it can be neglected. Besides, the variation $\frac{d}{dt} k_{iv}$ is exogeneous and mainly depends on $\bar{R}_i$, which can thus be included directly. As a consequence, we can write:

$$F_1\left(\bar{R}_i, \frac{\dot{K}_i}{K_i}\right) \rightarrow F_1\left(\bar{R}_i, \frac{\dot{K}_{ip}(t)}{K_{ip}(t)}\right)$$

Substracting the average $\left\langle \frac{\dot{K}_{ip}(t)}{K_{ip}(t)} \right\rangle$, it writes in first approximation:

$$F_1\left(\bar{R}_i, \frac{\dot{K}_i}{K_i}\right) \rightarrow F_1\left(\bar{R}(K_i, X_i)\right) + \tau\left(\bar{R}(K_i, X_i)\right)\left(\frac{\dot{K}_{ip}(t)}{K_{ip}(t)} - \left\langle \frac{\dot{K}_{ip}(t)}{K_{ip}(t)} \right\rangle\right)$$

and the return is:

$$r_i'\left(K_i\left(t\right),\frac{\dot{K}_i}{K_i}\right)=r_i+F_1\left(\bar{R}\left(K_i,X_i\right)\right)+\tau\left(\bar{R}\left(K_i,X_i\right)\right)\Delta f_1'\left(K_i\left(t\right)\right)$$

with:

$$\Delta f_1'\left(K_i\left(t\right)\right)=f_1'\left(K_i\left(t\right)\right)-\left\langle f_1'\left(K_i\left(t\right)\right)\right\rangle$$

where the average is taken over all firms. We assume that $F_1\left(\bar{R}\left(K_i,X_i\right)\right)$ is equal to 0 in average, so that we can write:

$$F_1\left(\bar{R}\left(K_i,X_i\right)\right)+\tau\left(\bar{R}\left(K_i,X_i\right)\right)\Delta f_1'\left(K_i\left(t\right)\right)=\Delta F_\tau\left(\bar{R}\left(K_i,X_i\right)\right)$$

We define also:

$$f_1'\left(K_i\left(t\right)\right)=\left(\left(1+\sum_\nu k_{2j\nu}\hat{K}_\nu\left(t\right)\right)f_1\left(\left(K_i\left(t\right)\right)\right)-\bar{r}\sum_v k_{2lv}\right)\hat{K}_v\left(t\right)\tag{197}$$

$$f_1\left(\left(K_i\left(t\right)\right)\right)=\frac{f_1'\left(K_i\left(t\right)\right)}{1+\sum_\nu k_{2j\nu}\hat{K}_\nu\left(t\right)}+\bar{r}\sum_v k_{2lv}\hat{K}_v\left(t\right)$$

and:

$$r_i'\left(K_i\left(t\right),\frac{\dot{K}_i}{K_i}\right)=\frac{f_1'\left(K_i\left(t\right)\right)}{1+\sum_\nu k_{2j\nu}\hat{K}_\nu\left(t\right)}+\bar{r}\sum_v k_{2lv}\hat{K}_v\left(t\right)+\Delta F_\tau\left(\bar{R}\left(K_i,X_i\right)\right)$$

A1.4 Investors returns and investors' disposable dynamics

The return equation writes in matricial form:

$$\sum_l\left(\delta_{jl}-\frac{\hat{k}_{1lj}\hat{K}_l\left(t\right)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v\left(t\right)}\right)R_l=R_j'\tag{198}$$

with solution:

$$R_j=\sum_l\left(1-\frac{\hat{k}_{1lj}\hat{K}_j\left(t\right)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v\left(t\right)}\right)^{-1}\frac{1+\sum_v\hat{k}_{2lv}\hat{K}_v\left(t\right)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v\left(t\right)}R_l'$$

The accumulation of disposable capital can be computed by starting first with the private capital $\hat{K}_{jp}\left(t\right)$.

By a reasoning similar to the firms, the dynamics for private funds is given by:

$$\begin{aligned}\hat{K}_{jp}\left(t+\varepsilon\right)-\hat{K}_{jp}\left(t\right) &= R_j\left(1+\sum_v\hat{k}_{2jv}\hat{K}_v\left(t\right)\right)\hat{K}_{jp}\left(t\right)-\bar{r}\sum_v\hat{k}_{2jv}\hat{K}_v\left(t\right)\hat{K}_{jp}\left(t\right)\\ &= \sum_l\left(1-\frac{\hat{k}_{1lj}\hat{K}_j\left(t\right)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v\left(t\right)}\right)^{-1}R_l'\left(\hat{K}_{jp}\left(t\right)+\hat{K}_{jp}\left(t\right)\sum_v\hat{k}_{2jv}\hat{K}_v\left(t\right)\right)\\ &\quad-\bar{r}\sum_v\hat{k}_{2jv}\hat{K}_{jp}\left(t\right)\hat{K}_v\left(t\right)\end{aligned}\tag{199}$$

written also:

$$\hat{K}_{jp}\left(t+\varepsilon\right)-\hat{K}_{jp}\left(t\right)=f_j\hat{K}_{jp}\left(t\right)$$

with:

$$\hat{f}_j = \left(1 + \sum_v \hat{k}_{2jv}\hat{K}_v(t)\right) R_j - \bar{r}\sum_v \hat{k}_{2jv}\hat{K}_v(t) \tag{200}$$

In the continuous approximation:

$$\hat{K}_{jp}(t+\varepsilon) - \hat{K}_{jp}(t) \rightarrow \frac{d}{dt}\hat{K}_{jp}(t) = \hat{f}_j\hat{K}_{jp}(t)$$

This can be transformed to obtain a dynamics for the disposable income. The differenciation of (37) yields:

$$\frac{d}{dt}\hat{K}_{jp}(t) = \frac{\frac{d}{dt}\hat{K}_j(t)}{1+\sum_l\left(\hat{k}_{1jl}+\hat{k}_{2jl}\right)\hat{K}_l(t)} - \sum_l \frac{\left(\hat{k}_{1jl}+\hat{k}_{2jl}\right)\hat{K}_l(t)}{\left(\sum_l\left(\hat{k}_{1jl}+\hat{k}_{2jl}\right)\hat{K}_l(t)^2\right)}\frac{d}{dt}\hat{K}_l(t) \tag{201}$$

Using (88) we obtain the dynamic equation for disposable capital (199) $\hat{K}_j(t)$:

$$\frac{\frac{d}{dt}\hat{K}_j(t)}{1+\sum_l\left(\hat{k}_{jl}\hat{K}_l(t)\right)} - \sum_l \frac{\hat{k}_{jl}\hat{K}_j(t)}{\left(1+\sum_l \hat{k}_{jl}\hat{K}_l(t)\right)^2}\frac{d}{dt}\hat{K}_l(t) = f_j\frac{\hat{K}_j(t)}{1+\sum_l\left(\hat{k}_{jl}\hat{K}_l(t)\right)}$$

This equation is used to write a system of differential equations for the $\frac{d}{dt}\hat{K}_j(t)$:

$$\frac{d}{dt}\hat{K}_j(t) = \sum_l \left(1 - \frac{\hat{k}_{1jl}\hat{K}_j(t)}{1+\sum_v \hat{k}_{jv}\hat{K}_v(t)}\right)^{-1} f_l\hat{K}_l(t) = \sum_l (1-M)^{-1}_{jl} f_l\hat{K}_l(t) \tag{202}$$

where:

$$M_{jm} = \frac{\hat{k}_{jm}\hat{K}_j(t)}{1+\sum_\nu \hat{k}_{j\nu}\hat{K}_\nu(t)} \tag{203}$$

A1.5 Investors return equation without default

Writing (200) as:

$$R_j = \frac{\hat{f}_j}{1+\sum_v \hat{k}_{2jv}\hat{K}_v(t)} + \bar{r}\frac{\sum_v \hat{k}_{2jv}\hat{K}_v(t)}{1+\sum_v \hat{k}_{2jv}\hat{K}_v(t)}$$

we can rewrite the return equation (198) as an equation for $\hat{f}_j$:

$$\sum_l \left(\delta_{jl} - \frac{\hat{k}_{1lj}\hat{K}_l(t)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v(t)}\right)\left(\frac{\hat{f}_l}{1+\sum_v \hat{k}_{2lv}\hat{K}_v(t)} + \bar{r}\frac{\sum_v \hat{k}_{2lv}\hat{K}_v(t)}{1+\sum_v \hat{k}_{2lv}\hat{K}_v(t)}\right) = R'_j \tag{204}$$

Substracting $\bar{r}$ on both side it writes:

$$\begin{aligned}&\frac{\hat{f}_j-\bar{r}}{1+\sum_v \hat{k}_{2jv}\hat{K}_v(t)} - \sum_l\left(\frac{\hat{k}_{1lj}\hat{K}_l(t)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v(t)}\right)\left(\frac{\hat{f}_l-\bar{r}}{1+\sum_v \hat{k}_{2lv}\hat{K}_v(t)}+\bar{r}\right) \\ &= R'_j - \bar{r}\end{aligned} \tag{205}$$

that is:

$$\begin{aligned}&\frac{\hat{f}_j-\bar{r}}{1+\sum_v \hat{k}_{2jv}\hat{K}_v(t)} - \sum_l\left(\frac{\hat{k}_{1lj}\hat{K}_l(t)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v(t)}\right)\left(\frac{\hat{f}_l-\bar{r}}{1+\sum_v \hat{k}_{2lv}\hat{K}_v(t)}\right) \\ &= R'_j - \bar{r} + \bar{r}\sum_l\left(\frac{\hat{k}_{1lj}\hat{K}_l(t)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v(t)}\right)\end{aligned} \tag{206}$$

Using (198):

$$\begin{aligned}
&R_j^{\prime}-\bar{r}+\bar{r}\sum_l\left(\frac{\hat{k}_{1lj}\hat{K}_l\left(t\right)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v\left(t\right)}\right)\\
=&\ \bar{r}\sum_l\frac{\hat{k}_{1lj}\hat{K}_l\left(t\right)+\hat{k}_{2lj}\hat{K}_l\left(t\right)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v\left(t\right)}+\bar{r}\sum_i\frac{k_{2ij}K_i\left(t\right)}{1+\sum_v\left(k_{1iv}+k_{2iv}\right)\hat{K}_v\left(t\right)}\\
&+\sum_i\left(r_i+F_1\left(\bar{R}_i,\frac{\dot{K}_i\left(t\right)}{K_i\left(t\right)}\right)+\tau\left(\bar{R}\left(K_i,X_i\right)\right)\Delta f_1^{\prime}\left(K_i\left(t\right)\right)\right)\frac{k_{1ij}K_i\left(t\right)}{1+\sum_v\left(k_{1iv}+k_{2iv}\right)\hat{K}_v\left(t\right)}-\bar{r}
\end{aligned}$$

The identity:

$$\sum_l\frac{\hat{k}_{1lj}\hat{K}_l\left(t\right)+\hat{k}_{2lj}\hat{K}_l\left(t\right)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v\left(t\right)}+\sum_i\frac{k_{2ij}K_i\left(t\right)}{1+\sum_v\left(k_{1iv}+k_{2iv}\right)\hat{K}_v\left(t\right)}+\frac{k_{1ij}K_i\left(t\right)}{1+\sum_v\left(k_{1iv}+k_{2iv}\right)\hat{K}_v\left(t\right)}=1$$

leads to:

$$\begin{aligned}
&R_j^{\prime}-\bar{r}+\bar{r}\sum_l\left(\frac{\hat{k}_{1lj}\hat{K}_l\left(t\right)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v\left(t\right)}\right)\\
=&\sum_i\frac{\left(r_i+F_1\left(\bar{R}_i,\frac{\dot{K}_i(t)}{K_i(t)}\right)+\tau\left(\bar{R}\left(K_i,X_i\right)\right)\Delta f_1^{\prime}\left(K_i\left(t\right)\right)-\bar{r}\right)k_{1ij}K_i\left(t\right)}{1+\sum_v\left(k_{1iv}+k_{2iv}\right)\hat{K}_v\left(t\right)}
\end{aligned}$$

and the return equation is:

$$\frac{\hat{f}_j-\bar{r}}{1+\sum_v\hat{k}_{2jv}\hat{K}_v\left(t\right)}-\sum_l\left(\frac{\hat{k}_{1lj}\hat{K}_l\left(t\right)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v\left(t\right)}\right)\left(\frac{\hat{f}_l-\bar{r}}{1+\sum_v\hat{k}_{2lv}\hat{K}_v\left(t\right)}\right)\tag{207}$$

$$=\sum_i\frac{\left(r_i+F_1\left(\bar{R}_i,\frac{\dot{K}_i(t)}{K_i(t)}\right)+\tau\left(\bar{R}\left(K_i,X_i\right)\right)\Delta f_1^{\prime}\left(K_i\left(t\right)\right)-\bar{r}\right)k_{1ij}K_i\left(t\right)}{1+\sum_v\left(k_{1iv}+k_{2iv}\right)\hat{K}_v\left(t\right)}\tag{208}$$

Appendix 2 From large number of agents to field formalism

This appendix summarizes the most useful steps of the method developed in Gosselin, Lotz and Wambst (2017, 2020, 2021), to switch from the probabilistic description of the model to the field theoretic formalism and summarizes the translation of a generalization of (10) involving different time variables. By convention and unless otherwise mentioned, the symbol of integration $\int$ refers to all the variables involved.

A2.1 Probabilistic formalism

The probabilistic formalism for a system with N identical economic agents in interaction is based on the minimization functions described in the text. Classically, the dynamics derives through the optimization problem of these functions. The probabilistic formalism relies on the contrary on the fact, that, due to uncertainties, shocks... agents do not optimize fully these functions. Moreover, given the large number of agents, there may be some discrepancy between agents minimization functions, and this fact may be translated in an uncertainty of behavior around some average minimization, or objective function.

We thus assume that each agent chooses for his action a path randomly distributed around the optimal path. The agent's behavior can be described as a weight that is an exponential of the intertemporal utility, that concentrates the probability around the optimal path. This feature models some internal uncertainty as well as non-measurable shocks. Gathering all agents, it yields a probabilistic description of the system in terms of a probabilistic weight.

In general, this weight includes utility functions and internalizes forward-looking behaviors, such as intertemporal budget constraints and interactions among agents. These interactions may for instance arise through constraints, since income flows depend on other agents demand. The probabilistic description then allows to compute the transition functions of the system, and in turn compute the probability for a system to evolve from an initial state to a final state within a given time span. They have the form of Euclidean path integrals.

In the context of the present paper, we have seen that the minimization functions for the system considered in this work have the form:

$$\int dt \left(\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right)\right)^2 + \sum_i \left(\sum_{j,k,l...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right)\right)\right) \tag{209}$$

The minimization of this function will yield a dynamic equation for N agents in interaction described by a set of dynamic variables $\mathbf{A}_i(t)$ during a given timespan T.

The probabilistic description is straightforwardly obtained from (209). The probability associated to a configuration $(\mathbf{A}_i(t))_{\substack{i=1,...,N \\ 0\leqslant t\leqslant T}}$ is directly given by:

$$\mathcal{N} \exp\left(-\frac{1}{\sigma^2}\int dt \left(\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right)\right)^2 + \sum_i \left(\sum_{j,k,l...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right)\right)\right)\right) \tag{210}$$

where $\mathcal{N}$ is a normalization factor and σ^2 is a variance whose magnitude describes the amplitude of deviations around the optimal path.

As in the paper, the system is in general modelled by several equations, and thus, several minimization function. The overall system is thus described by several functions, and the minimization function of the whole system is described by the set of functions:

$$\int dt \left(\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right)\right)^2 + \sum_i \left(\sum_{j,k,l...} g^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right)\right)\right) \tag{211}$$

where α runs over the set equations describing the system's dynamics. The associated weight is then:

$$\mathcal{N}\exp\left(-\left(\sum_{i,\alpha}\frac{1}{\sigma_{\alpha}^{2}}\int dt\left(\frac{d\mathbf{A}_{i}\left(t\right)}{dt}-\sum_{j,k,l...}f^{\left(\alpha\right)}\left(\mathbf{A}_{i}\left(t\right),\mathbf{A}_{j}\left(t\right),\mathbf{A}_{k}\left(t\right),\mathbf{A}_{l}\left(t\right)...\right)\right)^{2}\right.\right.$$
$$\left.\left.+\sum_{i,\alpha}\left(\sum_{j,k,l...}g^{\left(\alpha\right)}\left(\mathbf{A}_{i}\left(t\right),\mathbf{A}_{j}\left(t\right),\mathbf{A}_{k}\left(t\right),\mathbf{A}_{l}\left(t\right)...\right)\right)\right)\right) \tag{212}$$

The appearance of the sum of minimization functions in the probabilistic weight (212) translates the hypothesis that the deviations with respect to the optimization of the functions (211) are assumed to be independent.

For a large number of agents, the system described by (212) involves a large number of variables $K_{i}\left(t\right)$, $P_{i}\left(t\right)$ and $X_{i}\left(t\right)$ that are difficult to handle. To overcome this difficulty, we consider the space H of complex functions defined on the space of a single agent's actions. The space H describes the collective behavior of the system. Each function Ψ of H encodes a particular state of the system. We then associate to each function Ψ of H a statistical weight, i.e. a probability describing the state encoded in Ψ. This probability is written $\exp\left(-S\left(\Psi\right)\right)$, where $S\left(\Psi\right)$ is a functional, i.e. the function of the function Ψ. The form of $S\left(\Psi\right)$ is derived directly from the form of (212) as detailed in the text. As seen from (212), this translation can in fact be directly obtained from the sum of "classical" minimization functions weighted by the factors $\frac{1}{\sigma_{\alpha}^{2}}$:

$$\sum_{i,\alpha}\frac{1}{\sigma_{\alpha}^{2}}\int dt\left(\frac{d\mathbf{A}_{i}\left(t\right)}{dt}-\sum_{j,k,l...}f^{\left(\alpha\right)}\left(\mathbf{A}_{i}\left(t\right),\mathbf{A}_{j}\left(t\right),\mathbf{A}_{k}\left(t\right),\mathbf{A}_{l}\left(t\right)...\right)\right)^{2}$$
$$+\sum_{i,\alpha}\left(\sum_{j,k,l...}g^{\left(\alpha\right)}\left(\mathbf{A}_{i}\left(t\right),\mathbf{A}_{j}\left(t\right),\mathbf{A}_{k}\left(t\right),\mathbf{A}_{l}\left(t\right)...\right)\right)$$

This is this shortcut we used in the text.

A2.2 Interactions between agents at different times

A straightforward generalization of (10) involve agents interactions at different times. The terms considered have the form:

$$\sum_{i}\left(\frac{d\mathbf{A}_{i}\left(t\right)}{dt}-\sum_{j,k,l...}\int f\left(\mathbf{A}_{i}\left(t_{i}\right),\mathbf{A}_{j}\left(t_{j}\right),\mathbf{A}_{k}\left(t_{k}\right),\mathbf{A}_{l}\left(t_{l}\right)...,\mathbf{t}\right)\mathbf{dt}\right)^{2} \tag{213}$$
$$+\sum_{i}\sum_{j,k,l...}\int g\left(\mathbf{A}_{i}\left(t_{i}\right),\mathbf{A}_{j}\left(t_{j}\right),\mathbf{A}_{k}\left(t_{k}\right),\mathbf{A}_{l}\left(t_{l}\right)...,\mathbf{t}\right)\mathbf{dt}$$

where $\mathbf{t}$ stands for $\left(t_{i},t_{j},t_{k},t_{l}...\right)$ and $\mathbf{dt}$ stands for $dt_{i}dt_{j}dt_{k}dt_{l}...$

The translation is straightforward. We introduce a time variable θ on the field side and the fields write $\left|\Psi\left(\mathbf{A},\theta\right)\right|^{2}$ and $\left|\hat{\Psi}\left(\hat{\mathbf{A}},\hat{\theta}\right)\right|^{2}$. The second term in (213) becomes:

$$\sum_{i}\sum_{j}\sum_{j,k...}\int g\left(\mathbf{A}_{i}\left(t_{i}\right),\mathbf{A}_{j}\left(t_{j}\right),\mathbf{A}_{k}\left(t_{k}\right),\mathbf{A}_{l}\left(t_{l}\right)...,\mathbf{t}\right)\mathbf{dt}$$
$$\rightarrow\int g\left(\mathbf{A},\mathbf{A}',\mathbf{A}'',\hat{\mathbf{A}},\hat{\mathbf{A}}'...,\boldsymbol{\theta},\hat{\boldsymbol{\theta}}\right)\left|\Psi\left(\mathbf{A},\theta\right)\right|^{2}\left|\Psi\left(\mathbf{A}',\theta'\right)\right|^{2}\left|\Psi\left(\mathbf{A}'',\theta''\right)\right|^{2}d\mathbf{A}d\mathbf{A}'d\mathbf{A}'' \tag{214}$$
$$\times\left|\hat{\Psi}\left(\hat{\mathbf{A}},\hat{\theta}\right)\right|^{2}\left|\hat{\Psi}\left(\hat{\mathbf{A}}',\hat{\theta}'\right)\right|^{2}d\hat{\mathbf{A}}d\hat{\mathbf{A}}'\mathbf{d}\boldsymbol{\theta}\mathbf{d}\hat{\boldsymbol{\theta}}$$

where $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$ are the multivariables $(\theta, \theta', \theta''...)$ and $\left(\hat{\theta}, \hat{\theta}'...\right)$ respectively and $\mathbf{d}\boldsymbol{\theta}\mathbf{d}\hat{\boldsymbol{\theta}}$ stands for $d\theta d\theta' d\theta''...$ and $d\hat{\theta}d\hat{\theta}'...$

Similarly, the first term in (213) translates as:

$$\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} \int f\left(\mathbf{A}_i(t_i), \mathbf{A}_j(t_j), \mathbf{A}_k(t_k), \mathbf{A}_l(t_l)..., \mathbf{t}\right) \mathbf{dt}\right)^2 \tag{215}$$

$$\rightarrow \int \Psi^\dagger(\mathbf{A}, \theta)\left(-\nabla_{\mathbf{A}^{(\alpha)}}\left(\frac{\sigma^2_{\mathbf{A}^{(\alpha)}}}{2}\nabla_{\mathbf{A}^{(\alpha)}} - \Lambda(\mathbf{A}, \theta)\right)\right)\Psi(\mathbf{A}, \theta)\, d\mathbf{A} d\theta \tag{216}$$

by:

$$\begin{aligned}\Lambda(\mathbf{A}, \theta) &= \int f^{(\alpha)}\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'..., \boldsymbol{\theta}, \hat{\boldsymbol{\theta}}\right)\left|\Psi(\mathbf{A}', \theta')\right|^2 \left|\Psi(\mathbf{A}'', \theta'')\right|^2 d\mathbf{A}' d\mathbf{A}'' \\ &\quad \times \left|\hat{\Psi}\left(\hat{\mathbf{A}}, \theta\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}', \theta''\right)\right|^2 d\hat{\mathbf{A}} d\hat{\mathbf{A}}' \mathbf{d}\bar{\boldsymbol{\theta}} \mathbf{d}\hat{\boldsymbol{\theta}}\end{aligned} \tag{217}$$

with $\mathbf{d}\bar{\boldsymbol{\theta}} = d\theta' d\theta''$.

Ultimately, as in the text, additional terms (17):

$$\begin{aligned}&\Psi^\dagger(\mathbf{A}, \theta)\left(-\nabla_\theta\left(\frac{\sigma^2_\theta}{2}\nabla_\theta - 1\right)\right)\Psi(\mathbf{A}, \theta) \\ &+\hat{\Psi}^\dagger\left(\hat{\mathbf{A}}, \theta\right)\left(-\nabla_\theta\left(\frac{\sigma^2_\theta}{2}\nabla_\theta - 1\right)\right)\hat{\Psi}\left(\hat{\mathbf{A}}, \theta\right) + \alpha\left|\Psi(\mathbf{A})\right|^2 + \alpha\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2\end{aligned} \tag{218}$$

are included to the action functional to take into account for the time variable.

Appendix 3 Translation of the model minimization functions

We call $\hat{\Psi}$ the field describing the investors. It depends on the two variables $\hat{K}$ and $\hat{X}$ and We call Ψ the field describing the investors. It depends on the two variables $\hat{K}$ and $\hat{X}$

A3.1. Firms action functional and firms' return

A3.1.1 Translation of the minimization function: Physical capital accumulation

Let us start by translating in terms of fields the minimization function associated to (42):

$$\sum_i \left(\frac{d}{dt}K_{ip}(t) - f_1'(K_i(t))\, K_{ip}(t)\right)^2 \tag{219}$$

with:

$$f_1'(K_i(t)) = \left(1 + \sum_\nu k_{2j\nu}\hat{K}_\nu(t)\right) r_i - \bar{r}\sum_v k_{2lv}\hat{K}_v(t) \tag{220}$$

To translate the full term (219), we use the translation (17) of a type-(16) expression. The gradient term appearing in equation (17) is ∇_K. We thus obtain the translation:

$$\begin{aligned}&\sum_i \left(\frac{d}{dt}K_{ip}(t) - f_1'(K_i(t))\, K_{ip}(t)\right)^2 \\ &\rightarrow \int \Psi^\dagger(K, X)\left(-\nabla_K\left(\frac{\sigma^2_K}{2}\nabla_K + \Lambda(X, K)\right)\right)\Psi(K, X)\, dK dX\end{aligned} \tag{221}$$

Note that the variance σ_K^2 reflects the probabilistic nature of the model hidden behind the field formalism. This σ_K^2 represents the characteristic level of uncertainty of the sectors space dynamics. It is a parameter of the model. The term $\Lambda(X,K)$ is the translation of the term $f_1'(K_i(t))$ in the parenthesis of (219). This term is a function of one sole index "i". In that case, the term Λ is simply obtained by replacing (K_i, X_i) by (K, X). We use the translation (15) of (13)-type term. The term (220). This term contains no derivative. The form of the translation is given by formula (11).

The first step of the translation is to replace r_i by a function $r(K,X)$ and $\hat{K}_\nu(t)$ by the variable $\hat{K}$, and to replace:

$$k_{2j\nu} \rightarrow k_2\left(K, X, \hat{K}, \hat{X}\right)$$

Given our assumptions in the txt, k_2 is a function of the sole variable X:

$$k_{2j\nu} \rightarrow k_2(X)$$

The other one have been sorted out in a linear approximation. This means that $k_2(X)$ captures the average share of loans in investmnts in sector X. This leads to substitute:

$$\begin{aligned} & f_1'(K_i(t)) \\ \rightarrow \quad & \Lambda(X,K) = \left(1 + \int k_2(X) \int \left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\left(\hat{K},\hat{X}\right)\right) r(K,X) - \bar{r} \int k_2(X) \int \left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\left(\hat{K},\hat{X}\right) \end{aligned}$$

The action functional for the field of firms is:

$$-\Psi^\dagger(K,X)\left(\nabla_{K_p}\left(\sigma_K^2\nabla_{K_p} - f_1'(K,X)K_p\right)\right)\Psi(K,X)$$

where:

$$\begin{aligned} f_1'(X) &= \left(1 + \underline{k}_2(X)\right) f_1(X) - \bar{r}\underline{k}_2\left(\hat{X}\right) \\ &= f_1(X) + \left(f_1(X) - \bar{r}\right) \int k_2\left(X,\hat{X}_1\right) \hat{K}_1 \left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1 \end{aligned}$$

A3.1.1 Introducing fluctuation of number of firms

The description is completed by assuming that the number of firms is not constant in one sector, but that some rigidity prevents this number to change quickly. Rather than introducing some dynamics for the number of firms as in our previous works, we rather consider for the sake of simplicity that frms number in one sector fluctuate around some average quantities. An hypothesis that is sufficient in the medium run to study the impact of capital flows. Technically, this amounts to introduce a potential in the field action ensuring this type of rigidity. The potential has the frm:

$$\frac{1}{2\epsilon}\left(\left|\Psi(K,X)\right|^2 - \left|\Psi_0(X)\right|^2\right)^2$$

and the full action for firms wrts:

$$-\Psi^\dagger(K,X)\left(\nabla_{K_p}\left(\sigma_K^2\nabla_{K_p} - f_1'(K,X)K_p\right)\right)\Psi(K,X) + \frac{1}{2\epsilon}\left(\left|\Psi(K,X)\right|^2 - \left|\Psi_0(X)\right|^2\right)^2 \tag{222}$$

A3.2 Field action functional and return for financial markets

The functions to be translated are those of the financial capital dynamics We consider the minimisation function associatd to the dynamics (46):

$$\left(\frac{d}{dt}\hat{K}_j(t)-\sum_l\left(1-\frac{\hat{k}_{1jl}\hat{K}_j(t)}{1+\sum_v\hat{k}_{jv}\hat{K}_v(t)}\right)^{-1}f_l\hat{K}_l(t)\right)^2=\left(\frac{d}{dt}\hat{K}_j(t)-\sum_l(1-M)_{jl}^{-1}f_l\hat{K}_l(t)\right)^2 \tag{223}$$

where:

$$M_{jm}=\frac{\hat{k}_{jm}\hat{K}_j(t)}{1+\sum_\nu\hat{k}_{j\nu}\hat{K}_\nu(t)} \tag{224}$$

Both expressions include a time derivative and are thus of type (12). As for the real economy, the application of the translation rules is straightforward using the general translation formula of expression (16) in (17), into:

$$\int\hat{\Psi}^\dagger\left(\hat{K},\hat{X}\right)\left(-\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}+\Lambda\left(\hat{K},\hat{X}\right)\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right)d\hat{K}d\hat{X}$$

The function $\Lambda\left(\hat{K},\hat{X}\right)$ is obtained, as before, by translating the term following the derivative in the function (223) and b rplcng vrbl:

$$\begin{aligned}(K_i,X_i)&\rightarrow(K,X)\\(K_l,X_l)&\rightarrow(K',X')\\\left(\hat{K}_j,\hat{X}_j\right)&\rightarrow\left(\hat{K},\hat{X}\right)\end{aligned}$$

We also need to translate the return equation for investors (54):

$$\begin{aligned}&\sum_l\left(\delta_{jl}-\frac{\hat{k}_{1lj}\hat{K}_l(t)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v(t)}\right)\left(\frac{\hat{f}_j-\bar{r}}{1+\sum_v\hat{k}_{2jv}\hat{K}_v(t)}\right)\\&+\sum_l\left(\bar{r}-\frac{\left(1+\hat{f}\left(\hat{K}_{vp}(t)\right)\right)}{\sum_m\hat{k}_{2vm}\hat{K}_m}\right)H\left(-\left(1+\hat{f}\left(\hat{K}_{vp}(t)\right)\right)\right)\frac{\hat{k}_{2lj}\hat{K}_l(t)}{1+\sum_v\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_v(t)}\\&+\sum_i\left(\bar{r}-\frac{(1+f_1'(K_i(t)))}{\sum_m k_{2im}\hat{K}_m}\right)H\left(-(1+f_1'(K_i(t)))\right)\frac{k_{2ij}K_i(t)}{1+\sum_v(k_{1iv}+k_{2iv})\hat{K}_v(t)}\\=&\sum_i\left(\frac{f_1'(K_i(t))-\bar{r}}{1+\sum_\nu k_{2j\nu}\hat{K}_\nu(t)}+\Delta F_\tau\left(\bar{R}(K_i,X_i)\right)\right)\frac{k_{1ij}K_i(t)}{1+\sum_v(k_{1iv}+k_{2iv})\hat{K}_v(t)}\end{aligned} \tag{225}$$

This equation can be written as compctly as:

$$RT=0$$

which is implemented as a minimisation function:

$$\frac{1}{\epsilon}(RT)^2 \tag{226}$$

with $\epsilon<<1$. The translation of such a term will lead to a potential that will corresponds to a field contraints when $\epsilon\rightarrow0$.

The translation of (223) and (226) is done in several steps, by introducing some notations and performing some intermediate translations of the coefficients arising in these functions.

A3.2.1 Translations of shares of invested capital

Let us first define some parameters that will appear in the translation. We define:

$$\begin{aligned}
\underline{\hat{k}}_2\left(\hat{X}'\right) &= \int \hat{k}_2\left(\hat{X}',\hat{X}_1\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1 \\
\underline{\hat{k}}_1\left(\hat{X}'\right) &= \int \hat{k}_1\left(\hat{X}',\hat{X}_1\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1 \\
\underline{\hat{k}}\left(\hat{X}'\right) &= \int \hat{k}\left(\hat{X}',\hat{X}_1\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1 = \underline{\hat{k}}_1\left(\hat{X}'\right)+\underline{\hat{k}}_2\left(\hat{X}'\right)
\end{aligned}$$

These three parameters can also be writen in a more compact way as:

$$\underline{\hat{k}}_\eta\left(\hat{X}'\right) = \int \hat{k}_\eta\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2 d\hat{X}$$

$$\underline{\hat{k}}_1\left(\hat{X}'\right)+\underline{\hat{k}}_2\left(\hat{X}'\right) = \underline{\hat{k}}\left(\hat{X}'\right)$$

These parameters correspond to the amount of capital invested in the financial sector $\hat{X}'$, in terms of loans ($\underline{\hat{k}}_2$), participation ($\underline{\hat{k}}_1$), and their total sum ($\underline{\hat{k}}_1+\underline{\hat{k}}_2$). We also define the shares of capital outgoing from sector $\hat{X}'$ towards other financial sectors:

$$\begin{aligned}
\underline{\hat{k}}_{2E}\left(\hat{X}'\right) &= \int \hat{k}_2\left(\hat{X}_1,\hat{X}'\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1 \qquad (227)\\
\underline{\hat{k}}_{1E}\left(\hat{X}'\right) &= \int \hat{k}_1\left(\hat{X}_1,\hat{X}'\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1 \\
\underline{\hat{k}}_{E}\left(\hat{X}'\right) &= \int \hat{k}\left(\hat{X}_1,\hat{X}'\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1 = \underline{\hat{k}}_{1E}\left(\hat{X}'\right)+\underline{\hat{k}}_{2E}\left(\hat{X}'\right)
\end{aligned}$$

Similarly, we define the amount of capital invested in real sector $\hat{X}'$:

$$\begin{aligned}
\underline{k}_2\left(\hat{X}'\right) &= \int k_2\left(\hat{X}',\hat{X}_1\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1 \\
\underline{k}_1\left(\hat{X}'\right) &= \int k_1\left(\hat{X}',\hat{X}_1\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1 \\
\underline{k}\left(\hat{X}'\right) &= \int k\left(\hat{X}',\hat{X}_1\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1 = \underline{k}_1\left(\hat{X}'\right)+\underline{k}_2\left(\hat{X}'\right)
\end{aligned}$$

and the outgoing amount of capital outgoing from $\hat{X}'$ toward the real sectrs:

$$\begin{aligned}
\underline{k}_{2E}\left(\hat{X}'\right) &= \int k_2\left(\hat{X}_1,\hat{X}'\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1 \\
\underline{k}_{1E}\left(\hat{X}'\right) &= \int k_1\left(\hat{X}_1,\hat{X}'\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1 \\
\underline{k}_{E}\left(\hat{X}'\right) &= \int k\left(\hat{X}_1,\hat{X}'\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 d\hat{K}_1 d\hat{X}_1 = \underline{k}_{1E}\left(\hat{X}'\right)+\underline{k}_{2E}\left(\hat{X}'\right)
\end{aligned}$$

where :

$$\underline{k}_\eta\left(\hat{X}'\right) = \int k_\eta\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2 d\hat{X}$$

and:

$$\hat{k}_1\left(\hat{X},\hat{Y}\right)+\hat{k}_2\left(\hat{X},\hat{Y}\right)\rightarrow\hat{k}\left(\hat{X},\hat{Y}\right)$$

Once these quantities are defined, the translation method will lead to the following substitutions, from the micro system to the field model:

$$1+\sum_{v}\left(\hat{k}_{1lv}+\hat{k}_{2lv}\right)\hat{K}_{v}\left(t\right)\rightarrow 1+\int\left(\hat{k}\left(\hat{X},\hat{Y}\right)\right)\hat{K}''\left|\hat{\Psi}\left(\hat{K}'',Y\right)\right|^{2}d\hat{K}''Y$$

$$\frac{\left(\hat{k}_{1jl}+\hat{k}_{2jl}\right)}{1+\sum_{v}\left(\hat{k}_{1jv}+\hat{k}_{2jv}\right)\hat{K}_{v}\left(t\right)}\rightarrow\frac{\hat{k}\left(\hat{X},\hat{X}'\right)}{1+\int\hat{k}\left(\hat{X},\hat{Y}\right)\hat{K}''\left|\hat{\Psi}\left(\hat{K}'',Y\right)\right|^{2}d\hat{K}''Y}$$

$$\frac{\left(k_{1jl}+k_{2jl}\right)}{1+\sum_{v}\left(k_{1jv}+k_{2jv}\right)\hat{K}_{v}\left(t\right)}\rightarrow\frac{k\left(\hat{X},\hat{X}'\right)}{1+\int k\left(\hat{X},\hat{Y}\right)\hat{K}''\left|\hat{\Psi}\left(\hat{K}'',Y\right)\right|^{2}d\hat{K}''Y}$$

$$M_{jl}\rightarrow M\left(\hat{K},\hat{X},\hat{K}',\hat{X}'\right)=\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}}{1+\underline{\hat{k}}\left(\hat{X}\right)}$$

and, ultimately:

$$\begin{aligned}&\left(1+\sum_{\nu}\hat{k}_{j\nu}\hat{K}_{\nu}\left(t\right)\right)\left(1-M\right)_{jl}^{-1}\left(R_{l}-\bar{r}\sum_{v}\hat{k}_{2lv}\hat{K}_{v}\left(t\right)\right)\\ \rightarrow\;&\left(1+\underline{\hat{k}}\left(\hat{X}\right)\right)\left(1-M\left(\hat{K},\hat{X},\hat{K}',\hat{X}'\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}\right)^{-1}\left(R\left(\hat{X}'\right)-\bar{r}\underline{\hat{k}}_{2}\left(\hat{X}'\right)\right)\end{aligned}$$

A3.2.2 Translation of constraints and their averages

We consider that each investor j invest, each period, the entire amount of his capital, which translates in the constraint:

$$\sum_{\nu}\frac{\hat{k}_{vj}\hat{K}_{v}\left(t\right)}{1+\sum_{m}\hat{k}_{vm}\hat{K}_{m}\left(t\right)}+\sum_{i}\frac{k_{ij}K_{i}\left(t\right)}{1+\sum_{v}k_{iv}\hat{K}_{v}\left(t\right)}=1$$

that becomes in terms of fields:

$$\int\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}\frac{\hat{k}\left(\hat{X}',\hat{X}\right)\hat{K}'}{1+\underline{\hat{k}}\left(\hat{X}'\right)}+\int\left|\Psi\left(K',X'\right)\right|^{2}\frac{k\left(X',X\right)K'}{1+\underline{k}\left(\hat{X}'\right)}=1$$

that can rewritten, given our previous definition (227):

$$\frac{\underline{\hat{k}}_{E}\left(\hat{X}\right)}{1+\underline{\hat{k}}}+\frac{\underline{k}_{E}\left(X\right)}{1+\underline{k}}=1$$

We can consider the average of this equation by defining:

$$\int\underline{\hat{k}}_{i}\left(\hat{X}\right)\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}=\underline{\hat{k}}_{i}$$

$$\int\underline{k}_{i}\left(X\right)\left|\Psi\left(K,X\right)\right|^{2}=\underline{k}_{i}$$

$$\int\underline{\hat{k}}_{iE}\left(\hat{X}\right)\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}=\underline{\hat{k}}_{iE}$$

The average constraints thus write:

$$\begin{aligned}
\frac{\hat{\underline{k}}_E}{1+\hat{\underline{k}}}+\frac{\underline{k}_E}{1+\underline{k}}&=1\\
\frac{\hat{\underline{k}}}{1+\hat{\underline{k}}}+\frac{\underline{k}}{1+\underline{k}}&=1\\
\frac{\underline{k}}{1+\underline{k}}&=\frac{1}{1+\hat{\underline{k}}}
\end{aligned}\tag{228}$$

A3.2.3 Estimation of $(1-M_{jl})^{-1}$

The matrix $(1-M_{jl})^{-1}$ arising in (46) and (48) is averaged on some vector with coordinate j. In field theory, this corresponds to sum over the argument for the field, that is computing averages. Expanding in series the field translation of $(1-M_{jl})^{-1}$, leads to:

$$\begin{aligned}
(1-M_{jl})^{-1} &\rightarrow \frac{1}{1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}\\
&= 1+M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\\
&\quad +M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}_1',\hat{X}_1'\right)\right)\left|\hat{\Psi}\left(\hat{K}_1',\hat{X}_1'\right)\right|^2 M\left(\left(\hat{K}_1',\hat{X}_1'\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}_1',\hat{X}_1'\right)\right|^2\\
&\quad +M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}_1',\hat{X}_1'\right)\right)\left|\hat{\Psi}\left(\hat{K}_1',\hat{X}_1'\right)\right|^2\\
&\quad \times M\left(\left(\hat{K}_1',\hat{X}_1'\right),\left(\hat{K}_2',\hat{X}_2'\right)\right)\left|\hat{\Psi}\left(\hat{K}_2',\hat{X}_2'\right)\right|^2 M\left(\left(\hat{K}_2',\hat{X}_2'\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2
\end{aligned}$$

and the application to any vector $A\left(\left(\hat{K}',\hat{X}'\right)\right)$:

$$\int\frac{1}{1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}A\left(\left(\hat{K}',\hat{X}'\right)\right)$$

can be approximated by:

$$\begin{aligned}
&1+M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2 A\left(\left(\hat{K}',\hat{X}'\right)\right)\\
&+M\left(\left(\hat{K},\hat{X}\right),\left(\left\langle\hat{K}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)\left\|\hat{\Psi}\right\|^2 M\left(\left(\left\langle\hat{K}\right\rangle,\left\langle\hat{X}\right\rangle\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2 A\left(\left(\hat{K}',\hat{X}'\right)\right)\\
&+...\\
=\;&1+M\left(\left(\hat{K},\hat{X}\right),\left(\left\langle\hat{K}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)\left\|\hat{\Psi}\right\|^2\langle A\rangle\\
&+M\left(\left(\hat{K},\hat{X}\right),\left(\left\langle\hat{K}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)\left\|\hat{\Psi}\right\|^2 M\left(\left(\left\langle\hat{K}\right\rangle,\left\langle\hat{X}\right\rangle\right),\left(\left\langle\hat{K}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)\left\|\hat{\Psi}\right\|^2\langle A\rangle+...
\end{aligned}$$

This corresponds to replace:

$$\int\frac{1}{1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}A\left(\left(\hat{K}',\hat{X}'\right)\right)$$

with:

$$A\left(\hat{K},\hat{X}\right)+\frac{M\left(\left(\hat{K},\hat{X}\right),\left(\left\langle\hat{K}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)\left\|\hat{\Psi}\right\|^2\langle A\rangle}{1-M\left(\left(\left\langle\hat{K}\right\rangle,\left\langle\hat{X}\right\rangle\right),\left(\left\langle\hat{K}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)\left\|\hat{\Psi}\right\|^2}\tag{229}$$

$$A\left(\hat{K},\hat{X}\right)+\frac{\frac{\hat{\underline{k}}(\hat{X})}{1+\hat{\underline{k}}(\hat{X})}\langle A\rangle}{1-\frac{\hat{\underline{k}}(\langle\hat{X}\rangle)}{1+\hat{\underline{k}}(\langle\hat{X}\rangle)}}$$

$$A\left(\hat{K},\hat{X}\right)+\frac{1+\hat{\underline{k}}\left(\left\langle\hat{X}\right\rangle\right)}{1+\hat{\underline{k}}\left(\hat{X}\right)}\hat{\underline{k}}\left(\hat{X}\right)\langle A\rangle$$

$$\Delta M\hat{\underline{k}}\left(\hat{X}\right)$$

As a consequence, given (229) we obtain:

$$\begin{aligned}&\frac{1}{1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}A\\ \simeq\quad& A+\frac{M\left(\left(\hat{K},\hat{X}\right),\left(\left\langle\hat{K}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)\left\|\hat{\Psi}\right\|^2\langle A\rangle}{1-M\left(\left(\left\langle\hat{K}\right\rangle,\left\langle\hat{X}\right\rangle\right),\left(\left\langle\hat{K}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)\left\|\hat{\Psi}\right\|^2}\end{aligned}$$

A3.2.3 Action functional for the investors' field

We can use the translation method to derive the field version of (223). The derivation is straightforward:

$$-\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\left(\nabla_{\hat{K}}\left(\sigma_{\hat{K}}^2\nabla_{\hat{K}}-\hat{g}\left(\hat{K},\hat{X}\right)\hat{K}\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right)+\frac{1}{2\hat{\mu}}\left(\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2-\left|\hat{\Psi}_0\left(\hat{X}\right)\right|^2\right)^2$$

where:

$$\hat{g}\left(\hat{K},\hat{X}\right)=\left(1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\right)^{-1}\hat{f}\left(\hat{K}',\hat{X}'\right)$$

and

$$\begin{aligned}\hat{f}\left(\hat{X}\right)&=\left(1+\hat{\underline{k}}_2\left(\hat{X}\right)\right)\left(1+R_{\nu}\left(\hat{X}\right)\right)\\&=\left(1+\int\hat{k}_2\left(\hat{X},\hat{X}_1\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2d\hat{K}_1d\hat{X}_1\right)\left(1+R_{\nu}\left(\hat{X}\right)\right)\end{aligned}$$

:translates:

$$\hat{f}\left(\hat{K}_v\left(t\right)\right)=\left(1+\sum_m\hat{k}_{2vm}\hat{K}_m\right)\left(1+R_{\nu}\right)$$

A3.2.4 Investors' field returns translation

The translation of the return equations (80) for $R\left(\hat{X}\right)$ is obtained by dividing by $\hat{K}$. Imposing the constraint amounts to introduce the potential (226). This potential has no derivative and the translation is straightforward and given by formula (11):

$$\frac{\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}{\varepsilon^2}\left[\left(\delta\left(\hat{X}-\hat{X}'\right)-\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\hat{K}'}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\right)R\left(\hat{X}'\right)\right.$$
$$-\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\frac{\hat{k}_2\left(\hat{X}',\hat{X}\right)\hat{K}'}{1+\underline{\hat{k}}\left(\hat{X}'\right)}$$
$$\times\left(\bar{r}-\left(\bar{r}-\left(\frac{1+R\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\hat{X}'\right)}+R\left(\hat{X}'\right)\right)\right)H\left(-\left(\bar{r}-\left(\frac{1+R\left(\hat{X}\right)}{\underline{\hat{k}}_2\left(\hat{X}'\right)}+R\left(\hat{X}'\right)\right)\right)\right)\right)$$
$$-\left|\Psi\left(K',X'\right)\right|^2\frac{k_2\left(X',\hat{X}\right)K'}{1+\underline{k}\left(X'\right)}$$
$$\times\left(\left(\bar{r}-\left(\bar{r}-\left(\frac{1+f_1\left(X'\right)}{\underline{k}_2\left(X'\right)}+f_1\left(X'\right)\right)\right)H\left(-\left(\bar{r}-\left(\frac{1+f_1\left(X'\right)}{\underline{k}_2\left(X'\right)}+f_1\left(K',X'\right)\right)\right)\right)\right)\right.$$
$$\left.\left.+\frac{k_1\left(X',\hat{X}\right)K'}{1+\underline{k}\left(X'\right)}f_1\left(K',X',\Psi,\hat{\Psi}\right)\right)\right]$$

The function H is the Heaviside function $H(x)=1$ for $x\geqslant 0$ and $H(x)=0$ for $x\leqslant 0$. When $\varepsilon\to 0$, it implies an equation defining the returns:

$$\left(\delta\left(\hat{X}-\hat{X}'\right)-\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\hat{K}'}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\right)R\left(\hat{X}'\right) \tag{230}$$
$$=\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\frac{\hat{k}_2\left(\hat{X}',\hat{X}\right)\hat{K}'}{1+\underline{\hat{k}}\left(\hat{X}'\right)}$$
$$\times\left(\bar{r}-\left(\bar{r}-\left(\frac{1+R\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\hat{X}'\right)}+R\left(\hat{X}'\right)\right)\right)H\left(-\left(\bar{r}-\left(\frac{1+R\left(\hat{X}\right)}{\underline{\hat{k}}_2\left(\hat{X}'\right)}+R\left(\hat{X}'\right)\right)\right)\right)\right)$$
$$+\left|\Psi\left(K',X'\right)\right|^2\frac{k_2\left(X',\hat{X}\right)K'}{1+\underline{k}\left(X'\right)}$$
$$\times\left(\left(\bar{r}-\left(\bar{r}-\left(\frac{1+f_1\left(X'\right)}{\underline{k}_2\left(X'\right)}+f_1\left(X'\right)\right)\right)H\left(-\left(\bar{r}-\left(\frac{1+f_1\left(X'\right)}{\underline{k}_2\left(X'\right)}+f_1\left(K',X'\right)\right)\right)\right)\right)\right.$$
$$\left.+\frac{k_1\left(X',\hat{X}\right)K'}{1+\underline{k}\left(X'\right)}f_1\left(K',X',\Psi,\hat{\Psi}\right)\right)$$

and $f\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)$ is defined by:

$$R\left(\hat{X}\right)=\frac{f\left(\hat{X}\right)}{1+\underline{\hat{k}}_2\left(\hat{X}\right)}+\bar{r}\frac{\underline{\hat{k}}_2\left(\hat{X}\right)}{1+\underline{\hat{k}}_2\left(\hat{X}\right)}$$

This equation allows to rewrite the quantities arising in (230) :

$$\begin{aligned}
\bar{r}-\left(\frac{1+R\left(\hat{X}^{\prime}\right)}{\underline{\hat{k}}_{2}\left(\hat{X}^{\prime}\right)}+R\left(\hat{X}^{\prime}\right)\right) &= \bar{r}-\left(\frac{1+\frac{f\left(\hat{X}^{\prime}\right)}{1+\underline{\hat{k}}_{2}\left(\hat{X}^{\prime}\right)}+\bar{r}\frac{\underline{\hat{k}}_{2}\left(\hat{X}^{\prime}\right)}{1+\underline{\hat{k}}_{2}\left(\hat{X}^{\prime}\right)}}{\underline{\hat{k}}_{2}\left(\hat{X}^{\prime}\right)}+\frac{f\left(\hat{X}^{\prime}\right)}{1+\underline{\hat{k}}_{2}\left(\hat{X}^{\prime}\right)}+\bar{r}\frac{\underline{\hat{k}}_{2}\left(\hat{X}^{\prime}\right)}{1+\underline{\hat{k}}_{2}\left(\hat{X}^{\prime}\right)}\right) \\
&= -\frac{1+f\left(\hat{X}^{\prime}\right)}{\underline{\hat{k}}_{2}\left(\hat{X}^{\prime}\right)}
\end{aligned}$$

with:

$$f_{1}(X)=\frac{f_{1}^{\prime}(K,X)}{1+k_{2}\left(\hat{X}\right)}+\frac{\bar{r}k_{2}\left(\hat{X}\right)}{1+k_{2}\left(\hat{X}\right)}+\Delta F_{\tau}\left(\bar{R}(K,X)\right)$$

and:

$$\Delta F_{\tau}\left(\bar{R}(K,X)\right)=F_{1}\left(\bar{R}(K,X)\right)+\tau\left(\bar{R}(K,X)\right)\Delta f_{1}^{\prime}(K)$$

Using these relations, equation (230) becomes:

$$\begin{aligned}
&\frac{f\left(\hat{X}\right)-\bar{r}}{1+\underline{\hat{k}}_{2}\left(\hat{X}\right)}-\frac{\hat{k}_{1}\left(\hat{X}^{\prime},\hat{X}\right)\hat{K}^{\prime}\left|\hat{\Psi}\left(\hat{K}^{\prime},\hat{X}^{\prime}\right)\right|^{2}}{1+\underline{\hat{k}}\left(\hat{X}^{\prime}\right)}\frac{f\left(\hat{X}^{\prime}\right)-\bar{r}}{1+\underline{\hat{k}}_{2}\left(\hat{X}^{\prime}\right)} \\
= &\int\left(\bar{r}+\frac{1+f\left(\hat{X}^{\prime}\right)}{\underline{\hat{k}}_{2}\left(\hat{X}^{\prime}\right)}H\left(-\frac{1+f\left(\hat{X}^{\prime}\right)}{\underline{\hat{k}}_{2}\left(\hat{X}^{\prime}\right)}\right)\right)\frac{\hat{k}_{2}\left(\hat{X}^{\prime},\hat{X}\right)\hat{K}^{\prime}\left|\hat{\Psi}\left(\hat{K}^{\prime},\hat{X}^{\prime}\right)\right|^{2}}{1+\underline{\hat{k}}\left(\hat{X}^{\prime}\right)} \\
&+\int\left(\bar{r}+\frac{1+f_{1}^{\prime}(X^{\prime})}{\underline{k}_{2}(X^{\prime})}H\left(-\frac{1+f_{1}^{\prime}(X^{\prime})}{\underline{k}_{2}(X^{\prime})}\right)\right)\frac{k_{2}\left(X^{\prime},\hat{X}\right)\left|\Psi(K^{\prime},X^{\prime})\right|^{2}K^{\prime}}{1+\underline{k}(X^{\prime})} \\
&+\int\frac{\left|\Psi(K^{\prime},X^{\prime})\right|^{2}k_{1}\left(X^{\prime},\hat{X}\right)K^{\prime}}{1+\underline{k}(X^{\prime})}\left(\frac{f_{1}^{\prime}(K,X)-\bar{r}k_{2}\left(\hat{X}\right)}{1+k_{2}\left(\hat{X}\right)}+\Delta F_{\tau}\left(\bar{R}(K,X)\right)\right)
\end{aligned}$$

28 Appendix 4 Alternate description of investors return equation

We rewrite the returns equation for investors in terms of an alternate set of parameters. These parameters represents relative shares of investments for an agent. This reformulation will be used while studying the dynamics of interacting groups of investors.

Defining:

$$\hat{S}_{\eta}\left(\hat{X}^{\prime},\hat{K}^{\prime},\hat{X}\right)=\frac{\hat{k}_{\eta}\left(\hat{X}^{\prime},\hat{X}\right)\hat{K}^{\prime}\left|\hat{\Psi}\left(\hat{K}^{\prime},\hat{X}^{\prime}\right)\right|^{2}}{1+\underline{\hat{k}}\left(\hat{X}^{\prime}\right)}$$

and:

$$S_{\eta}(X^{\prime},K^{\prime},X)=\frac{k_{\eta}\left(X^{\prime},\hat{X}\right)\left|\Psi(K^{\prime},X^{\prime})\right|^{2}K^{\prime}}{1+\underline{k}(X^{\prime})}$$

the equation for returns becomes:

$$0 = \int \left(\Delta\left(\hat{X},\hat{X}'\right) - \hat{S}_1\left(\hat{X}',\hat{K}',\hat{X}\right)\right) \frac{f\left(\hat{X}'\right)-\bar{r}}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)} d\hat{X}'d\hat{K}' \tag{231}$$
$$-\int S_1\left(X',K',\hat{X}\right)\left(\frac{f_1'\left(\hat{X}'\right)-\bar{r}}{1+\underline{k}_2\left(X'\right)} + \Delta F_\tau\left(\bar{R}\left(K',X'\right)\right)\right) dX'dK'$$
$$-\int \frac{1+f\left(\hat{X}'\right)}{\hat{\underline{k}}_2\left(\hat{X}'\right)} H\left(-\frac{1+f\left(\hat{X}'\right)}{\hat{\underline{k}}_2\left(\hat{X}'\right)}\right)\hat{S}_2\left(\hat{X}',\hat{K}',\hat{X}\right)d\hat{X}'d\hat{K}'$$
$$-\int \frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)} H\left(-\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)}\right) S_2\left(X',K',\hat{X}\right)dX'd\hat{K}'$$

Integrating this equation over $\hat{K}'$ leads to define the following quantities:

$$\hat{S}_\eta\left(\hat{X}',\hat{K}',\hat{X}\right) \rightarrow \int \frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)} \equiv \hat{S}_\eta\left(\hat{X}',\hat{X}\right)$$

and:

$$S_\eta\left(X',K',\hat{X}\right) \rightarrow \frac{k_\eta\left(X',\hat{X}\right)K_{X'}\left|\Psi\left(X'\right)\right|^2}{1+\underline{k}\left(X'\right)} \equiv S_\eta\left(X',\hat{X}\right)$$

and it yields to replace (231) by:

$$0 = \int \left(\Delta\left(\hat{X},\hat{X}'\right) - \hat{S}_1\left(\hat{X}',\hat{X}\right)\right) \frac{f\left(\hat{X}'\right)-\bar{r}}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)} d\hat{X}' \tag{232}$$
$$-\int S_1\left(X',\hat{X}\right)\left(\frac{f_1'\left(\hat{X}'\right)-\bar{r}}{1+\underline{k}_2\left(X'\right)} + \Delta F_\tau\left(\bar{R}\left(K',X'\right)\right)\right) dX'$$
$$-\int \frac{1+f\left(\hat{X}'\right)}{\hat{\underline{k}}_2\left(\hat{X}'\right)} H\left(-\frac{1+f\left(\hat{X}'\right)}{\hat{\underline{k}}_2\left(\hat{X}'\right)}\right)\hat{S}_2\left(\hat{X}',\hat{X}\right)d\hat{X}' - \int \frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)} H\left(-\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)}\right) S_2\left(X',\hat{X}\right)dX'$$

with the constraint:

$$\int \left(\hat{S}_1\left(\hat{X}',\hat{X}\right) + \hat{S}_2\left(\hat{X}',\hat{X}\right)\right)d\hat{X}' + \int \left(S_1\left(X',\hat{X}\right) + S_2\left(X',\hat{X}\right)\right)dX' = 1$$

The coefficents can be expressed in terms of the $\hat{S}_\eta$ and S_η. Given that:

$$\hat{S}_\eta\left(\hat{X}',\hat{X}\right) = \int \frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)}$$

and:

$$\hat{S}\left(\hat{X}',\hat{X}\right) = \hat{S}_1\left(\hat{X}',\hat{X}\right) + \hat{S}_2\left(\hat{X}',\hat{X}\right)$$

we have:

$$\int \hat{S}\left(\hat{X}',\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X} = \frac{\hat{\underline{k}}\left(\hat{X}'\right)}{1+\hat{\underline{k}}\left(\hat{X}'\right)}$$

Then, defining averages:

$$\hat{S}_\eta\left(\hat{X}'\right)=\int \hat{S}_\eta\left(\hat{X}',\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X}$$

$$\hat{S}\left(\hat{X}'\right)=\int \hat{S}\left(\hat{X}',\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X}$$

$$S_\eta\left(X'\right)=\int S_\eta\left(X',\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\hat{X}$$

$$S\left(X'\right)=\int S\left(X',\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\hat{X}$$

we find the expression of the initial set of parameters as functions of the new parameters:

$$\frac{1}{1+\hat{\underline{k}}\left(\hat{X}'\right)}=1-\hat{S}\left(\hat{X}'\right)$$

$$\begin{aligned}\hat{\underline{k}}\left(\hat{X}'\right) &= \frac{\hat{S}\left(\hat{X}'\right)}{1-\hat{S}\left(\hat{X}'\right)}\\ \hat{\underline{k}}_\eta\left(\hat{X}'\right) &= \frac{\hat{S}\left(\hat{X}'\right)}{1-\hat{S}\left(\hat{X}'\right)}\end{aligned}$$

$$1+\hat{\underline{k}}_2\left(\hat{X}'\right)=\frac{1-\hat{S}_1\left(\hat{X}'\right)}{1-\hat{S}\left(\hat{X}'\right)}$$

$$\frac{\underline{k}\left(X'\right)}{1+\underline{k}\left(X'\right)}=S\left(X'\right)$$

$$\frac{1}{1+\underline{k}\left(X'\right)}=1-S\left(X'\right)$$

$$\begin{aligned}\underline{k}\left(X'\right) &= \frac{S\left(X'\right)}{1-S\left(X'\right)}\\ \underline{k}_\eta\left(X'\right) &= S\left(X'\right)\end{aligned}$$

$$1+\underline{k}_2\left(X'\right)=\frac{1-S_1\left(X'\right)}{1-S\left(X'\right)}$$

and equation (232) writes:

$$\begin{aligned}
0 &= \int \left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_1\left(\hat{X}',\hat{X}\right)\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{1-\hat{S}_1\left(\hat{X}'\right)}\left(f\left(\hat{X}'\right)-\bar{r}\right)d\hat{X}' \\
&-\int S_1\left(X',\hat{X}\right)\left(\frac{1-S\left(X'\right)}{1-S_1\left(X'\right)}\left(f_1'\left(\hat{X}'\right)-\bar{r}\right)+\Delta F_\tau\left(\bar{R}\left(K',X'\right)\right)\right)dX' \\
&-\int\left(1+f\left(\hat{X}'\right)\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{\hat{S}_2\left(\hat{X}'\right)}H\left(-\frac{1+f\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\hat{X}'\right)}\right)\hat{S}_2\left(\hat{X}',\hat{X}\right)d\hat{X}' \\
&-\int\left(1+f_1'\left(X'\right)\right)\frac{1-S\left(X'\right)}{S_2\left(X'\right)}H\left(-\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)}\right)S_2\left(X',\hat{X}\right)dX'
\end{aligned}$$

As in the text, we consider that investment take place to close location so that:

$$\begin{aligned}
S_1\left(X',\hat{X}\right) &= S_1\left(\hat{X},\hat{X}\right)\delta\left(X'-\hat{X}\right) \\
S_2\left(X',\hat{X}\right) &= S_2\left(\hat{X},\hat{X}\right)\delta\left(X'-\hat{X}\right)
\end{aligned}$$

and the constraint simplifies as:

$$\int\left(\hat{S}_1\left(\hat{X}',\hat{X}\right)+\hat{S}_2\left(\hat{X}',\hat{X}\right)\right)d\hat{X}'+S_1\left(\hat{X},\hat{X}\right)+S_2\left(\hat{X},\hat{X}\right)=1$$

whereas the return equation (232) becomes:

$$\begin{aligned}
0 &= \int \left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_1\left(\hat{X}',\hat{X}\right)\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{1-\hat{S}_1\left(\hat{X}'\right)}\left(f\left(\hat{X}'\right)-\bar{r}\right)d\hat{X}' \qquad (233)\\
&-\int\left(1+f\left(\hat{X}'\right)\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{\hat{S}_2\left(\hat{X}'\right)}H\left(-\left(1+f\left(\hat{X}'\right)\right)\right)\hat{S}_2\left(\hat{X}',\hat{X}\right)d\hat{X}' \\
&-\left(1+f_1'\left(\hat{X}\right)\right)\left(1-S\left(\hat{X}\right)\right)H\left(-\left(1+f_1'\left(X\right)\right)\right) \\
&-S_1\left(\hat{X}\right)\left(\frac{1-S\left(\hat{X}\right)}{1-S_1\left(\hat{X}\right)}\left(f_1'\left(\hat{X}\right)-\bar{r}\right)+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)
\end{aligned}$$

Remark that without defaut, this reduces to:

$$\begin{aligned}
0 &= \int \left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_1\left(\hat{X}',\hat{X}\right)\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{1-\hat{S}_1\left(\hat{X}'\right)}\left(f\left(\hat{X}'\right)-\bar{r}\right)d\hat{X}' \qquad (234)\\
&-S_1\left(\hat{X}\right)\left(\frac{1-S\left(\hat{X}\right)}{1-S_1\left(\hat{X}\right)}\left(f_1'\left(\hat{X}\right)-\bar{r}\right)+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)
\end{aligned}$$

with solution:

$$\begin{aligned}
f\left(\hat{X}'\right) &= \bar{r}+\frac{1-\hat{S}_1\left(\hat{X}'\right)}{1-\hat{S}\left(\hat{X}'\right)}\left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_1\left(\hat{X}',\hat{X}\right)\right)^{-1} \qquad (235)\\
&\times\left(S_1\left(\hat{X}\right)\left(\frac{1-S\left(\hat{X}\right)}{1-S_1\left(\hat{X}\right)}\left(f_1'\left(\hat{X}\right)-\bar{r}\right)+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)\right)
\end{aligned}$$

Coming back to the equation with default, the loss realized when some default arise is:

$$\max\left(-\left(1+\bar{r}\right),\frac{1+f_{1}'\left(X'\right)}{\underline{k}_{2}\left(X'\right)}\right)$$

which translates that an investor can not lose more than its amounts of loans plus the potential return on them.

When the loss is maximal equation (233) becomes:

$$\begin{aligned}
0 &= \int\left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_{1}\left(\hat{X}',\hat{X}\right)\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{1-\hat{S}_{1}\left(\hat{X}'\right)}\left(f\left(\hat{X}'\right)-\bar{r}\right)d\hat{X}' \\
&+\int\left(1+\bar{r}\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{\hat{S}_{2}\left(\hat{X}'\right)}\hat{S}_{2}\left(\hat{X}',\hat{X}\right)d\hat{X}'+\left(1+\bar{r}\right)\left(1-S\left(\hat{X}\right)\right) \\
&-S_{1}\left(\hat{X}\right)\left(\frac{1-S\left(\hat{X}\right)}{1-S_{1}\left(\hat{X}\right)}\left(f_{1}'\left(\hat{X}\right)-\bar{r}\right)+\Delta F_{\tau}\left(\bar{R}\left(K,X\right)\right)\right)
\end{aligned}$$

whereas, in the general case, the return equation (233) is:

$$\begin{aligned}
0 &= \int\left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_{1}\left(\hat{X}',\hat{X}\right)\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{1-\hat{S}_{1}\left(\hat{X}'\right)}f\left(\hat{X}'\right)d\hat{X}' \\
&-\int\max\left(-1,\left(1+f\left(\hat{X}'\right)\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{\hat{S}_{2}\left(\hat{X}'\right)}\right)H\left(-\left(1+f\left(\hat{X}'\right)\right)\right)\hat{S}_{2}\left(\hat{X}',\hat{X}\right)d\hat{X}' \\
&-\int\max\left(-1,\left(1+f\left(\hat{X}'\right)\right)\frac{\left(1-S\left(\hat{X}\right)\right)}{S_{2}\left(\hat{X}\right)}\right)H\left(-\left(1+f_{1}'\left(X\right)\right)\right)S_{2}\left(\hat{X}\right)-S_{1}\left(\hat{X}\right)\frac{1-S\left(\hat{X}\right)}{1-S_{1}\left(\hat{X}\right)}f_{1}'\left(\hat{X}\right)
\end{aligned}$$

Defining $\hat{S}_{-}$and S_{-} the default sets for investors and firms respectively, so that $1+f\left(\hat{X}\right)<0$, the equation also writes:

$$\begin{aligned}
0 &= \int\left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_{1}\left(\hat{X}',\hat{X}\right)\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{1-\hat{S}_{1}\left(\hat{X}'\right)}f\left(\hat{X}'\right)d\hat{X}' \\
&-\int_{\hat{S}_{-}}\max\left(-1,\left(1+f\left(\hat{X}'\right)\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{\hat{S}_{2}\left(\hat{X}'\right)}\right)\hat{S}_{2}\left(\hat{X}',\hat{X}\right)d\hat{X}' \\
&-\int_{S_{-}}\max\left(-1,\left(1+f_{1}'\left(\hat{X}\right)\right)\frac{\left(1-S\left(\hat{X}\right)\right)}{S_{2}\left(\hat{X}\right)}\right)H\left(-\left(1+f_{1}'\left(X\right)\right)\right)S_{2}\left(\hat{X}\right)-S_{1}\left(\hat{X}\right)\frac{1-S\left(\hat{X}\right)}{1-S_{1}\left(\hat{X}\right)}f_{1}'\left(\hat{X}\right)
\end{aligned}$$

with solution:

$$\begin{aligned}
f\left(\hat{X}\right) &= \left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_1\left(\hat{X}',\hat{X}\right)\right)^{-1} \\
&\left\{\int_{\hat{S}_-}\max\left(-1,\left(1+f\left(\hat{X}'\right)\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{\hat{S}_2\left(\hat{X}'\right)}\right)\hat{S}_2\left(\hat{X}',\hat{X}\right)d\hat{X}'\right. \\
&\left.+\int_{S_-}\max\left(-1,\left(1+f_1'\left(\hat{X}\right)\right)\frac{\left(1-S\left(\hat{X}\right)\right)}{S_2\left(\hat{X}\right)}\right)S_2\left(\hat{X}\right)\right\}+S_1\left(\hat{X}\right)\frac{1-S\left(\hat{X}\right)}{1-S_1\left(\hat{X}\right)}\max\left(f_1'\left(\hat{X}\right),0\right)
\end{aligned}$$

if $\max\left(f_1'\left(\hat{X}\right),0\right)=0$, we hav:

$$\begin{aligned}
f\left(\hat{X}\right) &= \left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_1\left(\hat{X}',\hat{X}\right)\right)^{-1} \\
&\left\{\int_{\hat{S}_-}\max\left(-1,\left(1+f\left(\hat{X}'\right)\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{\hat{S}_2\left(\hat{X}'\right)}\right)\hat{S}_2\left(\hat{X}',\hat{X}\right)d\hat{X}'\right. \\
&\left.+\int_{S_-}\max\left(-1,\left(1+f_1'\left(\hat{X}\right)\right)\frac{\left(1-S\left(\hat{X}\right)\right)}{S_2\left(\hat{X}\right)}\right)S_2\left(\hat{X}\right)\right\}
\end{aligned}$$

Appendix 5 Introducing normalisations

A5.1 Firms

For firms, the relevant variable K corresponds to the private capital $K_p(t)$ and the share k_{ajl} is normalised as:

$$k_{ajl}\rightarrow\frac{k_{ail}}{N\langle K(t)\rangle\left(1-\left(\langle k_1\rangle+\langle k_2\rangle\right)\frac{\hat{N}\langle\hat{K}_v(t)\rangle}{\langle K(t)\rangle}\right)}=\frac{k_{ail}}{N\langle K_p(t)\rangle}$$

so that the translation of these coefficients in terms of field becomes:

$$\begin{aligned}
k_1\left(X',\hat{X}\right) &\rightarrow \frac{k_1\left(X',\hat{X}\right)}{\|\Psi\|^2\langle K\rangle} \\
k_2\left(X',\hat{X}\right) &\rightarrow \frac{k_2\left(X',\hat{X}\right)}{\|\Psi\|^2\langle K\rangle}
\end{aligned}$$

and:

$$\begin{aligned}
\frac{k_\eta\left(X',\hat{X}\right)}{1+\underline{k}(X')} &\rightarrow \frac{k_\eta\left(X',\hat{X}\right)}{\|\Psi\|^2\langle K\rangle}\frac{1}{1+\int\frac{k\left(X',\hat{X}\right)}{\|\Psi\|^2\langle K\rangle}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2d\hat{X}} \\
&= \frac{\frac{k_\eta\left(X',\hat{X}\right)}{\|\Psi\|^2\langle K\rangle}}{1+\underline{k}(X')}
\end{aligned}$$

with:

$$\underline{k}(X')=\int\frac{k\left(X',\hat{X}\right)}{\|\Psi\|^2\langle K\rangle}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2d\hat{X}$$

A5.2 Investors

For investors, since we look at the dynamics for disposable capital, we consider the normalisation:

$$\hat{k}_{ajl} \rightarrow \frac{\hat{k}_{ajl}}{\hat{N}\left\langle \hat{K}(t)\right\rangle \left(1-\left(\left\langle \hat{k}_1\right\rangle + \left\langle \hat{k}_2\right\rangle\right)\right)} = \frac{\hat{k}_{ajl}}{\hat{N}\left\langle \hat{K}(t)\right\rangle \left(1-\left\langle \hat{k}\right\rangle\right)}$$

which modifies the coefficients:

$$\begin{aligned}
\hat{k}_1\left(\hat{X}',\hat{X}\right) &\rightarrow \frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle \hat{K}\right\rangle\left(1-\left\langle \hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)} \\
\hat{k}_2\left(\hat{X}',\hat{X}\right) &\rightarrow \frac{\hat{k}_2\left(\hat{X}',\hat{X}\right)}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle \hat{K}\right\rangle\left(1-\left\langle \hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)}
\end{aligned}$$

that is:

$$\hat{k}_\lambda\left(\hat{X}',\hat{X}\right) \rightarrow \frac{\hat{k}_\lambda\left(\hat{X}',\hat{X}\right)}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle \hat{K}\right\rangle\left(1-\frac{\left\langle \underline{k}\left(\hat{X}'\right)\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle \hat{K}\right\rangle}\right)}$$

This modifies the coefficients $\underline{\hat{k}}_\eta\left(\hat{X}'\right)$ as:

$$\underline{\hat{k}}_\eta\left(\hat{X}'\right) \rightarrow \int \frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle \hat{K}\right\rangle\left(1-\left\langle \hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2 d\hat{X}$$

and in average:

$$\left\langle \underline{\hat{k}}_\eta\left(\hat{X}'\right)\right\rangle \simeq \frac{\left\langle \hat{k}_\eta\left(\hat{X}',\hat{X}\right)\right\rangle}{1-\left\langle \hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle}$$

Moreover the terms $\frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)}$, $1+\underline{\hat{k}}_2\left(\hat{X}'\right)$ and $\frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)}$ arising in the various equations, are modified as follows:

$$\begin{aligned}
\frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)} &\rightarrow \frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{\left(\left\Vert\hat{\Psi}\right\Vert^2\left\langle \hat{K}\right\rangle\left(1-\left\langle \hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\right)}\frac{1}{1+\int\frac{\hat{k}\left(\hat{X}',\hat{X}\right)}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle \hat{K}\right\rangle\left(1-\left\langle \hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2 d\hat{X}} \\
&= \frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle \hat{K}\right\rangle}\frac{1}{1-\left\langle \hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle+\int\frac{\hat{k}\left(\hat{X}',\hat{X}\right)}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle \hat{K}\right\rangle}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2 d\hat{X}} \\
&= \frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle \hat{K}\right\rangle}\frac{1}{1+\int\frac{\hat{k}\left(\hat{X}',\hat{X}\right)-\left\langle \hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle \hat{K}\right\rangle}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2 d\hat{X}} \\
&\rightarrow \frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle \hat{K}\right\rangle\left(1+\underline{\hat{k}}\left(\hat{X}'\right)\right)}
\end{aligned}$$

with now:

$$\underline{\hat{k}}\left(\hat{X}'\right)=\int\frac{\hat{k}\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^{2}d\hat{X}$$

$$\begin{aligned}1+\underline{\hat{k}}_{2}\left(\hat{X}'\right) &\rightarrow 1+\int\frac{\hat{k}_{2}\left(\hat{X}',\hat{X}\right)}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^{2}d\hat{X}\\
&= \frac{1}{\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)}\left(\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)+\int\frac{\hat{k}_{2}\left(\hat{X}',\hat{X}\right)}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^{2}d\hat{X}\right)\\
&= \frac{1}{\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)}\left(\left(1-\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle\right)+\int\frac{\hat{k}_{2}\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}_{2}\left(\hat{X}',\hat{X}\right)\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^{2}d\hat{X}\right)\end{aligned}$$

$$1+\underline{\hat{k}}_{2}\left(\hat{X}'\right)\rightarrow\frac{1-\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle}{1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle}\left(1+\frac{\underline{\hat{k}}_{2}\left(\hat{X}'\right)}{1-\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle}\right)$$

where we define:

$$\underline{\hat{k}}_{2}\left(\hat{X}'\right)=\int\frac{\hat{k}_{2}\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}_{2}\left(\hat{X}',\hat{X}\right)\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^{2}d\hat{X}$$

Ultimately, expression $\frac{\hat{k}_{\eta}\left(\hat{X}',\hat{X}\right)}{1+\underline{\hat{k}}_{2}\left(\hat{X}'\right)}$ is replacd by:

$$\frac{\hat{k}_{\eta}\left(\hat{X}',\hat{X}\right)}{1+\underline{\hat{k}}_{2}\left(\hat{X}'\right)}\rightarrow\frac{\hat{k}_{\eta}\left(\hat{X}',\hat{X}\right)}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle\left(1-\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\left(1+\frac{\underline{\hat{k}}_{2}\left(\hat{X}'\right)}{1-\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle}\right)}$$

Appendix 6 Detailling the field firms' return

A6.1 Firms action functional

We have shown (see 222) that the action functional for the field of firms is:

$$-\Psi^{\dagger}\left(K,X\right)\left(\nabla_{K_{p}}\left(\sigma_{K}^{2}\nabla_{K_{p}}-f_{1}'\left(K,X\right)K_{p}\right)\right)\Psi\left(K,X\right)+\frac{1}{2\epsilon}\left(\left|\Psi\left(K,X\right)\right|^{2}-\left|\Psi_{0}\left(X\right)\right|^{2}\right)^{2}$$

where:

$$\begin{aligned}f_{1}'\left(K_{i}\left(t\right)\right) &= \left(1+\sum_{\nu}k_{2j\nu}\hat{K}_{\nu}\left(t\right)\right)f_{1}\left(K_{i}\left(t\right)\right)-\bar{r}\sum_{v}k_{2lv}\hat{K}_{\nu}\left(t\right)\\
&\rightarrow f_{1}'\left(X\right)=\left(1+\underline{k}_{2}\left(X\right)\right)f_{1}\left(X\right)-\bar{r}\underline{k}_{2}\left(\hat{X}\right)\\
&= f_{1}\left(X\right)+\left(f_{1}\left(X\right)-\bar{r}\right)\int k_{2}\left(X,\hat{X}_{1}\right)\hat{K}_{1}\left|\hat{\Psi}\left(\hat{K}_{1},\hat{X}_{1}\right)\right|^{2}d\hat{K}_{1}d\hat{X}_{1}\end{aligned}$$

We will consider the change of variables:

$$\begin{aligned}
\Psi(K,X) &\rightarrow \exp\left(-\frac{1}{\sigma_{\hat{K}}^{2}}\int f_1'(K_p,X)\,K_p dK_p\right)\Psi \\
\Psi^{\dagger}(K,X) &\rightarrow \exp\left(\frac{1}{\sigma_{\hat{K}}^{2}}\int f_1'(K_p,X)\,K_p dK_p\right)\Psi^{\dagger}
\end{aligned}$$

leading to the modified action:

$$\begin{aligned}
S(\Psi) &= -\Psi^{\dagger}(K,X)\,\sigma_{\hat{K}}^{2}\nabla_{K_p}^{2}\Psi(K,X) \\
&\quad+\Psi^{\dagger}(K,X)\left(\frac{\left(f_1'(K_p,X)\,K_p\right)^2}{2\sigma_{\hat{K}}^{2}}+\frac{1}{2}f_1'(K_p,X)\right)\Psi(K,X)+\frac{1}{2\epsilon}\left(|\Psi(K,X)|^2-|\Psi_0(X)|^2\right)^2
\end{aligned} \tag{236}$$

A6.1.1 Return for firms

The return for firms is computed as:

$$\begin{aligned}
f_1'(X) &= \left(1+\underline{k}_2(X)\right)\frac{f_1\left(\left(1+\underline{k}(X)\right)K_p,X\right)}{\left(1+\underline{k}(X)\right)K_p}-\bar{r}\underline{k}_2(X) \\
&\rightarrow \left(1+\underline{k}_2(X)\right)\frac{A(X)\left(f_1\left(1+\underline{k}(X)\right)K_p\right)^{\delta}-C}{\left(1+\underline{k}(X)\right)K_p}-\bar{r}\underline{k}_2(X)
\end{aligned}$$

where:

$$\underline{k}_2(X)=\frac{k_2\left(\hat{X},\hat{X}\right)}{N\langle K_p\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\rightarrow\frac{\underline{k}_2(X)}{\langle K_p\rangle}\hat{K}_X$$

and:

$$\begin{aligned}
&\left(1+\underline{k}_2(X)\right)A(X)\left(f_1\left(1+\underline{k}(X)\right)K_p\right)^{\delta-1} \\
=\ &\left(1+\frac{\underline{k}_2(X)}{\langle K_p\rangle}\hat{K}_X\right)A(X)\left(f_1\left(1+\frac{\underline{k}_2(X)}{\langle K\rangle}\hat{K}_X\right)K_{pX}\right)^{\delta-1}
\end{aligned}$$

so that:

$$f_1'(X)-r=\left(1+\underline{k}_2(X)\right)\left(A(X)\left(f_1\left(1+\underline{k}(X)\right)K_p\right)^{\delta-1}-\bar{r}\right)$$

writen also in the alternate formulation:

$$f_1'(X)-r=\frac{1-S_1\left(\hat{X},\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{K_{\hat{X}}\left|\Psi\left(\hat{X}\right)\right|^2}}{1-S\left(\hat{X},\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{K_{\hat{X}}\left|\Psi\left(\hat{X}\right)\right|^2}}\left(\left(f_1\frac{1}{1-S\left(\hat{X},\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{K_{\hat{X}}\left|\Psi\left(\hat{X}\right)\right|^2}}K_p\right)^{\delta-1}-\bar{r}\right)$$

This simplifies:

$$f_1'(X)-\bar{r}=\frac{1-S_1\left(\hat{X}\right)}{1-S\left(\hat{X}\right)}\left(\left(f_1\left(\hat{X}\right)\frac{K_p}{1-\frac{\hat{K}_X}{\langle K_p\rangle}S\left(\hat{X}\right)}\right)^{\delta-1}-\bar{r}\right)$$

and:

$$f_1'(X)=\frac{1-S_1\left(\hat{X}\right)}{1-S\left(\hat{X}\right)}\left(\left(f_1\left(\hat{X}\right)\frac{K_p}{1-\frac{\hat{K}_X}{\langle K_p\rangle}S\left(\hat{X}\right)}\right)^{\delta-1}\right)-\frac{S_2\left(\hat{X}\right)}{1-S\left(\hat{X}\right)}\bar{r}$$

In the constant return to scale approximation, this becomes:

$$\begin{aligned}
&K_{pX}\left(\frac{f_1(X)\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\right)K_p-C}{\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_{pX}}\right)\\
\rightarrow\quad&\frac{(1+f_1(X))\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_p-C-\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_p}{1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}\\
=\quad&f_1(X)K_p-\frac{C}{1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}
\end{aligned}$$

and we rewrite this expression by introducing an effectiv cost $\bar{C}(X)=\frac{C}{1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}$ depending on the amount of capital invested:

$$f_1(X)K_p-\frac{C}{1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\frac{\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{|\Psi(X)|^2}}=f_1(X)K_p-\bar{C}(X)$$

Note that in first approximation:

$$\frac{C}{1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\frac{\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{|\Psi(X)|^2}}\simeq\frac{C}{1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\frac{\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{|\Psi_0(X)|^2}}$$

The full return for the firms accounts for its total capital invested and can thus be computed as:

$$\begin{aligned}
&(1+\underline{k}_2(X))\left(f_1(X)K_p-\bar{C}(X)\right)-\underline{k}_2(X)K_p\bar{r}\\
=\quad&(1+\underline{k}_2(X))\left(\left(f_1(X)-\frac{\underline{k}_2(X)}{1+\underline{k}_2(X)}\bar{r}\right)K_p-\bar{C}(X)\right)
\end{aligned}$$

Appendix 7 Firms constant return computation of background field and capital

A7.1 Case one

Average capital per sector and firms returns

We start with the minimization of (236):

$$\begin{aligned}
0\quad=\quad&-\sigma_{\hat{K}}^2\nabla_{K_p}^2\Psi(K,X)\\
&+\left(\frac{\left(f_1'(K_p,X)K_p\right)^2}{2\sigma_{\hat{K}}^2}+\frac{1}{2}f_1'(K_p,X)\right)\Psi(K,X)+\frac{1}{\epsilon}\left(|\Psi(K,X)|^2-|\Psi_0(X)|^2\right)\Psi(K,X)
\end{aligned}\tag{237}$$

Neglecting the fluctuation term $\sigma_{\hat{K}}^2\nabla_{K_p}^2\Psi(K,X)$ in first approximation, leads to the field formula:

$$|\Psi(K,X)|^2=|\Psi_0(X)|^2-\epsilon\left(\frac{\left(f_1^{(e)}(X)K_p-\bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2}+\frac{f_1^{(e)}(X)}{2}\right)\tag{238}$$

We consider the case:

$$\left|\Psi_0(X)\right|^2-\epsilon\frac{f_1(X)}{2}-\epsilon\frac{\left(\bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2}>0$$

The field formula implies that the amount of capital for firms in the sector is bounded by the condition:

$$\left|\Psi_0(X)\right|^2-\epsilon\left(\frac{\left(f_1(X)K_p-\bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2}+\frac{f_1(X)}{2}\right)>0$$

The maximum level of capital for frm is thus defined by:

$$\left|\Psi_0(X)\right|^2-\epsilon\left(\frac{\left(f_1(X)K_0-\bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2}+\frac{f_1(X)}{2}\right)=0$$

which is solvd as:

$$K_0=\frac{\bar{C}(X)+\sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{f_1(X)}{2}\right)}}{f_1(X)}$$

This allows to compute $\left|\Psi(X)\right|^2$ by integrating frm 0 t K_0.

$$\begin{aligned}
\left|\Psi(X)\right|^2 &= \left(\left|\Psi_0(X)\right|^2-\frac{\epsilon f_1(X)}{2}\right)K_0-\epsilon\left(\frac{\left(f_1(X)K_0-\bar{C}(X)\right)^3}{3f_1(X)\sigma_{\hat{K}}^2}+\left(\frac{\left(\bar{C}(X)\right)^3}{3f_1(X)\sigma_{\hat{K}}^2}\right)\right)\\
&= \left(\left|\Psi_0(X)\right|^2-\frac{\epsilon f_1(X)}{2}\right)K_0-\frac{\epsilon K_0}{3\sigma_{\hat{K}}^2}\left(\left(f_1(X)K_0-\bar{C}(X)\right)^2+\left(\bar{C}(X)\right)^2-\bar{C}(X)\left(f_1(X)K_0-\bar{C}(X)\right)\right)\\
&= \left(\left|\Psi_0(X)\right|^2-\frac{\epsilon f_1(X)}{2}\right)K_0\\
&\quad -\frac{\epsilon K_0}{3\sigma_{\hat{K}}^2}\left(\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{f_1(X)}{2}\right)+\bar{C}(X)\left(\bar{C}(X)-\sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{f_1(X)}{2}\right)}\right)\right)\\
&= \left(\frac{2}{3}\left(\left|\Psi_0(X)\right|^2-\frac{\epsilon f_1(X)}{2}\right)+\frac{\epsilon}{3\sigma_{\hat{K}}^2}\left(\sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{f_1(X)}{2}\right)}-\bar{C}(X)\right)\bar{C}(X)\right)K_0
\end{aligned}$$

and the densty of firms in sector X, $\left|\Psi(X)\right|^2$, has the form:

$$\begin{aligned}
\left|\Psi(X)\right|^2 &= \left(\frac{2}{3}\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{f_1(X)}{2}\right)\right.\\
&\quad \left.+\frac{1}{3}\left(\sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{f_1(X)}{2}\right)}-\bar{C}(X)\right)\bar{C}(X)\right)\epsilon\frac{K_0}{\sigma_{\hat{K}}^2}
\end{aligned}\tag{239}$$

To find the amount of capital in sector X, we start with:

$$\left|\Psi(K,X)\right|^2=\left[\left|\Psi_0(X)\right|^2-\epsilon\frac{f_1(X)}{2}\right]-\epsilon\left(\frac{\left(f_1(X)K_p-\bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2}\right)$$

and:

$$K_p\left(\left(f_1(X)K_p-\bar{C}(X)\right)^2\right)=\frac{1}{f_1(X)}\left(\left(f_1(X)K_p-\bar{C}(X)\right)\left(\left(f_1(X)K_p-\bar{C}(X)\right)^2\right)+\bar{C}(X)\left(\left(f_1(X)K_p-\bar{C}(X)\right)^2\right)\right)$$

Integrating this relation leads to:

$$\begin{aligned}
K_X \left|\Psi(X)\right|^2 &= \left(\left|\Psi_0(X)\right|^2 - \frac{\epsilon f_1(X)}{2}\right)\frac{K_0^2}{2} - \frac{\epsilon}{4\left(f_1(X)\right)^2 \sigma_{\hat{K}}^2}\left(\left(f_1(X)K_0 - \bar{C}(X)\right)^4 - \left(\bar{C}(X)\right)^4\right) \\
&\quad - \frac{\epsilon \bar{C}(X)}{3\left(f_1(X)\right)^2 \sigma_{\hat{K}}^2}\left(\left(f_1(X)K_0 - \bar{C}(X)\right)^3 + \left(\bar{C}(X)\right)^3\right) \\
&= \left(\left|\Psi_0(X)\right|^2 - \frac{\epsilon f_1(X)}{2}\right)\frac{K_0^2}{2} - \frac{\epsilon K_0}{4 f_1(X)\sigma_{\hat{K}}^2}\left(f_1(X)K_0 - 2\bar{C}(X)\right)\left(\left(f_1(X)K_0 - \bar{C}(X)\right)^2 + \left(\bar{C}(X)\right)^2\right) \\
&= \left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon} - \frac{f_1(X)}{2}\right)\epsilon\frac{K_0^2}{2} \\
&\quad - \frac{\epsilon K_0}{4 f_1(X)\sigma_{\hat{K}}^2}\left(\sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon} - \frac{f_1(X)}{2}\right)} - \bar{C}(X)\right)\left(\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon} - \frac{f_1(X)}{2}\right) + \left(\bar{C}(X)\right)^2\right)
\end{aligned}$$

To simplify this expression, we use that:

$$\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon} - \frac{f_1(X)}{2}\right)\epsilon\frac{K_0^2}{2} = \sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon} - \frac{f_1(X)}{2}\right)\left(\bar{C}(X) + \sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon} - \frac{f_1(X)}{2}\right)}\right)\frac{\epsilon K_0}{2\sigma_{\hat{K}}^2 f_1(X)}$$

and rewrite:

$$\begin{aligned}
&-\frac{\epsilon \bar{C}(X)}{3\left(f_1(X)\right)^2 \sigma_{\hat{K}}^2}\left(\left(f_1(X)K_0 - \bar{C}(X)\right)^3 + \left(\bar{C}(X)\right)^3\right) \\
&= -\frac{\epsilon K_0 \bar{C}(X)}{3 f_1(X)\sigma_{\hat{K}}^2}\left(\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon} - \frac{f_1(X)}{2}\right) + \bar{C}(X)\left(\bar{C}(X) - \sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon} - \frac{f_1(X)}{2}\right)}\right)\right) \\
&= -\frac{\epsilon K_0 \bar{C}(X)}{3 f_1(X)\sigma_{\hat{K}}^2}\left(\sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon} - \frac{f_1(X)}{2}\right)}\left(\sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon} - \frac{f_1(X)}{2}\right)} - \bar{C}(X)\right) + \bar{C}^2(X)\right)
\end{aligned}$$

so that, definng:

$$X = \sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon} - \frac{f_1(X)}{2}\right)}$$

we find the amount of capital, $K_X \left|\Psi(X)\right|^2$:

$$K_X \left|\Psi(X)\right|^2 = \frac{\epsilon K_0}{\sigma_{\hat{K}}^2 f_1(X)}\left\{\frac{1}{4}\left(2X^2\left(\bar{C} + X\right) - \left(X - \bar{C}\right)\left(X^2 + \left(\bar{C}\right)^2\right)\right) - \frac{\bar{C}}{3}\left(X\left(X - \bar{C}\right) + C^2\right)\right\} \tag{240}$$

The average capital in sector X is given by the ratio of (240) and (239):

$$\begin{aligned}
K_X &= \frac{1}{f_1(X)}\frac{\frac{1}{4}\left(2X^2(C+X) - (X-C)\left(X^2 + C^2\right)\right) - \frac{C}{3}\left(X(X-C) + C^2\right)}{\frac{2}{3}X^2 + \frac{1}{3}(X-C)C} \\
&= \frac{1}{4 f_1(X)}(3X - C)\frac{(C+X)}{2X - C}
\end{aligned}$$

Rewriting:

$$\left(\frac{2}{3}X^2+\frac{1}{3}(X-C)C\right)\left(\frac{1}{4}(3X-C)\frac{(C+X)}{2X-C}-C\right)=\epsilon\frac{K_0}{\sigma_{\hat{K}}^2}\frac{1}{4}(C-X)^2(C+X)=\frac{\epsilon}{\sigma_{\hat{K}}^2}\frac{1}{4}\left(X^2-C^2\right)^2$$

and using that:

$$X^2=\sigma_{\hat{K}}^2\left(\frac{|\Psi_0(X)|^2}{\epsilon}-\frac{f_1(X)}{2}\right)$$

leads to the expression for $|\Psi(X)|^2$:

$$\begin{aligned}|\Psi(X)|^2 &= \left(\frac{2}{3}\sigma_{\hat{K}}^2\left(\frac{|\Psi_0(X)|^2}{\epsilon}-\frac{f_1(X)}{2}\right)+\frac{1}{3}\left(\sqrt{\sigma_{\hat{K}}^2\left(\frac{|\Psi_0(X)|^2}{\epsilon}-\frac{f_1(X)}{2}\right)}-\bar{C}(X)\right)\bar{C}(X)\right)\\ &\times\epsilon\frac{\bar{C}(X)+\sqrt{\sigma_{\hat{K}}^2\left(\frac{|\Psi_0(X)|^2}{\epsilon}-\frac{f_1(X)}{2}\right)}}{\sigma_{\hat{K}}^2 f_1(X)}\end{aligned}$$

The overall return for a firm was given by:

$$\begin{aligned}&(1+\underline{k}_2(X))\left(f_1(X)K_p-\bar{C}(X)\right)-\underline{k}_2(X)K_p\bar{r}\\ =\ &(1+\underline{k}_2(X))\left(\left(f_1(X)-\frac{\underline{k}_2(X)}{1+\underline{k}_2(X)}\bar{r}\right)K_p-\bar{C}(X)\right)\\ =\ &f_1^{(e)}(X)K_p-\bar{C}^{(e)}(X)\end{aligned}$$

with:

$$f_1^{(e)}(X)=(1+\underline{k}_2(X))f_1(X)-\underline{k}_2(X)\bar{r}$$

and the associated return per unit of capital is:

$$f_1'(X)=(1+\underline{k}_2(X))\left(\left(f_1(X)-\frac{\underline{k}_2(X)}{1+\underline{k}_2(X)}\bar{r}\right)-\frac{\bar{C}(X)}{K_p}\right)$$

so that the relative return with respect to the interest rates is:

$$\frac{f_1'(X)-\bar{r}}{1+\underline{k}_2(X)}=f_1(X)-\bar{r}-\frac{\bar{C}(X)}{K_p}$$

A7.2.2 Return of the firm for investors

The return provided by the firms of the sector X is given by:

$$|\Psi(X)|^2K_X\frac{f_1'(X)-\bar{r}}{1+\underline{k}_2(X)}=|\Psi(X)|^2\left((f_1(X)-\bar{r})K_X-\bar{C}(X)\right)$$

where:

$$K_X\rightarrow\frac{1}{4f_1^{(e)}(X)}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}$$

and we have:

$$\begin{aligned}&|\Psi(X)|^2\left((f_1(X)-\bar{r})K_X-\bar{C}(X)\right)\\ =\ &\frac{\left(C^{(e)}+X^{(e)}\right)\left(\frac{2}{3}X^2+\frac{1}{3}\left(X-C^{(e)}\right)C^{(e)}\right)\epsilon}{\sigma_{\hat{K}}^2f_1^{(e)}(X)}\left(\frac{(f_1(X)-\bar{r})}{4f_1^{(e)}(X)}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}-C\right)\\ =\ &\frac{\left(C^{(e)}+X^{(e)}\right)\left(\frac{2}{3}X^2+\frac{1}{3}\left(X-C^{(e)}\right)C^{(e)}\right)\epsilon}{\sigma_{\hat{K}}^2(1+\underline{k}_2(X))f_1^{(e)}(X)}\left\{\frac{(f_1(X)-\bar{r})}{4\left(f_1(X)-\frac{\underline{k}_2(X)}{1+\underline{k}_2(X)}\bar{r}\right)}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}-C^{(e)}\right\}\end{aligned}$$

A7.2 Case two

A7.2.1 Average capital per sector and firms returns

In this case, we consider that all firms have a minimum capital, so that:

$$|\Psi_0(X)|^2 - \epsilon\frac{f_1(X)}{2} - \epsilon\frac{\left(\bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2} < 0$$

This exquation also writes:

$$|\Psi_0(X)|^2 - \epsilon\left(\frac{\left(f_1(X)K_p - \bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2} + \frac{f_1(X)}{2}\right) > 0$$

and there are two solutions:

$$|\Psi_0(X)|^2 - \epsilon\left(\frac{\left(f_1(X)K_0 - \bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2} + \frac{f_1(X)}{2}\right) = 0$$

$$K_{0\pm} = \frac{\bar{C}(X) \pm \sqrt{\sigma_{\hat{K}}^2\left(\frac{|\Psi_0(X)|^2}{\epsilon} - \frac{f_1(X)}{2}\right)}}{f_1(X)}$$

These are the two bound to the level of capital in one sector. We will write:

$$\begin{aligned} K_0 &= K_{0+} \\ K_{0+} - K_{0-} &= \frac{2\sqrt{\sigma_{\hat{K}}^2\left(\frac{|\Psi_0(X)|^2}{\epsilon} - \frac{f_1(X)}{2}\right)}}{f_1(X)} \end{aligned}$$

The computation of $|\Psi(X)|^2$ is similar to the case one:

$$\begin{aligned} |\Psi(X)|^2 &= \left(|\Psi_0(X)|^2 - \frac{\epsilon f_1(X)}{2}\right)\Delta K_0 - \epsilon\left(\frac{\left(f_1(X)K_{0+} - \bar{C}(X)\right)^3}{3f_1(X)\sigma_{\hat{K}}^2} - \frac{\left(f_1(X)K_{0-} - \bar{C}(X)\right)^3}{3f_1(X)\sigma_{\hat{K}}^2}\right) \\ &= \left(|\Psi_0(X)|^2 - \frac{\epsilon f_1(X)}{2}\right)\Delta K_0 \\ &\quad -\frac{\epsilon\Delta K_0}{3\sigma_{\hat{K}}^2}\left(\left(f_1(X)K_{0+} - \bar{C}(X)\right)^2 + \left(f_1(X)K_{0-} - \bar{C}(X)\right)^2\right. \\ &\quad \left.+\left(f_1(X)K_{0+} - \bar{C}(X)\right)\left(f_1(X)K_{0-} - \bar{C}(X)\right)\right) \\ &= \left(|\Psi_0(X)|^2 - \frac{\epsilon f_1(X)}{2}\right)\Delta K_0 - \frac{\epsilon\Delta K_0}{3\sigma_{\hat{K}}^2}\left(\sigma_{\hat{K}}^2\left(\frac{|\Psi_0(X)|^2}{\epsilon} - \frac{f_1(X)}{2}\right)\right) \\ &= \frac{2}{3}\left(\frac{|\Psi_0(X)|^2}{\epsilon} - \frac{f_1(X)}{2}\right)\epsilon\Delta K_0 \end{aligned}$$

and we find:

$$|\Psi(X)|^2 = \frac{2}{3}\left(\frac{|\Psi_0(X)|^2}{\epsilon} - \frac{f_1(X)}{2}\right)\epsilon\Delta K_0$$

We find $K_X|\Psi(X)|^2$ using:

$$|\Psi(K,X)|^2 \rightarrow \left[|\Psi_0(X)|^2 - \epsilon\frac{f_1(X)}{2}\right] - \epsilon\left(\frac{\left(f_1(X)K_p - \bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2}\right)$$

Multiplying by K_p:

$$\begin{aligned}
& K_p\left(\left(f_1(X)K_p-\bar{C}(X)\right)^2\right) \\
= & \frac{1}{f_1(X)}\left(\left(f_1(X)K_p-\bar{C}(X)\right)\left(\left(f_1(X)K_p-\bar{C}(X)\right)^2\right)+\bar{C}(X)\left(\left(f_1(X)K_p-\bar{C}(X)\right)^2\right)\right)
\end{aligned}$$

Integration between the capital bound yields:

$$\begin{aligned}
K_X\left|\Psi(X)\right|^2 &= \left(\left|\Psi_0(X)\right|^2-\frac{\epsilon f_1(X)}{2}\right)\frac{K_{0+}^2-K_{0-}^2}{2} \\
&\quad -\frac{\epsilon}{4\left(f_1(X)\right)^2\sigma_{\hat{K}}^2}\left(\left(f_1(X)K_{0+}-\bar{C}(X)\right)^4-\left(f_1(X)K_{0-}-\bar{C}(X)\right)^4\right) \\
&\quad -\frac{\epsilon\bar{C}(X)}{3\left(f_1(X)\right)^2\sigma_{\hat{K}}^2}\left(\left(f_1(X)K_{0+}-\bar{C}(X)\right)^3-\left(f_1(X)K_{0-}-\bar{C}(X)\right)^3\right) \\
&= \left(\left|\Psi_0(X)\right|^2-\frac{\epsilon f_1(X)}{2}\right)\frac{K_{0+}^2-K_{0-}^2}{2} \\
&\quad -\frac{\epsilon\Delta K_0\left(f_1(X)\left(K_{0+}+K_{0-}\right)-2\bar{C}(X)\right)}{4f_1(X)\sigma_{\hat{K}}^2}\left(\left(f_1(X)K_{0+}-\bar{C}(X)\right)^2+\left(f_1(X)K_{0-}-\bar{C}(X)\right)^2\right)
\end{aligned}$$

We write:

$$\left(\left|\Psi_0(X)\right|^2-\frac{\epsilon f_1(X)}{2}\right)\frac{K_{0+}^2-K_{0-}^2}{2}=\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{f_1(X)}{2}\right)\epsilon\Delta K_0\frac{\bar{C}(X)}{f_1(X)}$$

and:

$$\begin{aligned}
& -\frac{\epsilon\bar{C}(X)}{3\left(f_1(X)\right)^2\sigma_{\hat{K}}^2}\left(\left(f_1(X)K_{0+}-\bar{C}(X)\right)^3-\left(f_1(X)K_{0-}-\bar{C}(X)\right)^3\right) \\
= & -\frac{\epsilon\bar{C}(X)\Delta K_0}{3f_1(X)\sigma_{\hat{K}}^2}\left(\left(f_1(X)K_{0+}-\bar{C}(X)\right)^2+\left(f_1(X)K_{0-}-\bar{C}(X)\right)^2+\left(f_1(X)K_{0+}-\bar{C}(X)\right)\left(f_1(X)K_{0-}-\bar{C}(X)\right)\right) \\
= & -\frac{\epsilon\bar{C}(X)\Delta K_0}{3f_1(X)\sigma_{\hat{K}}^2}\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{f_1(X)}{2}\right)
\end{aligned}$$

so that ultimately:

$$\begin{aligned}
K_X\left|\Psi(X)\right|^2 &\rightarrow \sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{f_1(X)}{2}\right)\frac{\epsilon\Delta K_0\bar{C}(X)}{f_1(X)\sigma_{\hat{K}}^2}-\frac{\epsilon\bar{C}(X)\Delta K_0}{3f_1(X)\sigma_{\hat{K}}^2}\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{f_1(X)}{2}\right) \\
&= \frac{2}{3}\frac{\epsilon\Delta K_0\bar{C}(X)}{f_1(X)\sigma_{\hat{K}}^2}\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{f_1(X)}{2}\right)
\end{aligned}$$

denoting:

$$X=\sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{f_1(X)}{2}\right)}$$

we obtain:

$$K_X\left|\Psi(X)\right|^2=\frac{2\epsilon\Delta K_0\bar{C}(X)}{3f_1(X)\sigma_{\hat{K}}^2}X^2$$

given that:

$$\left|\Psi\left(X\right)\right|^{2}=2\frac{\epsilon\Delta K_{0}}{3\sigma_{\hat{K}}^{2}}X^{2}$$

The average capital is:

$$K_{X}=\frac{\bar{C}\left(X\right)}{f_{1}\left(X\right)}$$

A7.2.2 Return of the firm for investors

The return of the firms in sector X for the investors is:

$$\left|\Psi\left(X\right)\right|^{2}K_{X}\frac{f_{1}^{\prime}\left(X\right)-\bar{r}}{1+\underline{k}_{2}\left(X\right)}=\left|\Psi\left(X\right)\right|^{2}\left(\left(f_{1}\left(X\right)-\bar{r}\right)K_{X}-\bar{C}\left(X\right)\right)$$

We use:

$$\begin{aligned}\left(f_{1}-\bar{r}\right)K_{X}-C &= \left(f_{1}\left(X\right)-\bar{r}\right)\frac{C^{(e)}}{f_{1}^{(e)}\left(X\right)}-\frac{C^{(e)}}{\left(1+\underline{k}_{2}\left(X\right)\right)}\\ &= \frac{C}{\left(1+\underline{k}_{2}\left(X\right)\right)\left(1+\underline{k}\left(X\right)\right)}\left(\frac{f_{1}\left(X\right)-\bar{r}}{f_{1}\left(X\right)-\frac{\underline{k}_{2}\left(X\right)}{\left(1+\underline{k}_{2}\left(X\right)\right)}\bar{r}}-1\right)\end{aligned}$$

so that:

$$\left|\Psi\left(X\right)\right|^{2}\left(f_{1}K_{X}-C\right)=2\frac{\epsilon\Delta K_{0}C^{(e)}}{3\sigma_{\hat{K}}^{2}}X^{2}\left(\left(f_{1}\left(X\right)-\bar{r}\right)\frac{\left(3X^{(e)}-1\right)}{f_{1}^{(e)}\left(X\right)}-\frac{C^{(e)}}{\left(1+\underline{k}_{2}\left(X\right)\right)}\right)$$

and this becmes:

$$\left|\Psi\left(X\right)\right|^{2}K_{X}\frac{f_{1}^{\prime}\left(X\right)-\bar{r}}{1+\underline{k}_{2}\left(X\right)}=4\frac{\epsilon\left(C^{(e)}\right)^{2}}{3\sigma_{\hat{K}}^{2}f_{1}^{(e)}\left(X\right)}X^{2}\left(\frac{\left(f_{1}\left(X\right)-\bar{r}\right)}{f_{1}^{(e)}\left(X\right)}\left(3X^{(e)}-1\right)-C\right)$$

Appendix 8 Estimation of the functionl derivative $\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K}_{1},\hat{X}_{1}\right)\right|^{2}}\hat{g}\left(\hat{K},\hat{X}\right)$

We decompose:

$$\begin{aligned}&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K}_{1},\hat{X}_{1}\right)\right|^{2}}\hat{g}\left(\hat{K},\hat{X}\right)\\
=\;&\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_{1},\hat{X}_{1}\right)\right|^{2}}\frac{1}{1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}\hat{f}\left(\hat{K}',\hat{X}'\right)\\
=\;&\frac{1}{1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}\\
&\times\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_{1},\hat{X}_{1}\right)\right|^{2}}\left(M\left(\left(\hat{K}',\hat{X}'\right),\left(\hat{K}'',\hat{X}''\right)\right)\left|\hat{\Psi}\left(\hat{K}'',\hat{X}''\right)\right|^{2}\right)\\
&\times\frac{1}{1-M\left(\left(\hat{K}'',\hat{X}''\right),\left(\hat{K}''',\hat{X}'''\right)\right)\left|\hat{\Psi}\left(\hat{K}''',\hat{X}'''\right)\right|^{2}}\hat{f}\left(\hat{K}''',\hat{X}'''\right)\\
&+\frac{1}{1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_{1},\hat{X}_{1}\right)\right|^{2}}\hat{f}\left(\hat{K}',\hat{X}'\right)\end{aligned}$$

and estimate each term separately.

A8.1 Estimation of derivative of $M\left(\hat{K},\hat{X},\hat{K}',\hat{X}'\right)$

$$M\left(\hat{K},\hat{X},\hat{K}',\hat{X}'\right)=\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}}{1+\underline{\hat{k}}\left(\hat{X}\right)}\rightarrow\frac{\hat{k}\left(\hat{X}',\hat{X}\right)\hat{K}}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle\left(1+\underline{\hat{k}}\left(\hat{X}'\right)\right)}$$

$$\underline{\hat{k}}\left(\hat{X}'\right)=\int\frac{\hat{k}\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^{2}d\hat{X}$$

$$\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^{2}}\left(1+\underline{\hat{k}}\left(\hat{X}\right)\right)\rightarrow\int\frac{\hat{k}\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^{2}d\hat{X}\left(-\frac{\hat{K}_1}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)$$

in averages:

$$\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^{2}}\left(1+\underline{\hat{k}}\left(\hat{X}\right)\right)<<1$$

$$\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}\hat{K}=\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle$$

for:

$$k\left(X,\hat{X}'\right)=\frac{k\left(X,\hat{X}'\right)}{\left|\Psi\left(X\right)\right|^{2}\left\langle K\right\rangle}$$

$$1+\underline{k}\left(X\right)=1+\int\frac{k\left(X,\hat{X}'\right)}{\left\|\Psi\right\|^{2}\left\langle K\right\rangle}\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}>>1$$

$$\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^{2}}\frac{1}{1+\underline{k}\left(X\right)}=-\frac{1}{1+\underline{k}\left(X\right)}\frac{\frac{k\left(X,\hat{X}'\right)}{\left\|\Psi\right\|^{2}\left\langle K\right\rangle}\hat{K}'}{1+\underline{k}\left(X\right)}<<\frac{1}{1+\underline{k}\left(X\right)}$$

$$\begin{aligned}
&\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^{2}}\left(M\left(\left(\hat{K}',\hat{X}'\right),\left(\hat{K}'',\hat{X}''\right)\right)\left|\hat{\Psi}\left(\hat{K}'',\hat{X}''\right)\right|^{2}\right)\\
=\ & M\left(\left(\hat{K}',\hat{X}'\right),\left(\hat{K}_1,\hat{X}_1\right)\right)+\frac{\partial M\left(\left(\hat{K}',\hat{X}'\right),\left(\hat{K}'',\hat{X}''\right)\right)}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^{2}}\left|\hat{\Psi}\left(\hat{K}'',\hat{X}''\right)\right|^{2}\\
=\ & M\left(\left(\hat{K}'',\hat{X}''\right),\left(\hat{K}_1,\hat{X}_1\right)\right)+\frac{1}{\left(1+\underline{\hat{k}}\left(\hat{X}'\right)\right)}\frac{\partial\left(\frac{\hat{k}\left(\hat{X}',\hat{X}''\right)}{\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}\hat{K}}\hat{K}'\right)}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^{2}}\left|\hat{\Psi}\left(\hat{K}'',\hat{X}''\right)\right|^{2}\\
\simeq\ & M\left(\left(\hat{K}'',\hat{X}''\right),\left(\hat{K}_1,\hat{X}_1\right)\right)-\frac{\hat{k}\left(\hat{X}',\hat{X}''\right)\hat{K}'\hat{K}_1}{\left(\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}\hat{K}\right)^{2}}\frac{1}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\left|\hat{\Psi}\left(\hat{K}'',\hat{X}''\right)\right|^{2}
\end{aligned}$$

$$\frac{1}{1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}$$

$$\times\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_{1},\hat{X}_{1}\right)\right|^{2}}\left(M\left(\left(\hat{K}',\hat{X}'\right),\left(\hat{K}'',\hat{X}''\right)\right)\left|\hat{\Psi}\left(\hat{K}'',\hat{X}''\right)\right|^{2}\right)\hat{g}\left(\hat{K}'',\hat{X}''\right)$$

$$\simeq \frac{1}{1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}M\left(\left(\hat{K}',\hat{X}'\right),\left(\hat{K}_{1},\hat{X}_{1}\right)\right)\hat{g}\left(\hat{K}_{1},\hat{X}_{1}\right)$$

$$-\frac{1}{1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}\frac{\hat{k}\left(\hat{X}',\hat{X}''\right)\hat{K}'\hat{K}_{1}}{\left(\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}\hat{K}\right)^{2}}\frac{1}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\left|\hat{\Psi}\left(\hat{K}'',\hat{X}''\right)\right|^{2}\hat{g}\left(\hat{K}'',\hat{X}''\right)$$

$$\simeq \frac{1}{1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}M\left(\left(\hat{K}',\hat{X}'\right),\hat{X}_{1}\right)\hat{g}\left(\hat{K}_{1},\hat{X}_{1}\right)$$

$$-\frac{1}{1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}M\left(\left(\hat{K}',\hat{X}'\right),\left\langle\hat{X}\right\rangle\right)\langle\hat{g}\rangle\frac{\hat{K}_{1}}{\left\langle\hat{K}\right\rangle}$$

We estimate:

$$\left(1+\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle\right)\right)\frac{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_{1}\right)\left\langle\hat{K}\right\rangle\hat{g}\left(\hat{X}_{1}\right)}{\left(1+\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle\right)\right)\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}\hat{K}}$$

$$-\left(1+\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle\right)\right)\frac{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\left\langle\hat{K}\right\rangle}{\left(1+\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle\right)\right)\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}\hat{K}}\langle\hat{g}\rangle\frac{\hat{K}_{1}}{\left\langle\hat{K}\right\rangle}$$

$$\simeq \frac{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_{1}\right)}{\left\|\hat{\Psi}\right\|^{2}}\hat{g}\left(\hat{X}_{1}\right)-\frac{\hat{K}_{1}}{\left\langle\hat{K}\right\rangle}\frac{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)}{\left\|\hat{\Psi}\right\|^{2}}\langle\hat{g}\rangle$$

and:

$$\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_{1},\hat{X}_{1}\right)\right|^{2}}\hat{g}\left(\hat{K},\hat{X}\right) \tag{241}$$

$$\simeq \frac{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_{1}\right)}{\left\|\hat{\Psi}\right\|^{2}}\hat{g}\left(\hat{X}_{1}\right)-\frac{\hat{K}_{1}}{\left\langle\hat{K}\right\rangle}\frac{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)}{\left\|\hat{\Psi}\right\|^{2}}\langle\hat{g}\rangle\frac{1}{1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}\frac{\partial\hat{f}\left(\hat{K}',\hat{X}'\right)}{\partial\left|\hat{\Psi}\left(\hat{K}_{1},\hat{X}_{1}\right)\right|^{2}}$$

A8.2 Estimation of $\frac{\partial \hat{f}\left(\hat{K}',\hat{X}'\right)}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}$

We first estimate the second term in the RHS of (241) $\frac{\partial \hat{f}\left(\hat{K}',\hat{X}'\right)}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}$, which is given by return equation (235):

$$\begin{aligned}&\frac{\partial \hat{f}\left(\hat{K}',\hat{X}'\right)}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\\ =\;&\frac{\partial\left\{\frac{1-\hat{S}_1\left(\hat{X}\right)}{1-\hat{S}\left(\hat{X}\right)}\left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_1\left(\hat{X}',\hat{X}\right)\right)^{-1}S_1\left(\hat{X}',\hat{X}'\right)\frac{1-S\left(\hat{X}'\right)}{1-S_1\left(\hat{X}'\right)}\left(\left(f_1'\left(\hat{X}\right)-\bar{r}\right)+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)\right\}}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\end{aligned}$$

In average:

$$\frac{1-\hat{S}_1\left(\hat{X}\right)}{1-\hat{S}\left(\hat{X}\right)}\left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_1\left(\hat{X}',\hat{X}\right)\right)^{-1}\simeq\frac{1}{1-\hat{S}\left(\hat{X}\right)}=1+\underline{\hat{k}}\left(X'\right)$$

so that:

$$\frac{\partial\left(\frac{1-\hat{S}_1\left(\hat{X}\right)}{1-\hat{S}\left(\hat{X}\right)}\left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_1\left(\hat{X}',\hat{X}\right)\right)^{-1}\right)}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}<<1$$

and:

$$\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\left(\left(f_1'\left(\hat{X}\right)-\bar{r}\right)+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)=\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}f_1'\left(\hat{X}\right)$$

so that:

$$\begin{aligned}\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}f_1' &= \frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\left(f_1\left(X\right)K_p-\frac{C}{1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\frac{\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{|\Psi(X)|^2}}\right)\\ &\simeq \frac{Ck\left(X,X\right)\frac{\hat{K}}{\langle K\rangle}+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\frac{\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{|\Psi(X)|^2}}{\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\frac{\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{|\Psi(X)|^2}\right)^2}<k\end{aligned}$$

and:

$$\frac{\partial \hat{f}\left(\hat{K}',\hat{X}'\right)}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}<<1$$

A8.3 Gathering contribution

Summing the contribution leads to:

$$\frac{\partial}{\partial\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\hat{g}\left(\hat{K},\hat{X}\right)\simeq\frac{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)}{\left\|\hat{\Psi}\right\|^2}\hat{g}\left(\hat{X}_1\right)-\frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle}\frac{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)}{\left\|\hat{\Psi}\right\|^2}\left\langle\hat{g}\right\rangle \tag{242}$$

Appendix 9 Stability of the saddle point

Starting with (69) the second derivative of the action functional is:

$$\begin{aligned}
&\frac{\partial^2 S}{\partial \left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 \partial \left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2} \qquad (243)\\
\simeq\ & \frac{\partial}{\partial \left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(\frac{\hat{g}^2\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)}{\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(\left(\hat{K}_1,\hat{X}_1\right),\Psi,\hat{\Psi}\right)}{2\hat{K}}\right)(1+\delta)\\
&\times\left(1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}_1,\hat{X}_1\right)\right)\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2\right)^{-1}+\hat{\mu}\\
&+\left(\frac{\hat{g}^2\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)}{\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(\left(\hat{K}_1,\hat{X}_1\right),\Psi,\hat{\Psi}\right)}{2\hat{K}}\right)(1+\delta)\\
&\times\frac{\partial}{\partial \left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}_1,\hat{X}_1\right)\right)\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2\right)^{-1}
\end{aligned}$$

where:

$$\delta=\left\langle\frac{\partial}{\partial \left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\hat{g}\left(\left(\hat{K}',\hat{X}'\right),\Psi,\hat{\Psi}\right)\right\rangle$$

In average, we can replace:

$$\left(1-M\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}_1,\hat{X}_1\right)\right)\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2\right)^{-1}=1+\underline{\hat{k}}\left(\hat{X}\right)$$

and:

$$\frac{\partial}{\partial \left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\underline{\hat{k}}\left(\hat{X}\right)\simeq\hat{k}\left(\hat{X}\right)$$

$$\left(\frac{\hat{K}_{\hat{X}_1}\hat{f}\left(\hat{K}_{\hat{X}_1}\frac{\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2}{\left\|\Psi\left(\hat{X}_1\right)\right\|^2},\hat{X}_1\right)}{\sigma_{\hat{K}}}\right)^2=X\Delta\hat{M}\left(\hat{K}_{\hat{X}_1},\hat{X}_1\right)\frac{\underline{\hat{k}}\left(\hat{X}\right)}{1+\underline{\hat{k}}\left(\hat{X}\right)}Y \qquad (244)$$

so that we are led to:

$$\frac{\partial}{\partial \left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(\frac{\hat{K}_{\hat{X}_1}\hat{f}\left(\hat{K}_{\hat{X}_1}\frac{\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2}{\left\|\Psi\left(\hat{X}_1\right)\right\|^2},\hat{X}_1\right)}{\sigma_{\hat{K}}}\right)^2\simeq\frac{X\underline{\hat{k}}\left(\hat{X}\right)}{\left(1+\underline{\hat{k}}\left(\hat{X}\right)\right)^2}Y+X\frac{\underline{\hat{k}}\left(\hat{X}\right)}{1+\underline{\hat{k}}\left(\hat{X}\right)}\frac{\hat{\mu}}{1+\delta}$$

Given that:

$$-X\underline{\hat{k}}\left(\hat{X}\right)\simeq 1$$

we find:

$$\begin{aligned}
&\frac{\partial^2 S}{\partial \left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 \partial \left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\\
=\ &\hat{\mu}-\frac{Y}{\left(1+\underline{\hat{k}}\left(\hat{X}\right)\right)^2}+\left(\frac{f^2\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)}{\sigma_{\hat{K}}^2}+\frac{f\left(\left(\hat{K}_1,\hat{X}_1\right),\Psi,\hat{\Psi}\right)}{2\hat{K}}\right)(1+\delta)\,\hat{k}\left(\hat{X}\right)>0
\end{aligned}$$

and th saddle point is a minimm.

Appendix 10 Field and capital for investors

A10.1 Expression for the field

Solving for $\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2$ yields:

$$\begin{aligned}\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 &= \left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}\left\{\left(\frac{\hat{K}_1^2\hat{g}^2\left(\hat{X}_1\right)}{2\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(\hat{X}_1\right)}{2}\right)\right. \\ &\left.+\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left(\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)\hat{g}\left(\hat{X}_1\right)-\frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle}\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\langle\hat{g}\rangle\right)\right\}\end{aligned} \tag{245}$$

A10.1.1 Finding the maximal capital

The maximal value for $\hat{K}$, written $\hat{K}_0$ is found by settng $\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2=0$.

$$\begin{aligned}0 &= \left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}\left\{\left(\frac{\hat{K}_0^2\hat{g}^2\left(\hat{X}_1\right)}{2\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(\hat{X}_1\right)}{2}\right)\right. \\ &\left.+\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left(\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)\hat{g}\left(\hat{X}_1\right)-\frac{\hat{K}_0}{\left\langle\hat{K}\right\rangle}\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\langle\hat{g}\rangle\right)\right\}\end{aligned} \tag{246}$$

leading to:

$$\begin{aligned}\hat{K}_0^2 &\simeq \frac{2\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}}\right. \\ &\left.-\left(\frac{\hat{g}\left(\hat{X}_1\right)}{2}+\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left(\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)\hat{g}\left(\hat{X}_1\right)-\frac{\left\langle\hat{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\langle\hat{g}\rangle\right)\right)\right) \\ &\simeq 2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}}-\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left(\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)\hat{g}\left(\hat{X}_1\right)-\frac{\left\langle\hat{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\langle\hat{g}\rangle\right)\right) \\ &\simeq 2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}}+\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle^2}{\sigma_{\hat{K}}^2}+\frac{\langle\hat{g}\rangle}{2}\right)\left(\left(\frac{\left\langle\hat{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}-\frac{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)}{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)}\right)\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)\right)\end{aligned} \tag{247}$$

A10.1.2 Expression for $\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2$

Integrating (245) over $\hat{K}$ yields:

$$\begin{aligned}\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2 &= \hat{K}_0\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}\left\{\left(\frac{\hat{K}_0^3\hat{g}^2\left(\hat{X}_1\right)}{6\sigma_{\hat{K}}^2}+\frac{\hat{K}_0\hat{g}\left(\hat{X}_1\right)}{2}\right)\right. \\ &\left.+\hat{K}_0\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left(\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)\hat{g}\left(\hat{X}_1\right)-\frac{\hat{K}_0}{2\left\langle\hat{K}\right\rangle}\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\langle\hat{g}\rangle\right)\right\}\end{aligned}$$

$\hat{K}_0^2$ satisfies (246) and:

$$\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2 = \hat{\mu}\hat{K}_0^3\frac{\hat{g}^2\left(\hat{X}_1\right)}{3\sigma_{\hat{K}}^2} - \hat{\mu}\frac{\hat{K}_0^2}{2\left\langle\hat{K}\right\rangle}\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\langle\hat{g}\rangle \tag{248}$$

$$\simeq \hat{\mu}\frac{\hat{K}_0^2}{\sigma_{\hat{K}}^2}\left(\frac{\hat{K}_0\hat{g}^2\left(\hat{X}_1\right)}{3}-\frac{\left\langle\hat{K}\right\rangle\langle\hat{g}\rangle^2}{2}\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)$$

multiplip b $\hat{K}$ and integrtng:

$$\hat{K}_{\hat{X}}\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2 = \frac{\hat{K}_0^2}{2}\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}\hat{K}_0\left\{\left(\frac{\hat{K}_0^3\hat{g}\left(\hat{X}_1\right)}{8\sigma_{\hat{K}}^2}+\frac{\hat{K}_0\hat{g}\left(\hat{X}_1\right)}{4}\right)\right.$$
$$\left.+\frac{\hat{K}_0^2}{2}\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left(\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)\hat{g}\left(\hat{X}_1\right)-\frac{2\hat{K}_0}{3\left\langle\hat{K}\right\rangle}\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\langle\hat{g}\rangle\right)\right\}$$

since $\hat{K}_0$ satisfies (246):

$$\hat{K}_{\hat{X}}\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2 = \hat{\mu}\frac{\hat{K}_0^4\hat{g}^2\left(\hat{X}_1\right)}{8\sigma_{\hat{K}}^2} - \hat{\mu}\frac{\hat{K}_0^2}{6\left\langle\hat{K}\right\rangle}\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\langle\hat{g}\rangle \tag{249}$$

$$\simeq \hat{\mu}\frac{\hat{K}_0^3}{2\sigma_{\hat{K}}^2}\left(\frac{\hat{K}_0\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\left\langle\hat{K}\right\rangle\langle\hat{g}\rangle^2}{3}\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)$$

$$= \hat{\mu}\frac{\hat{K}_0^4}{2\sigma_{\hat{K}}^2}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\left\langle\hat{K}\right\rangle\langle\hat{g}\rangle^2}{3\hat{K}_0}\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)$$

A10.1.3 Averages and ratio $\frac{\langle\hat{K}\rangle}{\langle\hat{K}_0\rangle}$

Averages for (248) and (249) lead to find the norm $\left\|\hat{\Psi}\right\|^2$ of the field, computing the total number of investrs and the total disposable capital for investors $\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2$:

$$\left\|\hat{\Psi}\right\|^2 \simeq \hat{\mu}V\frac{\left\langle\hat{K}_0\right\rangle^3}{\sigma_{\hat{K}}^2}\left(\frac{1}{3}-\frac{\left\langle\hat{K}\right\rangle}{2\left\langle\hat{K}_0\right\rangle}\hat{k}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)\langle\hat{g}\rangle^2 \tag{250}$$

$$\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2 = \hat{\mu}V\frac{\left\langle\hat{K}_0\right\rangle^4}{2\sigma_{\hat{K}}^2}\left(\frac{1}{4}-\frac{\left\langle\hat{K}\right\rangle}{3\left\langle\hat{K}_0\right\rangle}\hat{k}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)\right)\langle\hat{g}\rangle^2$$

where V is the volume of the sectors space. These formula allow to derive the ratio $\frac{\langle\hat{K}\rangle}{\langle\hat{K}_0\rangle}$:

$$\frac{\left\langle\hat{K}\right\rangle}{\left\langle\hat{K}_0\right\rangle} = \frac{1}{2}\frac{\frac{1}{4}-\frac{\langle\hat{K}\rangle}{3\langle\hat{K}_0\rangle}\hat{k}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)}{\frac{1}{3}-\frac{\langle\hat{K}\rangle}{2\langle\hat{K}_0\rangle}\hat{k}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)} \tag{251}$$

with solution:

$$\frac{\left\langle \hat{K}\right\rangle}{\left\langle \hat{K}_0\right\rangle}=\frac{2+\hat{k}-\sqrt{\left(2+\hat{k}\right)^2-\hat{k}}}{6\underline{\hat{k}}}=r \tag{252}$$

where:

$$\hat{k}=\hat{k}\left(\left\langle \hat{X}\right\rangle,\left\langle \hat{X}\right\rangle\right)$$

This enables to obtain the previous quantities in terms of $\hat{k}$:

$$\left\langle \hat{K}_0\right\rangle^2=2\frac{\sigma_{\hat{K}}^2}{\langle\hat{g}\rangle^2}\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}}+\left(\frac{r^2\left\langle \hat{K}_0\right\rangle^2\langle\hat{g}\rangle^2}{\sigma_{\hat{K}}^2}+\frac{\langle\hat{g}\rangle}{2}\right)\left(\frac{1-r}{r}\right)\hat{k}\right)$$

that is;

$$\left\langle \hat{K}_0\right\rangle^2=2\frac{\sigma_{\hat{K}}^2\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}}+\frac{\langle\hat{g}\rangle}{2}\left(\frac{1-r}{r}\right)\hat{k}\right)}{\langle\hat{g}\rangle^2\left(1-2r\left(1-r\right)\hat{k}\right)}$$

and for $\left\|\hat{\Psi}\right\|^2$ and $\left\langle \hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2$:

$$\left\|\hat{\Psi}\right\|^2\simeq\frac{\hat{\mu}V}{3\sigma_{\hat{K}}^2}\left(2\frac{\sigma_{\hat{K}}^2\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}}+\frac{\langle\hat{g}\rangle}{2}\left(\frac{1-r}{r}\right)\hat{k}\right)}{\langle\hat{g}\rangle^2\left(1-2r\left(1-r\right)\hat{k}\right)}\right)^{\frac{3}{2}}\left(1-\frac{2+\hat{k}-\sqrt{\left(2+\hat{k}\right)^2-\hat{k}}}{4\hat{k}}\hat{k}\right)\langle\hat{g}\rangle^2 \tag{253}$$

$$\left\langle \hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2=\frac{\hat{\mu}V\sigma_{\hat{K}}^2}{2\langle\hat{g}\rangle^2}\left(\frac{\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}}+\frac{\langle\hat{g}\rangle}{2}\left(\frac{1-r}{r}\right)\hat{k}}{1-2r\left(1-r\right)\hat{k}}\right)^2\left(1-2\frac{2+\hat{k}-\sqrt{\left(2+\hat{k}\right)^2-\hat{k}}}{9}\right) \tag{254}$$

The average capital per investor is obtained by the ratio of (254) and (253):

$$\left\langle \hat{K}\right\rangle=\frac{3}{4\langle\hat{g}\rangle}\sqrt{\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}}\frac{1-2\frac{2+\hat{k}-\sqrt{\left(2+\hat{k}\right)^2-\hat{k}}}{9}}{1-\frac{2+\hat{k}-\sqrt{\left(2+\hat{k}\right)^2-\hat{k}}}{4\hat{k}}}\sqrt{\frac{\left\|\hat{\Psi}_0\right\|^2+\hat{\mu}\frac{\langle\hat{g}\rangle}{2}\left(\frac{1-r}{r}\right)\hat{k}}{1-2r\left(1-r\right)\hat{k}}} \tag{255}$$

A10.1.4 Expression for $\hat{K}_0^2$

Having obtained the average quantities, we can come back to the quantities per sector. We find $\hat{K}_0^2$:

$$\begin{aligned}\hat{K}_0^2 &= 2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}}-\left(\frac{\left\langle \hat{K}\right\rangle^2\langle\hat{g}\rangle^2}{\sigma_{\hat{K}}^2}+\frac{\langle\hat{g}\rangle}{2}\right)\left(\frac{\hat{k}\left(\left\langle \hat{X}\right\rangle,\hat{X}_1\right)}{\hat{k}\left(\left\langle \hat{X}\right\rangle,\left\langle \hat{X}\right\rangle\right)}-\frac{6\hat{k}}{2+\hat{k}-\sqrt{\left(2+\hat{k}\right)^2-\hat{k}}}\right)\hat{k}\right)\\ &= 2\frac{\sigma_{\hat{K}}^2}{\hat{\mu}\hat{g}^2\left(\hat{X}_1\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\end{aligned}$$

where:

$$D\left(\hat{X}_1\right)=\left(\frac{\left\langle \hat{K}\right\rangle^2\langle\hat{g}\rangle^2}{\sigma_{\hat{K}}^2}+\frac{\langle\hat{g}\rangle}{2}\right)\left(\frac{\hat{k}\left(\left\langle \hat{X}\right\rangle,\hat{X}_1\right)}{\hat{k}\left(\left\langle \hat{X}\right\rangle,\left\langle \hat{X}\right\rangle\right)}-\frac{6\hat{k}}{2+\hat{k}-\sqrt{\left(2+\hat{k}\right)^2-\hat{k}}}\right)\hat{k}$$

A10.1.5 Expression for field and capital

Having obtained the expression for the field $\left\Vert \hat{\Psi}\left(\hat{X}_{1}\right) \right\Vert ^{2}$ and the amount $\hat{K}\left[\hat{X}_{1}\right]$ of financial capital in sector $\hat{X}_{1}$ are given by:

$$\left\Vert \hat{\Psi}\left(\hat{X}_{1}\right) \right\Vert ^{2} \simeq \hat{\mu}\frac{\hat{K}_{0}^{3}}{\sigma_{\hat{K}}^{2}}\left(\frac{\hat{g}^{2}\left(\hat{X}_{1}\right) }{3}-\frac{\left\langle \hat{K}\right\rangle \left\langle \hat{g}\right\rangle ^{2}}{2\left\langle \hat{K}_{0}\right\rangle }\underline{\hat{k}}\left(\left\langle \hat{X}\right\rangle ,\left\langle \hat{X}\right\rangle \right) \right) \tag{256}$$

$$= \frac{\hat{\mu}}{\sigma_{\hat{K}}^{2}}\left(2\frac{\sigma_{\hat{K}}^{2}}{\hat{g}^{2}\left(\hat{X}_{1}\right) }\left(\frac{\left\Vert \hat{\Psi}_{0}\left(\hat{X}_{1}\right) \right\Vert ^{2}}{\hat{\mu}}-D\left(\hat{X}_{1}\right) \right) \right) ^{\frac{3}{2}}\left(\frac{\hat{g}^{2}\left(\hat{X}_{1}\right) }{3}-\frac{\left\langle \hat{K}\right\rangle \left\langle \hat{g}\right\rangle ^{2}}{2\left\langle \hat{K}_{0}\right\rangle }\hat{k}\right)$$

$$\hat{K}\left[\hat{X}_{1}\right] = \hat{K}_{\hat{X}}\left\Vert \hat{\Psi}\left(\hat{X}_{1}\right) \right\Vert ^{2}\simeq \hat{\mu}\frac{\hat{K}_{0}^{4}}{2\sigma_{\hat{K}}^{2}}\left(\frac{\hat{g}^{2}\left(\hat{X}_{1}\right) }{4}-\frac{\left\langle \hat{K}\right\rangle \left\langle \hat{g}\right\rangle ^{2}}{3\left\langle \hat{K}_{0}\right\rangle }\hat{k}\left(\left\langle \hat{X}\right\rangle ,\left\langle \hat{X}\right\rangle \right) \right) \tag{257}$$

$$= \frac{\hat{\mu}}{2\sigma_{\hat{K}}^{2}}\left(2\frac{\sigma_{\hat{K}}^{2}}{\hat{g}^{2}\left(\hat{X}_{1}\right) }\left(\frac{\left\Vert \hat{\Psi}_{0}\left(\hat{X}_{1}\right) \right\Vert ^{2}}{\hat{\mu}}-D\left(\hat{X}_{1}\right) \right) \right) ^{2}\left(\frac{\hat{g}^{2}\left(\hat{X}_{1}\right) }{4}-\frac{r\left\langle \hat{g}\right\rangle ^{2}}{3}\underline{\hat{k}}\right)$$

and the average capital per investor in sector $\hat{X}_{1}$ is:

$$\hat{K}_{\hat{X}_{1}}=\frac{\hat{K}\left[\hat{X}_{1}\right] }{\left\Vert \hat{\Psi}\left(\hat{X}_{1}\right) \right\Vert ^{2}}=\frac{\sqrt{2\frac{\sigma_{\hat{K}}^{2}}{\hat{g}^{2}\left(\hat{X}_{1}\right) }\left(\frac{\left\Vert \hat{\Psi}_{0}\left(\hat{X}_{1}\right) \right\Vert ^{2}}{\hat{\mu}}-D\left(\hat{X}_{1}\right) \right) }\left(\frac{\hat{g}^{2}\left(\hat{X}_{1}\right) }{4}-\frac{r\left\langle \hat{g}\right\rangle ^{2}}{3}\hat{k}\right) }{2\left(\frac{\hat{g}^{2}\left(\hat{X}_{1}\right) }{3}-\frac{\left\langle \hat{K}\right\rangle \left\langle \hat{g}\right\rangle ^{2}}{2\left\langle \hat{K}_{0}\right\rangle }\hat{k}\right) } \tag{258}$$

Appendix 11 Return equation for investors

We present the details for the derivation of the expanded form of the return equation (80):

$$\int \left(\frac{\Delta \left(\hat{X},\hat{X}^{\prime }\right) }{1+\underline{\hat{k}}_{2}\left(\hat{X}\right) }-\frac{\hat{k}_{1}\left(\hat{X}^{\prime },\hat{X}\right) \hat{K}^{\prime }\left\vert \hat{\Psi}\left(\hat{K}^{\prime },\hat{X}^{\prime }\right) \right\vert ^{2}}{1+\underline{\hat{k}}_{2}\left(\hat{X}^{\prime }\right) }\right) \left(1-M\right) \left(g-\bar{r}^{\prime }\right) d\hat{X}^{\prime }=\frac{k_{1}\left(X\right) }{1+\underline{k}\left(X\right) }f_{1}^{\prime } \tag{259}$$

A11.1 Computation of left side of (80)

Define:

$$A=\int \left(\frac{\Delta \left(\hat{X},\hat{X}^{\prime }\right) }{1+\underline{\hat{k}}_{2}\left(\hat{X}\right) }-\frac{\hat{k}_{1}\left(\hat{X}^{\prime },\hat{X}\right) \hat{K}^{\prime }\left\vert \hat{\Psi}\left(\hat{K}^{\prime },\hat{X}^{\prime }\right) \right\vert ^{2}}{1+\underline{\hat{k}}_{2}\left(\hat{X}^{\prime }\right) }\right) \left(1-M\right)$$

it expands as:

$$\frac{1}{1+\underline{\hat{k}}_{2}\left(\hat{X}\right) }\left(1-\frac{\hat{k}\left(\hat{X},\hat{X}^{\prime }\right) \hat{K}_{X}}{1+\hat{k}\left(\hat{X},\hat{X}^{\prime }\right) \hat{K}_{X^{\prime }}\left\Vert \hat{\Psi}\left(\hat{X}^{\prime }\right) \right\Vert ^{2}}\left\Vert \hat{\Psi}\left(\hat{X}^{\prime }\right) \right\Vert ^{2}\right)$$

$$-\hat{S}_{1}\left(\hat{X}^{\prime },\hat{X}\right) \frac{1}{1+\underline{\hat{k}}_{2}\left(\hat{X}^{\prime }\right) }+\hat{S}_{1}\left(\hat{X}^{\prime },\hat{X}\right) \frac{1}{1+\underline{\hat{k}}_{2}\left(\hat{X}^{\prime }\right) }\frac{\hat{k}\left(X^{\prime },X^{\prime \prime }\right) \hat{K}_{X^{\prime }}}{1+\hat{k}\left(X^{\prime },X^{\prime \prime }\right) \hat{K}_{X^{\prime \prime }}\left\Vert \hat{\Psi}\left(\hat{X}^{\prime \prime }\right) \right\Vert ^{2}}\left\Vert \hat{\Psi}\left(\hat{X}^{\prime \prime }\right) \right\Vert ^{2}$$

Then, replacing:

$$1+\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}_{X'}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^{2}\rightarrow 1+\underline{\hat{k}}\left(\hat{X}\right)$$

yields:

$$\begin{aligned}A &= \frac{1}{1+\underline{\hat{k}}_{2}\left(\hat{X}\right)}\left(1-\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}_{X}}{1+\underline{\hat{k}}\left(\hat{X}\right)}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^{2}\right)\\ &\quad -\frac{\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\hat{K}_{X'}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^{2}}{1+\underline{\hat{k}}\left(\hat{X}\right)}\frac{1}{1+\underline{\hat{k}}_{2}\left(\hat{X}'\right)}\\ &\quad +\frac{\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\hat{K}_{X'}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^{2}}{1+\underline{\hat{k}}\left(\hat{X}\right)}\frac{1}{1+\underline{\hat{k}}_{2}\left(\hat{X}'\right)}\frac{\hat{k}\left(\hat{X}',\hat{X}''\right)\hat{K}_{X'}}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\left\|\hat{\Psi}\left(\hat{X}''\right)\right\|^{2}\end{aligned}$$

wh:

$$1+\underline{\hat{k}}_{2}\left(\hat{X}\right)\simeq 1+\frac{\hat{k}_{2}\left(\hat{X},\left\langle\hat{X}\right\rangle\right)}{1-\hat{k}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)}=1+\frac{\hat{k}_{2}\left(\hat{X},\left\langle\hat{X}\right\rangle\right)}{1-\left\langle\hat{k}\right\rangle}=1+\hat{k}_{2}^{n}\left(\hat{X},\left\langle\hat{X}\right\rangle\right)$$

$$1+\underline{\hat{k}}\left(\hat{X}\right)=1+\hat{k}\left(\hat{X},\left\langle\hat{X}\right\rangle\right)-\hat{k}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)$$

The last term, can be approximated:

$$\begin{aligned}&\frac{\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\hat{K}_{X}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^{2}}{1+\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}_{X'}}\frac{1}{1+\underline{\hat{k}}_{2}\left(\hat{X}'\right)}\frac{\hat{k}\left(X',X''\right)\hat{K}_{X'}}{1+\underline{\hat{k}}\left(\hat{X}'\right)}\left\|\hat{\Psi}\left(\hat{X}''\right)\right\|^{2}\\ \simeq\;&\frac{\hat{k}_{1}\left(\left\langle X\right\rangle,\hat{X}\right)\hat{K}_{X}\left\|\hat{\Psi}\left(\left\langle\hat{X}\right\rangle\right)\right\|^{2}}{1+\underline{\hat{k}}\left(\hat{X}\right)}\frac{1}{1+\underline{\hat{k}}_{2}\left(\left\langle X\right\rangle\right)}\frac{\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)\left\langle K\right\rangle}{1+\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle\right)}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^{2}\\ =\;&\frac{\hat{k}_{1}\left(\left\langle X\right\rangle,\hat{X}\right)\hat{K}_{X}\left\|\hat{\Psi}\left(\left\langle\hat{X}\right\rangle\right)\right\|^{2}}{1+\underline{\hat{k}}\left(\hat{X}\right)}\frac{1}{1+\underline{\hat{k}}_{2}\left(\left\langle X\right\rangle\right)}\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)\left\langle K\right\rangle\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^{2}\end{aligned}$$

and rewriting A as:

$$A=\frac{1}{1+\underline{\hat{k}}_{2}\left(\hat{X}\right)}-\hat{S}_{1}^{E}\left(\hat{X},\hat{X}'\right)$$

we find:

$$
\begin{aligned}
&-\hat{S}_1^E\left(\hat{X},\hat{X}'\right)\\
\rightarrow\quad & -\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}_X}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)\hat{K}_{X'}}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)}\right)\frac{\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2}{1+\hat{k}\left(\hat{X}\right)}\\
&+\frac{\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\langle K\rangle\left\Vert\hat{\Psi}\left(\left\langle\hat{X}\right\rangle\right)\right\Vert^2}{1+\hat{k}\left(\hat{X}\right)}\frac{\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)\langle K\rangle}{1+\hat{\underline{k}}_2\left(\langle X\rangle\right)}\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2\\
=\quad & -\left(\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}_X}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)\hat{K}_{X'}}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)}\right)-\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\langle K\rangle\left\Vert\hat{\Psi}\left(\left\langle\hat{X}\right\rangle\right)\right\Vert^2\frac{\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)\langle K\rangle}{1+\hat{\underline{k}}_2\left(\langle X\rangle\right)}\right)\frac{\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2}{1+\hat{k}\left(\hat{X}\right)}\\
=\quad & -\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}_X-\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\langle K\rangle\left\Vert\hat{\Psi}\left(\left\langle\hat{X}\right\rangle\right)\right\Vert^2\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)\langle K\rangle}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)\hat{K}_{X'}}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)}\right)\frac{\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2}{1+\hat{k}\left(\hat{X}\right)}
\end{aligned}
$$

In average, we sum over $\hat{X}'$ and:

$$
-\hat{S}_1^E\left(\hat{X},\hat{X}'\right)\rightarrow-\frac{1}{1+\hat{k}\left(\hat{X}\right)}\left(\frac{\hat{k}\left(\hat{X}\right)\frac{\hat{K}_X}{\langle K\rangle}-\hat{k}_{1E}\left(\hat{X}\right)\hat{k}_E\left(\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}+\frac{\hat{k}_{1E}\left(\hat{X}\right)}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)}\right)
$$

then, we define:

$$
\hat{\underline{k}}_1\left(\langle X\rangle,\hat{X}\right)=\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\left\Vert\hat{\Psi}\left(\left\langle\hat{X}\right\rangle\right)\right\Vert^2\langle K\rangle
$$

and we can rewrite:

$$
\begin{aligned}
&\int\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}-\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)K'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)}\right)(1-M) \qquad (260)\\
=\quad & \frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}-\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\hat{\underline{k}}_1\left(\langle X\rangle,\hat{X}\right)\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)}\right)\frac{\hat{K}_X\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2}{1+\hat{k}\left(\hat{X}\right)}\\
\equiv\quad & \frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}-\hat{S}_1^E\left(\hat{X}',\hat{X}\right)
\end{aligned}
$$

The differnts contributions to $\hat{S}_1^E\left(\hat{X},\hat{X}'\right)$ are interpreted in the text.

A11.2 Computation of the RHS of (80)

The right hand side of (87) is obtained by noting that:

$$
S_1\left(\hat{X}',\hat{X}'\right)=\frac{k_1\left(\hat{X}',\hat{X}'\right)}{1+\underline{k}(X)}
$$

and using that:

$$
|\Psi(X)|^2K_X\frac{f_1'(X)-\bar{r}}{1+\underline{k}_2(X)}=|\Psi(X)|^2\left((f_1(X)-\bar{r})K_X-\bar{C}(X)\right)
$$

so that:

$$\begin{aligned}
& S_1\left(\hat{X}',\hat{X}'\right)\frac{1-S\left(\hat{X}'\right)}{1-S_1\left(\hat{X}'\right)}\left(f_1'\left(X'\right)-\bar{r}\right) \\
= \; & S_1\left(\hat{X}',\hat{X}'\right)\left|\Psi\left(X'\right)\right|^2 K_{X'}\frac{f_1'\left(X'\right)-\bar{r}}{1+\underline{k}_2\left(X'\right)} \\
= \; & \frac{k_1\left(\hat{X}',\hat{X}'\right)}{1+\underline{k}\left(X\right)}\left|\Psi\left(X\right)\right|^2\left(\left(f_1\left(X\right)-\bar{r}\right)K_X-\bar{C}\left(X\right)\right)
\end{aligned}$$

We write:

$$\begin{aligned}
\underline{k}_2\left(\hat{X}\right) &= \beta\underline{k}\left(\hat{X}\right) \\
\underline{k}_2\left(\hat{X}\right) &= \left(1-\beta\right)\underline{k}\left(\hat{X}\right)
\end{aligned}$$

and assume th:

$$\underline{k}\left(\hat{X}\right)>>1$$

Under this assumtion

$$\frac{\underline{k}_1\left(\hat{X}\right)}{\left(1+\underline{k}\left(\hat{X}\right)\right)}\rightarrow 1-\beta$$

and the overall return for firms:

$$\left|\Psi\left(X\right)\right|^2\left(\left(f_1\left(X\right)-\bar{r}\right)K_X-\bar{C}\left(X\right)\right)$$

rewrites:

$$\frac{\left(\beta C+X^{(e)}\right)\left(\frac{2}{3}X^2+\frac{1}{3}\left(X-\beta C\right)\beta C\right)\epsilon}{\sigma_{\hat{K}}^2\left(f_1\left(X\right)+\beta\underline{k}\left(X\right)\left(\left(f_1\left(X\right)-\bar{r}\right)\right)\right)}\left(\frac{\left(f_1\left(X\right)-\bar{r}\right)}{4\left(f_1\left(X\right)+\beta\underline{k}\left(X\right)\left(\left(f_1\left(X\right)-\bar{r}\right)\right)\right)}\frac{\left(3X^{(e)}-\beta C\right)\left(\beta C+X^{(e)}\right)}{2X^{(e)}-\beta C}-\frac{C}{1+\underline{k}\left(X\right)}\right)$$

or equivalently:

$$\frac{1}{3}\frac{\left(X+C\beta\right)^2\epsilon}{\sigma_{\hat{K}}^2\left(f_1\left(X\right)+\beta\underline{k}\left(X\right)\left(f_1\left(X\right)-\bar{r}\right)\right)}\left(\frac{\left(f_1\left(X\right)-\bar{r}\right)\left(3X^{(e)}-\beta C\right)\left(\beta C+X^{(e)}\right)}{4\left(f_1\left(X\right)+\beta\underline{k}\left(X\right)\left(f_1\left(X\right)-\bar{r}\right)\right)}-\frac{C\left(2X-C\beta\right)}{1+\underline{k}\left(X\right)}\right)$$

where we can approximat:

$$X\rightarrow\sqrt{\frac{\left|\Psi_0\left(X\right)\right|^2}{\epsilon}-\frac{1}{2}f_1\left(X\right)-\frac{1}{2}\left(\beta\underline{k}\left(X\right)\left(f_1\left(X\right)-\bar{r}\right)\right)}\simeq\sqrt{\frac{\left|\Psi_0\left(X\right)\right|^2}{\epsilon}-\frac{1}{2}f_1\left(X\right)}$$

$$X=\sqrt{\sigma_{\hat{K}}^2\left(\frac{\left|\Psi_0\left(X\right)\right|^2}{\epsilon}-\frac{f_1\left(X\right)}{2}\right)}$$

Defining

$$R=f_1\left(X\right)-\bar{r}$$

A11.3 Estimation of $\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$

We have obtand bv:

$$\hat{S}_1^E\left(\hat{X}',\hat{X}\right)=\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\hat{\underline{k}}_1\left(\langle X\rangle,\hat{X}\right)\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)}\right)\frac{\hat{K}_X\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2}{1+\hat{k}\left(\hat{X}\right)}$$

so that:

$$\begin{aligned}&\left(1+\hat{\underline{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\\ \rightarrow\quad&\frac{\left(1+\hat{\underline{k}}_2\left(\hat{X}_1\right)\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)}\right)\frac{\hat{K}_X\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2}{1+\hat{k}\left(\hat{X}\right)}\end{aligned}$$

This is computed using (247), (249) and (345). at the lowest order $\hat{g}\left(\left\langle\hat{X}\right\rangle\right)\rightarrow\bar{r}'$.

$$\hat{K}_{\hat{X}}=\frac{\sqrt{\frac{\sigma_{\hat{K}}^2}{2}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2}{\hat{\mu}}-D\left(\hat{X}\right)\right)}\left(\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}\right)}{\left(\frac{1}{3}-\frac{\left\langle\hat{K}\right\rangle}{2\left\langle\hat{K}_0\right\rangle}\hat{\underline{k}}\right)\bar{r}'}\tag{261}$$

$$\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2\simeq\frac{2\sqrt{2\sigma_{\hat{K}}^2}\hat{\mu}}{\bar{r}'}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2}{\hat{\mu}}-D\left(\hat{X}'\right)\right)^{\frac{3}{2}}\left(\frac{1}{3}-\frac{\left\langle\hat{K}\right\rangle}{2\left\langle\hat{K}_0\right\rangle}\hat{\underline{k}}\right)\tag{262}$$

$$D\left(\hat{X}_1\right)\rightarrow\left(\frac{\left\langle\hat{K}\right\rangle^2\bar{r}'^2}{\sigma_{\hat{K}}^2}+\frac{\bar{r}'}{2}\right)\left(1-\frac{6\hat{\underline{k}}}{2+\hat{\underline{k}}-\sqrt{\left(2+\hat{\underline{k}}\right)^2-\hat{\underline{k}}}}\right)\hat{\underline{k}}$$

$$\hat{K}_{\hat{X}}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2\rightarrow\frac{2\sqrt{2}\sigma_{\hat{K}}^2}{\hat{\mu}\bar{r}'^2}\sqrt{\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2-\hat{\mu}D\left(\hat{X}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)^{\frac{3}{2}}$$

$$\frac{\left\langle\hat{K}\right\rangle}{\left\langle\hat{K}_0\right\rangle}=\frac{1}{2}\frac{\frac{1}{4}-\frac{\left\langle\hat{K}\right\rangle}{3\left\langle\hat{K}_0\right\rangle}\hat{\underline{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)}{\frac{1}{3}-\frac{\left\langle\hat{K}\right\rangle}{2\left\langle\hat{K}_0\right\rangle}\hat{\underline{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)}\tag{263}$$

$$r\rightarrow\frac{\left\langle\hat{K}\right\rangle}{\left\langle\hat{K}_0\right\rangle}=\frac{2+\hat{\underline{k}}-\sqrt{\left(2+\hat{\underline{k}}\right)^2-\hat{\underline{k}}}}{6\hat{\underline{k}}}\tag{264}$$

As a consequence, the estimation of $\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$:

$$\begin{aligned}
&\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\\
\rightarrow\quad&\frac{1+\underline{\hat{k}}_2\left(\hat{X}_1\right)}{1+\underline{\hat{k}}\left(\hat{X}\right)}\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)}\right)\frac{2\sqrt{2}\sigma_{\hat{K}}^2}{\hat{\mu}\bar{r}'^2}\\
&\times\frac{\sqrt{\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2-\hat{\mu}D\left(\hat{X}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)^{\frac{3}{2}}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\\
\rightarrow\quad&\frac{1+\underline{\hat{k}}_2\left(\hat{X}_1\right)}{1+\underline{\hat{k}}\left(\hat{X}\right)}\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)}\right)\\
&\times\frac{\sqrt{\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2-\hat{\mu}D\left(\hat{X}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)^{\frac{3}{2}}}{\left(\left\|\hat{\Psi}_0\right\|^2-\hat{\mu}\langle D\rangle\right)^2}
\end{aligned}$$

intrducing the normalizatn, in the averages, we can replac:

$$1+\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle\right)\rightarrow 1$$

and:

$$1+\underline{\hat{k}}_2\left(\hat{X}'\right)\rightarrow 1+\frac{\hat{k}_2\left(\hat{X},\left\langle\hat{X}\right\rangle\right)}{1-\left\langle\hat{k}\right\rangle}$$

so that:

$$\begin{aligned}
\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\quad\rightarrow\quad&\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)}{1+\hat{k}\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{\left(1+\hat{k}\left(\hat{X}\right)\right)\left(1+\frac{\hat{k}_2\left(\hat{X},\left\langle\hat{X}\right\rangle\right)}{1-\left\langle\hat{k}\right\rangle}\right)}\right)\\
&\times\frac{\sqrt{\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2-\hat{\mu}D\left(\hat{X}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)^{\frac{3}{2}}}{\left(\left\|\hat{\Psi}_0\right\|^2-\hat{\mu}\langle D\rangle\right)^2}
\end{aligned}$$

with:

$$\underline{\hat{k}}_2\left(\hat{X}'\right)=\int\frac{\hat{k}_2\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}_2\left(\hat{X}',\hat{X}\right)\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}K_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2d\hat{X}$$

leading in vr:

$$\begin{aligned}
\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\quad\rightarrow\quad&\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\left\langle\hat{k}_1\right\rangle\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)}{1+\hat{k}\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{\left(1+\hat{k}\left(\hat{X}\right)\right)1+\frac{\hat{k}_2\left(\hat{X},\left\langle\hat{X}\right\rangle\right)}{1-\left\langle\hat{k}\right\rangle}}\right)\\
&\times\frac{\sqrt{\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2-\hat{\mu}D\left(\hat{X}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)^{\frac{3}{2}}}{\left(\left\|\hat{\Psi}_0\right\|^2-\hat{\mu}\langle D\rangle\right)^2}
\end{aligned}$$

A11.4 Equation for $\hat{g}$

Given (344):

$$\begin{aligned}
\hat{K}\left[\hat{X}_1\right] &= \frac{2\sigma_{\hat{K}}^2}{\hat{\mu}}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)}{\hat{g}^2\left(\hat{X}_1\right)}\right)^2\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{r\left\langle\hat{g}\right\rangle^2}{3}\hat{\underline{k}}\right) \\
&\simeq \frac{2\sigma_{\hat{K}}^2}{\hat{\mu}\hat{g}^2\left(\hat{X}_1\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)^2\left(\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}\right)
\end{aligned} \tag{265}$$

We can find the return as a function of $\hat{K}\left[\hat{X}_1\right]$ the total capital in sector $\hat{X}_1$ by writing the equation:

$$0=\frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{2\sigma_{\hat{K}}^2}\left(\hat{g}^2\left(\hat{X}_1\right)\right)^2-\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)^2\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}+\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)^2\frac{r\left\langle\hat{g}\right\rangle^2}{3}\hat{\underline{k}}$$

so that:

$$\begin{aligned}
\hat{g}\left(\hat{X}_1\right) &= \frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)^2-\sqrt{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)^2-2\frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{\sigma_{\hat{K}}^2}\frac{r\left\langle\hat{g}\right\rangle^2}{3}\hat{\underline{k}}}}{\frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{\sigma_{\hat{K}}^2}} \\
&\simeq \frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}}}{\sqrt{\frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{2\sigma_{\hat{K}}^2}}}
\end{aligned}$$

where:

$$D\left(\hat{X}_1\right)=\left(\frac{\left\langle\hat{K}\right\rangle^2\left\langle\hat{g}\right\rangle^2}{\sigma_{\hat{K}}^2}+\frac{\left\langle\hat{g}\right\rangle}{2}\right)\left(\frac{\hat{\underline{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)}{\hat{\underline{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)}-\frac{6\hat{\underline{k}}}{2+\hat{\underline{k}}-\sqrt{\left(2+\hat{\underline{k}}\right)^2-\hat{\underline{k}}}}\right)\hat{\underline{k}}$$

This allows to rewrite (80):

$$\begin{aligned}
&\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}-\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)\left(\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}}}{\sqrt{\frac{\hat{\mu}\hat{K}\left[\hat{X}'\right]}{2\sigma_{\hat{K}}^2}}}-\bar{r}'\right) \\
&= \frac{1}{3}\frac{(1-\beta)(X+C\beta)^2\epsilon}{\sigma_{\hat{K}}^2\left(f_1(X)+\beta\underline{k}(X)R\right)}\left(\frac{R(3X-\beta C)(\beta C+X)}{4\left(f_1(X)+\beta\underline{k}(X)R\right)}-\frac{C(2X-C\beta)}{1+\underline{k}(X)}\right)
\end{aligned} \tag{266}$$

where $\hat{S}_1^E\left(\hat{X}',\hat{X}\right)$ has been computed previously in (260).

A11.5 Solving equation for $\hat{g}$

Equation (266) is solved by defining:

$$H=\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\left(\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}}}{\sqrt{\frac{\hat{\mu}\hat{K}\left[\hat{X}'\right]}{2\sigma_{\hat{K}}^2}}}-\bar{r}'\right)$$

$$\underline{k}(X)=\frac{k(X)}{\langle K\rangle}\hat{K}_X\frac{\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{|\Psi_0(X)|^2}$$

$$\hat{K}\left[\hat{X}\right]=\frac{\underline{k}(X)}{k(X)}\langle K\rangle|\Psi_0(X)|^2\rightarrow\frac{\underline{k}(X)}{k}$$

so that:

$$\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}}}{\left(1+\hat{\underline{k}}_2\left(\hat{X}_1\right)\right)\sqrt{\frac{\hat{\mu}\hat{K}\left[\hat{X}'\right]}{2\sigma_{\hat{K}}^2}}}\rightarrow\frac{\sqrt{\frac{2\sigma_{\hat{K}}^2k}{\hat{\mu}}\left(\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)}{\left(1+\hat{\underline{k}}_2\left(\hat{X}_1\right)\right)\sqrt{\underline{k}\left(\hat{X}_1\right)}}$$

and then, by isolating the terms not acted by $\hat{S}_1^E\left(\hat{X}',\hat{X}\right)$ on, to write (80) as:

$$\begin{aligned}&\frac{1}{3}\frac{(1-\beta)(X+C\beta)^2\epsilon}{\sigma_{\hat{K}}^2\left(f_1(X)+\beta\underline{k}(X)R\right)}\left(\frac{R(3X-\beta C)(\beta C+X)}{4\left(f_1(X)+\beta\underline{k}(X)R\right)}-\frac{C(2X-C\beta)}{1+\underline{k}(X)}\right)\\&-\frac{\sqrt{\frac{2\sigma_{\hat{K}}^2k}{\hat{\mu}}\left(\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)}{\left(1+\hat{\underline{k}}_2\left(\hat{X}_1\right)\right)\sqrt{\underline{k}\left(\hat{X}_1\right)}}\\=&\left(-H-\frac{\bar{r}'}{1+\hat{\underline{k}}_2\left(\hat{X}_1\right)}\right)\end{aligned}$$

Then a first order expansion in R leads to:

$$\begin{aligned}&\left(1-\frac{\beta\underline{k}(X)R}{f_1^2(X)}\right)\left(\frac{R(3X-\beta C)(\beta C+X)}{4\left(f_1(X)+\beta\underline{k}(X)R\right)}-\frac{C(2X-C\beta)}{1+\underline{k}(X)}\right)\\&-\frac{3\sigma_{\hat{K}}^2f_1(X)}{(1-\beta)(X+C\beta)^2\epsilon}\frac{\sqrt{\frac{2\sigma_{\hat{K}}^2k}{\hat{\mu}}\left(\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)}{\left(1+\hat{\underline{k}}_2\left(\hat{X}_1\right)\right)\sqrt{\underline{k}\left(\hat{X}_1\right)}}\\=&\frac{3\sigma_{\hat{K}}^2f_1(X)}{(1-\beta)(X+C\beta)^2\epsilon}\left(-H-\frac{\bar{r}'}{1+\hat{\underline{k}}_2\left(\hat{X}_1\right)}\right)\end{aligned}$$

$$\begin{aligned}&\left(1-\frac{\beta\underline{k}(X)R}{f_1^2(X)}\right)\left(-\frac{C(2X-C\beta)}{1+\underline{k}(X)}\right)-\frac{3\sigma_{\hat{K}}^2f_1(X)}{(1-\beta)(X+C\beta)^2\epsilon}\frac{\sqrt{\frac{2\sigma_{\hat{K}}^2k}{\hat{\mu}}\left(\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)}{\left(1+\hat{\underline{k}}_2\left(\hat{X}_1\right)\right)\sqrt{\underline{k}\left(\hat{X}_1\right)}}\\=&\frac{3\sigma_{\hat{K}}^2f_1(X)}{(1-\beta)(X+C\beta)^2\epsilon}\left(-H-\frac{\bar{r}'}{1+\hat{\underline{k}}_2\left(\hat{X}_1\right)}\right)-\frac{R(3X-\beta C)(\beta C+X)}{4\left(f_1(X)+\beta\underline{k}(X)R\right)}\end{aligned}$$

and this equation leads to:

$$\left(\frac{1}{1+\underline{k}(X)}\right)\left(1-\frac{\beta\underline{k}(X)R}{f_1^2(X)}\right)+\frac{3\sigma_{\hat{K}}^2 f_1(X)}{(1-\beta)C(2X-C\beta)(X+C\beta)^2\epsilon}\frac{\sqrt{\frac{2\sigma_{\hat{K}}^2 k}{\hat{\mu}}\left(\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)}{\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\sqrt{\underline{k}\left(\hat{X}_1\right)}}$$

$$=\frac{3\sigma_{\hat{K}}^2 f_1(X)}{(1-\beta)C(2X-C\beta)(X+C\beta)^2\epsilon}\left(H+\frac{\bar{r}'}{1+\underline{\hat{k}}_2\left(\hat{X}_1\right)}\right)+\frac{R(3X-\beta C)(\beta C+X)}{4f_1(X)C(2X-C\beta)}$$

$$\frac{1}{1+\underline{k}(X)}\left(1+\frac{\beta R}{f_1^2(X)}\right)+\frac{3\sigma_{\hat{K}}^2 f_1(X)}{C(2X-C\beta)(1-\beta)(X+C\beta)^2\epsilon}\frac{\sqrt{\frac{2\sigma_{\hat{K}}^2 k}{\hat{\mu}}\left(\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)}{\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\sqrt{\underline{k}\left(\hat{X}_1\right)}}$$

$$=\frac{3\sigma_{\hat{K}}^2 f_1(X)}{C(2X-C\beta)(1-\beta)(X+C\beta)^2\epsilon}\left(H+\frac{\bar{r}'}{1+\underline{\hat{k}}_2\left(\hat{X}_1\right)}\right)+\frac{R(3X-\beta C)(\beta C+X)}{4f_1(X)C(2X-C\beta)}+\frac{\beta R}{f_1^2(X)}$$

Then we define:

$$a=\frac{3\sigma_{\hat{K}}^2 f_1(X)}{C(2X-C\beta)(1-\beta)(X+C\beta)^2\epsilon}\frac{\sqrt{\frac{2\sigma_{\hat{K}}^2 k}{\hat{\mu}}\left(\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)}{\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)}$$

and:

$$c=\left(\frac{3\sigma_{\hat{K}}^2 f_1(X)}{C(2X-C\beta)(1-\beta)(X+C\beta)^2\epsilon}\left(H+\frac{\bar{r}'}{1+\underline{\hat{k}}_2\left(\hat{X}_1\right)}\right)+\frac{R(3X-\beta C)(\beta C+X)}{4f_1(X)C(2X-C\beta)}+\frac{\beta R}{f_1^2(X)}\right)$$

so that the equation reduces in first approximation:

$$0=\frac{1}{1+\underline{k}(X)}\left(1+\frac{\beta R}{f_1^2(X)}\right)+\frac{a}{\sqrt{\underline{k}(X)}}-c$$

for $\underline{k}(X)>>1$ and $\epsilon<\sigma_{\hat{K}}^2 f_1(X)$ this reduces to:

$$\frac{a}{\sqrt{\underline{k}(X)}+b}-c=0$$

with solution:

$$\sqrt{\underline{k}(X)}=\frac{a}{c}$$

so that $\underline{k}(X)$ is:

$$\underline{k}(X)=\left(\frac{\sqrt{\frac{2\sigma_{\hat{K}}^2 k}{\hat{\mu}}\left(\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)}{\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\left[H+\frac{\bar{r}'}{1+\underline{\hat{k}}_2\left(\hat{X}_1\right)}+\frac{\epsilon(1-\beta)(X+C\beta)^2}{3\sigma_{\hat{K}}^2}\left(\frac{R(3X-\beta C)(\beta C+X)}{4f_1^2(X)C(2X-C\beta)}+\frac{\beta RC(2X-C\beta)}{f_1^3(X)}\right)\right]}-\sqrt{\frac{\sigma_{\hat{K}}^2\hat{\mu}k}{2}}D\left(\hat{X}_1\right)\right)^2$$

$$\begin{aligned}
\frac{\hat{g}\left(\hat{X}_1\right)-\bar{r}'}{\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)} &= \frac{\sqrt{\frac{2\sigma_{\hat{K}}^2 k}{\hat{\mu}}\left(\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)}{\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\sqrt{\underline{k}\left(\hat{X}_1\right)}} \\
&\simeq H+\frac{\bar{r}'}{1+\underline{\hat{k}}_2\left(\hat{X}_1\right)}+\frac{\frac{(3X-\beta C)(\beta C+X)}{4f_1(X)C(2X-C\beta)}+\frac{\beta}{f_1^2(X)}}{\frac{3\sigma_{\hat{K}}^2 f_1(X)}{C(2X-C\beta)(1-\beta)(X+C\beta)^2\epsilon}}\left(R+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right) \\
&= H+\frac{\bar{r}'}{1+\underline{\hat{k}}_2\left(\hat{X}_1\right)} \\
&\quad +\frac{\epsilon\left(1-\beta\right)\left(X+C\beta\right)^2}{3\sigma_{\hat{K}}^2}\left(\frac{\left(3X-\beta C\right)\left(\beta C+X\right)}{4f_1^2\left(X\right)C\left(2X-C\beta\right)}+\frac{\beta C\left(2X-C\beta\right)}{f_1^3\left(X\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)
\end{aligned}$$

and this rewrts:

$$\begin{aligned}
&\hat{g}\left(\hat{X}_1\right)-\bar{r}' \\
&= \left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\left(H+\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(X'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(X\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)\right)
\end{aligned}$$

where:

$$\begin{aligned}
A\left(\hat{X}'\right) &= \frac{\epsilon\left(1-\beta\right)\left(X+C\beta\right)^2}{3\sigma_{\hat{K}}^2}\frac{\left(3X-\beta C\right)\left(\beta C+X\right)}{4C\left(2X-C\beta\right)} \\
B\left(\hat{X}'\right) &= \frac{\epsilon\left(1-\beta\right)\left(X+C\beta\right)^2}{3\sigma_{\hat{K}}^2}\beta C\left(2X-C\beta\right)
\end{aligned}$$

We replace:

$$H\rightarrow\int\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\left(\frac{\sqrt{\frac{2\sigma_{\hat{K}}^2 k}{\hat{\mu}}\left(\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}}-\bar{r}'\right)$$

and the equation for return rewrites:

$$\begin{aligned}
\hat{g}\left(\hat{X}_1\right)-\bar{r}' &= \left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\int\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\left(\frac{\sqrt{\frac{2\sigma_{\hat{K}}^2 k}{\hat{\mu}}\left(\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}}-\bar{r}'\right) \\
&\quad +\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(X'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(X\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)
\end{aligned}$$

with solution:

$$\begin{aligned}
\hat{g}\left(\hat{X}_1\right)-\bar{r}' &= \int\left(1-\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1} \\
&\quad \times\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(X'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(X\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)
\end{aligned}$$

A11.6 Estimation of the derivativ $\frac{\partial}{\partial f_1(X)}\hat{K}\left[\hat{X}\right]$ for decreasing return to scale correction

$$\frac{\partial}{\partial f_1(X)}\hat{K}\left[\hat{X}\right]$$

$$= \frac{\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)\left(\tau F(X)+A\frac{\left(f_1^r(X)-C_0\right)-2r\left(f_1^r(X)-C_0-\bar{r}'\right)}{\left(f_1^r(X)-C_0\right)^3}+B\frac{\left(f_1^r(X)-C_0\right)-3r\left(f_1^r(X)-C_0-\bar{r}'\right)}{\left(f_1^r(X)-C_0\right)^4}\right)}{f_1(X)\left(\tau F(X)+A\frac{\left(f_1^r(X)-C_0\right)-2r\left(f_1^r(X)-C_0-\bar{r}'\right)}{\left(f_1^r(X)-C_0\right)^3}+B\frac{\left(f_1^r(X)-C_0\right)-3r\left(f_1^r(X)-C_0-\bar{r}'\right)}{\left(f_1^r(X)-C_0\right)^4}\right)-\frac{1}{2}\frac{\left(\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)^{1+r}\right)H\left(\hat{X}_1\right)}{\left(\hat{K}\left[\hat{X}_1\right]\right)^{\frac{3}{2}}}}$$

$$= \frac{\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)^{1-r}\left(\tau F(X)+A\frac{\left(f_1^r(X)-C_0\right)-2r\left(f_1^r(X)-C_0-\bar{r}'\right)}{\left(f_1^r(X)-C_0\right)^3}+B\frac{\left(f_1^r(X)-C_0\right)-3r\left(f_1^r(X)-C_0-\bar{r}'\right)}{\left(f_1^r(X)-C_0\right)^4}\right)}{\frac{f_1(X)}{\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)^r}\left(\tau F(X)+A\frac{\left(f_1^r(X)-C_0\right)-2r\left(f_1^r(X)-C_0-\bar{r}'\right)}{\left(f_1^r(X)-C_0\right)^3}+B\frac{\left(f_1^r(X)-C_0\right)-3r\left(f_1^r(X)-C_0-\bar{r}'\right)}{\left(f_1^r(X)-C_0\right)^4}\right)-\frac{1}{2}\frac{\left(\frac{1}{\hat{K}\left[\hat{X}\right]}+\underline{k}(X)\right)H\left(\hat{X}_1\right)}{\sqrt{\hat{K}\left[\hat{X}_1\right]}}}$$

The denominator is estimated with the return equation and writes:

$$\frac{H\left(\hat{X}_1\right)}{\sqrt{\hat{K}\left[\hat{X}_1\right]}}-\bar{r}'+\left(C_0+\bar{r}'\right)\left(\frac{A}{\left(f_1^r(X)-C_0\right)^2}+\frac{B}{\left(f_1^r(X)-C_0\right)^3}\right)+\tau F(X)\left\langle\frac{f_1(X)}{\left(\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)\right)^r}\right\rangle \tag{267}$$

$$-\frac{f_1(X)}{\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)^r}\left(\tau F(X)+A\frac{2r\left(f_1^r(X)-C_0-\bar{r}'\right)}{\left(f_1^r(X)-C_0\right)^3}+B\frac{3r\left(f_1^r(X)-C_0-\bar{r}'\right)}{\left(f_1^r(X)-C_0\right)^4}\right)-\frac{1}{2}\frac{\left(\frac{1}{\hat{K}\left[\hat{X}\right]}+\underline{k}(X)\right)H\left(\hat{X}_1\right)}{\sqrt{\hat{K}\left[\hat{X}_1\right]}}$$

that is:

$$\frac{1}{2}\frac{H\left(\hat{X}_1\right)}{\sqrt{\hat{K}\left[\hat{X}_1\right]}}-\bar{r}'+\left(C_0+\bar{r}'\right)\left(\frac{A}{\left(f_1^r(X)-C_0\right)^2}+\frac{B}{\left(f_1^r(X)-C_0\right)^3}\right)+\tau F(X)\left\langle\frac{f_1(X)}{\left(\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)\right)^r}\right\rangle$$

$$-\frac{f_1(X)}{\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)^r}\left(\tau F(X)+A\frac{2r\left(f_1^r(X)-C_0-\bar{r}'\right)}{\left(f_1^r(X)-C_0\right)^3}+B\frac{3r\left(f_1^r(X)-C_0-\bar{r}'\right)}{\left(f_1^r(X)-C_0\right)^4}\right)$$

Using the return equation:

$$\frac{1}{2}\left(\frac{H\left(\hat{X}_1\right)}{\sqrt{\hat{K}\left[\hat{X}_1\right]}}-\bar{r}'\right) = \frac{1}{2}\left(\frac{A}{\left(f_1^r(X)-C_0\right)^2}+\frac{B}{\left(f_1^r(X)-C_0\right)^3}\right)\left(f_1^r(X)-C_0-\bar{r}'\right)$$

$$+\frac{1}{2}\tau F(X)\left(\frac{f_1(X)}{\left(\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)\right)^r}-\left\langle\frac{f_1(X)}{\left(\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)\right)^r}\right\rangle\right)$$

term (267) becoms:

$$\frac{1}{2}\left(-\bar{r}'+(C_0+\bar{r}')\left(\frac{A}{(f_1^r(X)-C_0)^2}+\frac{B}{(f_1^r(X)-C_0)^3}\right)+\tau F(X)\left\langle\frac{f_1(X)}{\left(\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)\right)^r}\right\rangle\right)$$
$$+\frac{1}{2}\left(\frac{A}{(f_1^r(X)-C_0)^2}+\frac{B}{(f_1^r(X)-C_0)^3}\right)\frac{f_1(X)}{\left(\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)\right)^r}+\frac{1}{2}\tau F(X)\frac{f_1(X)}{\left(\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)\right)^r}$$
$$-\frac{f_1(X)}{\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)^r}\left(\tau F(X)+A\frac{2r(f_1^r(X)-C_0-\bar{r}')}{(f_1^r(X)-C_0)^3}+B\frac{3r(f_1^r(X)-C_0-\bar{r}')}{(f_1^r(X)-C_0)^4}\right)$$
$$=\frac{1}{2}\left(-\bar{r}'+(C_0+\bar{r}')\left(\frac{A}{(f_1^r(X)-C_0)^2}+\frac{B}{(f_1^r(X)-C_0)^3}\right)\right)+\frac{1}{2}\tau F(X)\left(\langle f_1^r(X)\rangle-f_1^r(X)\right)$$
$$+\frac{1}{2}\left(\frac{A}{(f_1^r(X)-C_0)^2}+\frac{B}{(f_1^r(X)-C_0)^3}\right)f_1^r(X)-f_1^r(X)\left(A\frac{2r(f_1^r(X)-C_0-\bar{r}')}{(f_1^r(X)-C_0)^3}+B\frac{3r(f_1^r(X)-C_0-\bar{r}')}{(f_1^r(X)-C_0)^4}\right)$$

In the main cases:

$$|f_1^r(X)-C_0-\bar{r}'|<<|f_1^r(X)-C_0|$$

so that (267) is in frst approximation:

$$\frac{1}{2}\left(-\bar{r}'+(C_0+\bar{r}')\left(\frac{A}{(f_1^r(X)-C_0)^2}+\frac{B}{(f_1^r(X)-C_0)^3}\right)\right)+\frac{1}{2}\tau F(X)\left(\langle f_1^r(X)\rangle-f_1^r(X)\right)$$
$$+\frac{1}{2}\left(\frac{A}{(f_1^r(X)-C_0)^2}+\frac{B}{(f_1^r(X)-C_0)^3}\right)f_1^r(X)$$

and:

$$\frac{1}{2}\left(-\bar{r}'+(C_0+\bar{r}')\left(\frac{A}{(\bar{r}')^2}+\frac{B}{(\bar{r}')^3}\right)\right)+\frac{1}{2}\tau F(X)\left(\langle f_1^r(X)\rangle-f_1^r(X)\right)+\frac{1}{2}\left(\frac{A}{(\bar{r}')^2}+\frac{B}{(\bar{r}')^3}\right)f_1^r(X)>0$$

A11.7 Averaged equations

We compute the various averages arising in the model. Recal that:

$$\underline{\hat{k}}_2\left(\hat{X}'\right)=\int\hat{k}_2\left(\hat{X}',\hat{X}_1\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2d\hat{K}_1d\hat{X}_1 \quad (268)$$
$$\underline{\hat{k}}\left(\hat{X}'\right)=\int\hat{k}\left(\hat{X}',\hat{X}_1\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2d\hat{K}_1d\hat{X}_1$$

w sm:

$$\hat{k}\left(\hat{X}',\hat{X}_1\right)\rightarrow\frac{\hat{k}\left(\hat{X}',\hat{X}_1\right)}{\left\langle\hat{K}\right\rangle}$$

$$\underline{\hat{k}}\left(\hat{X}'\right)=\int\hat{k}\left(\hat{X}',\hat{X}_1\right)\hat{K}_1\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2d\hat{K}_1d\hat{X}_1\simeq\hat{k}\left(\hat{X}',\left\langle\hat{X}\right\rangle\right)\left\|\hat{\Psi}\right\|^2 \quad (269)$$

A11.7.1 Firm average return and average capital

In average, and given the normalisations:

$$\begin{aligned}
\frac{1}{1+\left\langle \underline{\hat{k}}(X)\right\rangle} &= \frac{1}{1+\left\langle \left(\hat{k}(X,X')-\hat{k}(\langle X\rangle,\langle X\rangle)\right)\frac{\hat{K}_{X'}\left\|\hat{\Psi}(\hat{X}')\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}\right\rangle}\rightarrow 1 \\
\langle\underline{k}(X)\rangle &= \left\langle k(X,X')\frac{\hat{K}_{X'}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2}{\langle K\rangle\left\|\Psi\right\|^2}\right\rangle \\
&\simeq k\left(\langle X\rangle,\left\langle\hat{X}\right\rangle\right)\frac{\left\langle\hat{K}\right\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}\right\|^2}{\left\|\Psi\right\|^2}\simeq\langle\underline{k}\rangle\frac{\left\langle\hat{K}\right\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\left\|\Psi_0\right\|^2}
\end{aligned}$$

to this order we replace $\hat{g}\left(\left\langle\hat{X}\right\rangle\right)\rightarrow\bar{r}'$ so that using: (255):

$$\left\langle\hat{K}\right\rangle=\frac{3}{4\bar{r}'}\sqrt{\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}}\frac{1-2\frac{2+\underline{\hat{k}}-\sqrt{\left(2+\underline{\hat{k}}\right)^2-\underline{\hat{k}}}}{9}}{1-\frac{2+\underline{\hat{k}}-\sqrt{\left(2+\underline{\hat{k}}\right)^2-\underline{\hat{k}}}}{4\underline{\hat{k}}}}\sqrt{\frac{\left\|\hat{\Psi}_0\right\|^2+\hat{\mu}\frac{\bar{r}'}{2}\left(\frac{1-r}{r}\right)\underline{\hat{k}}}{1-2r(1-r)\underline{\hat{k}}}} \tag{270}$$

with:

$$r=\frac{2+\underline{\hat{k}}-\sqrt{\left(2+\underline{\hat{k}}\right)^2-\underline{\hat{k}}}}{6\underline{\hat{k}}}$$

We compute the average firm capital by using:

$$\langle K\rangle=\frac{1}{4\left\langle f_1^{(e)}\right\rangle}\left\langle\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}\right\rangle$$

and use:

$$\begin{aligned}
\left\langle X^{(e)}\right\rangle &\rightarrow \sqrt{\frac{\left|\Psi_0\right|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle-\frac{1}{2}\left(\beta\underline{k}\left(\langle f_1\rangle-\bar{r}\right)\right)} \\
&\simeq \sqrt{\frac{\left|\Psi_0\right|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\frac{\beta\underline{k}\left(\langle f_1\rangle-\bar{r}\right)}{4\sqrt{\frac{\left|\Psi_0\right|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}}=\sqrt{\frac{\left|\Psi_0\right|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\frac{\beta\langle\underline{k}\rangle\frac{\langle\hat{K}\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\left\|\Psi_0\right\|^2}\left(\langle f_1\rangle-\bar{r}\right)}{4\sqrt{\frac{\left|\Psi_0\right|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}}
\end{aligned}$$

with:

$$\left\langle C^{(e)}\right\rangle\simeq\beta C$$

The effective average return $\left\langle f_1^{(e)}\right\rangle$ for firms is:

$$\begin{aligned}
&\left(\left(1+\frac{k(X)}{K[X]}\hat{K}\left[\hat{X}\right]\right)^r K_X^r\right)\left\langle f_1^{(e)}\right\rangle \\
=&\ \langle f_1\rangle - C_0\left\langle\left(1+\frac{k(X)}{K[X]}\hat{K}\left[\hat{X}\right]\right)^r K_X^r\right\rangle + \beta\underline{k}\left(\langle f_1\rangle - C_0\left\langle\left(1+\frac{k(X)}{K[X]}\hat{K}\left[\hat{X}\right]\right)^r K_X^r\right\rangle - \bar{r}\right) \\
=&\ \langle f_1\rangle - C_0\left(1+\langle\underline{k}\rangle\frac{\left\langle\hat{K}\right\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\right)^r\langle K\rangle^r + \beta\langle\underline{k}\rangle\frac{\left\langle\hat{K}\right\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\left(\langle f_1\rangle - C_0\left(1+\langle\underline{k}\rangle\frac{\left\langle\hat{K}\right\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\right)^r\langle K\rangle^r - \bar{r}\right) \\
\simeq&\ \langle f_1\rangle - C_0\left(1+\langle\underline{k}\rangle\frac{\left\langle\hat{K}\right\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\right)^r\langle K\rangle^r + \beta\langle\underline{k}\rangle\frac{\left\langle\hat{K}\right\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\left(\langle f_1\rangle - C_0\langle K\rangle^r - \bar{r}\right) \\
=&\ \langle f_1\rangle - C_0\left(1+\langle\underline{k}\rangle\frac{\left\langle\hat{K}\right\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\right)^r\langle K\rangle^r
\end{aligned}$$

and:

$$\left\langle f_1^{(e)}\right\rangle \simeq \frac{\langle f_1\rangle}{\left(\left(1+\frac{k(X)}{K[X]}\hat{K}\left[\hat{X}\right]\right)^r K_X^r\right)} - C_0$$

leading to the average cp fm:

$$\langle K\rangle \simeq \frac{\left(3\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)\left(\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}+\beta C\right)}{4\left(\langle f_1\rangle - C_0\langle K\rangle^r\right)\left(2\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)}$$

$$\langle K\rangle\left(\langle f_1\rangle - C_0\left(1+\langle\underline{k}\rangle\frac{\left\langle\hat{K}\right\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\right)^r\langle K\rangle^r\right) \simeq \frac{\left(3\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)\left(\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}+\beta C\right)}{4\left(2\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)}$$

Define:

$$\begin{aligned}
\langle K\rangle_1^r &\simeq \frac{\langle f_1\rangle}{C_0\left(1+\langle\underline{k}\rangle\frac{\left\langle\hat{K}\right\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\right)^r} \\
&\simeq \frac{\langle f_1\rangle}{C_0\left(1+\langle\underline{k}\rangle\frac{\left\langle\hat{K}\right\rangle}{\langle f_1\rangle^r}C_0^r\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\right)^r} = \frac{\langle f_1\rangle^{1+r^2}}{C_0\left(\langle f_1\rangle^r + \langle\underline{k}\rangle\left\langle\hat{K}\right\rangle C_0^r\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\right)^r}
\end{aligned}$$

that is:

$$\langle K\rangle_1 = \frac{\langle f_1\rangle^{\frac{1+r^2}{r}}}{C_0^{\frac{1}{r}}\left(\langle f_1\rangle^r + \langle\underline{k}\rangle\left\langle\hat{K}\right\rangle C_0^r\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\right)}$$

and the average capital for firms is obtained as correction of $\langle K\rangle_1^r$:

$$\langle K\rangle^r = \langle K\rangle_1^r - \frac{\left(3\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)\left(\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}+\beta C\right)\langle K\rangle_1^r}{4\langle f_1\rangle\langle K\rangle_1\left(2\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)}$$

that writes:

$$\langle K\rangle \simeq \langle K\rangle_1 \left(1-\frac{\left(3\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)\left(\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}+\beta C\right)}{4\langle f_1\rangle \langle K\rangle_1 \left(2\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)}\right)^{\frac{1}{r}}$$

and the effective return:

$$\begin{aligned}\left\langle f_1^{(e)}\right\rangle &= \langle f_1\rangle \left(1-\beta\langle\underline{k}\rangle \frac{\left(\frac{1}{4}-\frac{r}{3}\hat{k}\right)}{\left(\frac{1}{3}-\frac{r}{2}\hat{k}\right)\bar{r}'} \frac{4\left(2\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)\sqrt{\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(\left\|\hat{\Psi}_0\right\|^2-\hat{\mu}\langle D\rangle\right)}}{\left(3\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)\left(\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}+\beta C\right)} \frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}(\langle f_1\rangle-\bar{r})\right)^{-1} \\
&\simeq \langle f_1\rangle+\beta\langle\underline{k}\rangle \frac{\left(\frac{1}{4}-\frac{r}{3}\hat{k}\right)}{\left(\frac{1}{3}-\frac{r}{2}\hat{k}\right)\bar{r}'} \\
&\quad \times \frac{4\left\langle f_1^{(e)}\right\rangle\left(2\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)\sqrt{\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(\left\|\hat{\Psi}_0\right\|^2-\hat{\mu}\langle D\rangle\right)}}{\left(3\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\beta C\right)\left(\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}+\beta C\right)} \frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}(\langle f_1\rangle-\bar{r})\end{aligned}$$

A11.7.2 Investors average return

We consider the average for investors' return by starting with $\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$:

$$\begin{aligned}&\left(1+\hat{\underline{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right) \\
\rightarrow\ & \frac{1+\hat{\underline{k}}_2\left(\hat{X}_1\right)}{1+\hat{k}\left(\hat{X}\right)}\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\hat{\underline{k}}_1(\langle X\rangle)\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)}\right) \\
&\times\frac{\sqrt{\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2-\hat{\mu}D\left(\hat{X}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)^{\frac{3}{2}}}{\left(\left\|\hat{\Psi}_0\right\|^2-\hat{\mu}\langle D\rangle\right)^2}\end{aligned}$$

so that:

$$\begin{aligned}&\left\langle\left(1+\beta\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle\right)\right)\hat{S}_1^E\right\rangle \\
\simeq\ & \left\langle\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)}{1+\hat{\underline{k}}\left(\hat{X}\right)}-\frac{(1-\beta)\hat{k}\left(\langle X\rangle,\hat{X}'\right)}{\left(1+\hat{\underline{k}}\left(\left\langle\hat{X}\right\rangle\right)\right)^2}+\frac{(1-\beta)\hat{k}_1^E\left(\hat{X},\left\langle\hat{X}\right\rangle\right)-(1-\beta)\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}'\right)}{1+\beta\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle\right)}\right)\right\rangle \\
\simeq\ & \left\langle\hat{\underline{k}}\right\rangle+(1-\beta)\left\langle\hat{\underline{k}}\right\rangle\end{aligned}$$

As a consequence:

$$\left\langle\left(1-\left(1+\hat{\underline{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1}\right\rangle=\left(1-(2-\beta)\left\langle\hat{k}\right\rangle\right)^{-1}=\frac{1}{1-(2-\beta)\left\langle\hat{k}\right\rangle}$$

and the average return is given by:

$$\begin{aligned}
\langle\hat{g}\rangle &= \frac{\left(1-(2-\beta)\left\langle\underline{\hat{k}}\right\rangle\right)^{-1}}{\left(1+\left\langle\underline{\hat{k}}_2\right\rangle\right)} \\
&\times\left\langle\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\left(\frac{\epsilon\left(1-\beta\right)\left(X+C\beta\right)^2\left(\frac{(3X-\beta C)(\beta C+X)}{4f_1^2(X)C(2X-C\beta)}+\frac{\beta C(2X-C\beta)}{f_1^3(X)}\right)}{3\sigma_{\hat{K}}^2}\left(R+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)\right)\right\rangle \\
&= \frac{1}{1-(2-\beta)\left\langle\underline{\hat{k}}\right\rangle}\left\langle\Delta\right\rangle+\bar{r}'
\end{aligned}$$

given the normalization:

$$\left\langle\underline{\hat{k}}\right\rangle\rightarrow 0$$

this leads to:

$$\langle\hat{g}\rangle=\langle\Delta\rangle+\bar{r}'$$

with:

$$\begin{aligned}
\langle\Delta\rangle &= \left\langle\left(\frac{A}{\left(f_1^{(r)}\left(X\right)-C_0\right)^2}+\frac{B}{\left(f_1^{(r)}\left(X\right)-C_0\right)^3}\right)\right. \\
&\quad\left.\times\left(f_1^{(r)}\left(X\right)+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)\right\rangle
\end{aligned}$$

with:

$$f_1^{(r)}\left(X\right)=\frac{f_1\left(X\right)}{\left(1+\langle\underline{k}\rangle\left\langle\hat{K}\right\rangle\frac{\left\|\hat{\Psi}_0\right\|^2}{\left\|\Psi_0\right\|^2}\right)^r}-C_0$$

A11.7.3 Investors capital

Having found an expression for $\langle\hat{g}\rangle$ we can correct the previous expression for $\left\langle\hat{K}\right\rangle$ at first order. Using (253) (254) we find:

$$\left\|\hat{\Psi}\right\|^2\simeq\frac{\hat{\mu}V}{3\sigma_{\hat{K}}^2\langle\hat{g}\rangle}\left(2\frac{\sigma_{\hat{K}}^2\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}}+\frac{\langle\hat{g}\rangle}{2}\left(\frac{1-r}{r}\right)\underline{\hat{k}}\right)}{\left(1-2r\left(1-r\right)\underline{\hat{k}}\right)}\right)^{\frac{3}{2}}\left(1-\frac{2+\underline{\hat{k}}-\sqrt{\left(2+\underline{\hat{k}}\right)^2-\underline{\hat{k}}}}{4\underline{\hat{k}}}\underline{\hat{k}}\right) \tag{271}$$

$$\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2=\frac{\hat{\mu}V\sigma_{\hat{K}}^2}{2\left\langle\hat{g}\right\rangle^2}\left(\frac{\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}}+\frac{\langle\hat{g}\rangle}{2}\left(\frac{1-r}{r}\right)\underline{\hat{k}}\right)}{\left(1-2r\left(1-r\right)\underline{\hat{k}}\right)}\right)^2\left(1-2\frac{2+\underline{\hat{k}}-\sqrt{\left(2+\underline{\hat{k}}\right)^2-\underline{\hat{k}}}}{9}\right) \tag{272}$$

where:

$$\langle\hat{g}\rangle=\langle\Delta\rangle+\bar{r}'$$

A11.7.4 Dependency in productivity

The derivative of (272) with respect to $\langle f_1(X)\rangle$ is:

$$\frac{\partial\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}{\partial\left\langle f_1(X)\right\rangle}=-\hat{\mu}V\sigma_{\hat{K}}^2\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}\left\langle\hat{g}\right\rangle^2}\left(\frac{\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}\left\langle\hat{g}\right\rangle}+\frac{1}{2}\left(\frac{1-r}{r}\right)\underline{\hat{k}}}{\left(1-2r(1-r)\underline{\hat{k}}\right)}\right)\left(1-2\frac{2+\underline{\hat{k}}-\sqrt{\left(2+\underline{\hat{k}}\right)^2-\underline{\hat{k}}}}{9}\right)\frac{\partial\left\langle\Delta\right\rangle}{\partial\left\langle f_1(X)\right\rangle}$$

Assuming that $\left\langle\Delta F_\tau\left(\bar{R}(K,X)\right)\right\rangle=0$, define:

$$f_1^{(r)}(X)=\frac{f_1(X)}{\left(1+\left\langle\underline{k}\right\rangle\left\langle\hat{K}\right\rangle\frac{\left\|\hat{\Psi}_0\right\|^2}{\left\|\Psi_0\right\|^2}\right)^r}$$

$\langle\Delta\rangle$ has the form:

$$\langle\Delta\rangle=\left\langle\left(\frac{A}{\left(f_1^{(r)}(X)-C_0\right)^2}+\frac{B}{\left(f_1^{(r)}(X)-C_0\right)^3}\right)\left(f_1^{(r)}(X)-C_0-\bar{r}\right)\right\rangle$$

we obtn:

$$\begin{aligned}\frac{\partial\langle\Delta\rangle}{\partial\left\langle f_1(X)\right\rangle}&=-\left(\frac{1-rf_1^{(r)}(X)\frac{\partial\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}{\partial\left\langle f_1(X)\right\rangle}\frac{\left\langle\underline{k}\right\rangle\left\langle K\right\rangle^r}{\left\langle K\right\rangle\left\|\Psi_0\right\|^2}}{1+\left\langle\underline{k}\right\rangle\frac{\left\langle\hat{K}\right\rangle}{\left\langle K\right\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\left\|\Psi_0\right\|^2}}\right)\\&\quad\times\left(\frac{A\left(\left(f_1^{(r)}(X)-C_0\right)-2\bar{r}\right)}{\left(f_1^{(r)}(X)-C_0\right)^3}+\frac{B\left(2\left(f_1^{(r)}(X)-C_0\right)-3\bar{r}\right)}{\left(f_1^{(r)}(X)-C_0\right)^4}\right)\end{aligned}\tag{273}$$

We set:

$$W=\hat{\mu}V\sigma_{\hat{K}}^2\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}\left\langle\hat{g}\right\rangle^2}\left(\frac{\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}\left\langle\hat{g}\right\rangle}+\frac{1}{2}\left(\frac{1-r}{r}\right)\underline{\hat{k}}}{\left(1-2r(1-r)\underline{\hat{k}}\right)}\right)\left(1-2\frac{2+\underline{\hat{k}}-\sqrt{\left(2+\underline{\hat{k}}\right)^2-\underline{\hat{k}}}}{9}\right)$$

so that

$$\frac{\partial\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}{\partial\left\langle f_1(X)\right\rangle}=\frac{W\left(A\left(\left(f_1^{(r)}(X)-C_0\right)-2\bar{r}\right)+\frac{B\left(2\left(f_1^{(r)}(X)-C_0\right)-3\bar{r}\right)}{f_1^{(r)}(X)-C_0}\right)}{\left(1+W\frac{rf_1^{(r)}(X)\frac{\partial\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}{\partial\left\langle f_1(X)\right\rangle}\frac{\left\langle\underline{k}\right\rangle\left\langle K\right\rangle^r}{\left\langle K\right\rangle\left\|\Psi_0\right\|^2}}{1+\left\langle\underline{k}\right\rangle\frac{\left\langle\hat{K}\right\rangle}{\left\langle K\right\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\left\|\Psi_0\right\|^2}}\right)\left(A\left(\left(f_1^{(r)}(X)-C_0\right)-2\bar{r}\right)+\frac{B\left(2\left(f_1^{(r)}(X)-C_0\right)-3\bar{r}\right)}{f_1^{(r)}(X)-C_0}\right)}$$

leading to:

$$\frac{\partial\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}{\partial\left\langle f_1(X)\right\rangle}>0$$

in the main case. Actually, the standard marginal return condition writes here:

$$(1-r)f_1^{(r)}(X)-C_0=\bar{r}$$

So that the condition:

$$\left(f_1^{(r)}(X)-C_0\right)-2\bar{r}>0$$

writes:

$$\frac{r}{(1-r)}C_0+\left(\frac{1}{(1-r)}-2\right)\bar{r}>0$$

which is statisfied in general.

Similarly, writing:

$$\left\|\hat{\Psi}\right\|^2\simeq\frac{\hat{\mu}V}{3\sigma_{\hat{K}}^2\langle\hat{g}\rangle}\left(2\frac{\sigma_{\hat{K}}^2\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}}+\frac{\langle\hat{g}\rangle}{2}\left(\frac{1-r}{r}\right)\hat{\underline{k}}\right)}{\left(1-2r(1-r)\hat{\underline{k}}\right)}\right)^{\frac{3}{2}}\left(1-\frac{2+\hat{\underline{k}}-\sqrt{\left(2+\hat{\underline{k}}\right)^2-\hat{\underline{k}}}}{4\hat{\underline{k}}}\hat{\underline{k}}\right)\tag{274}$$

we hav:

$$\frac{\partial\left\|\hat{\Psi}\right\|^2}{\partial\langle f_1(X)\rangle}=H\frac{\partial\langle\Delta\rangle}{\partial\langle f_1(X)\rangle}$$

where:

$$H=\frac{\partial}{\partial\langle\hat{g}\rangle}\left[\frac{\hat{\mu}V}{3\sigma_{\hat{K}}^2\langle\hat{g}\rangle}\left(2\frac{\sigma_{\hat{K}}^2\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}}+\frac{\langle\hat{g}\rangle}{2}\left(\frac{1-r}{r}\right)\hat{\underline{k}}\right)}{\left(1-2r(1-r)\hat{\underline{k}}\right)}\right)^{\frac{3}{2}}\left(1-\frac{2+\hat{\underline{k}}-\sqrt{\left(2+\hat{\underline{k}}\right)^2-\hat{\underline{k}}}}{4\hat{\underline{k}}}\hat{\underline{k}}\right)\right]$$

$$\frac{(10+x)^{\frac{3}{2}}}{x}$$

so that, using (273):

$$\frac{\partial\left\|\hat{\Psi}\right\|^2}{\partial\langle f_1(X)\rangle}>0$$

for $\langle\hat{g}\rangle$ in usual ranges.

A11.8 Approximate solutions to (90)

To find approximate solutions to (90), we rewrite the equation as:

$$\begin{aligned}
&\left(1-\left(1+\hat{\underline{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)\left(\pm\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}-\bar{r}'\right)\\
=\;&\left(1+\hat{\underline{k}}_2\left(\hat{X}\right)\right)\left(\frac{A}{\left(\frac{f_1(X')}{\left(\left(1+\underline{k}(X)\hat{K}\left[\hat{X}'\right]\right)\right)^r}-C_0\right)^2}+\frac{B}{\left(\frac{f_1(X')}{\left(\left(1+\underline{k}(X)\hat{K}\left[\hat{X}'\right]\right)\right)^r}-C_0\right)^3}\right)\\
&\times\left[\left(\frac{f_1(X')}{\left(\left(1+\underline{k}(X')\hat{K}\left[\hat{X}\right]\right)\right)^r}-C_0-\bar{r}'\right)+\tau F(X')\left(\frac{f_1(X')}{\left(\left(1+\underline{k}(X)\hat{K}\left[\hat{X}'\right]\right)\right)^r}-\left\langle\frac{f_1(X)}{\left(\left(1+\underline{k}(X)\hat{K}\left[\hat{X}\right]\right)\right)^r}\right\rangle\right)\right]
\end{aligned}$$

We define $\hat{K}_1\left[\hat{X}\right]$ the solution without interactions between sectrs, and consider the difference between interaction/no interaction cases by:

$$\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}=\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}_1\left[\hat{X}\right]}}+\Delta\left(\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\right)$$

so that the equation for $\frac{N(\hat{X})}{\sqrt{\hat{K}[\hat{X}]}}$ becomes:

$$\begin{aligned}&\left(1-\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)\left(\pm\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}_1\left[\hat{X}\right]}}-\bar{r}'\pm\Delta\left(\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\right)\right)\\ =\;&\pm\left(\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}_1\left[\hat{X}\right]}}+H_1\Delta\left(\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\right)+\frac{1}{2}H_2\left(\Delta\left(\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\right)\right)^2\right)\end{aligned}$$

where the coefficients H_l are given by:

$$\begin{aligned}H_l\;=\;&\frac{\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)}{\left(N\left(\hat{X}\right)\right)^l}\frac{\partial^l}{\partial\left(\frac{1}{\sqrt{\hat{K}[\hat{X}]}}\right)^l}\left(\frac{A}{\left(\frac{f_1(X')}{\left((1+\underline{k}(X)\hat{K}[\hat{X}'])\right)^r}-C_0\right)^2}+\frac{B}{\left(\frac{f_1(X')}{\left((1+\underline{k}(X)\hat{K}[\hat{X}'])\right)^r}-C_0\right)^3}\right)\\ &\times\left[\left(\frac{f_1\left(X'\right)}{\left(\left(1+\underline{k}\left(X'\right)\hat{K}\left[\hat{X}\right]\right)\right)^r}-C_0-\bar{r}'\right)+\tau F\left(X'\right)\left(\frac{f_1\left(X'\right)}{\left(\left(1+\underline{k}\left(X\right)\hat{K}\left[\hat{X}'\right]\right)\right)^r}-\left\langle\frac{f_1\left(X\right)}{\left(\left(1+\underline{k}\left(X\right)\hat{K}\left[\hat{X}\right]\right)\right)^r}\right\rangle\right)\right]\end{aligned}$$

We thus find a second order expansion for (90):

$$\begin{aligned}0=\;&\frac{1}{2}H_2\left(\Delta\left(\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\right)\right)^2-\left(1-H_1\right)\Delta\left(\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\right)\\ &+\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}_1\left[\hat{X}'\right]}}+\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}\left[\hat{X}'\right]}}\right)\end{aligned}$$

with solution:

$$\Delta\left(\frac{N\left(\hat{X}\right)}{\sqrt{\hat{K}\left[\hat{X}\right]}}\right)=\frac{\left(1-H_1\right)\pm\sqrt{\left(1-H_1\right)^2-2H_2\left(\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\left(\frac{N(\hat{X}')}{\sqrt{\hat{K}_1[\hat{X}']}}+\Delta\left(\frac{N(\hat{X}')}{\sqrt{\hat{K}[\hat{X}']}}\right)\right)\right)}}{H_2}$$

expanding the square root:

$$\sqrt{1-2H_2\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}_1\left[\hat{X}'\right]}}}\simeq 1-\frac{2H_2\left(\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\Delta\left(\frac{N(\hat{X}')}{\sqrt{\hat{K}[\hat{X}']}}\right)\right)}{\left(\left(1-H_1\right)^2-2H_2\left(1+\underline{\hat{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\frac{N(\hat{X}')}{\sqrt{\hat{K}_1[\hat{X}']}}\right)^2}$$

leads to the lowest order equation:

$$\left(\delta\left(\hat{X}-\hat{X}'\right)\pm_{\hat{X}}\frac{2H_2\left(\left(1+\hat{\underline{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)}{\left((1-H_1)^2-2H_2\left(1+\hat{\underline{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}_1\left[\hat{X}'\right]}}\right)^{\frac{3}{2}}}\right)\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}\left[\hat{X}'\right]}}\right)$$

$$=\frac{(1-H_1)\pm_{\hat{X}}\sqrt{(1-H_1)^2-\left(2H_2\left(1+\hat{\underline{k}}_2\left(\hat{X}\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\frac{N\left(\hat{X}'\right)}{\sqrt{\hat{K}_1\left[\hat{X}'\right]}}\right)}}{H_2}$$

where the sign $\pm_{\hat{X}}$ reminds that the choice of sign depends on the sector considered. This double possibility per sector implies multiplicity of solutn.

Appendix 12 Firms decreasing return to scale: computation of returns and capital for investors

A12.1 Firm returns with decreasing return to scale: computation of field and capital

We correct the linear approximation by assuming decreasing return to scale. This amounts to replace the return $f_1(X)$ by a factor depending on K:

$$f_1(X)\rightarrow\frac{f_1(X)}{K^r}$$

To include constant cost and depreciation of capital we thus globally replace:

$$f_1(X)K\rightarrow\left(\frac{f_1(X)}{K^r}-C_0\right)K$$

which is also rewritten in terms of private capital:

$$f_1(X)\left(\left(1+\frac{\underline{k}(X)}{\langle K_p\rangle}\hat{K}_X\right)K_p\right)^{1-r}-C_0\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\right)K_p$$

Including the constant cost, the return of the total capital invested writes:

$$K_{pX}\frac{f_1(X)\left(\left(1+\frac{\underline{k}(X)}{\langle K_p\rangle}\hat{K}_X\right)K_p\right)^{1-r}-C_0\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\right)K_p-C}{\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\right)K_{pX}}$$

$$\rightarrow\quad\frac{f_1(X)\left(\left(1+\frac{\underline{k}(X)}{\langle K_p\rangle}\hat{K}_X\right)K_p\right)^{1-r}-C_0\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_p-C}{\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)}$$

$$=\quad\left(\frac{f_1(X)}{\left(\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_p\right)^r}-C_0\right)K_p-\frac{C}{1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}$$

and compared to the constant return to scale case, this amounts to replace $f_1(X)$ by:

$$\frac{f_1(X)}{\left(\left(1+\frac{k(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_p\right)^r}-C_0\rightarrow\frac{f_1(X)}{\left(\left(1+\frac{k(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_X\right)^r}-C_0$$

and the field $|\Psi(K,X)|^2$ is given by:

$$|\Psi(K,X)|^2\rightarrow|\Psi_0(X)|^2-\epsilon\left(\frac{\left(f_1^{(e)}(X)K_p-\bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2}+\frac{f_1^{(e)}(X)}{2}\right)$$

where $f_1^{(e)}(X)$ is the return of the firm, corresponding to the net return of production, from which the paiements of loans is substracted:

$$\begin{aligned}
f_1^{(e)}(X) &= (1+\underline{k}_2(X))\left(\frac{f_1(X)}{\left(\left(1+\frac{k(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_X\right)^r}-C_0\right)-\underline{k}_2(X)\bar{r}\\
\bar{C}^{(e)}(X) &= (1+\underline{k}_2(X))\bar{C}(X)
\end{aligned}$$

A12.1.1 Case one

A12.1.1.1 Average capital per sector and return The first case corresponds to the condition:

$$|\Psi_0(X)|^2-\epsilon\frac{f_1(X)}{2}-\epsilon\frac{\left(\bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2}>0$$

That is, we assume there are some firms with 0, or very low private capital.

Resolution is similar to part 1. We find K_X as a function of $f_1^{(e)}(X)$:

$$K_X\rightarrow\frac{1}{4f_1^{(e)}(X)}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}} \tag{275}$$

where we replace:

$$f_1^{(e)}(X)\rightarrow\frac{(1+\underline{k}_2(X))f_1(X)}{\left(\left(1+\frac{k(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_X\right)^r}-C_0^{(e)}$$

and:

$$C_0^{(e)}=(1+\underline{k}_2(X))C_0+\underline{k}_2(X)\bar{r}$$

Thus (275), is an equation for average capital K_X insector X :

$$0=K_X\left(\frac{(1+\underline{k}_2(X))f_1(X)}{\left(\left(1+\frac{k(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_X\right)^r}-C_0^{(e)}\right)-\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}$$

with lowest order solution in $C_0^{(e)}$:

$$K_X\simeq\frac{1}{\left(1+\frac{k(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)}\left(\frac{(1+\underline{k}_2(X))f_1(X)}{C_0^{(e)}}\right)^{\frac{1}{r}} \tag{276}$$

The correction at the first order are obtained by using:

$$\frac{\left(1+\underline{k}_2\left(X\right)\right)f_1\left(X\right)}{\left(\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_X\right)^r\left(1+\frac{c}{K_X}\right)^r}-C_0^{(e)}$$
$$\simeq -C_0^{(e)}r\frac{c}{K_X}$$

so that, the correction to the solution (276) satisfies:

$$C_0^{(e)}rc=-\left(3X-C\right)\frac{\left(C+X\right)}{2X-C}$$

and the average capital per sector becomes:

$$K_X\simeq\frac{1}{\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)}\left(\frac{\left(1+\underline{k}_2\left(X\right)\right)f_1\left(X\right)}{C_0^{(e)}}\right)^{\frac{1}{r}}-\frac{1}{C_0^{(e)}r}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}$$

Compared to the constant return to scale case, we replace $f_1^{(e)}\left(X\right)$ by:

$$\frac{\left(1+\underline{k}_2\left(X\right)\right)f_1\left(X\right)}{\left(\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_X\right)^r}-C_0^{(e)}$$
$$=\frac{1}{\frac{1}{\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)}\left(\frac{\left(1+\underline{k}_2(X)\right)f_1(X)}{C_0^{(e)}}\right)^{\frac{1}{r}}-\frac{1}{C_0^{(e)}r}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}$$
$$=\frac{\left(\left(1+\underline{k}_2\left(X\right)\right)C_0+\underline{k}_2\left(X\right)\bar{r}\right)^{\frac{1}{r}}r\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}}{r\left(\left(1+\underline{k}_2\left(X\right)\right)f_1\left(X\right)\right)^{\frac{1}{r}}-\left(\left(1+\underline{k}_2\left(X\right)\right)C_0+\underline{k}_2\left(X\right)\bar{r}\right)^{\frac{1}{r}-1}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)}$$

Using that in first approximation:

$$\frac{f_1\left(X\right)}{\left(\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_X\right)^r}-C_0\simeq\frac{f_1\left(X\right)}{\left(1+\underline{k}\left(X\right)\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)^r}-C_0$$

and:

$$\frac{f_1\left(X\right)}{\left(1+\underline{k}\left(X\right)\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2K_X\right)^r}-C_0$$
$$\simeq\frac{f_1\left(X\right)}{\left(\left(\frac{\left(1+\underline{k}_2(X)\right)f_1(X)}{C_0^{(e)}}\right)^{\frac{1}{r}}-\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)\frac{1}{C_0^{(e)}r}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}\right)^r}-C_0$$

$$\frac{C_0^{(e)}}{\left(1+\underline{k}_2(X)\right)}\left(1+\frac{1}{\left(C_0^{(e)}\right)^{1-\frac{1}{r}}\left(\left(1+\underline{k}_2(X)\right)f_1(X)\right)^{\frac{1}{r}}}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}\right)-C_0$$

$$\begin{aligned}
&= \frac{C_0^{(e)}}{\left(1+\underline{k}_2(X)\right)}\left(\frac{1}{\left(C_0^{(e)}\right)^{1-\frac{1}{r}}\left(\left(1+\underline{k}_2(X)\right)f_1(X)\right)^{\frac{1}{r}}}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}\right)+\frac{\underline{k}_2(X)}{1+\underline{k}_2(X)}\bar{r}\\
&= \frac{1}{\left(1+\underline{k}_2(X)\right)}\left(\frac{\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)}{\left(\frac{1+\underline{k}_2(X)}{C_0^{(e)}}f_1(X)\right)^{\frac{1}{r}}}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}\right)+\frac{\underline{k}_2(X)}{1+\underline{k}_2(X)}\bar{r}\\
&= \frac{1}{\left(1+\underline{k}_2(X)\right)}\left(\left(\frac{C_0^{(e)}}{\left(1+\underline{k}_2(X)\right)f_1(X)}\right)^{\frac{1}{r}}\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}\right)+\frac{\underline{k}_2(X)}{1+\underline{k}_2(X)}\bar{r}\\
&= \frac{1}{\left(1+\underline{k}_2(X)\right)}\left(\left(\frac{C_0+\frac{\underline{k}_2(X)}{1+\underline{k}_2(X)}\bar{r}}{f_1(X)}\right)^{\frac{1}{r}}\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}\right)+\frac{\underline{k}_2(X)}{1+\underline{k}_2(X)}\bar{r}
\end{aligned}$$

The following relation is satisfied:

$$\frac{f_1(X)}{\left(\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_X\right)^r}-C_0-\bar{r}=\frac{f_1^{(e)}(X)-\bar{r}}{\left(1+\underline{k}_2(X)\right)}$$

and:

$$\begin{aligned}
f_1^{(e)}(X) &= \left(1+\underline{k}_2(X)\right)\left(\frac{f_1(X)}{\left(\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_X\right)^r}-C_0\right)-\underline{k}_2(X)\bar{r}\\
&\simeq \left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)\left(\frac{C_0+\frac{\underline{k}_2(X)}{1+\underline{k}_2(X)}\bar{r}}{f_1(X)}\right)^{\frac{1}{r}}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}\\
&\simeq \left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)\left(\frac{C_0+\bar{r}}{f_1(X)}\right)^{\frac{1}{r}}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}
\end{aligned}$$

A12.1.1.2 Returns to investrs In the sequel, we need to compute the return of sector X firms to investors. This amnts to compute:

$$\left|\Psi(X)\right|^2 K_X\frac{f_1'(X)-\bar{r}}{1+\underline{k}_2(X)} \tag{277}$$

Replacing the previous expression yields:

$$\begin{aligned}
&\left|\Psi(X)\right|^2\left(\left(f_1(X)-\bar{r}\right)K_X-\bar{C}(X)\right)\\
\rightarrow\ &\frac{\left(C^{(e)}+X^{(e)}\right)\left(\frac{2}{3}X^2+\frac{1}{3}\left(X-C^{(e)}\right)C^{(e)}\right)\epsilon}{\sigma_{\hat{K}}^2 f_1^{(e)}(X)}\left(\frac{\left(f_1(X)-\bar{r}\right)}{4f_1^{(e)}(X)}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}-\bar{C}\right)
\end{aligned}$$

$$|\Psi(X)|^2\left((f_1(X)-\bar{r})K_X-\bar{C}(X)\right)$$

$$\rightarrow \frac{\left(C^{(e)}+X^{(e)}\right)\left(\frac{2}{3}X^2+\frac{1}{3}\left(X-C^{(e)}\right)C^{(e)}\right)\epsilon}{\sigma_{\hat{K}}^2 f_1^{(e)}(X)}\left(\frac{(f_1(X)-\bar{r})}{4f_1^{(e)}(X)}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}-\bar{C}\right)$$

and this becomes:

$$\frac{\left(C^{(e)}+X^{(e)}\right)\left(\frac{2}{3}X^2+\frac{1}{3}\left(X-C^{(e)}\right)C^{(e)}\right)\epsilon}{\sigma_{\hat{K}}^2 f_1^{(e)}(X)\left(1+\underline{k}_2(X)\right)}\left(\frac{\left(f_1^{(e)}(X)-\bar{r}\right)}{4f_1^{(e)}(X)}\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}-C^{(e)}\right)$$

so that (277) writes:

$$\begin{aligned}
&\frac{\frac{1}{3}(2X-C^e)^2(X+C^e)\epsilon}{\sigma_{\hat{K}}^2\left(1+\underline{k}(X)\right)\left(\frac{C_0+\bar{r}}{f_1(X)}\right)^{\frac{1}{r}}\left(3X^{(e)}-C^{(e)}\right)\left(1+\underline{k}_2(X)\right)}\left(\frac{\left(f_1^{(e)}(X)-\bar{r}\right)}{4\left(1+\underline{k}(X)\right)\left(\frac{C_0+\bar{r}}{f_1(X)}\right)^{\frac{1}{r}}}-C^{(e)}\right)\\
=\;&\frac{\frac{1}{3}(2X-C^e)^2(X+C^e)\epsilon}{\sigma_{\hat{K}}^2\left(1+\underline{k}(X)\right)\left(\frac{C_0+\bar{r}}{f_1(X)}\right)^{\frac{1}{r}}\left(3X^{(e)}-C^{(e)}\right)}\left(\frac{\left(f_1^{(e)}(X)-\bar{r}\right)}{4\left(1+\underline{k}(X)\right)\left(\frac{C_0+\bar{r}}{f_1(X)}\right)^{\frac{1}{r}}\left(1+\underline{k}_2(X)\right)}-\bar{C}\right)\\
=\;&\frac{\frac{1}{3}(2X-C^e)^2(X+C^e)\epsilon}{\sigma_{\hat{K}}^2\left(1+\underline{k}(X)\right)\left(3X^{(e)}-C^{(e)}\right)}\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}}\left(\frac{\left(1+\underline{k}(X)\right)\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}-\bar{r}\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}}}{4\left(1+\underline{k}(X)\right)\left(1+\underline{k}_2(X)\right)}-\bar{C}\right)\\
=\;&\frac{\frac{1}{3}(2X-C^e)^2(X+C^e)\epsilon}{\sigma_{\hat{K}}^2\left(1+\underline{k}(X)\right)^2\left(3X^{(e)}-C^{(e)}\right)}\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}}\left(\frac{\left(1+\underline{k}(X)\right)\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}-\bar{r}\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}}}{4\left(1+\underline{k}_2(X)\right)}-C\right)
\end{aligned}$$

and the return (277) can be written as:

$$\frac{1}{12}\frac{\left(C^{(e)}+X^{(e)}\right)^2\epsilon}{\sigma_{\hat{K}}^2\left(1+\underline{k}_2(X)\right)f_1^{(e)}(X)}\left\{3\left(X^{(e)}-C^{(e)}\right)^2-\frac{\bar{r}\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{f_1^{(e)}(X)}\right\}$$

A12.1.2 Case two

In this case, we consider that all firms have a minimum capital, so that:

$$|\Psi_0(X)|^2-\epsilon\frac{\frac{f_1^{(e)}(X)}{\left(\left(1+\frac{k(X)}{\langle K\rangle}\hat{K}_X|\hat{\Psi}(\hat{X})|^2\right)K_X\right)^r}-C_0^{(e)}}{2}-\epsilon\frac{\left(\bar{C}(X)\right)^2}{\sigma_{\hat{K}}^2}<0$$

Computation are similar to the constant return to scale case leads to compute the minimal and maximal level of capital in sector X, written in a compact form $K_{0\pm}$:

$$K_{0\pm}=\frac{\bar{C}(X)\pm\sqrt{\sigma_{\hat{K}}^2\left(\frac{|\Psi_0(X)|^2}{\epsilon}-\frac{\frac{f_1^{(e)}(X)}{\left(\left(1+\frac{k(X)}{\langle K\rangle}\hat{K}_X|\hat{\Psi}(\hat{X})|^2\right)K_X\right)^r}-C_0^{(e)}}{2}\right)}}{\frac{f_1^{(e)}(X)}{\left(\left(1+\frac{k(X)}{\langle K\rangle}\hat{K}_X|\hat{\Psi}(\hat{X})|^2\right)K_X\right)^r}-C_0^{(e)}}$$

leading to:

$$K_X \left|\Psi(X)\right|^2 = \frac{2\epsilon \Delta K_0 \bar{C}(X)}{3\left(\frac{f_1^{(e)}(X)}{\left(\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}(\hat{X})\right|^2\right)K_X\right)^r} - C_0^{(e)}\right)\sigma_{\hat{K}}^2} X^2$$

and for the field

$$\left|\Psi(X)\right|^2 = 2\frac{\epsilon \Delta K_0}{3\sigma_{\hat{K}}^2} X^2$$

where:

$$X = \sqrt{\sigma_{\hat{K}}^2 \left(\frac{\left|\Psi_0(X)\right|^2}{\epsilon} - \frac{\frac{f_1^{(e)}(X)}{\left(\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}(\hat{X})\right|^2\right)K_X\right)^r} - C_0^{(e)}}{2}\right)}$$

These formula allow to find the average capital per frm n the sector.

$$K_X = \frac{\bar{C}(X)}{\frac{f_1^{(e)}(X)}{\left(\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}(\hat{X})\right|^2\right)K_X\right)^r} - C_0^{(e)}}$$

As in case one, this is an equation for K_X:

$$K_X \left(\frac{f_1^{(e)}(X)}{\left(\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_X\right)^r} - C_0^{(e)}\right) = \bar{C}(X)$$

with solutions to first order:

$$K_X \simeq \frac{1}{\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)}\left(\frac{\left(1+\underline{k}_2(X)\right)f_1(X)}{C_0^{(e)}}\right)^{\frac{1}{r}} - \frac{1}{C_0^{(e)}r}\bar{C}(X)$$

Using this expression allows to find the return $f_1^{(e)}(X)$ of the firm:

$$\begin{aligned}
f_1^{(e)}(X) &= \frac{\left(1+\underline{k}_2(X)\right)f_1(X)}{\left(\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)K_X\right)^r} - C_0^{(e)} \\
&= \frac{1}{\frac{1}{\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}(\hat{X})\right|^2\right)}\left(\frac{\left(1+\underline{k}_2(X)\right)f_1(X)}{C_0^{(e)}}\right)^{\frac{1}{r}} - \frac{1}{C_0^{(e)}r}\bar{C}(X)}\bar{C}(X) \\
&= \frac{\left(\left(1+\underline{k}_2(X)\right)C_0+\underline{k}_2(X)\bar{r}\right)^{\frac{1}{r}} r\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)\bar{C}(X)}{r\left(\left(1+\underline{k}_2(X)\right)f_1(X)\right)^{\frac{1}{r}} - \left(\left(1+\underline{k}_2(X)\right)C_0+\underline{k}_2(X)\bar{r}\right)^{\frac{1}{r}-1}\bar{C}(X)\left(1+\frac{\underline{k}(X)}{\langle K\rangle}\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2\right)} \\
&\simeq \frac{\left(\left(1+\underline{k}_2(X)\right)C_0+\underline{k}_2(X)\bar{r}\right)^{\frac{1}{r}} rC(X)}{r\left(\left(1+\underline{k}_2(X)\right)f_1(X)\right)^{\frac{1}{r}} - \left(\left(1+\underline{k}_2(X)\right)C_0+\underline{k}_2(X)\bar{r}\right)^{\frac{1}{r}-1}C(X)}
\end{aligned}$$

$$\begin{aligned}
\left|\Psi(X)\right|^2 K_X \frac{f_1'(X)-\bar{r}}{1+\underline{k}_2(X)} &= \left|\Psi(X)\right|^2\left(\left(f_1(X)-\bar{r}\right)K_X - \bar{C}(X)\right) \\
&= 4\frac{\epsilon\left(C^{(e)}\right)^2}{3\sigma_{\hat{K}}^2 f_1^{(e)}(X)\left(1+\underline{k}_2(X)\right)}X^2\left(\left(1-\frac{\bar{r}}{f_1^{(e)}(X)}\right)\left(3X^{(e)}-1\right) - C^{(e)}\right)
\end{aligned}$$

A12.2 Investors side

A12.2.1 Computation of $\hat{g}$

Given (344):

$$\hat{K}\left[\hat{X}_1\right]=\frac{2\sigma_{\hat{K}}^2}{\hat{\mu}}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)}{\hat{g}^2\left(\hat{X}_1\right)}\right)^2\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{r\left\langle\hat{g}\right\rangle^2}{3}\underline{\hat{k}}\right) \tag{278}$$

This formula for the amount of capital in sector $\hat{X}_1$ allows to derive the return $\hat{g}\left(\hat{X}_1\right)$ as a function of capital. The following equation stands:

$$0=\frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{2\sigma_{\hat{K}}^2}\left(\hat{g}^2\left(\hat{X}_1\right)\right)^2-\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)^2\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}+\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)^2\frac{r\left\langle\hat{g}\right\rangle^2}{3}\underline{\hat{k}}$$

and we have:

$$\hat{g}\left(\hat{X}_1\right)=\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)^2-\sqrt{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)^2-2\frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{\sigma_{\hat{K}}^2}\frac{r\left\langle\hat{g}\right\rangle^2}{3}\underline{\hat{k}}}}{\frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{\sigma_{\hat{K}}^2}}$$

In first approximation, we can also write:

$$\begin{aligned}\hat{K}\left[\hat{X}_1\right] &\simeq \frac{2\sigma_{\hat{K}}^2}{\hat{\mu}\hat{g}^2\left(\hat{X}_1\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)^2\left(\frac{1}{4}-\frac{r\left\langle\hat{g}\right\rangle^2}{3\hat{g}^2\left(\hat{X}_1\right)}\underline{\hat{k}}\right) \\ &\simeq \frac{2\sigma_{\hat{K}}^2}{\hat{\mu}\hat{g}^2\left(\hat{X}_1\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)^2\left(\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}\right)\end{aligned} \tag{279}$$

and $\hat{g}\left(\hat{X}_1\right)$ is given in this approximation b:

$$\hat{g}\left(\hat{X}_1\right)\rightarrow\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}}}{\sqrt{\frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{2\sigma_{\hat{K}}^2}}}$$

where:

$$D\left(\hat{X}_1\right)=\left(\frac{\left\langle\hat{K}\right\rangle^2\left\langle\hat{g}\right\rangle^2}{\sigma_{\hat{K}}^2}+\frac{\left\langle\hat{g}\right\rangle}{2}\right)\left(\frac{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)}{\underline{\hat{k}}\left(\left\langle\hat{X}\right\rangle,\left\langle\hat{X}\right\rangle\right)}-\frac{6\underline{\hat{k}}}{2+\underline{\hat{k}}-\sqrt{\left(2+\underline{\hat{k}}\right)^2-\underline{\hat{k}}}}\right)\underline{\hat{k}}$$

The equation for $\hat{g}$ is obtained by rewriting (80):

$$g\left(\hat{X}_1\right)=(1-M)^{-1}\frac{1-\hat{S}_1\left(\hat{X}\right)}{1-\hat{S}\left(\hat{X}\right)}\left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_1\left(\hat{X}',\hat{X}\right)\right)^{-1}S_1\left(\hat{X}',\hat{X}'\right)\frac{1-S\left(\hat{X}'\right)}{1-S_1\left(\hat{X}'\right)}f_1'$$

which writes as:

$$\left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_1\left(\hat{X}',\hat{X}\right)\right)\frac{1-\hat{S}\left(\hat{X}\right)}{1-\hat{S}_1\left(\hat{X}\right)}\left((1-M)\,\hat{g}\left(\hat{X}'\right)-\bar{r}\right)$$
$$=\quad S_1\left(\hat{X}',\hat{X}'\right)\frac{1-S\left(\hat{X}'\right)}{1-S_1\left(\hat{X}'\right)}\left|\Psi(X)\right|^2K_X\frac{f_1'(X)-\bar{r}}{1+\underline{k}_2(X)}$$

and then by replacing the left hand side by:

$$\left(\frac{1}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}-\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)\left(\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{\frac{1}{4}-\frac{r}{3}\hat{\underline{k}}}}{\sqrt{\frac{\hat{\mu}\hat{K}\left[\hat{X}'\right]}{2\sigma_{\hat{K}}^2}}}-\bar{r}'\right)$$

The right hand side is derived by using that:

$$S_1\left(\hat{X}',\hat{X}'\right)=\frac{k_1\left(\hat{X}',\hat{X}'\right)}{1+\underline{k}(X)}$$

and that:

$$\left|\Psi(X)\right|^2K_X\frac{f_1'(X)-\bar{r}}{1+\underline{k}_2(X)}=\left|\Psi(X)\right|^2\left((f_1(X)-\bar{r})\,K_X-\bar{C}(X)\right)$$

As a consequence, we have

$$S_1\left(\hat{X}',\hat{X}'\right)\frac{1-S\left(\hat{X}'\right)}{1-S_1\left(\hat{X}'\right)}\left(f_1'(X')-\bar{r}\right)$$
$$=\quad S_1\left(\hat{X}',\hat{X}'\right)\left|\Psi(X')\right|^2K_{X'}\frac{f_1'(X')-\bar{r}}{1+\underline{k}_2(X')}$$
$$=\quad \frac{k_1\left(\hat{X}',\hat{X}'\right)}{1+\underline{k}(X)}\left|\Psi(X)\right|^2\left((f_1(X)-\bar{r})\,K_X-\bar{C}(X)\right)$$

$$\left|\Psi(X)\right|^2\left((f_1(X)-\bar{r})\,K_X-\bar{C}(X)\right)$$

has been estimated before, and we write:

$$\underline{k}_2\left(\hat{X}\right)\quad=\quad\beta\underline{k}\left(\hat{X}\right)$$
$$\underline{k}_1\left(\hat{X}\right)\quad=\quad(1-\beta)\,\underline{k}\left(\hat{X}\right)$$

for:

$$\underline{k}\left(\hat{X}\right)>>1$$

$$\frac{\underline{k}_1\left(\hat{X}\right)}{\left(1+\underline{k}\left(\hat{X}\right)\right)}\rightarrow 1-\beta$$

and

$$X^{(e)}\rightarrow\sqrt{\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{1}{2}f_1(X)-\frac{1}{2}\left(\beta\underline{k}(X)\left(f_1(X)-\bar{r}\right)\right)}\simeq X=\sqrt{\frac{\left|\Psi_0(X)\right|^2}{\epsilon}-\frac{1}{2}f_1(X)}$$

$$C^{e}=\frac{\left(1+\underline{k}_{2}\left(X\right)\right)C}{\left(1+\underline{k}\left(X\right)\right)}\simeq\beta C$$

Ultimately, the right hand side becomes:

$$\begin{aligned}
\rightarrow &\quad \frac{\underline{k}_{1}\left(\hat{X}\right)}{\left(1+\underline{k}\left(\hat{X}\right)\right)}\frac{\frac{1}{3}\left(2X-C^{e}\right)^{2}\left(X+C^{e}\right)\epsilon}{\sigma_{\hat{K}}^{2}\left(1+\underline{k}\left(X\right)\right)^{2}\left(3X^{(e)}-C^{(e)}\right)}\left(\frac{f_{1}\left(X\right)}{C_{0}+\bar{r}}\right)^{\frac{1}{r}}\\
&\times\left(\frac{\left(1+\underline{k}\left(X\right)\right)\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}-\bar{r}\left(\frac{f_{1}(X)}{C_{0}+\bar{r}}\right)^{\frac{1}{r}}}{4\left(1+\underline{k}_{2}\left(X\right)\right)}-C\right)\\
\simeq &\quad (1-\beta)\frac{\frac{1}{3}\left(2X-C\beta\right)^{2}\left(\left(X+C\beta\right)\right)\epsilon}{\sigma_{\hat{K}}^{2}\left(1+\underline{k}\left(X\right)\right)^{2}\left(3X-C\beta\right)}\left(\frac{f_{1}\left(X\right)}{C_{0}+\bar{r}}\right)^{\frac{1}{r}}\left(\frac{\left(1+\underline{k}\left(X\right)\right)\frac{(3X-C\beta)(X+C\beta)}{2X-C\beta}-\bar{r}\left(\frac{f_{1}(X)}{C_{0}+\bar{r}}\right)^{\frac{1}{r}}}{4\left(1+\underline{k}_{2}\left(X\right)\right)}-C\right)
\end{aligned}$$

and the equation for $\hat{g}$ is:

$$\begin{aligned}
&\int\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\hat{X}\right)}-\hat{S}_{1}^{E}\left(\hat{X}',\hat{X}\right)\right)\left(\frac{\left(\left\|\hat{\Psi}_{0}\left(\hat{X}'\right)\right\|^{2}-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}}}{\sqrt{\frac{\hat{\mu}\hat{K}\left[\hat{X}'\right]}{2\sigma_{\hat{K}}^{2}}}}-\bar{r}'\right)\\
= &\quad (1-\beta)\frac{\frac{1}{3}\left(2X-C\beta\right)^{2}\left(\left(X+C\beta\right)\right)\epsilon}{\sigma_{\hat{K}}^{2}\left(1+\underline{k}\left(X\right)\right)^{2}\left(3X-C\beta\right)}\left(\frac{f_{1}\left(X\right)}{C_{0}+\bar{r}}\right)^{\frac{1}{r}}\left(\frac{\left(1+\underline{k}\left(X\right)\right)\frac{(3X-C\beta)(X+C\beta)}{2X-C\beta}-\bar{r}\left(\frac{f_{1}(X)}{C_{0}+\bar{r}}\right)^{\frac{1}{r}}}{4\left(1+\underline{k}_{2}\left(X\right)\right)}-C\right)
\end{aligned}$$

Rewritten as an equation for $\underline{k}\left(X\right)$ and using as before:

$$\hat{K}\left[\hat{X}'\right]=\frac{\underline{k}\left(X\right)}{k}$$

this becomes:

$$\begin{aligned}
&\int\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\hat{X}\right)}-\hat{S}_{1}^{E}\left(\hat{X}',\hat{X}\right)\right)\left(\frac{\left(\left\|\hat{\Psi}_{0}\left(\hat{X}'\right)\right\|^{2}-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{\frac{2\sigma_{\hat{K}}^{2}k}{\hat{\mu}}}\sqrt{\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}}}{\sqrt{\underline{k}\left(\hat{X}'\right)}}-\bar{r}'\right)\\
= &\quad (1-\beta)\frac{\frac{1}{3}\left(2X-C\beta\right)^{2}\left(\left(X+C\beta\right)\right)\epsilon}{\sigma_{\hat{K}}^{2}\left(1+\underline{k}\left(X\right)\right)^{2}\left(3X-C\beta\right)}\left(\frac{f_{1}\left(X\right)}{C_{0}+\bar{r}}\right)^{\frac{1}{r}}\left(\frac{\left(1+\underline{k}\left(X\right)\right)\frac{(3X-C\beta)(X+C\beta)}{2X-C\beta}-\bar{r}\left(\frac{f_{1}(X)}{C_{0}+\bar{r}}\right)^{\frac{1}{r}}}{4\left(1+\beta\underline{k}\left(X\right)\right)}-C\right)
\end{aligned}$$

A12.3 Solutions for investors without connections

$$\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}}-\bar{r}'=(1-\beta)\frac{\frac{1}{3}\left(2X-C\beta\right)^{2}\left(\left(X+C\beta\right)\right)\epsilon}{\sigma_{\hat{K}}^{2}\left(1+\underline{k}\left(X\right)\right)^{2}\left(3X-C\beta\right)}\left(\frac{f_{1}\left(X\right)}{C_{0}+\bar{r}}\right)^{\frac{1}{r}}\left(\frac{\left(1+\underline{k}\left(X\right)\right)\frac{(3X-C\beta)(X+C\beta)}{2X-C\beta}-\bar{r}\left(\frac{f_{1}(X)}{C_{0}+\bar{r}}\right)^{\frac{1}{r}}}{4\left(1+\beta\underline{k}\left(X\right)\right)}-C\right)$$

with:

$$N\left(\hat{X}'\right)=\left(\left\|\hat{\Psi}_{0}\left(\hat{X}'\right)\right\|^{2}-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{\frac{2\sigma_{\hat{K}}^{2}k}{\hat{\mu}}}\sqrt{\frac{1}{4}-\frac{r}{3}\underline{\hat{k}}}$$

Defining:

$$\begin{aligned} A &= \frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}} - \bar{r}' \\ B &= (1-\beta)\frac{\frac{1}{3}(2X-C\beta)^2((X+C\beta))\epsilon}{\sigma_{\hat{K}}^2(3X-C\beta)} \\ F &= \frac{(3X-C\beta)(X+C\beta)}{4(2X-C\beta)} \\ D &= \frac{\bar{r}}{4} \end{aligned}$$

We can express $\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}}$ as a function of $\underline{k}(X)$. Two cases arise

$$\begin{aligned} \left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}} &= \frac{(1+\beta\underline{k}(X))}{2D} \\ &\times \left(\sqrt{\left(\frac{(1+\underline{k}(X))F}{4(1+\beta\underline{k}(X))}-C\right)^2 - 4\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}\right)}} - \bar{r}'\right)\frac{(1+\underline{k}(X))^2 D}{B(1+\beta\underline{k}(X))}} + \left(\frac{(1+\underline{k}(X))F}{4(1+\beta\underline{k}(X))}-C\right)\right) \end{aligned}$$

We find:

$$\frac{d\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}}}{d\underline{k}(X)} > 0$$

so that:

$$\frac{d\underline{k}(X)}{d\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}}} > 0$$

as in the cases studied in the text.

For the case $\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}} < 0$ corresponding to negative retrn, the soltn wrt:

$$\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}} = \frac{(1+\beta\underline{k}(X))}{2D}\left(\sqrt{\left(\frac{(1+\underline{k}(X))F}{4(1+\beta\underline{k}(X))}-C\right)^2 + 4\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}\right)}} + \bar{r}'\right)\frac{(1+\underline{k}(X))^2 D}{B(1+\beta\underline{k}(X))}} + \left(\frac{(1+\underline{k}(X))F}{4(1+\beta\underline{k}(X))}-C\right)\right)$$

Then, consider $\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}} - \bar{r}'\right) > 0$. Then the equation to consider is:

$$-\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}} - \bar{r}' = (1-\beta)\frac{\frac{1}{3}(2X-C\beta)^2((X+C\beta))\epsilon}{\sigma_{\hat{K}}^2(1+\underline{k}(X))^2(3X-C\beta)}\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}}\left(\frac{(1+\underline{k}(X))\frac{(3X-C\beta)(X+C\beta)}{2X-C\beta} - \bar{r}\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}}}{4(1+\beta\underline{k}(X))} - C\right)$$

corresponding to negative returns.

The solution is thus:

$$\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}}=\frac{(1+\beta\underline{k}(X))}{2D}\left(\sqrt{\left(\frac{(1+\underline{k}(X))F}{4(1+\beta\underline{k}(X))}-C\right)^2+4\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}\right)}}+\bar{r}'\right)\frac{(1+\underline{k}(X))^2 D}{B(1+\beta\underline{k}(X))}}-\left(\frac{(1+\underline{k}(X))F}{4(1+\beta\underline{k}(X))}-C\right)\right)$$

with also:

$$\frac{d\underline{k}(X)}{d\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}}}>0$$

A12.4 Solutions for investors with connections

Looking for apprximate solutions of:

$$\begin{aligned}&\int\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}-\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}}-\bar{r}'\right)\\
=\;&\frac{B}{(1+\underline{k}(X))^2}\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}}\left(\frac{(1+\underline{k}(X))F}{4(1+\beta\underline{k}(X))}-\frac{D}{(1+\beta\underline{k}(X))}\left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}}-C\right)\end{aligned}$$

where $\hat{S}_1^E\left(\hat{X}',\hat{X}\right)$ has been computed previousl:

$$\begin{aligned}\hat{S}_1^E\left(\hat{X}',\hat{X}\right)&=\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\hat{\underline{k}}_1\left(\langle X\rangle,\hat{X}\right)k\left(\langle X\rangle,X'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)}\right)\frac{\hat{K}_X\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2}{1+\hat{k}\left(\hat{X}\right)}\\
&=\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\hat{\underline{k}}_1\left(\langle X\rangle,\hat{X}\right)k\left(\langle X\rangle,X'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)}\right)\frac{\underline{k}(X)}{k\left(1+\hat{k}\left(\hat{X}\right)\right)}\end{aligned}$$

As in the case of slowly decreasing returns, we define:

$$\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}}=\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}'\left(\hat{X}'\right)}}+\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}}\right)$$

where $\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}'\left(\hat{X}'\right)}}$ is the solution without interactions. This leads to the following qt:

$$\begin{aligned}&\int\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}-\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)\left(\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}}\right)\right)-\int\left(\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}'\left(\hat{X}'\right)}}\right)\\
=\;&H_1\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}}\right)+\frac{1}{2}H_2\Delta\left(\frac{N\left(\hat{X}'\right)}{\sqrt{\underline{k}\left(\hat{X}'\right)}}\right)^2\end{aligned}$$

where:

$$H_l = \frac{\partial^l}{\partial \underline{k}^l \left(\hat{X}'\right)} \left(\frac{B}{\left(1+\underline{k}(X)\right)^2} \left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}} \left(\frac{(1+\underline{k}(X)) F}{4\left(1+\beta \underline{k}(X)\right)} - \frac{D}{\left(1+\beta \underline{k}(X)\right)} \left(\frac{f_1(X)}{C_0+\bar{r}}\right)^{\frac{1}{r}} - C \right) \right)$$

an equation similar to that presented in the text. The analysis of the solutions is thus similar.

Appendix 13 Several groups

As a benchmark and to introduce some notations, we first reconsider the case of an homogeneous group.

A13.1 Homogeneous group

We reconsider the previous computations, considering the system as an homogeneous group of agents, with identical average return. The return equation for such homogeneous group is:

$$\begin{aligned}
&\left(\frac{f\left(\hat{X}\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)} + \bar{r}\frac{\hat{\underline{k}}_2\left(\hat{X}\right)}{1+\hat{\underline{k}}_2\left(\hat{X}\right)}\right) - \frac{\hat{K}\hat{k}_{1E}\left(\hat{X}\right)}{1+\hat{\underline{k}}}\left(\frac{f}{1+\hat{\underline{k}}_2} + \bar{r}\frac{\hat{\underline{k}}_2}{1+\hat{\underline{k}}_2}\right) \\
= &\left\{\left(\bar{r} + \frac{1+f}{\hat{\underline{k}}_2} H\left(-\frac{1+f}{\hat{\underline{k}}_2}\right)\right) \frac{\hat{k}_{2E}\left(\hat{X}\right)}{1+\hat{\underline{k}}\left(\hat{X}'\right)}\right. \\
&\left. + \left(\bar{r} + \frac{1+f_1'(X')}{\underline{k}_2(X')} H\left(-\frac{1+f_1'(X')}{\underline{k}_2(X')}\right)\right) \frac{k_{2E}\left(\hat{X}\right)}{1+\underline{k}(X')} + \frac{k_{1E}\left(\hat{X}\right)}{1+\underline{k}(X')} f_1\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)\right\}
\end{aligned}$$

$\hat{\underline{k}}_2$ and $\hat{\underline{k}}_1$ are the averages of $\hat{\underline{k}}_2\left(\hat{X}\right)$ and $\hat{\underline{k}}_2\left(\hat{X}\right)$. In average, using that $\hat{k}_{\eta E} = \hat{k}_{\eta 1}$, we fnd:

$$\begin{aligned}
&\left(\frac{f}{1+\hat{\underline{k}}_2} + \bar{r}\frac{\hat{\underline{k}}_2}{1+\hat{\underline{k}}_2}\right) - \frac{\hat{\underline{k}}_1}{1+\hat{\underline{k}}}\left(\frac{f}{1+\hat{\underline{k}}_2} + \bar{r}\frac{\hat{\underline{k}}_2}{1+\hat{\underline{k}}_2}\right) \\
= &\left\{\left(\bar{r} + \left(\frac{1+f}{\hat{\underline{k}}_2}\right) H\left(-(1+f)\right)\right) \frac{\hat{\underline{k}}_2}{1+\hat{\underline{k}}}\right. \\
&\left. + \left(\bar{r} + \left(\frac{1+f_1'}{\underline{k}_2}\right) H\left(-(1+f_1')\right)\right) \frac{k_2}{1+\underline{k}} + \frac{k_1}{1+\underline{k}} f_1\right\}
\end{aligned}$$

leading to:

$$\frac{f}{1+\hat{\underline{k}}} = \left(\frac{1+f}{\hat{\underline{k}}_2}\right) H\left(-(1+f)\right)\frac{\hat{\underline{k}}_2}{1+\hat{\underline{k}}} + \left(\frac{1+f_1'}{\underline{k}_2}\right) H\left(-(1+f_1')\right)\frac{k_2}{1+\underline{k}} + \bar{r}\frac{k_2}{1+\underline{k}} + \frac{k_1}{1+\underline{k}} f_1$$

Using the constraint (228):

$$\frac{\underline{k}}{1+\underline{k}} = \frac{1}{1+\hat{\underline{k}}}$$

this reduces to:

$$\frac{f}{1+\hat{\underline{k}}} = \left(\frac{1+f}{\hat{\underline{k}}_2}\right) H\left(-(1+f)\right)\frac{\hat{\underline{k}}_2}{1+\hat{\underline{k}}} + \left(\frac{1+f_1'}{\underline{k}_2}\right) H\left(-(1+f_1')\right)\frac{k_2}{k\left(1+\hat{\underline{k}}\right)} + \bar{r}\frac{k_2}{k\left(1+\hat{\underline{k}}\right)} + \frac{k_1}{k\left(1+\hat{\underline{k}}\right)} f_1$$

that simplifies using (228)

$$\frac{f}{1+\underline{\hat{k}}}=\left(\frac{1+f}{\underline{\hat{k}}_2}\right)H\left(-\left(1+f\right)\right)\frac{\underline{\hat{k}}_2}{1+\underline{\hat{k}}}+\left(\frac{1+f_1'}{\underline{k}_2}\right)H\left(-\left(1+f_1'\right)\right)\frac{k_2}{1+\underline{k}}+\bar{r}\frac{k_2}{1+\underline{k}}+\frac{k_1}{1+\underline{k}}f_1$$

A13.2 Several groups

comng back to the return equation, it can be written for several homogeneous groups:

$$\begin{aligned}
&\left(\frac{f^{[i]}}{1+\underline{\hat{k}}_2^{[i]}}+\bar{r}\frac{\underline{\hat{k}}_2^{[i]}}{1+\underline{\hat{k}}_2^{[i]}}\right)-\frac{\hat{k}_1^{[ji]}}{1+\underline{\hat{k}}^{[j]}}\left(\frac{f^{[j]}}{1+\underline{\hat{k}}_2^{[j]}}+\bar{r}\frac{\underline{\hat{k}}_2^{[j]}}{1+\underline{\hat{k}}_2^{[j]}}\right)\\
=&\left(\bar{r}+\frac{1+f^{[i]}}{\underline{\hat{k}}_2^{[i]}}H\left(-\left(1+f^{[i]}\right)\right)\right)\frac{\underline{\hat{k}}_2^{[ii]}}{1+\underline{\hat{k}}^{[i]}}+\left(\bar{r}+\frac{1+f^{[j]}}{\underline{\hat{k}}_2^{[j]}}H\left(-\left(1+f^{[j]}\right)\right)\right)\frac{\underline{\hat{k}}_2^{[ji]}}{1+\underline{\hat{k}}^{[j]}}\\
&+\left(\bar{r}+\frac{1+f_1'^{[i]}}{\underline{k}_2^{[i]}}H\left(-\frac{1+f_1'^{[i]}}{\underline{k}_2^{[i]}}\right)\right)\frac{k_2^{[ii]}}{1+\underline{k}^{i}}+\frac{k_1^{[ii]}}{1+\underline{k}^{i}}f_1^{[i]}+\left(\bar{r}+\frac{1+f_1'^{[j]}}{\underline{k}_2^{[j]}}H\left(-\frac{1+f_1'^{[j]}}{\underline{k}_2^{[j]}}\right)\right)\frac{k_2^{[ji]}}{1+\underline{k}^{j}}+\frac{k_1^{[ij]}}{1+\underline{k}^{j}}f_1^{[j]}
\end{aligned}$$

The following constraints arise, translating that investors share their investors in loans and participation in firms or investrs:

$$\frac{\underline{\hat{k}}^{[ji]}}{1+\underline{\hat{k}}^{[j]}}+\frac{\underline{\hat{k}}^{[ii]}}{1+\underline{\hat{k}}^{[i]}}+\frac{\underline{k}^{[ji]}}{1+\underline{k}^{[j]}}+\frac{\underline{k}^{[ii]}}{1+\underline{k}^{[i]}}=1$$

This can be rewritten matrcial:

$$\begin{aligned}
0 =& \begin{pmatrix} \frac{1}{1+\underline{\hat{k}}_2^{[i]}}-\frac{\underline{\hat{k}}_1^{[ii]}}{\left(1+\underline{\hat{k}}^{[i]}\right)\left(1+\underline{\hat{k}}_2^{[i]}\right)} & -\frac{\underline{\hat{k}}_1^{[ji]}}{\left(1+\underline{\hat{k}}^{[j]}\right)\left(1+\underline{\hat{k}}_2^{[j]}\right)} \\ -\frac{\underline{\hat{k}}_1^{[ij]}}{\left(1+\underline{\hat{k}}^{[i]}\right)\left(1+\underline{\hat{k}}_2^{[i]}\right)} & \frac{1}{1+\underline{\hat{k}}_2^{[j]}}-\frac{\underline{\hat{k}}_1^{[jj]}}{\left(1+\underline{\hat{k}}^{[j]}\right)\left(1+\underline{\hat{k}}_2^{[j]}\right)} \end{pmatrix}\begin{pmatrix} f^{[i]}+\bar{r}\underline{\hat{k}}_2^{[i]} \\ f^{[j]}+\bar{r}\underline{\hat{k}}_2^{[j]} \end{pmatrix} \qquad (280)\\
&-\begin{pmatrix} \frac{1+f^{[i]}}{\underline{\hat{k}}_2^{[i]}}\frac{\underline{\hat{k}}_2^{[ii]}}{1+\underline{\hat{k}}^{[i]}}H\left(-\left(1+f^{[i]}\right)\right)+\frac{1+f^{[j]}}{\underline{\hat{k}}_2^{[j]}}\frac{\underline{\hat{k}}_2^{[ji]}}{1+\underline{\hat{k}}^{[j]}}H\left(-\left(1+f^{[j]}\right)\right) \\ \frac{1+f^{[j]}}{\underline{\hat{k}}_2^{[j]}}\frac{\underline{\hat{k}}_2^{[jj]}}{1+\underline{\hat{k}}^{[j]}}H\left(-\left(1+f^{[j]}\right)\right)+\frac{1+f^{[i]}}{\underline{\hat{k}}_2^{[i]}}\frac{\underline{\hat{k}}_2^{[ij]}}{1+\underline{\hat{k}}^{[i]}}H\left(-\left(1+f^{[i]}\right)\right) \end{pmatrix}\\
&-\begin{pmatrix} \frac{1+f_1'^{[i]}}{\underline{k}_2^{[i]}}\frac{k_2^{[ii]}}{1+\underline{k}^{[i]}}H\left(-\left(1+f_1'^{[i]}\right)\right)+\frac{1+f_1'^{[j]}}{\underline{k}_2^{[j]}}\frac{k_2^{[ji]}}{1+\underline{k}^{[j]}}H\left(-\left(1+f_1'^{[j]}\right)\right) \\ \frac{1+f_1'^{[i]}}{\underline{k}_2^{[i]}}\frac{k_2^{[ij]}}{1+\underline{k}^{[i]}}H\left(-\frac{1+f_1'^{[i]}}{\underline{k}_2^{[i]}}\right)+\frac{1+f_1'^{[j]}}{\underline{k}_2^{[j]}}\frac{k_2^{[jj]}}{1+\underline{k}^{[j]}}H\left(-\left(1+f_1'^{[j]}\right)\right) \end{pmatrix}-\begin{pmatrix} \bar{r}_i \\ \bar{r}_j \end{pmatrix}
\end{aligned}$$

whr:

$$\begin{aligned}
\bar{r}_i &= \left(\frac{\underline{\hat{k}}_2^{[ji]}}{1+\underline{\hat{k}}^{[j]}}+\frac{\underline{\hat{k}}_2^{[ii]}}{1+\underline{\hat{k}}^{[i]}}+\frac{k_2^{[ii]}}{1+\underline{k}^{i}}+\frac{k_2^{[ji]}}{1+\underline{k}^{j}}\right)\bar{r}+\frac{k_1^{[ii]}}{1+\underline{k}^{[i]}}f_1^{[i]}+\frac{k_1^{[ji]}}{1+\underline{k}^{[j]}}f_1^{[j]}\\
&= \left(\frac{\underline{\hat{k}}_2^{[ji]}}{1+\underline{\hat{k}}^{[j]}}+\frac{\underline{\hat{k}}_2^{[ii]}}{1+\underline{\hat{k}}^{[i]}}+\frac{k^{[ii]}}{1+\underline{k}^{i}}+\frac{k^{[ji]}}{1+\underline{k}^{j}}\right)\bar{r}+\frac{k_1^{[ii]}}{1+\underline{k}^{[i]}}\left(f_1^{[i]}-\bar{r}\right)+\frac{k_1^{[ji]}}{1+\underline{k}^{[j]}}\left(f_1^{[j]}-\bar{r}\right)
\end{aligned}$$

Usng:

$$\frac{\underline{\hat{k}}^{[ji]}}{1+\underline{\hat{k}}^{[j]}}+\frac{\underline{\hat{k}}^{[ii]}}{1+\underline{\hat{k}}^{[i]}}+\frac{\underline{k}^{[ji]}}{1+\underline{k}^{[j]}}+\frac{\underline{k}^{[ii]}}{1+\underline{k}^{[i]}}=1$$

this becms:

$$\left(1-\frac{\hat{\underline{k}}^{[ii]}}{1+\hat{\underline{k}}^{[i]}}-\frac{\hat{\underline{k}}^{[ji]}}{1+\hat{\underline{k}}^{[j]}}+\frac{\hat{\underline{k}}_2^{[ji]}}{1+\hat{\underline{k}}^{[j]}}+\frac{\hat{\underline{k}}_2^{[ii]}}{1+\hat{\underline{k}}^{[i]}}\right)\bar{r}+\frac{k_1^{[ii]}}{1+\underline{k}^{[i]}}\left(f_1^{[i]}-\bar{r}\right)+\frac{k_1^{[ji]}}{1+\underline{k}^{[j]}}\left(f_1^{[j]}-\bar{r}\right)$$

$$=\quad\left(1-\frac{\hat{\underline{k}}_1^{[ii]}}{1+\hat{\underline{k}}^{[i]}}-\frac{\hat{\underline{k}}_1^{[ji]}}{1+\hat{\underline{k}}^{[j]}}\right)\bar{r}+\frac{k_1^{[ii]}}{1+\underline{k}^{[i]}}\left(f_1^{[i]}-\bar{r}\right)+\frac{k_1^{[ji]}}{1+\underline{k}^{[j]}}\left(f_1^{[j]}-\bar{r}\right)$$

$$1-\frac{\hat{\underline{k}}_1^{[ii]}}{1+\hat{\underline{k}}^{[i]}}-\frac{\hat{\underline{k}}_1^{[ji]}}{1+\hat{\underline{k}}^{[j]}}$$

Then we reformulate (280) by using the alternative description corresponding here to define:

$$\hat{\underline{S}}_\eta^{[ii]} = \frac{\hat{\underline{k}}_\eta^{[ii]}}{1+\hat{\underline{k}}^{[i]}}$$

$$\hat{\underline{S}}_\eta^{[ij]} = \frac{\hat{\underline{k}}_\eta^{[ij]}}{1+\hat{\underline{k}}^{[i]}}$$

$$\hat{\underline{S}}^{[ii]} = \hat{\underline{S}}_1^{[ii]}+\hat{\underline{S}}_2^{[ii]}$$

$$\hat{\underline{S}}^{[ij]} = \hat{\underline{S}}_1^{[ij]}+\hat{\underline{S}}_2^{[ij]}$$

$$\underline{S}_\eta^{[ii]} = \frac{k_\eta^{[ii]}}{1+\underline{k}^{[i]}}$$

$$\underline{S}_\eta^{[ij]} = \frac{k_\eta^{[ij]}}{1+\underline{k}^{[i]}}$$

$$\underline{S}^{[ii]} = \underline{S}_1^{[ii]}+S_2^{[ii]}$$

$$\underline{S}^{[ij]} = \underline{S}_1^{[ij]}+S_2^{[ij]}$$

$$\hat{\underline{S}}_\eta^{[ii]}+\hat{\underline{S}}_\eta^{[ij]}=\frac{\hat{\underline{k}}_\eta^{[i]}}{1+\hat{\underline{k}}^{[i]}}$$

and using that these coeficients satisfy the identitites

$$\hat{\underline{S}}^{[ii]}+\hat{\underline{S}}^{[ji]}+\underline{S}^{[ii]}+\underline{S}^{[ji]}=1$$

$$\hat{\underline{S}}_\eta^{[ii]}+\hat{\underline{S}}_\eta^{[ij]}=\frac{\hat{\underline{k}}_\eta^{[i]}}{1+\hat{\underline{k}}^{[i]}}$$

$$\hat{\underline{S}}^{[ii]}+\hat{\underline{S}}^{[ij]}=\frac{\hat{\underline{k}}^{[i]}}{1+\hat{\underline{k}}^{[i]}}$$

$$\frac{1}{1+\hat{\underline{k}}^{[i]}}=1-\left(\hat{\underline{S}}^{[ii]}+\hat{\underline{S}}^{[ij]}\right)$$

$$\underline{\hat{S}}_{\eta}^{[ii]}+\underline{\hat{S}}_{\eta}^{[ij]}=\frac{\underline{\hat{k}}_{\eta}^{[i]}}{1+\underline{\hat{k}}^{[i]}}=\underline{\hat{k}}_{\eta}^{[i]}\left(1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)\right)$$

$$\underline{\hat{k}}_{\eta}^{[i]}=\frac{\underline{\hat{S}}_{\eta}^{[ii]}+\underline{\hat{S}}_{\eta}^{[ij]}}{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}$$

$$1+\underline{\hat{k}}_{\eta}^{[i]}=\frac{1-\left(\underline{\hat{S}}_{3-\eta}^{[ii]}+\underline{\hat{S}}_{3-\eta}^{[ij]}\right)}{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}$$

$$\frac{1}{1+\underline{k}^{[i]}}=1-\left(\underline{S}^{[ii]}+\underline{S}^{[ij]}\right)$$

$$\underline{k}_{\eta}^{[i]}=\frac{\underline{S}_{\eta}^{[ii]}+\underline{S}_{\eta}^{[ij]}}{1-\left(\underline{S}^{[ii]}+\underline{S}^{[ij]}\right)}$$

$$1+\underline{k}_{\eta}^{[i]}=\frac{1-\left(\underline{S}_{3-\eta}^{[ii]}+\underline{S}_{3-\eta}^{[ij]}\right)}{1-\left(\underline{S}^{[ii]}+\underline{S}^{[ij]}\right)}$$

we find the return equations for the system of several group:

$$\begin{aligned}0 \quad = \quad & \begin{pmatrix} 1-\underline{\hat{S}}_1^{[ii]} & -\underline{\hat{S}}_1^{[ji]} \\ -\underline{\hat{S}}_1^{[ij]} & 1-\underline{\hat{S}}_1^{[jj]} \end{pmatrix}\begin{pmatrix} \frac{f^{[i]}-\bar{r}}{1+\underline{\hat{k}}_2^{[i]}} \\ \frac{f^{[j]}-\bar{r}}{1+\underline{\hat{k}}_2^{[j]}} \end{pmatrix}-\begin{pmatrix} \underline{\hat{S}}_2^{[ii]} & \underline{\hat{S}}_2^{[ji]} \\ \underline{\hat{S}}_2^{[ij]} & \underline{\hat{S}}_2^{[jj]} \end{pmatrix}\begin{pmatrix} \frac{1+f^{[i]}}{\underline{\hat{k}}_2^{[i]}}H\left(-\left(1+f^{[i]}\right)\right) \\ \frac{1+f^{[j]}}{\underline{\hat{k}}_2^{[j]}}H\left(-\left(1+f^{[j]}\right)\right) \end{pmatrix} \\ & -\begin{pmatrix} \underline{S}_2^{[ii]} & \underline{S}_2^{[ji]} \\ \underline{S}_2^{[ij]} & \underline{S}_2^{[jj]} \end{pmatrix}\begin{pmatrix} \frac{1+f_1'^{[i]}}{\underline{k}_2^{[i]}}H\left(-\left(1+f_1'^{[i]}\right)\right) \\ \frac{1+f_1'^{[j]}}{\underline{k}_2^{[j]}}H\left(-\left(1+f_1'^{[j]}\right)\right) \end{pmatrix}-\begin{pmatrix} \underline{S}_1^{[ii]} & \underline{S}_1^{[ji]} \\ \underline{S}_1^{[ij]} & \underline{S}_1^{[jj]} \end{pmatrix}\begin{pmatrix} f_1^{[i]}-\bar{r} \\ f_1^{[j]}-\bar{r} \end{pmatrix}\end{aligned}$$

with:

$$f_1^{[i]}-\bar{r}=\frac{f_1'^{[i]}-\bar{r}}{1+k_2^{[i]}}$$

or by fully replacing the coeffcients:

$$\begin{aligned}0 \quad = \quad & \begin{pmatrix} 1-\underline{\hat{S}}_1^{[ii]} & -\underline{\hat{S}}_1^{[ji]} \\ -\underline{\hat{S}}_1^{[ij]} & 1-\underline{\hat{S}}_1^{[jj]} \end{pmatrix}\begin{pmatrix} \left(f^{[i]}-\bar{r}\right)\frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}\right)} \\ \left(f^{[j]}-\bar{r}\right)\frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[jj]}+\underline{\hat{S}}_1^{[ji]}\right)} \end{pmatrix} \\ & -\begin{pmatrix} \underline{\hat{S}}_2^{[ii]} & \underline{\hat{S}}_2^{[ji]} \\ \underline{\hat{S}}_2^{[ij]} & \underline{\hat{S}}_2^{[jj]} \end{pmatrix}\begin{pmatrix} \left(1+f^{[i]}\right)\frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{\underline{\hat{S}}_2^{[ii]}+\underline{\hat{S}}_2^{[ij]}}H\left(-\left(1+f^{[i]}\right)\right) \\ \left(1+f^{[j]}\right)\frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ji]}\right)}{\underline{\hat{S}}_2^{[jj]}+\underline{\hat{S}}_2^{[ji]}}H\left(-\left(1+f^{[j]}\right)\right) \end{pmatrix} \\ & -\begin{pmatrix} \underline{S}_2^{[ii]} & \underline{S}_2^{[ji]} \\ \underline{S}_2^{[ij]} & \underline{S}_2^{[jj]} \end{pmatrix}\begin{pmatrix} \left(1+f_1'^{[i]}\right)\frac{1-\left(\underline{S}^{[ii]}+\underline{S}^{[ij]}\right)}{\underline{S}_2^{[ii]}+\underline{S}_2^{[ij]}}H\left(-\left(1+f_1'^{[i]}\right)\right) \\ \left(1+f_1'^{[j]}\right)\frac{1-\left(\underline{S}^{[jj]}+\underline{S}^{[ji]}\right)}{\underline{S}_2^{[jj]}+\underline{S}_2^{[ji]}}H\left(-\left(1+f_1'^{[j]}\right)\right) \end{pmatrix} \\ & -\begin{pmatrix} \underline{S}_1^{[ii]} & \underline{S}_1^{[ji]} \\ \underline{S}_1^{[ij]} & \underline{S}_1^{[jj]} \end{pmatrix}\begin{pmatrix} \left(f_1'^{[i]}-\bar{r}\right)\frac{1-\left(\underline{S}^{[ii]}+\underline{S}^{[ij]}\right)}{1-\left(\underline{S}_1^{[ii]}+\underline{S}_1^{[ij]}\right)} \\ \left(f_1'^{[j]}-\bar{r}\right)\frac{1-\left(\underline{S}^{[jj]}+\underline{S}^{[ij]}\right)}{1-\left(\underline{S}_1^{[jj]}+\underline{S}_1^{[ji]}\right)} \end{pmatrix}\end{aligned}$$

Appendix 14 Computation of Green functions

We consider the operator:

$$-\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^{2}}{2}\nabla_{\hat{K}}-\hat{K}f\left(\hat{X},K_{\hat{X}}\right)+\int\delta f_{1}\left|\Xi\left(\hat{X},\delta f_{1}\right)\right|^{2}d\left(\delta f_{1}\right)\right)$$

whose Green function will be computed approximatively betwwen an initial state δf_1 and $\delta f_1'$. This amounts to consider that the capital dynamics adapt to a slower variable, the excess return. Consider the average as fixed:

$$\int\delta f_{1}\left|\Xi\left(\hat{X},\delta f_{1}\right)\right|^{2}d\left(\delta f_{1}\right)$$

the Green function should be formally given by:

$$\sqrt{\left|\frac{f_{\Xi}\left(\hat{X},K_{\hat{X}}\right)}{\sigma_{\hat{K}}^{2}\left(1-\exp\left(2f_{\Xi}\left(\hat{X},K_{\hat{X}}\right)\Delta t\right)\right)}\right|}\times\exp\left(\frac{f_{\Xi}\left(\hat{X},K_{\hat{X}}\right)}{\sigma_{\hat{K}}^{2}\left(1-\exp\left(2f_{\Xi}\left(\hat{X},K_{\hat{X}}\right)\Delta t\right)\right)}\left(\hat{K}-\exp\left(f_{\Xi}\left(\hat{X},K_{\hat{X}}\right)\Delta t\right)\hat{K}'\right)^{2}\right)\tag{281}$$

with:

$$f_{\Xi}\left(\hat{X},K_{\hat{X}}\right)=f\left(\hat{X},K_{\hat{X}}\right)+\int\delta f_{1}\left|\Xi\left(\hat{X},\delta f_{1}\right)\right|^{2}d\left(\delta f_{1}\right)$$

To estimate the Green function between two values δf_1 and $\delta f_1'$, we expand (281) as a series expansion of $\left|\Xi\left(\hat{X},\delta f_{1}\right)\right|^{2}$ and estimate the results between the two states using the transitions functions (98):

$$G\left(\delta f_{1},\delta f_{1}'\right)=\sqrt{\frac{1}{\sigma_{\delta f_{1}}^{2}}}\exp\left(-\frac{\left(\delta f_{1}-\delta f_{1}'\right)^{2}}{\sigma_{\delta f_{1}}^{2}}+J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\delta f_{1}-J\left(\hat{X}',K_{\hat{X}'},\mathbf{E}'\right)\delta f_{1}'\right)\tag{282}$$

and normalize by diving by $G\left(\delta f_{1},\delta f_{1}'\right)$.

For $\mathbf{E}=\mathbf{E}'$, that is, if the external parameters are constant aver a perd of time, the estimation of a product:

$$\int d\left(\delta f_{1}\right)_{1}...d\left(\delta f_{1}\right)_{k}\left(\delta f_{1}\right)_{1}\left|\Xi\left(\hat{X},\left(\delta f_{1}\right)_{1}\right)\right|^{2}...\left(\delta f_{1}\right)_{1}\left|\Xi\left(\hat{X},\left(\delta f_{1}\right)_{k}\right)\right|^{2}$$

Without fixing the initial and final value for δf_1, this integral is equal to 0. By imposing

$$\frac{\sum\left(\delta f_{1}\right)_{j}}{k}=\frac{\delta f_{1}+\delta f_{1}'}{2}$$

we can by change of variable replace $\left(\delta f_{1}\right)_{k}$ in the integral by the average $\frac{\delta f_{1}+\delta f_{1}'}{2}$. The estimation of the prdct bcms:

$$\begin{aligned}&\int d\left(\delta f_{1}\right)_{1}...d\left(\delta f_{1}\right)_{k}\left(\delta f_{1}\right)_{1}\left|\Xi\left(\hat{X},\left(\delta f_{1}\right)_{1}\right)\right|^{2}...\left(\delta f_{1}\right)_{1}\left|\Xi\left(\hat{X},\left(\delta f_{1}\right)_{k}\right)\right|^{2}\\ \simeq&\left(\frac{\delta f_{1}+\delta f_{1}'}{2}\right)^{k}\int d\left(\delta f_{1}\right)_{1}...d\left(\delta f_{1}\right)_{k}G\left(\delta f_{1},\left(\delta f_{1}'\right)_{1}\right)...G\left(\left(\delta f_{1}\right)_{k},\delta f_{1}'\right)\\ =&\left(\frac{\delta f_{1}+\delta f_{1}'}{2}\right)^{k}\sqrt{\frac{1}{\sigma_{\delta f_{1}}^{2}}}\exp\left(-\frac{\left(\delta f_{1}-\delta f_{1}'\right)^{2}}{\sigma_{\delta f_{1}}^{2}}+J\left(\hat{X},K_{\hat{X}},\mathbf{E}\right)\delta f_{1}-J\left(\hat{X}',K_{\hat{X}'},\mathbf{E}'\right)\delta f_{1}'\right)\\ =&\left(\frac{\delta f_{1}+\delta f_{1}'}{2}\right)^{k}G\left(\delta f_{1},\delta f_{1}'\right)\end{aligned}$$

The product of integrls is a convolutn:

$$\int d(\delta f_1)_1 ... d(\delta f_1)_k \, G(\delta f_1, (\delta f_1')_1) ... G((\delta f_1)_k, \delta f_1') = \sqrt{\frac{1}{\sigma^2_{\delta f_1}}} \exp\left(-\frac{(\delta f_1 - \delta f_1')^2}{\sigma^2_{\delta f_1}} + J\left(\hat{X}, K_{\hat{X}}, \mathbf{E}\right)\delta f_1 - J\left(\hat{X}', K_{\hat{X}'}, \mathbf{E}'\right)\delta f_1'\right)$$

Leading to:

$$\begin{aligned} & \int d(\delta f_1)_1 ... d(\delta f_1)_k (\delta f_1)_1 \left|\Xi\left(\hat{X}, (\delta f_1)_1\right)\right|^2 ... (\delta f_1)_1 \left|\Xi\left(\hat{X}, (\delta f_1)_k\right)\right|^2 \\ = & \left(\frac{\delta f_1 + \delta f_1'}{2}\right)^k G(\delta f_1, \delta f_1') \end{aligned}$$

After normaliztn, the correction becomes:

$$\int d(\delta f_1)_1 ... d(\delta f_1)_k (\delta f_1)_1 \left|\Xi\left(\hat{X}, (\delta f_1)_1\right)\right|^2 ... (\delta f_1)_1 \left|\Xi\left(\hat{X}, (\delta f_1)_k\right)\right|^2 \simeq \left(\frac{\delta f_1 + \delta f_1'}{2}\right)^k$$

Ultimately, the Green function conditionned by the return is:

$$\begin{aligned} & \sqrt{\left|\frac{f\left(\hat{X}, K_{\hat{X}}\right) + \left(\frac{\delta f_1 + \delta f_1'}{2}\right)}{\sigma^2_{\hat{K}}\left(1 - \exp\left(2\left(f\left(\hat{X}, K_{\hat{X}}\right) + \left(\frac{\delta f_1 + \delta f_1'}{2}\right)\right)\Delta t\right)\right)}\right|} \\ & \times \exp\left(\frac{\left(f\left(\hat{X}, K_{\hat{X}}\right) + \left(\frac{\delta f_1 + \delta f_1'}{2}\right)\right)}{\sigma^2_{\hat{K}}\left(1 - \exp\left(2\left(f\left(\hat{X}, K_{\hat{X}}\right) + \left(\frac{\delta f_1 + \delta f_1'}{2}\right)\right)\Delta t\right)\right)}\left(\hat{K} - \exp\left(\left(f\left(\hat{X}, K_{\hat{X}}\right) + \left(\frac{\delta f_1 + \delta f_1'}{2}\right)\right)\Delta t\right)\hat{K}'\right)^2\right) \end{aligned} \tag{283}$$

Appendix 15 interactions in an homogeneous group

In computing the transition functions between several agents the insertion of terms:

$$\prod_i V\left(\delta\left(\hat{X}' - \hat{X}\right) - \frac{\hat{k}_1\left(\hat{X}', \hat{X}\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}', \hat{X}'\right)\right|^2}{1 + \underline{\hat{k}}\left(\hat{X}'\right)}\right)\frac{\int\left|\Xi\left(\hat{X}, \delta f_1\right)\right|^2\left|\Xi\left(\hat{X}, \delta f_1'\right)\right|^2 d(\delta f_1)\, d(\delta f_1')}{1 + \underline{\hat{k}}_2\left(\hat{X}'\right)}d\hat{X}'$$

for an arbitrary potential, induces contributions:

$$\langle(\delta f_1)_i| \prod_i V\left(\delta\left(\hat{X}' - \hat{X}\right) - \frac{\hat{k}_1\left(\hat{X}', \hat{X}\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}', \hat{X}'\right)\right|^2}{1 + \underline{\hat{k}}\left(\hat{X}'\right)}\right)\frac{\int\left|\Xi\left(\hat{X}, \delta f_1\right)\right|^2\left|\Xi\left(\hat{X}, \delta f_1'\right)\right|^2 d(\delta f_1)\, d(\delta f_1')}{1 + \underline{\hat{k}}_2\left(\hat{X}'\right)}d\hat{X}' \,|(\delta f_1'')_i\rangle$$

Contractn of the flds wth th stts, lds t trm:

$$\prod_i G((\delta f_1)_i, (\delta f_1')_i)\left\{\prod_i V\left((\delta f_1)_i - \sum_j \frac{\hat{k}_1\left(\hat{X}', \hat{X}\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}', \hat{X}'\right)\right|^2}{1 + \underline{\hat{k}}\left(\hat{X}'\right)}(\delta f_1')_j\right)\right\}\prod_i G((\delta f_1')_i, (\delta f_1'')_i)$$

If we choose a Dirac function, this imposes the constraint in the transitions:

$$\prod G((\delta f_1)_i, (\delta f_1')_i)\,\delta\left(\delta f_{1i}'\left(\hat{X}_i\right) - \sum_j \frac{\hat{k}_1\left(\hat{X}_i, \hat{X}_j\right)\left\langle\hat{K}_j\right\rangle\left|\hat{\Psi}\left(\hat{K}_j, \hat{X}_j\right)\right|^2}{1 + \underline{\hat{k}}\left(\hat{X}_j\right)}\delta f_{1j}\left(\hat{X}_j\right)\right)\prod G((\delta f_1')_i, (\delta f_1'')_i)$$

as in the text.

Appendix 16 Blocks interactions

Consider (104) without default:

$$0 = \begin{pmatrix} 1-\underline{\hat{S}}_1^{[ii]} & -\underline{\hat{S}}_1^{[ji]} \\ -\underline{\hat{S}}_1^{[ij]} & 1-\underline{\hat{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \left(f^{[i]}-\bar{r}\right)\frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}\right)} \\ \left(f^{[j]}-\bar{r}\right)\frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[jj]}+\underline{\hat{S}}_1^{[ji]}\right)} \end{pmatrix} - \begin{pmatrix} \underline{S}_1^{[ii]} & \underline{S}_1^{[ji]} \\ \underline{S}_1^{[ij]} & \underline{S}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \left(f_1'^{[i]}-\bar{r}\right)\frac{1-\left(\underline{S}^{[ii]}+\underline{S}^{[ij]}\right)}{1-\left(\underline{S}_1^{[ii]}+\underline{S}_1^{[ij]}\right)} \\ \left(f_1'^{[j]}-\bar{r}\right)\frac{1-\left(\underline{S}^{[jj]}+\underline{S}^{[ij]}\right)}{1-\left(\underline{S}_1^{[jj]}+\underline{S}_1^{[ji]}\right)} \end{pmatrix} \tag{284}$$

and replace $f^{[i]}+\delta f^{[i]}$. This leads to:

$$0 = \begin{pmatrix} -\underline{\hat{S}}_1^{[ii]} & -\underline{\hat{S}}_1^{[ji]} \\ -\underline{\hat{S}}_1^{[ij]} & -\underline{\hat{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \left(f^{[i]}-\bar{r}\right)\frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}\right)} \\ \left(f^{[j]}-\bar{r}\right)\frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[jj]}+\underline{\hat{S}}_1^{[ji]}\right)} \end{pmatrix} - \begin{pmatrix} 0 & \underline{S}_1^{[ji]} \\ \underline{S}_1^{[ij]} & 0 \end{pmatrix} \begin{pmatrix} \left(f_1'^{[i]}-\bar{r}\right)\frac{1-\left(\underline{S}^{[ii]}+\underline{S}^{[ij]}\right)}{1-\left(\underline{S}_1^{[ii]}+\underline{S}_1^{[ij]}\right)} \\ \left(f_1'^{[j]}-\bar{r}\right)\frac{1-\left(\underline{S}^{[jj]}+\underline{S}^{[ij]}\right)}{1-\left(\underline{S}_1^{[jj]}+\underline{S}_1^{[ji]}\right)} \end{pmatrix} + \begin{pmatrix} 1-\underline{\hat{S}}_1^{[ii]} & -\underline{\hat{S}}_1^{[ji]} \\ -\underline{\hat{S}}_1^{[ij]} & 1-\underline{\hat{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \delta f^{[i]}\frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}\right)} \\ \delta f^{[j]}\frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[jj]}+\underline{\hat{S}}_1^{[ji]}\right)} \end{pmatrix} \tag{285}$$

The two first trms in the right hand side corresponds to a shift in the average return of the block introduced by the interaction. It can be absorbed in a redefinition of these averages, and we are left with:

$$0 = \begin{pmatrix} 1-\underline{\hat{S}}_1^{[ii]} & -\underline{\hat{S}}_1^{[ji]} \\ -\underline{\hat{S}}_1^{[ij]} & 1-\underline{\hat{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \delta f^{[i]}\frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}\right)} \\ \delta f^{[j]}\frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[jj]}+\underline{\hat{S}}_1^{[ji]}\right)} \end{pmatrix} \tag{286}$$

that corresponds to intrdc ptntl:

$$\sum_i V_i\left(\hat{\Psi},\hat{X},K,\delta f_{1i}\right) = \sum_i \delta\left(\left(1-\underline{\hat{S}}_1^{[ii]}\right)\frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}\right)}\frac{\int\left|\Xi\left(\hat{X},\delta f^{[i]}\right)\right|^2 d\left(\delta f^{[i]}\right)d\hat{X}}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)} - \underline{\hat{S}}_1^{[ji]}\frac{1-\left(\underline{\hat{S}}^{[jj]}+\underline{\hat{S}}^{[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[jj]}+\underline{\hat{S}}_1^{[ji]}\right)}\frac{\int\left|\Xi\left(\hat{X},\delta f^{[j]}\right)\right|^2 d\left(\delta f^{[j]}\right)d\hat{X}}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)}\right)$$

replacing as before:

$$\frac{\int\left|\Xi\left(\hat{X},\delta f^{[i]}\right)\right|^2 d\left(\delta f^{[i]}\right)d\hat{X}}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)} \rightarrow \frac{\delta f^{[i]}+\delta f^{[i]\prime}}{2\left(1+\underline{\hat{k}}_2\left(\hat{X}'\right)\right)}$$

in the transitions, leads to the constraint in the text.

Part 2 Appendices

Appendix 17 Details for the micro framework

A17.1 Investors capital accumulation

The dynmcs for investors private capital is gven by:

$$\begin{aligned}
\hat{K}_{jp}(t+\varepsilon)-\hat{K}_{jp}(t) &= f_j\hat{K}_{jp}(t)=\sum_l\left(1+\sum_v\hat{k}_{2jv}\hat{K}_v(t)+\sum_v\hat{k}_{2jv}^B\bar{K}_v(t)\right)R_j\hat{K}_{jp}(t) \\
&\quad -\bar{r}\left(\sum_v\hat{k}_{2jv}\hat{K}_v(t)+\sum_v\hat{k}_{2jv}^B\bar{K}_v(t)\right)\hat{K}_{jp}(t) \\
&= \sum_l\left(1+\sum_v\hat{k}_{2lv}\hat{K}_v(t)+\kappa\sum_v\hat{k}_{2jv}^B\frac{\bar{K}_{v0}(t)}{1+\sum_m\bar{k}_{vm}\bar{K}_{m0}(t)}\right)R_j\hat{K}_{jp}(t) \\
&\quad -\bar{r}\left(\sum_v\hat{k}_{2jv}\hat{K}_v(t)+\kappa\sum_v\hat{k}_{2jv}^B\frac{\bar{K}_{v0}(t)}{1+\sum_m\bar{k}_{vm}\bar{K}_{m0}(t)}\right)\hat{K}_{jp}(t)
\end{aligned} \tag{287}$$

We can write:

$$\hat{K}_{jp}(t+\varepsilon)-\hat{K}_{jp}(t)\rightarrow\frac{d}{dt}\hat{K}_{jp}(t)$$

in the continuouss approximation. By differentiation of (115):

$$\begin{aligned}
\frac{d}{dt}\hat{K}_{jp}(t) &= \frac{\frac{d}{dt}\hat{K}_j(t)}{1+\sum_l\hat{k}_{jl}\hat{K}_l(t)+\sum_l\hat{k}_{1jl}^B\bar{K}_{l0}(t)+\kappa\sum_l\hat{k}_{2jl}^B\frac{\bar{K}_{l0}(t)}{1+\sum_m\bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}} \\
&\quad -\sum_l\frac{\hat{k}_{jl}\hat{K}_j(t)\frac{d}{dt}\hat{K}_l(t)}{\left(1+\sum_l\hat{k}_{jl}\hat{K}_l(t)+\sum_l\hat{k}_{1jl}^B\bar{K}_{l0}(t)+\kappa\sum_l\hat{k}_{2jl}^B\frac{\bar{K}_{l0}(t)}{1+\sum_m\bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}\right)^2} \\
&\quad -\sum_l\frac{\left(\hat{k}_{1jl}^B+\kappa\hat{k}_{2jl}^B\frac{1}{1+\sum_m\bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}\right)\frac{d}{dt}\bar{K}_{l0}(t)-\kappa\frac{\hat{k}_{2jl}^B\bar{K}_{l0}(t)\sum_m\bar{\bar{k}}_{lm}\frac{d}{dt}\bar{K}_{m0}(t)}{\left(1+\sum_m\bar{\bar{k}}_{lm}\bar{K}_{m0}(t)\right)^2}}{\left(1+\sum_l\hat{k}_{jl}\hat{K}_l(t)+\sum_l\hat{k}_{1jl}^B\bar{K}_{l0}(t)+\kappa\sum_l\hat{k}_{2jl}^B\frac{\bar{K}_{j0}(t)}{1+\sum_m\bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}\right)^2}
\end{aligned} \tag{288}$$

To obtain the dynamics for $\hat{K}_j(t)$ use (88), to wrt (288):

$$\begin{aligned}
&\frac{\frac{d}{dt}\hat{K}_j(t)}{1+\sum_l \hat{k}_{jl}\hat{K}_l(t)+\sum_l \hat{k}^B_{1jl}\bar{K}_{l0}(t)+\kappa\sum_l \hat{k}^B_{2jl}\frac{\bar{K}_{l0}(t)}{1+\sum_m \bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}} \\
&-\sum_l \frac{\hat{k}_{jl}\hat{K}_j(t)\frac{d}{dt}\hat{K}_l(t)}{\left(1+\sum_l \hat{k}_{jl}\hat{K}_l(t)+\sum_l \hat{k}^B_{1jl}\bar{K}_{l0}(t)+\kappa\sum_l \hat{k}^B_{2jl}\frac{\bar{K}_{l0}(t)}{1+\sum_m \bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}\right)^2} \\
&-\sum_l \frac{\hat{K}_j\left\{\hat{k}^B_{1jl}+\kappa\hat{k}^B_{2jl}\frac{1}{1+\sum_m \bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}-\kappa\frac{\sum_m \hat{k}^B_{2jm}\bar{K}_{m0}(t)\bar{\bar{k}}_{ml}}{\left(1+\sum_n \bar{\bar{k}}_{mn}\bar{K}_{n0}(t)\right)^2}\right\}\frac{d}{dt}\bar{K}_{l0}(t)}{\left(1+\sum_l \hat{k}_{jl}\hat{K}_l(t)+\sum_l \hat{k}^B_{1jl}\bar{K}_{l0}(t)+\kappa\sum_l \hat{k}^B_{2jl}\frac{\bar{K}_{j0}(t)}{1+\sum_m \bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}\right)^2} \\
= &\sum_l\left(1+\sum_v \hat{k}_{2lv}\hat{K}_v(t)+\kappa\sum_v \hat{k}^B_{2jv}\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{\bar{k}}_{vm}\bar{K}_{m0}(t)}\right)R_j\hat{K}_{jp}(t) \\
&-\bar{r}\left(\sum_v \hat{k}_{2jv}\hat{K}_v(t)+\kappa\sum_v \hat{k}^B_{2jv}\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{\bar{k}}_{vm}\bar{K}_{m0}(t)}\right)\hat{K}_{jp}(t) \\
\rightarrow\ &f_j\hat{K}_{jp}(t)
\end{aligned} \tag{289}$$

This is solved together with banks capital accumulation.

A17.2 Banks capital accumulation

We start with the dynamics for $\bar{K}_{jp}$:

$$\bar{K}_{jp}(t+\varepsilon)-\bar{K}_{jp}(t)=\left(1+\sum_v \bar{k}_{2jv}\bar{K}_{v0}(t)\right)\bar{R}_j\bar{K}_{jp}(t)-\bar{r}\sum_v \bar{k}_{2jv}\bar{K}_{v0}(t)\bar{K}_{jp}(t)$$

and its continuous approximation:

$$\frac{d}{dt}\bar{K}_{jp}(t)=\left(1+\sum_v \bar{k}_{2jv}\bar{K}_{v0}(t)\right)\bar{R}_j\bar{K}_{jp}(t)-\bar{r}\sum_v \bar{k}_{2jv}\bar{K}_{v0}(t)\bar{K}_{jp}(t)$$

leading to, for the disposable capital:

$$\begin{aligned}
&\frac{\frac{d}{dt}\bar{K}_{j0}(t)}{1+\sum_l\left(\bar{k}_{1jl}+\bar{\bar{k}}_{2jl}\right)\bar{K}_{l0}(t)}-\sum_l \frac{\bar{K}_{j0}(t)\left(\bar{k}_{1jl}+\bar{k}_{2jl}\right)}{\left(1+\sum_l\left(\bar{k}_{1jl}+\bar{k}_{2jl}\right)\bar{K}_{l0}(t)\right)^2}\frac{d}{dt}\bar{K}_{l0}(t) \\
= &\left(1+\sum_v \bar{k}_{2jv}\bar{K}_{v0}(t)\right)\bar{R}_j\bar{K}_{jp}(t)-\bar{r}\sum_v \bar{k}_{2jv}\bar{K}_{v0}(t)\bar{K}_{jp}(t)
\end{aligned}$$

where:

$$\hat{k}_{1ab}+\hat{k}_{2ab}=\hat{k}_{ab}$$

and this can be set in matricial form, as in the text.

A17.3 Return equations with default

A17.3.2 Investors return

Including default for investors and firms modifies the return equation for investors by:

$$\sum_l \left(\delta_{jl} - \frac{\hat{k}_{1lj}\hat{K}_l(t)}{1+\sum_v \hat{k}_{jv}\hat{K}_l(t) + \sum_v \hat{k}^B_{1jv}\bar{K}_{l0}(t) + \kappa\sum_v \hat{k}^B_{2jv}\frac{\bar{K}_{j0}(t)}{1+\sum_m \hat{k}_{vm}\bar{K}_{m0}(t)}}\right) \tag{290}$$

$$\times \frac{\hat{f}_l - \bar{r}}{1+\sum_v \hat{k}_{2lv}\hat{K}_v(t) + \kappa\sum_v \hat{k}^B_{2lv}\frac{\bar{K}_{\nu 0}(t)}{1+\sum_m \hat{k}_{vm}\bar{K}_{m0}(t)}} \tag{291}$$

$$+\sum_l \left(\bar{r} - \frac{\left(1+\hat{f}_\nu\right)}{\sum_m \hat{k}_{2vm}\hat{K}_m + \kappa\sum_m \hat{k}^B_{2vm}\frac{\bar{K}_{m0}(t)}{1+\sum_s \hat{k}_{ms}\bar{K}_{s0}(t)}}\right)$$

$$\times \frac{H\left(-\left(1+\hat{f}_\nu\right)\right)\hat{k}_{2lj}\hat{K}_l(t)}{1+\sum_\nu \hat{k}_{l\nu}\hat{K}_l(t) + \sum_\nu \hat{k}^B_{1l\nu}\bar{K}_{l0}(t) + \kappa\sum_\nu \hat{k}^B_{2l\nu}\frac{\bar{K}_{j0}(t)}{1+\sum_m \hat{k}_{vm}\bar{K}_{m0}(t)}} \tag{292}$$

$$+\sum_i \left(\bar{r} - \frac{\left(1+f_1'\left(K_i(t)\right)\right)}{\left(\sum_v k^{(B)}_{2iv}\kappa\frac{\bar{K}_{v0}(t)}{1+\sum_m \hat{k}_{vm}\bar{K}_{m0}(t)}\right) + \left(\sum_v k_{2iv}\hat{K}_v(t)\right)}\right)$$

$$\times \frac{H\left(-\left(1+f_1'\left(K_i(t)\right)\right)\right)k_{2ij}K_i(t)}{1+\left(\sum_v k^{(B)}_{1iv}\bar{K}_{v0}(t) + k^{(B)}_{2iv}\kappa\frac{\bar{K}_{v0}(t)}{1+\sum_m \hat{k}_{vm}\bar{K}_{m0}(t)}\right) + \left(\sum_v k_{iv}\hat{K}_v(t)\right)} \tag{293}$$

$$= \sum_i \frac{\left(r_i + F_1\left(\bar{R}_i, \frac{\dot{K}_i(t)}{K_i(t)}\right) + \tau\left(\bar{R}\left(K_i, X_i\right)\right)\Delta f_1'\left(K_i(t)\right) - \bar{r}\right)k_{1ij}K_i(t)}{1+\sum_v \hat{k}_{jv}\hat{K}_v(t) + \sum_v \hat{k}^B_{1jv}\bar{K}_{l0}(t) + \kappa\sum_v \hat{k}^B_{2jv}\frac{\bar{K}_{v0}(t)}{1+\sum_m \hat{k}_{vm}\bar{K}_{m0}(t)}}$$

A17.3.3 Banks return

Possibility of default for investors and firms modifies the return equation for banks by:

$$\sum_{l}\left(\delta_{jl}-\frac{\hat{k}_{1lj}\bar{K}_{l0}(t)}{1+\sum_{v}\bar{k}_{lv}\bar{K}_{v0}(t)}\right)\left(\frac{\bar{f}_{l}-\bar{r}}{1+\sum_{v}\bar{k}_{2lv}\bar{K}_{v0}(t)}\right) \tag{294}$$

$$\begin{aligned}
&-\sum_{l}\frac{\hat{k}_{1lj}^{B}\hat{K}_{l}(t)}{1+\sum_{v}\hat{k}_{lv}\hat{K}_{l}(t)+\sum_{v}\hat{k}_{1lv}^{B}\bar{K}_{l0}(t)+\kappa\sum_{v}\hat{k}_{2lv}^{B}\frac{\bar{K}_{l0}(t)}{1+\sum_{m}\bar{k}_{vm}\bar{K}_{m0}(t)}}\\
&\times\frac{\hat{f}_{l}-\bar{r}}{1+\sum_{v}\hat{k}_{2lv}\hat{K}_{v}(t)+\kappa\sum_{v}\hat{k}_{2lv}^{B}\frac{\bar{K}_{\nu0}(t)}{1+\sum_{m}\bar{k}_{vm}\bar{K}_{m0}(t)}}\\
&+\sum_{l}\left(\bar{r}-\frac{\left(1+\bar{f}_{\nu}\right)}{\sum_{m}\bar{k}_{2vm}\bar{K}_{m}}\right)\frac{H\left(-\left(1+\bar{f}_{\nu}\right)\right)\bar{k}_{2lj}\bar{K}_{l}(t)}{1+\sum_{v}\left(\bar{k}_{1lv}+\bar{k}_{2lv}\right)\bar{K}_{v}(t)}\\
&+\sum_{l}\left(\bar{r}-\frac{\left(1+\hat{f}_{\nu}\right)}{\sum_{m}\hat{k}_{2vm}\hat{K}_{m}+\kappa\sum_{m}\hat{k}_{2vm}^{B}\frac{\bar{K}_{m0}(t)}{1+\sum_{s}\bar{k}_{ms}\bar{K}_{s0}(t)}}\right)\\
&\times\frac{H\left(-\left(1+\hat{f}_{\nu}\right)\right)\hat{k}_{2lj}^{B}\hat{K}_{l}(t)}{1+\sum_{\nu}\hat{k}_{l\nu}\hat{K}_{l}(t)+\sum_{\nu}\hat{k}_{1l\nu}^{B}\bar{K}_{l0}(t)+\kappa\sum_{\nu}\hat{k}_{2l\nu}^{B}\frac{\bar{K}_{j0}(t)}{1+\sum_{m}\bar{k}_{vm}\bar{K}_{m0}(t)}}\\
&\sum_{i}\left(\bar{r}-\frac{\left(1+f_{1}'\left(K_{i}(t)\right)\right)}{\left(\sum_{v}k_{2iv}^{(B)}\kappa\frac{\bar{K}_{v0}(t)}{1+\sum_{m}\bar{k}_{vm}\bar{K}_{m0}(t)}\right)+\left(\sum_{v}k_{2iv}\hat{K}_{v}(t)\right)}\right)\\
&\times\frac{H\left(-\left(1+f_{1}'\left(K_{i}(t)\right)\right)\right)k_{2ij}^{(B)}K_{i}(t)}{1+\left(\sum_{v}k_{1iv}^{(B)}\bar{K}_{v0}(t)+k_{2iv}^{(B)}\kappa\frac{\bar{K}_{v0}(t)}{1+\sum_{m}\bar{k}_{vm}\bar{K}_{m0}(t)}\right)+\left(\sum_{v}k_{iv}\hat{K}_{v}(t)\right)}\\
=&\sum_{i}\frac{\left(r_{i}+F_{1}\left(\bar{R}_{i},\frac{\dot{K}_{i}(t)}{K_{i}(t)}\right)+\tau\left(\bar{R}\left(K_{i},X_{i}\right)\right)\Delta f_{1}'\left(K_{i}(t)\right)-\bar{r}\right)k_{1ij}^{B}K_{i}(t)}{1+\sum_{v}\hat{k}_{jv}\hat{K}_{v}(t)+\sum_{v}\hat{k}_{1jv}^{B}\bar{K}_{l0}(t)+\kappa\sum_{v}\hat{k}_{2jv}^{B}\frac{\bar{K}_{v0}(t)}{1+\sum_{m}\bar{k}_{vm}\bar{K}_{m0}(t)}}
\end{aligned} \tag{295}$$

Appendix 18 translation in field description

A18.1 Translation of coefficents

Translations are direct:

A18.1.1 Banks

$$\underline{\bar{k}}_{\eta}\left(\bar{X}'\right)=\int_{\eta}\bar{k}\left(\bar{X}',\bar{X}\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^{2}d\hat{X}$$

$$\underline{\bar{k}}\left(\bar{X}'\right)=\underline{\bar{k}}_{1}\left(\bar{X}'\right)+\kappa\frac{\underline{\bar{k}}_{2}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}$$

$$\underline{\bar{k}}=\int\bar{k}\left(\left\langle\bar{X}\right\rangle,\bar{Y}\right)\bar{K}_{1}d\bar{Y}d\bar{K}_{1}$$

$$1+\int\bar{k}\left(\bar{X}'',\bar{Y}\right)\bar{K}_{1}d\bar{Y}d\bar{K}_{1}\rightarrow1+\underline{\bar{k}}$$

A18.1.2 Investors

$$\sum_v \hat{k}_{lv}\hat{K}_v(t) \rightarrow \int \hat{k}_\nu\left(\hat{X}',\hat{X}''\right)\hat{K}''\left|\hat{\Psi}\left(\hat{K}'',\hat{X}''\right)\right|^2 d\hat{X}''d\hat{K}'' = \underline{\hat{k}}_\nu\left(\hat{X}'\right)$$

$$\sum_v \hat{k}_{lv}\hat{K}_v(t) \rightarrow \underline{\hat{k}}\left(\hat{X}'\right)$$

$$\begin{aligned}
&\sum_v \hat{k}^B_{1lv}\bar{K}_{v0}(t) + \kappa\sum_v \hat{k}^B_{2lv}\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{\bar{k}}_{vm}\bar{K}_{m0}(t)} \\
\rightarrow\ & \int \hat{k}^B_1\left(\bar{X}',\bar{X}''\right)\bar{K}''\left|\hat{\Psi}\left(\bar{K}'',\bar{X}''\right)\right|^2 d\bar{X}''d\bar{K}'' + \kappa\int\frac{\hat{k}^B_2\left(\bar{X}',\bar{X}''\right)\bar{K}''\left|\hat{\Psi}\left(\bar{K}'',\bar{X}''\right)\right|^2}{1+\int\bar{\bar{k}}\left(\bar{X}'',\bar{Y}\right)\bar{K}_1 d\bar{Y}d\bar{K}_1}d\bar{X}''d\bar{K}'' \\
=\ & \underline{\hat{k}}^B_1\left(\hat{X}'\right) + \kappa\left[\frac{\underline{\hat{k}}^B_2}{1+\bar{\underline{k}}}\right]\left(\hat{X}'\right)
\end{aligned}$$

and:

$$\begin{aligned}
&1+\sum_v \hat{k}_{lv}\hat{K}_v(t) + \sum_v \bar{k}_{1lv}\bar{K}_{v0}(t) + \kappa\sum_v \bar{k}_{2lv}\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)} \\
\rightarrow\ & 1+\underline{\hat{k}}\left(\hat{X}'\right) + \kappa\left[\frac{\underline{\hat{k}}^B_2}{1+\bar{\underline{k}}}\right](X')
\end{aligned}$$

A18.1.3 Firms

$$\begin{aligned}
&1+\sum_v k_{iv}\hat{K}_v(t) + \left(\sum_v k^{(B)}_{1iv}\bar{K}_{v0}(t) + k^{(B)}_{2iv}\kappa\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{k}_{vm}\bar{K}_{m0}(t)}\right) \\
\rightarrow\ & 1+\underline{k}\left(\hat{X}'\right) + \underline{k}^{(B)}_1\left(\bar{X}'\right) + \kappa\left[\frac{\underline{k}^B_2}{1+\bar{\underline{k}}}\right](X') \\
=\ & \left(1+\int k\left(X,,\hat{X}'\right)\hat{K}'_{\hat{X}'}\frac{\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}{\langle K_p\rangle\left|\Psi(X)\right|^2} + \int\left(\underline{k}^{(B)}_1\left(X,\bar{X}'\right)\bar{K}_{\bar{X}'} + \kappa\frac{\underline{k}^{(B)}_2\left(X,\bar{X}'\right)}{1+\underline{k}\left(\bar{X}\right)}\bar{K}_{\bar{X}'}\right)\frac{\left|\bar{\Psi}\left(\hat{X}'\right)\right|^2}{\langle K_p\rangle\left|\Psi(X)\right|^2}\right)
\end{aligned}$$

wth:

$$\begin{aligned}
\underline{k}\left(\hat{X}'\right) &= \int k\left(X,,\hat{X}'\right)\hat{K}'_{\hat{X}'}\frac{\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}{\langle K_p\rangle\left|\Psi(X)\right|^2} \\
\underline{k}^{(B)}_1\left(\bar{X}'\right) + \kappa\left[\frac{\underline{k}^B_2}{1+\bar{\underline{k}}}\right](X') &= \int\left(\underline{k}^{(B)}_1\left(X,\bar{X}'\right)\bar{K}_{\bar{X}'} + \kappa\frac{\underline{k}^{(B)}_2\left(X,\bar{X}'\right)}{1+\underline{k}\left(\bar{X}\right)}\bar{K}_{\bar{X}'}\right)\frac{\left|\bar{\Psi}\left(\hat{X}'\right)\right|^2}{\langle K_p\rangle\left|\Psi(X)\right|^2}
\end{aligned}$$

and:

$$\begin{aligned}
&1+\underline{\hat{k}}\left(\hat{X}'\right) + \underline{\hat{k}}^B_1\left(\bar{X}'\right) + \kappa\left[\frac{\underline{\hat{k}}^B_2}{1+\bar{\underline{k}}}\right]\left(\hat{X}'\right) \\
=\ & \left(1+\int\hat{k}\left(X,,\hat{X}'\right)\hat{K}'_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2 + \int\left(\hat{k}^{(B)}_1\left(X,\bar{X}'\right)\bar{K}_{\bar{X}'} + \kappa\frac{\hat{k}^{(B)}_2\left(X,\bar{X}'\right)}{1+\underline{k}\left(\bar{X}\right)}\bar{K}_{\bar{X}'}\right)\left|\bar{\Psi}\left(\hat{X}'\right)\right|^2\right)
\end{aligned}$$

A18.2 Translation of matrix elements

Three matrices arise in the micro-framework. The first one:

$$\bar{M}_{jm}=\frac{\bar{\bar{k}}_{jm}\bar{K}_{j0}\left(t\right)}{1+\sum_{\nu}\bar{\bar{k}}_{jv}\hat{K}_{\nu}\left(t\right)}$$

is translated in the matrix $\bar{M}\left(\left(\bar{K},\bar{X}\right),\left(\bar{K}',\bar{X}'\right)\right)$ which has the form:

$$\bar{M}\left(\left(\bar{K},\bar{X}\right),\left(\bar{K}',\bar{X}'\right)\right)=\frac{\bar{\bar{k}}\left(\bar{X},\bar{X}'\right)\bar{K}_{0}}{1+\int\bar{\bar{k}}\left(\bar{X},\bar{X}'\right)\bar{K}_{0}'\left|\bar{\Psi}\left(\bar{K}_{0}',\bar{X}'\right)\right|^{2}}$$

The matrix:

$$\hat{M}_{jm}=\frac{\hat{k}_{jm}\hat{K}_{j}\left(t\right)}{1+\sum_{l}\hat{k}_{jl}\hat{K}_{l}\left(t\right)+\sum_{l}\bar{k}_{1jl}\bar{K}_{l0}\left(t\right)+\kappa\sum_{l}\bar{k}_{2jl}\frac{\bar{K}_{l0}(t)}{1+\sum_{m}\bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}}$$

is translated into:

$$\begin{aligned}&\hat{M}\left(\left(\hat{K}',\hat{X}'\right),\left(\hat{K},\hat{X}\right)\right)\\=&\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}}{1+\int\hat{k}\left(\hat{X},\hat{X}'\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}+\int\hat{k}_{1}^{B}\left(\hat{X},\bar{X}'\right)\bar{K}_{0}'\left|\bar{\Psi}\left(\bar{K}_{0}',\bar{X}'\right)\right|^{2}+\int\hat{k}_{2}^{B}\left(\hat{X},\bar{X}'\right)\frac{\bar{K}_{0}'\left|\bar{\Psi}\left(\bar{K}_{0}',\bar{X}'\right)\right|^{2}}{1+\int\bar{\bar{k}}\left(\bar{X}',\bar{X}''\right)\bar{K}_{0}''\left|\bar{\Psi}\left(\bar{K}_{0}'',\bar{X}''\right)\right|^{2}}}\end{aligned}$$

and ultimately:

$$\begin{aligned}\bar{N}\quad=&\quad\frac{\hat{K}_{j}\left(\bar{k}_{1jl}+\kappa\bar{k}_{2jl}\frac{1}{1+\sum_{m}\bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}-\kappa\frac{\sum_{m}\bar{k}_{2jm}\bar{K}_{m0}(t)\bar{\bar{k}}_{ml}}{\left(1+\sum_{n}\bar{\bar{k}}_{mn}\bar{K}_{n0}(t)\right)^{2}}\right)}{1+\sum_{l}\hat{k}_{jl}\hat{K}_{l}\left(t\right)+\sum_{l}\bar{k}_{1jl}\bar{K}_{l0}\left(t\right)+\kappa\sum_{l}\bar{k}_{2jl}\frac{\bar{K}_{j0}(t)}{1+\sum_{m}\bar{\bar{k}}_{lm}\bar{K}_{m0}(t)}}\\\rightarrow&\quad\frac{\left(\hat{k}_{1}^{B}\left(\hat{X},\bar{X}'\right)+\kappa\frac{\hat{k}_{2}^{B}\left(\hat{X},\bar{X}'\right)}{1+\int\bar{\bar{k}}\left(\bar{X}',\bar{X}''\right)\bar{K}_{0}''\left|\bar{\Psi}\left(\bar{K}_{0}'',\bar{X}''\right)\right|^{2}}-\kappa\int\frac{\hat{k}_{2}^{B}\left(\bar{X},\bar{X}''\right)\bar{K}_{0}''\bar{\bar{k}}\left(\bar{X}'',\bar{X}'\right)}{\left(1+\int\bar{\bar{k}}\left(\bar{X}'',\bar{Y}\right)\bar{K}_{0}^{Y}\left|\bar{\Psi}\left(\bar{K}_{0}^{Y},\bar{Y}\right)\right|^{2}\right)^{2}}\right)\hat{K}}{1+\int\hat{k}\left(\hat{X},\hat{X}'\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}+\int\hat{k}_{1}^{B}\left(\hat{X},\bar{X}'\right)\bar{K}_{0}'\left|\bar{\Psi}\left(\bar{K}_{0}',\bar{X}'\right)\right|^{2}+\kappa\int\hat{k}_{2}^{B}\left(\hat{X},\bar{X}'\right)\frac{\bar{K}_{0}'\left|\bar{\Psi}\left(\bar{K}_{0}',\bar{X}'\right)\right|^{2}}{1+\int\bar{\bar{k}}\left(\bar{X}',\bar{X}''\right)\bar{K}_{0}''\left|\bar{\Psi}\left(\bar{K}_{0}'',\bar{X}''\right)\right|^{2}}}\end{aligned}$$

A18.3 Translation of return equations no-default scenario

The translations of the returns equations, without default, are:

$$\begin{aligned}&\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\frac{\hat{f}\left(\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}\qquad(296)\\=&\quad\frac{k_{1}\left(\hat{X}',X\right)\hat{K}'}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_{1}^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_{2}^{B}}{1+\bar{k}}\right]\left(X'\right)}\frac{f_{1}'\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)}{1+\underline{k}_{2}\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_{2}^{B}}{1+\bar{k}}\right]\left(X'\right)}\end{aligned}$$

for the investors, with solution:

$$\hat{f}\left(\hat{X}'\right) = \left(1+\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)^{-1}$$

$$\times\frac{k_1\left(\hat{X}',X\right)\hat{K}'}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}\frac{f_1'\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}$$

and:

$$\begin{aligned}
&\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\bar{f}\left(\hat{X}'\right)}{1+\bar{\underline{\bar{k}}}_2\left(\bar{X}'\right)} \\
&-\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\frac{\hat{f}\left(\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\frac{\hat{\underline{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}\left(\bar{X}\right)}} \\
= &\frac{\underline{k}_1^{(B)}\left(X',\bar{X}\right)}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}\frac{\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}
\end{aligned} \tag{297}$$

for banks. The solution for $\bar{f}\left(\hat{X}'\right)$ is straightforward:

$$\begin{aligned}
\bar{f}\left(\hat{X}'\right) = &\left(1+\bar{\underline{\bar{k}}}_2\left(\bar{X}'\right)\right)\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)^{-1} \\
&\left\{\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)^{-1}\right. \\
&\times\frac{k_1\left(\hat{X}',X\right)\hat{K}'}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}\frac{f_1'\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)} \\
&\left.+\frac{\underline{k}_1^{(B)}\left(X',\bar{X}\right)}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}\frac{\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}\right\}
\end{aligned}$$

The text rewrite these equations in terms of $\hat{g}\left(\hat{K}_1,\hat{X}_1\right)$ and $\bar{g}\left(\bar{K}_1,\bar{X}_1\right)$ using the relations

$$\begin{aligned}
\hat{g}\left(\hat{K}_1,\hat{X}_1\right) &= \left(1-\hat{M}\right)^{-1}\hat{f}\left(\hat{K}',\hat{X}'\right)+\left(1-\hat{M}\right)^{-1}\bar{N}\left(1-\bar{M}\right)^{-1}\bar{f}\left(\hat{K}',\hat{X}'\right) \\
&= \left(1-\hat{M}\right)^{-1}\hat{f}\left(\hat{K}',\hat{X}'\right)+\left(1-\hat{M}\right)^{-1}\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)
\end{aligned}$$

$$\bar{g}\left(\bar{K}_1,\bar{X}_1\right)=\left(1-\bar{M}\right)^{-1}\bar{f}\left(\hat{K}',\hat{X}'\right)$$

allow to rewrite the returns as:

$$\bar{f}\left(\bar{K}_1,\bar{X}_1\right)=\left(1-\bar{M}\right)\bar{g}\left(\hat{K}',\hat{X}'\right)$$

$$\hat{f}\left(\hat{K}_1,\hat{X}_1\right)=\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)-\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)$$

leading to the formula in the text.

A18.4 Translation of return equations with default

A18.4.1 Investors' field return equation

$$0<1+R_j+(R_j-\bar{r})\left(\sum_v \hat{k}_{2jv}\hat{K}_v(t)+\kappa\sum_v \hat{k}_{2jv}^B\frac{\bar{K}_{v0}(t)}{1+\sum_m \bar{\bar{k}}_{vm}\bar{K}_{m0}(t)}\right)$$

$$1+R\left(\hat{X}'\right)+\left(R\left(\hat{X}'\right)-\bar{r}\right)\left(\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\frac{\bar{\underline{k}}_2\left(\bar{X}'\right)}{1+\bar{k}}\right)>0$$

$$\frac{1+R\left(\hat{X}'\right)}{\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\frac{\bar{\underline{k}}_2\left(\bar{X}'\right)}{1+\bar{k}}}+R\left(\hat{X}'\right)>\bar{r}$$

$$\begin{aligned}
&\left(1-\frac{\hat{K}\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\frac{\hat{\underline{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}}}\right)R\left(\hat{X}'\right)\\
=&\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\frac{\hat{k}_2\left(\hat{X}',\hat{X}\right)\hat{K}'}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\frac{\hat{\underline{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}}}\\
&\times\left(\bar{r}-\left(\bar{r}-\left(\frac{1+R\left(\hat{X}'\right)}{\hat{\underline{k}}_2\left(\hat{X}'\right)+\kappa\frac{\hat{\underline{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}}}+R\left(\hat{X}'\right)\right)\right)H\left(-\left(\bar{r}-\left(\frac{1+R\left(\hat{X}\right)}{\hat{\underline{k}}_2\left(\hat{X}'\right)+\kappa\frac{\hat{\underline{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}}}+R\left(\hat{X}'\right)\right)\right)\right)\right)\\
&+\left|\Psi\left(K',X'\right)\right|^2\frac{\hat{k}_2\left(\hat{X}',X\right)\hat{K}'}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}\\
&\times\left(\bar{r}-\left(\bar{r}-\left(\frac{1+f_1\left(K',X'\right)}{\underline{k}_2\left(X'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(X'\right)}{1+\underline{\bar{k}}}}+f_1\left(K',X'\right)\right)\right)H\left(-\left(\bar{r}-\left(\frac{1+f_1\left(K',X'\right)}{\underline{k}_2\left(X'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(X'\right)}{1+\underline{\bar{k}}}}+f_1\left(K',X'\right)\right)\right)\right)\right)\\
&+\hat{f}_1\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)
\end{aligned}$$

Yhis is written in terms of excss retrns using (117) nd (118). Given that:

$$R\left(\hat{X}'\right)=\frac{\hat{f}\left(\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\frac{\hat{\underline{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}}}+\bar{r}\frac{\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\frac{\hat{\underline{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}}}{1+\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\frac{\hat{\underline{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}}}$$

whr:

$$f_1'\left(K',X'\right)=\frac{f_1\left(K',X'\right)}{1+\underline{k}_2\left(X'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}+\bar{r}\frac{\underline{k}_2\left(X'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}{1+\underline{k}_2\left(X'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}$$

The retun equation becoms:

$$\begin{aligned}
&\left(1-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right) \qquad (298)\\
&\times\left(\frac{\hat{f}\left(\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}+\bar{r}\frac{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\frac{\underline{\hat{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}}}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\\
&+\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\left(\bar{r}+\frac{1+\hat{f}\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}H\left(-\frac{1+\hat{f}\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\right)\\
&\times\frac{\hat{k}_2\left(\hat{X}',\hat{X}\right)\hat{K}'}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\\
&+\left|\Psi\left(K',X'\right)\right|^2\left(\bar{r}+\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}H\left(-\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}\right)\right)\\
&\times\frac{k_2\left(\hat{X}',X\right)\hat{K}'}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}\\
=\;&\frac{k_1\left(\hat{X}',X\right)\hat{K}'\left|\Psi\left(K',X'\right)\right|^2}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}\hat{f}_1\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)
\end{aligned}$$

with:

$$\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)=\int\frac{\underline{\hat{k}}_2^B\left(\hat{X}',\bar{X}\right)}{1+\bar{k}\left(\bar{X}\right)}\bar{K}\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2d\bar{K}d\bar{X}$$

$$\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)=\int\frac{\underline{k}_2^B\left(X',\bar{X}\right)}{1+\bar{k}\left(\bar{X}\right)}\bar{K}\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2d\bar{K}d\bar{X}$$

In terms of $\hat{g}\left(\hat{K}_1,\hat{X}_1\right)$ and $\bar{g}\left(\bar{K}_1,\bar{X}_1\right)$ this rewrites:

$$\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

$$+\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\left(\bar{r}+\frac{1+\hat{f}\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}H\left(-\frac{1+\hat{f}\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\right)$$

$$\times\frac{\hat{k}_2\left(\hat{X}',\hat{X}\right)\hat{K}'}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

$$+\left|\Psi\left(K',X'\right)\right|^2\left(\bar{r}+\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}H\left(-\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}\right)\right)$$

$$\times\frac{k_2\left(\hat{X}',X\right)\hat{K}'}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}$$

$$=\frac{k_1\left(X',X'\right)\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)}{\left(1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)\left(1+\underline{k}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)}$$

A18.4.2 Banks returns equation

$$\left(1-\frac{\bar{K}\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\bar{R}\left(\bar{X}'\right)-\frac{\hat{K}\bar{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}R\left(\hat{X}'\right) \tag{299}$$

$$=\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2\left(\bar{r}-\left(\bar{r}-\left(\frac{1+\bar{R}\left(\bar{X}'\right)}{\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}+\bar{R}\left(\bar{X}'\right)\right)\right)H\left(-\left(\bar{r}-\left(\frac{1+\bar{R}\left(\bar{X}'\right)}{\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}+\bar{R}\left(\bar{X}'\right)\right)\right)\right)\right)\frac{\underline{\bar{k}}_2\left(\bar{X}'\right)}{1+\underline{\bar{\bar{k}}}}$$

$$+\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\left(\bar{r}-\left(\bar{r}-\left(\frac{1+R\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\hat{X}'\right)}+R\left(\hat{X}'\right)\right)\right)H\left(-\left(\bar{r}-\left(\frac{1+R\left(\hat{X}\right)}{\underline{\hat{k}}_2\left(\hat{X}'\right)}+R\left(\hat{X}'\right)\right)\right)\right)\right)$$

$$\times\frac{\frac{\kappa\underline{\hat{k}}_2\left(\bar{X}\right)}{1+\underline{\bar{k}}}}{\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}$$

$$+\left|\Psi\left(K',X'\right)\right|^2\left(\bar{r}-\left(\bar{r}-\left(\frac{1+f_1\left(K',X'\right)}{\underline{k}_2\left(X'\right)}+f_1\left(K',X'\right)\right)\right)H\left(-\left(\bar{r}-\left(\frac{1+f_1\left(K',X'\right)}{\underline{k}_2\left(X'\right)}+f_1\left(K',X'\right)\right)\right)\right)\right)$$

$$\times\frac{\kappa\underline{k}_2^{(B)}\left(\bar{X}\right)}{\left(1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{\bar{k}}}\right]\left(X'\right)\right)\left(1+\underline{\bar{k}}\right)}+\bar{f}_1\left(\bar{K},\bar{X},\Psi,\hat{\Psi}\right)$$

$$\bar{R}\left(\bar{X}\right)=\frac{\bar{f}\left(\bar{X}\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}\right)}+\bar{r}\frac{\underline{\bar{\bar{k}}}_2\left(\bar{X}\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}\right)}$$

$$\left(1-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\left(\frac{\bar{f}\left(\bar{X}'\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}+\bar{r}\frac{\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}\right) \tag{300}$$

$$-\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\left(\frac{\hat{f}\left(\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\frac{\underline{\hat{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}\left(\bar{X}\right)}}+\bar{r}\frac{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\frac{\underline{\hat{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}\left(\bar{X}\right)}}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)$$

$$=\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2\left(\bar{r}+\left(\frac{1+\bar{f}\left(\bar{X}'\right)}{\underline{\bar{\bar{k}}}_2\left(\hat{X}'\right)}\right)H\left(-\left(1+\bar{f}\left(\bar{X}'\right)\right)\right)\right)\frac{\bar{K}'\underline{\hat{k}}_2^B\left(\bar{X}',\bar{X}\right)}{1+\underline{\bar{\bar{k}}}\left(\bar{X}\right)}$$

$$+\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\left(\bar{r}+\frac{1+\hat{f}\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}H\left(-\frac{1+\hat{f}\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\right)$$

$$\times\frac{\hat{K}'\kappa\frac{\underline{\hat{k}}_2^B\left(\hat{X}',\bar{X}\right)}{1+\bar{k}\left(\bar{X}\right)}}{\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}$$

$$+\left|\Psi\left(K',X'\right)\right|^2\left(\bar{r}-\frac{\left(\bar{r}-\left(\frac{1+f_1\left(K',X'\right)}{\underline{k}_2\left(X'\right)}+f_1\left(K',X'\right)\right)\right)}{1+\exp\left(-\xi\left(\bar{r}-\left(\frac{1+f_1\left(K',X'\right)}{\underline{k}_2\left(X'\right)}+f_1\left(K',X'\right)\right)\right)\right)}\right)$$

$$\times\frac{K'\kappa\frac{\underline{k}_2^{(B)}\left(X',\bar{X}\right)}{1+\underline{\bar{k}}\left(\bar{X}\right)}}{\left(1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{\bar{k}}}\right]\left(X'\right)\right)}+\frac{K'\underline{k}_1^{(B)}\left(X',\bar{X}\right)}{\left(1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}\right)}f_1\left(\bar{K},\bar{X},\Psi,\hat{\Psi}\right)$$

$$\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(X'\right)=\int\frac{\underline{\hat{k}}_2^B\left(X',\bar{X}\right)}{1+\bar{k}\left(\bar{X}\right)}\bar{K}\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2d\bar{K}d\bar{X}$$

In terms of $\bar{g}$, the derivation including default is similar to that for investors and leads to:

$$\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\bar{k}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\bar{k}}_2\left(\bar{X}'\right)} \tag{301}$$

$$-\int\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\left(\bar{N}\bar{g}\right)\left(\bar{K}',\bar{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\frac{\underline{\hat{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}\left(\bar{X}\right)}}d\bar{X}'$$

$$+\int\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2\left(\bar{r}+\frac{1+\bar{f}\left(\bar{X}'\right)}{\underline{\bar{k}}_2\left(\bar{X}'\right)}H\left(-\frac{1+\bar{f}\left(\bar{X}'\right)}{\underline{\bar{k}}_2\left(\bar{X}'\right)}\right)\right)\frac{\hat{k}_2^B\left(\hat{X}',\hat{X}\right)\hat{K}'}{1+\bar{k}\left(\hat{X}'\right)}$$

$$+\int\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\left(\bar{r}+\frac{1+\hat{f}\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}H\left(-\frac{1+\hat{f}\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\right)$$

$$\times\frac{\hat{k}_2^B\left(\hat{X}',\hat{X}\right)\hat{K}'}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

$$\int\left|\Psi\left(K',X'\right)\right|^2\left(\bar{r}+\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}H\left(-\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}\right)\right)$$

$$\times\frac{k_2^B\left(\hat{X}',X\right)\hat{K}'}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}$$

$$=\quad\frac{\underline{k}_1^{(B)}\left(X',\bar{X}\right)}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{k}}}\frac{\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{k}}}$$

A18.5 Normalization of coefficients

A18.5.1 Normalization in the micro framework

As before, the coefficients will be normalized implicitely:

$$\bar{k}_{\eta jl}\rightarrow\frac{\bar{k}_{\eta jl}}{\bar{N}\left\langle\bar{K}_{0p}\right\rangle}\rightarrow\frac{\bar{k}_{\eta jl}}{\bar{N}\left\langle\bar{K}_{0p}\right\rangle\left(1-\left\langle\bar{k}\right\rangle\right)}$$

$$\hat{k}_{\eta jl}^B\rightarrow\frac{\hat{k}_{\eta jl}^B}{\hat{N}\left\langle\hat{K}_p\right\rangle}=\frac{\hat{k}_{\eta jl}^B}{\hat{N}\left\langle\hat{K}\right\rangle\left(1-\left\langle\hat{k}\right\rangle-\left\langle\hat{k}_1^B\right\rangle+\kappa\frac{\left\langle\hat{k}_2^B\right\rangle}{1+\left\langle\bar{k}\right\rangle}\frac{\bar{N}\left\langle\bar{K}_0\right\rangle}{\hat{N}\left\langle\hat{K}\right\rangle}\right)}$$

$$\hat{k}_{\eta jl}^B\rightarrow\frac{\hat{k}_{\eta jl}^B}{\hat{N}\left\langle\hat{K}_p\right\rangle}=\frac{\hat{k}_{\eta jl}^B}{\hat{N}\left\langle\hat{K}\right\rangle\left(1-\left\langle\hat{k}\right\rangle-\left(\left\langle\hat{k}_1^B\right\rangle+\kappa\frac{\left\langle\hat{k}_2^B\right\rangle}{1+\left\langle\bar{k}\right\rangle\left\langle\bar{K}_0\right\rangle}\frac{\bar{N}\left\langle\bar{K}\right\rangle}{\hat{N}\left\langle\hat{K}\right\rangle}\right)\right)}$$

A18.5.2 Normalization in the field model

We can consider the normalization of coeficients for investors and banks. The principle is the same as in part one, this amounts to replace:

$$\hat{k}_{\eta}\left(\hat{X}',\hat{X}\right)\rightarrow\frac{\hat{k}_{\eta}\left(\hat{X}',\hat{X}\right)}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle-\left(\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle+\kappa\frac{\left\langle\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle}{1+\left\langle\bar{k}\right\rangle\left\langle\bar{K}\right\rangle}\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\right)\right)}$$

$$\underline{\hat{k}}_{\eta}\left(\hat{X}'\right)\rightarrow\int\frac{\hat{k}_{\eta}\left(\hat{X}',\hat{X}\right)}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle-\left(\left(\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle+\kappa\frac{\left\langle\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle}{1+\left\langle\bar{k}\right\rangle\left\langle\bar{K}\right\rangle}\right)\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\right)\right)}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^{2}d\hat{X}$$

$$\bar{k}_{\eta}\left(\bar{X}',\bar{X}\right)\rightarrow\frac{\bar{k}_{\eta}\left(\bar{X}',\bar{X}\right)}{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle\left(1-\left\langle\bar{k}\left(\bar{X}',\bar{X}\right)\right\rangle\right)}$$

$$\underline{\bar{k}}_{\eta}\left(\bar{X}'\right)\rightarrow\int\frac{\bar{k}_{\eta}\left(\bar{X}',\bar{X}\right)}{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle\left(1-\left\langle\bar{k}\left(\bar{X}',\bar{X}\right)\right\rangle\right)}\bar{K}_{\bar{X}}\left|\hat{\Psi}\left(\bar{X}\right)\right|^{2}d\bar{X}$$

and:

$$\begin{aligned}&\frac{\hat{k}_{\eta}\left(\hat{X}',\hat{X}\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}\\
\rightarrow\;&\frac{\hat{k}_{\eta}\left(\hat{X}',\hat{X}\right)}{\left(\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle\left(1+\left(\underline{\hat{k}}\left(\hat{X}'\right)-\left\langle\underline{\hat{k}}\left(\hat{X}'\right)\right\rangle\right)+\left(\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)-\left\langle\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)\right\rangle\right)+\kappa\left(\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)-\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right\rangle\right)\right)\right)}\\
\rightarrow\;&\frac{\hat{k}_{\eta}\left(\hat{X}',\hat{X}\right)}{\left(\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)\right)}\end{aligned}$$

with now:

$$\underline{\hat{k}}\left(\hat{X}'\right)\rightarrow\underline{\hat{k}}\left(\hat{X}'\right)-\left\langle\underline{\hat{k}}\left(\hat{X}'\right)\right\rangle=\int\frac{\hat{k}\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^{2}d\hat{X}$$

$$\begin{aligned}\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)&\rightarrow\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)-\left\langle\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)\right\rangle=\int\frac{\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^{2}d\bar{X}\\
\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)&\rightarrow\kappa\left(\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)-\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right\rangle\right)=\kappa\int\frac{\underline{\hat{k}}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\underline{\hat{k}}_{2}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^{2}d\bar{X}\end{aligned}$$

$$
\begin{aligned}
1+\underline{\bar{k}}_2\left(\bar{X}'\right) &\rightarrow 1+\int \frac{\bar{k}_2\left(\bar{X}',\bar{X}\right)}{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle\left(1-\left\langle\bar{k}\left(\bar{X}',\bar{X}\right)\right\rangle\right)}\bar{K}_{\bar{X}}\left|\hat{\Psi}\left(\bar{X}\right)\right|^2 d\bar{X} \\
&= \frac{1}{\left(1-\left\langle\bar{k}\left(\bar{X}',\bar{X}\right)\right\rangle\right)}\left(\left(1-\left\langle\bar{k}\left(\bar{X}',\bar{X}\right)\right\rangle\right)+\int \frac{\bar{k}_2\left(\bar{X}',\bar{X}\right)}{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\bar{K}_{\bar{X}}\left|\hat{\Psi}\left(\bar{X}\right)\right|^2 d\bar{X}\right) \\
&= \frac{1}{\left(1-\left\langle\bar{k}\left(\bar{X}',\bar{X}\right)\right\rangle\right)}\left(\left(1-\left\langle\bar{k}_1\left(\bar{X}',\bar{X}\right)\right\rangle\right)+\int \frac{\bar{k}_2\left(\bar{X}',\bar{X}\right)-\left\langle\bar{k}_2\left(\bar{X}',\bar{X}\right)\right\rangle}{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2 d\hat{X}\right)
\end{aligned}
$$

$$
1+\underline{\bar{k}}_2\left(\bar{X}'\right)\rightarrow\frac{1-\left\langle\bar{k}_1\left(\bar{X}',\bar{X}\right)\right\rangle}{1-\left\langle\bar{k}\left(\bar{X}',\bar{X}\right)\right\rangle}\left(1+\frac{\underline{\bar{k}}_2\left(\hat{X}'\right)}{1-\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle}\right)
$$

with:

$$
\underline{\bar{k}}_2\left(\bar{X}'\right)=\int \frac{\bar{k}_2\left(\bar{X}',\bar{X}\right)-\left\langle\bar{k}_2\left(\bar{X}',\bar{X}\right)\right\rangle}{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2 d\bar{X}
$$

$$
\frac{\bar{k}_\eta\left(\bar{X}',\bar{X}\right)}{1+\underline{\bar{k}}_2\left(\bar{X}'\right)}\rightarrow\frac{\bar{k}_\eta\left(\bar{X}',\bar{X}\right)}{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle\left(1-\left\langle\bar{k}_1\left(\bar{X}',\bar{X}\right)\right\rangle\right)\left(1+\frac{\underline{\bar{k}}_2\left(\hat{X}'\right)}{1-\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle}\right)}
$$

$$
\begin{aligned}
&\frac{1}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)} \\
\rightarrow\; &\frac{1-\left(\left\langle\underline{\hat{k}}\left(\hat{X}',\hat{X}\right)\right\rangle+\left(\left\langle\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B\left(\hat{X}',\bar{X}\right)}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)}{\left(1-\left(\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_1^B\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left(1+\frac{\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1-\left(\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_1^B\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)} \\
\rightarrow\; &\frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle\left(1-\left(\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_1^B\left(\hat{X}',\bar{X}'\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left(1+\frac{\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1-\left(\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_1^B\left(\hat{X}',\bar{X}'\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)}\right)}
\end{aligned}
$$

Appendix 19 Alternate description. Exprssn in terms of relative coefficients

A19.1 Investors returns equation

We write investors'return equation:

$$0 = \int\left(1-\hat{S}_1\left(\hat{X}',\hat{K}',\hat{X}\right)\right)\frac{\hat{f}\left(\hat{X}'\right)-\bar{r}}{1+\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\frac{\hat{\underline{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}}}d\hat{X}'d\hat{K}' \tag{302}$$
$$-\int\left(\frac{1+\hat{f}\left(\hat{X}'\right)}{\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}H\left(-\frac{1+\hat{f}\left(\hat{X}'\right)}{\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\right)\hat{S}_2\left(\hat{X}',\hat{K}',\hat{X}\right)d\hat{X}'d\hat{K}'$$
$$-\int\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}H\left(-\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}\right)S_2\left(X',K',\hat{X}\right)-\int S_1\left(X',K',\hat{X}\right)\left(\hat{f}_1\left(\hat{K},\hat{X}\right)-\bar{r}\right)$$

where we define:

$$\hat{S}_\eta\left(\hat{X}',\hat{K}',\hat{X}\right) = \frac{\hat{K}'\hat{k}_\eta\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)} \tag{303}$$
$$\hat{S}_\eta\left(\hat{X}',\hat{X}\right) = \int\frac{\hat{K}'\hat{k}_\eta\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}d\hat{K}' = \frac{\hat{K}_{\hat{X}'}\hat{k}_\eta\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

and:

$$S_\eta\left(X',K',\hat{X}\right) = \frac{k_\eta\left(X',\hat{X}\right)K'\left|\Psi\left(K',X'\right)\right|^2}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)} \tag{304}$$
$$S_\eta\left(X',\hat{X}\right) = \int\frac{k_\eta\left(X',\hat{X}\right)K'\left|\Psi\left(K',X'\right)\right|^2}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}dK' = \frac{k_\eta\left(X',\hat{X}\right)K_{X'}\left|\Psi\left(X'\right)\right|^2}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}$$

After averaging over $\hat{K}'$, this writes:

$$0 = \int\left(1-\hat{S}_1\left(\hat{X}',\hat{X}\right)\right)\frac{\hat{f}\left(\hat{X}'\right)-\bar{r}}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}d\hat{X}' \tag{305}$$
$$-\int\left(\frac{1+\hat{f}\left(\hat{X}'\right)}{\hat{\underline{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}H\left(-\frac{1+\hat{f}\left(\hat{X}'\right)}{\hat{\underline{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\right)\hat{S}_2\left(\hat{X}',\hat{X}\right)d\hat{X}'$$
$$-\int\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}H\left(-\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}\right)S_2\left(X',\hat{X}\right)-\int S_1\left(X',\hat{X}\right)\left(\hat{f}_1\left(\hat{X}\right)-\bar{r}\right)$$

or alternatively, by rewriting all parameters in the new variables:

$$
\begin{aligned}
0 &= \int \left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_1\left(\hat{X}',\hat{X}\right)\right)\frac{1-\int \hat{S}\left(\hat{X}',\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}(\hat{X})\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}(\hat{X}')\right|^2}d\hat{X}}{1-\int \hat{S}_1\left(\hat{X}',\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}(\hat{X})\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}(\hat{X}')\right|^2}d\hat{X}}\left(f\left(\hat{X}'\right)-\bar{r}\right)d\hat{X}' \\
&\quad -\int S_1\left(X',\hat{X}\right)\frac{1-\int S\left(X',\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}(\hat{X})\right|^2}{K_{X'}\left|\Psi(X')\right|^2}d\hat{X}}{1-\int S_1\left(X',\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}(\hat{X})\right|^2}{K_{X'}\left|\Psi(X')\right|^2}dX}\left(f_1'\left(\hat{X}'\right)-\bar{r}\right)dX' \\
&\quad -\int \frac{1+f\left(\hat{X}'\right)}{\hat{\underline{k}}_2\left(\hat{X}'\right)}H\left(-\frac{1+f\left(\hat{X}'\right)}{\hat{\underline{k}}_2\left(\hat{X}'\right)}\right)\hat{S}_2\left(\hat{X}',\hat{X}\right)d\hat{X}'-\int \frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)}H\left(-\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)}\right)S_2\left(X',\hat{X}\right)dX'
\end{aligned}
$$

The rates satisfy the following constrain:

$$
\int\left(\hat{S}_1\left(\hat{X}',\hat{X}\right)+\hat{S}_2\left(\hat{X}',\hat{X}\right)\right)d\hat{X}'+\int\left(S_1\left(X',\hat{X}\right)+S_2\left(X',\hat{X}\right)\right)dX'=1
$$

A19.1.1 Banks returns equation

We can express the rtrns equations in terms of rtes by defining:

$$
\begin{aligned}
\bar{S}_\eta\left(\bar{X}',\bar{K}',\bar{X}\right) &= \frac{\bar{K}'\bar{k}_\eta\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)} \\
\bar{S}_\eta\left(\bar{X}',\bar{X}\right) &= \int \frac{\bar{K}'\bar{k}_\eta\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}d\bar{K}'=\frac{\bar{K}_{\bar{X}'}\bar{k}_\eta\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{S}_1^B\left(\hat{X}',\hat{K}',\bar{X}\right) &= \frac{\hat{K}'\hat{\underline{k}}_1^B\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)} \\
\hat{S}_1^B\left(\hat{X}',\bar{X}\right) &= \frac{\hat{K}_{\hat{X}'}\hat{\underline{k}}_1^B\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}
\end{aligned}
$$

$$
\begin{aligned}
\hat{S}_2^B\left(\hat{X}',\hat{K}',\bar{X}\right) &= \frac{\hat{K}'\frac{\kappa\hat{\underline{k}}_2^B\left(\hat{X}',\bar{X}\right)}{1+\underline{\bar{k}}\left(\bar{X}\right)}\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)} \\
\hat{S}_2^B\left(\hat{X}',\bar{X}\right) &= \frac{\hat{K}_{\hat{X}'}\frac{\kappa\hat{\underline{k}}_2^B\left(\hat{X}',\bar{X}\right)}{1+\underline{\bar{k}}\left(\bar{X}\right)}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\frac{\hat{\underline{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}}}
\end{aligned}
$$

$$
\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)=\int\frac{\underline{k}_2^B\left(\hat{X}',\bar{X}\right)}{1+\bar{k}\left(\bar{X}\right)}\bar{K}\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2d\bar{K}d\bar{X}
$$

$$S_1^B\left(X',K',\hat{X}\right) = \frac{K'\underline{k}_1^{(B)}\left(X',\bar{X}\right)\left|\Psi\left(K',X'\right)\right|^2}{1+\underline{k}\left(X'\right)+\underline{k}_1^{(B)}\left(X'\right)+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{\bar{k}}}\right]\left(X'\right)} \tag{306}$$

$$S_2^B\left(X',K',\hat{X}\right) = \frac{\frac{\underline{k}_2^{(B)}\left(X',\hat{X}\right)}{1+\underline{\bar{k}}\left(\bar{X}\right)}K'\left|\Psi\left(K',X'\right)\right|^2}{1+\underline{k}\left(X'\right)+\underline{k}_1^{(B)}\left(X'\right)+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{\bar{k}}}\right]\left(X'\right)}$$

$$S_1^B\left(X',\hat{X}\right) = \frac{K_{X'}\underline{k}_1^{(B)}\left(X',\bar{X}\right)\left|\Psi\left(K',X'\right)\right|^2}{1+\underline{k}\left(X'\right)+\underline{k}_1^{(B)}\left(X'\right)+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{\bar{k}}}\right]\left(X'\right)}\left|\Psi\left(X'\right)\right|^2$$

$$S_2^B\left(X',\hat{X}\right) = \frac{\frac{\underline{k}_2^{(B)}\left(X',\hat{X}\right)}{1+\underline{\bar{k}}\left(\bar{X}\right)}K_{X'}\left|\Psi\left(X'\right)\right|^2}{1+\underline{k}\left(X'\right)+\underline{k}_1^{(B)}\left(X'\right)+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{\bar{k}}}\right]\left(X'\right)}$$

and the equation writes with this parametrization:

$$\begin{aligned}0 = {} & \left(1-\bar{S}_1\left(\bar{X}',\bar{X}\right)\right)\frac{\bar{f}\left(\bar{X}'\right)-\bar{r}}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}-\hat{S}_1^B\left(\hat{X}',\bar{X}\right)\left(\frac{\hat{f}\left(\hat{X}'\right)-\bar{r}}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\frac{\underline{\hat{k}}_2^B\left(\bar{X}'\right)}{1+\underline{\bar{k}}\left(\bar{X}\right)}}\right) \\ & -\frac{\left(1+\bar{f}\left(\bar{X}'\right)\right)H\left(-\left(1+\bar{f}\left(\bar{X}'\right)\right)\right)}{\underline{\bar{\bar{k}}}_2\left(\hat{X}'\right)}\bar{S}_2\left(\bar{X}',\bar{X}\right)-\frac{\left(1+\hat{f}\left(\hat{X}'\right)\right)H\left(-\left(1+\hat{f}\left(\hat{X}'\right)\right)\right)}{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{\hat{k}}}\right]\left(\hat{X}'\right)}\hat{S}_2^B\left(\hat{X}',\bar{X}\right) \\ & -\frac{\left(1+f_1'\left(X'\right)\right)H\left(1+f_1'\left(K',X'\right)\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{\bar{k}}}\right]\left(X'\right)}S_1^B\left(\hat{X}',\bar{X}\right)-S_1^B\left(\hat{X}',\bar{X}\right)\left(\frac{\left(f_1'\left(\bar{K},\bar{X}\right)-\bar{r}\right)}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)\end{aligned} \tag{307}$$

with constrant:

$$\int\left(\hat{S}_1\left(\hat{X}',\hat{X}\right)+\hat{S}_2\left(\hat{X}',\hat{X}\right)\right)d\hat{X}'+\int\left(S_1\left(X',\hat{X}\right)+S_2\left(X',\hat{X}\right)\right)dX'=1$$

$$\int\bar{S}\left(\bar{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\bar{K}_{\bar{X}'}\left|\bar{\Psi}\left(\bar{X}'\right)\right|^2}d\bar{X}=\frac{\underline{\bar{k}}\left(\hat{X}'\right)}{1+\underline{\bar{k}}\left(\hat{X}'\right)}$$

$$\bar{S}\left(\bar{X}',\bar{X}\right)=\bar{S}_1\left(\bar{X}',\bar{X}\right)+\bar{S}_2\left(\bar{X}',\bar{X}\right)$$

We can replace all coefficients in function of the new parametrization. Writing:

$$\frac{1}{1+\underline{\bar{k}}\left(\hat{X}'\right)}=1-\int\bar{S}\left(\bar{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\bar{K}_{\bar{X}'}\left|\bar{\Psi}\left(\bar{X}'\right)\right|^2}d\bar{X}$$

we find:

$$\begin{aligned}\underline{\bar{k}}\left(\hat{X}'\right) &= \int\bar{S}\left(\bar{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\bar{K}_{\bar{X}'}\left|\bar{\Psi}\left(\bar{X}'\right)\right|^2}d\bar{X} \\ \underline{\bar{k}}_\eta\left(\hat{X}'\right) &= \frac{\int\bar{S}_\eta\left(\bar{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\bar{K}_{\bar{X}'}\left|\bar{\Psi}\left(\bar{X}'\right)\right|^2}d\bar{X}}{1-\int\bar{S}\left(\bar{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\bar{K}_{\bar{X}'}\left|\bar{\Psi}\left(\bar{X}'\right)\right|^2}d\bar{X}}\end{aligned}$$

$$1+\underline{\bar{k}}_2\left(\hat{X}'\right)=\frac{1-\int \bar{S}_1\left(\bar{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\bar{K}_{\bar{X}'}\left|\bar{\Psi}\left(\bar{X}'\right)\right|^2}d\bar{X}}{1-\int \bar{S}\left(\bar{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\bar{K}_{\bar{X}'}\left|\bar{\Psi}\left(\bar{X}'\right)\right|^2}d\bar{X}}$$

Ultimately, using:

$$\hat{S}_\eta\left(\hat{X}',\hat{X}\right)=\frac{\hat{K}_{\hat{X}'}\hat{k}_\eta\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

$$S_\eta\left(X',\hat{X}\right)=\frac{k_\eta\left(X',\hat{X}\right)K_{X'}\left|\Psi\left(X'\right)\right|^2}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}$$

and the following relations:

$$\int \hat{S}\left(\hat{X}',\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X}=\frac{\underline{\hat{k}}\left(\hat{X}'\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

$$\int \hat{S}_1^B\left(\hat{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\bar{X}=\frac{\underline{\bar{k}}_1\left(\hat{X}'\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

$$\int \hat{S}_2^B\left(\hat{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}=\frac{\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

writing:

$$\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)=\int\frac{\underline{\hat{k}}_2^B\left(\hat{X}',\bar{X}\right)}{1+\bar{k}\left(\bar{X}\right)}\bar{K}\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2d\bar{K}d\bar{X}$$

leads to:

$$\begin{aligned}&\int\frac{\hat{S}\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X}+\int\frac{\left(\hat{S}_1^B\left(\hat{X}',\bar{X}\right)+\hat{S}_2^B\left(\hat{X}',\bar{X}\right)\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\bar{X}\\=&\frac{\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\end{aligned}$$

This allows to rewrite the parameters in terms of rates. First for the investors:

$$\begin{aligned}&\frac{1}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\\=&1-\left(\int\frac{\hat{S}\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X}+\int\frac{\left(\hat{S}_1^B\left(\hat{X}',\bar{X}\right)+\hat{S}_2^B\left(\hat{X}',\bar{X}\right)\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\bar{X}\right)\end{aligned}$$

$$\underline{\hat{k}}\left(\hat{X}'\right) = \frac{\int \hat{S}\left(\hat{X}',\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X}}{1-\left(\int \frac{\hat{S}\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X}+\int \frac{\left(\hat{S}_1^B\left(\hat{X}',\bar{X}\right)+\hat{S}_2^B\left(\hat{X}',\bar{X}\right)\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\bar{X}\right)}$$

$$\underline{\hat{k}}_1^B\left(\bar{X}'\right) = \frac{\int \hat{S}_1^B\left(\hat{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\bar{X}}{1-\left(\int \frac{\hat{S}\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X}+\int \frac{\left(\hat{S}_1^B\left(\hat{X}',\bar{X}\right)+\hat{S}_2^B\left(\hat{X}',\bar{X}\right)\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\bar{X}\right)}$$

$$\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right) = \frac{\int \hat{S}_2^B\left(\hat{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\bar{X}}{1-\left(\int \frac{\hat{S}\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X}+\int \frac{\left(\hat{S}_1^B\left(\hat{X}',\bar{X}\right)+\hat{S}_2^B\left(\hat{X}',\bar{X}\right)\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\bar{X}\right)}$$

and then for the frms bu using:

$$\int S\left(X',\hat{X}\right)\frac{K_X\left|\Psi\left(X\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}dX = \frac{\underline{\bar{k}}\left(X'\right)}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}$$

$$\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right) = \int \frac{\underline{k}_2^B\left(\hat{X}',\bar{X}\right)}{1+\bar{k}\left(\bar{X}\right)}\bar{K}\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2 d\bar{K}d\bar{X}$$

and (306), we find:

$$\underline{k}\left(X'\right) = \frac{\int \frac{S\left(X',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\hat{X}}{1-\left(\int \frac{S\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\hat{X}+\int \frac{\left(S_1^B\left(\hat{X}',\bar{X}\right)+S_2^B\left(\hat{X}',\bar{X}\right)\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\bar{X}\right)}$$

$$\underline{k}_1^{(B)}\left(X'\right) = \frac{\int S_1^B\left(\hat{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\bar{X}}{1-\left(\int \frac{S\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\hat{X}+\int \frac{\left(S_1^B\left(\hat{X}',\bar{X}\right)+S_2^B\left(\hat{X}',\bar{X}\right)\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\bar{X}\right)}$$

$$\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right) = \frac{\int S_2^B\left(\hat{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\bar{X}}{1-\left(\int \frac{S\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\hat{X}+\int \frac{\left(S_1^B\left(\hat{X}',\bar{X}\right)+S_2^B\left(\hat{X}',\bar{X}\right)\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\bar{X}\right)}$$

leading to:

$$
\begin{aligned}
0 \quad = \quad & \left(1-\bar{S}_1\left(\bar{X}',\bar{X}\right)\right)\left(\bar{f}\left(\bar{X}'\right)-\bar{r}\right)\frac{1-\int \bar{S}\left(\bar{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\bar{K}_{\bar{X}'}\left|\bar{\Psi}\left(\bar{X}'\right)\right|^2}d\bar{X}}{1-\int \bar{S}_1\left(\bar{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\bar{K}_{\bar{X}'}\left|\bar{\Psi}\left(\bar{X}'\right)\right|^2}d\bar{X}} \qquad (308) \\
& -\hat{S}_1^B\left(\hat{X}',\bar{X}\right)\left(\hat{f}\left(\hat{X}'\right)-\bar{r}\right) \\
& \times\frac{1-\left(\int\frac{\hat{S}\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X}+\int\frac{\left(\hat{S}_1^B\left(\hat{X}',\bar{X}\right)+\hat{S}_2^B\left(\hat{X}',\bar{X}\right)\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\bar{X}\right)}{1-\left(\int\frac{\hat{S}_1\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X}+\int\frac{\hat{S}_1^B\left(\hat{X}',\bar{X}\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\bar{X}\right)} \\
& -\left(1+\bar{f}\left(\bar{X}'\right)\right)H\left(-\left(1+\bar{f}\left(\bar{X}'\right)\right)\right)\bar{S}_2\left(\bar{X}',\bar{X}\right)\frac{\left(1-\int \bar{S}\left(\bar{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\bar{K}_{\bar{X}'}\left|\bar{\Psi}\left(\bar{X}'\right)\right|^2}d\bar{X}\right)}{\int \bar{S}_2\left(\bar{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\bar{K}_{\bar{X}'}\left|\bar{\Psi}\left(\bar{X}'\right)\right|^2}d\bar{X}} \\
& -\left(1+\hat{f}\left(\hat{X}'\right)\right)H\left(-\left(1+\hat{f}\left(\hat{X}'\right)\right)\right)\hat{S}_2^B\left(\hat{X}',\bar{X}\right) \\
& \times\frac{1-\left(\int\frac{S\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\hat{X}+\int\frac{\left(S_1^B\left(\hat{X}',\bar{X}\right)+S_2^B\left(\hat{X}',\bar{X}\right)\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\bar{X}\right)}{\int\frac{S_2\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\hat{X}+\int S_2^B\left(\hat{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\bar{X}} \\
& -\left(1+f_1'\left(X'\right)\right)H\left(-\left(1+f_1'\left(K',X'\right)\right)\right)S_2^B\left(\hat{X}',\bar{X}\right) \\
& \times\frac{1-\left(\int\frac{\hat{S}\left(\hat{X}',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X}+\int\frac{\left(\hat{S}_1^B\left(\hat{X}',\bar{X}\right)+\hat{S}_2^B\left(\hat{X}',\bar{X}\right)\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\bar{X}\right)}{\int \hat{S}_2\left(\hat{X}',\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X}+\int \hat{S}_2^B\left(\hat{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\bar{X}} \qquad (309) \\
& -S_1^B\left(X',\bar{X}\right)\left(f_1'\left(X'\right)-\bar{r}+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right) \\
& \times\frac{1-\left(\int\frac{S\left(X',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\hat{X}+\int\frac{\left(S_1^B\left(X',\bar{X}\right)+S_2^B\left(X',\bar{X}\right)\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\bar{X}\right)}{1-\int\frac{S_1\left(X',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\hat{X}-\int S_1^B\left(X',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{K_{X'}\left|\Psi\left(X'\right)\right|^2}d\bar{X}}
\end{aligned}
$$

$$\int\left(\hat{S}_1\left(\hat{X}',\hat{X}\right)+\hat{S}_2\left(\hat{X}',\hat{X}\right)\right)d\hat{X}'+\int\left(S_1\left(X',\hat{X}\right)+S_2\left(X',\hat{X}\right)\right)dX'=1$$

$$\int\left(\bar{S}_1\left(\bar{X}',\bar{X}\right)+\bar{S}_2\left(\bar{X}',\bar{X}\right)\right)d\bar{X}'+\int\hat{S}_1^B\left(\hat{X}',\bar{X}\right)d\hat{X}'+\int S_1^B\left(X',\bar{X}\right)dX'=1$$

$$\int\hat{S}_2^B\left(\hat{X}',\bar{X}\right)d\hat{X}'+\int S_2^B\left(X',\bar{X}\right)dX'=1$$

$$
\begin{aligned}
\hat{S}_\eta\left(\hat{X}'\right) &= \int\hat{S}_\eta\left(\hat{X}',\hat{X}\right)\frac{\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}d\hat{X} \\
\hat{S}\left(\hat{X}'\right) &= \hat{S}_1\left(\hat{X}'\right)+\hat{S}_2\left(\hat{X}'\right)
\end{aligned}
$$

$$\hat{S}_{\eta}^{B}\left(\hat{X}'\right) = \int \frac{\hat{S}_{\eta}^{B}\left(\hat{X}',\bar{X}\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^{2}}{\hat{K}_{\hat{X}'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^{2}}d\bar{X}$$

$$\hat{S}^{B}\left(\hat{X}'\right) = \hat{S}_{1}^{B}\left(\hat{X}'\right)+\hat{S}_{2}^{B}\left(\hat{X}'\right)$$

$$\bar{S}_{\eta}\left(\bar{X}'\right) = \int \bar{S}_{\eta}\left(\bar{X}',\bar{X}\right)\frac{\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^{2}}{\bar{K}_{\bar{X}'}\left|\bar{\Psi}\left(\bar{X}'\right)\right|^{2}}d\bar{X}$$

$$\bar{S}\left(\bar{X}'\right) = \bar{S}_{1}\left(\bar{X}'\right)+\bar{S}_{2}\left(\bar{X}'\right)$$

$$S_{\eta}\left(X'\right) = \int \frac{S_{\eta}\left(X',\hat{X}\right)\hat{K}_{\hat{X}}\left|\hat{\Psi}\left(\hat{X}\right)\right|^{2}}{K_{X'}\left|\Psi\left(X'\right)\right|^{2}}d\hat{X}$$

$$S\left(X'\right) = S_{1}\left(X'\right)+S_{2}\left(X'\right)$$

$$S_{\eta}^{B}\left(X'\right) = \int \frac{S_{\eta}^{B}\left(X',\bar{X}\right)\bar{K}_{\bar{X}}\left|\bar{\Psi}\left(\bar{X}\right)\right|^{2}}{K_{X'}\left|\Psi\left(X'\right)\right|^{2}}d\bar{X}$$

$$S^{B}\left(X'\right) = S_{1}^{B}\left(X'\right)+S_{2}^{B}\left(X'\right)$$

This lds:

$$\begin{aligned}
0 = & \int\left(\Delta\left(\hat{X},\hat{X}'\right)-\hat{S}_{1}\left(\hat{X}',\hat{X}\right)\right)\frac{1-\hat{S}\left(\hat{X}'\right)}{1-\hat{S}_{1}\left(\hat{X}'\right)}\left(f\left(\hat{X}'\right)-\bar{r}\right)d\hat{X}' \\
& -\int S_{1}\left(X',\hat{X}\right)\frac{1-S\left(X'\right)}{1-S_{1}\left(X'\right)}\left(\left(f_{1}'\left(\hat{X}'\right)-\bar{r}\right)+\Delta F_{\tau}\left(\bar{R}\left(K,X\right)\right)\right)dX' \\
& -\int\frac{1-\left(\hat{S}\left(\hat{X}'\right)+\hat{S}_{1}^{B}\left(\hat{X}'\right)+\hat{S}_{2}^{B}\left(\hat{X}'\right)\right)}{\hat{S}_{2}\left(\hat{X}'\right)}\left(1+f\left(\hat{X}'\right)\right)H\left(-\left(1+f\left(\hat{X}'\right)\right)\right)\hat{S}_{2}\left(\hat{X}',\hat{X}\right)d\hat{X}' \\
& -\int\frac{1-\left(S\left(X'\right)+S_{1}^{B}\left(X'\right)+S_{2}^{B}\left(X'\right)\right)}{S_{2}\left(X'\right)}\left(1+f_{1}'\left(X'\right)\right)H\left(-\left(1+f_{1}'\left(X'\right)\right)\right)S_{2}\left(X',\hat{X}\right)dX'
\end{aligned}$$

for nvstrs

$$0 = \left(1-\bar{S}_1\left(\bar{X}',\bar{X}\right)\right)\left(\bar{f}\left(\bar{X}'\right)-\bar{r}\right)\frac{1-\bar{S}\left(\bar{X}'\right)}{1-\bar{S}_1\left(\bar{X}'\right)} \tag{310}$$

$$-\hat{S}_1^B\left(\hat{X}',\bar{X}\right)\left(\hat{f}\left(\hat{X}'\right)-\bar{r}\right)\frac{1-\left(\hat{S}\left(\hat{X}',\hat{X}\right)+\hat{S}_1^B\left(\hat{X}'\right)+\hat{S}_2^B\left(\hat{X}'\right)\right)}{1-\left(\hat{S}_1\left(\hat{X}'\right)+\hat{S}_1^B\left(\hat{X}'\right)\right)}$$

$$-\left(1+\bar{f}\left(\bar{X}'\right)\right)H\left(-\left(1+\bar{f}\left(\bar{X}'\right)\right)\right)\bar{S}_2\left(\bar{X}',\bar{X}\right)\frac{\left(1-\bar{S}\left(\bar{X}'\right)\right)}{\bar{S}_2\left(\bar{X}'\right)}$$

$$-\left(1+\hat{f}\left(\hat{X}'\right)\right)H\left(-\left(1+\hat{f}\left(\hat{X}'\right)\right)\right)\hat{S}_2^B\left(\hat{X}',\bar{X}\right)\frac{1-\left(S\left(\hat{X}'\right)+\left(S_1^B\left(\hat{X}'\right)+S_2^B\left(\hat{X}'\right)\right)\right)}{S_2\left(\hat{X}'\right)+S_2^B\left(\hat{X}'\right)}$$

$$-\left(1+f_1'\left(X'\right)\right)H\left(-\left(1+f_1'\left(X'\right)\right)\right)S_2^B\left(\hat{X}',\bar{X}\right)\frac{1-\left(\hat{S}\left(\hat{X}',\hat{X}\right)+\left(\hat{S}_1^B\left(\hat{X}'\right)+\hat{S}_2^B\left(\hat{X}'\right)\right)\right)}{\hat{S}_2\left(\hat{X}'\right)+\hat{S}_2^B\left(\hat{X}'\right)}$$

$$-S_1^B\left(X',\bar{X}\right)\left\{\left(f_1'\left(X'\right)-\bar{r}\right)\frac{1-\left(S\left(X'\right)+\left(S_1^B\left(X'\right)+S_2^B\left(X'\right)\right)\right)}{1-S_1\left(X'\right)-S_1^B\left(X'\right)}+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right\}$$

fr bnk.

Appendix 20 minimization equations

To compute the saddle point equation we need to derive the functional derivative of the action functional of the full system for investors and banks with respect to field of investors and field of bank:

$$-\hat{\Psi}^\dagger\left(\hat{K},\hat{X}\right)\nabla^2\hat{\Psi}\left(\hat{K},\hat{X}\right)+\left(\frac{\hat{g}^2\left(\hat{K},\hat{X}\right)}{2\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(\hat{K},\hat{X}\right)}{2\hat{K}}\right)\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2+\frac{1}{2\hat{\mu}}\left(\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2-\left|\hat{\Psi}_0\left(\hat{X}\right)\right|^2\right)^2$$

$$-\bar{\Psi}^\dagger\left(\bar{K},\bar{X}\right)\nabla^2\bar{\Psi}\left(\bar{K},\bar{X}\right)+\left(\frac{\bar{g}^2\left(\bar{K},\bar{X}\right)}{2\sigma_{\bar{K}}^2}+\frac{\bar{g}\left(\bar{K},\bar{X}\right)}{2\bar{K}}\right)\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2+\frac{1}{2\bar{\mu}}\left(\left|\bar{\Psi}\left(\bar{K}_1,\bar{X}_1\right)\right|^2-\left|\bar{\Psi}_0\left(\bar{X}_1\right)\right|^2\right)^2$$

A20.1 Investors minimization equations

The derivative of the investors field functional with respect to $\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)$ are:

$$0 = \frac{\hat{K}_1^2\hat{g}^2\left(\hat{K}_1,\hat{X}_1\right)}{2\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(\hat{K}_1,\hat{X}_1\right)}{2}$$

$$+\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2\left(\frac{\hat{K}^2\hat{g}\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\frac{\delta\hat{g}\left(\hat{K},\hat{X}\right)}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}$$

$$+\int\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2\left(\frac{\hat{K}^2\bar{g}\left(\bar{K},\bar{X}\right)}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\frac{\delta\bar{g}\left(\bar{K},\bar{X}\right)}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}+\frac{1}{\hat{\mu}}\left(\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2-\left|\hat{\Psi}_0\left(\hat{X}\right)\right|^2\right)$$

The derivatives are estimatd in the next appendx and w fnd:

$$
\begin{aligned}
&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}}\bar{g}\left(\hat{K}',\hat{X}'\right)\\
\simeq\;& \frac{\left(\left(\hat{k}_1^B\left(\hat{X},\left\langle\bar{X}\right\rangle\right)+\left\langle\underline{\hat{k}}_1^B\right\rangle\frac{\hat{k}_1\left(\hat{X}_1,\left\langle\hat{X}\right\rangle\right)}{1-\left\langle\hat{k}_1\right\rangle}\right)A-\left\langle\hat{k}_1^B\left(\hat{X},\left\langle\bar{X}\right\rangle\right)+\left\langle\underline{\hat{k}}_1^B\right\rangle\frac{\hat{k}_1\left(\hat{X}_1,\left\langle\hat{X}\right\rangle\right)}{1-\left\langle\hat{k}_1\right\rangle}\right\rangle\left\langle A\right\rangle\right)}{\left(1-\left\langle\bar{k}\left(\bar{X}',\bar{X}\right)\right\rangle\right)^2}\frac{\bar{K}}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle\hat{K}\right\rangle}\\
=\;& \Delta\left(\hat{k}^B\left(\hat{X},\left\langle\bar{X}\right\rangle\right)A\right)\frac{\bar{K}}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle\hat{K}\right\rangle}
\end{aligned}
$$

so that:

$$
\left\langle\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}}\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle=0
$$

nd:

$$
\begin{aligned}
&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}}\hat{g}\left(\hat{K},\hat{X}\right)\\
\simeq\;& -\frac{\left(\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\hat{k}\right\rangle\right)+\left\langle\hat{k}_1^B\right\rangle\left\langle\hat{k}_2\right\rangle\right)\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\frac{1}{1-\left\langle\underline{\hat{k}}\right\rangle}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)\frac{\left\Vert\bar{\Psi}\right\Vert^2\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle\hat{K}\right\rangle}\frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle}}{\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^2\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle\hat{K}\right\rangle}\right)\right)\left(1-\left(\left\langle\underline{\hat{k}}\right\rangle+\left(\left\langle\underline{\hat{k}}_1^B\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\Vert\bar{\Psi}\right\Vert^2\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle\hat{K}\right\rangle}\right)\right)\left\Vert\hat{\Psi}\right\Vert^2}
\end{aligned}
$$

wth:

$$
\bar{N}\rightarrow\left\langle\hat{k}_1^B\right\rangle+\kappa\frac{\left\langle\hat{k}_2^B\right\rangle}{1+\left\langle\bar{k}\right\rangle}\left(1-\frac{\left\langle\bar{k}\right\rangle}{\left(1+\left\langle\bar{k}\right\rangle\right)^2}\right)
$$

and the equation reduces to:

$$
\begin{aligned}
0 =\;& \frac{\hat{K}_1^2\hat{g}^2\left(\hat{K}_1,\hat{X}_1\right)}{2\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(\hat{K}_1,\hat{X}_1\right)}{2}\\
&+\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2\left(\frac{\hat{K}^2\hat{g}\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\frac{\delta\hat{g}\left(\hat{K},\hat{X}\right)}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}+\frac{1}{\hat{\mu}}\left(\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2-\left|\hat{\Psi}_0\left(\hat{X}\right)\right|^2\right)
\end{aligned}
$$

A20.2 Banks minimization equations

The minimzation with respect to:

$$
\begin{aligned}
0 =\;& \left(\frac{\bar{K}_1^2\bar{g}^2\left(\bar{K}_1,\bar{X}_1\right)}{\sigma_{\hat{K}}^2}+\frac{\bar{g}\left(\bar{K}_1,\bar{X}_1\right)}{2}\right)\\
&+\int\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2\left(\frac{\hat{K}^2\bar{g}\left(\bar{K},\bar{X}\right)}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\frac{\delta\bar{g}\left(\bar{K},\bar{X}\right)}{\delta\left|\bar{\Psi}\left(\bar{K}_1,\bar{X}_1\right)\right|^2}\\
&+\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2\left(\frac{\hat{K}^2\hat{g}\left(\hat{K},\hat{X}\right)}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\frac{\delta\hat{g}\left(\hat{K},\hat{X}\right)}{\delta\left|\bar{\Psi}\left(\bar{K}_1,\bar{X}_1\right)\right|^2}+\frac{1}{\hat{\mu}}\left(\left|\bar{\Psi}\left(\bar{K}_1,\bar{X}_1\right)\right|^2-\left|\bar{\Psi}_0\left(\bar{X}_1\right)\right|^2\right)
\end{aligned}
$$

with:

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\hat{g}\left(\hat{K},\hat{X}\right)$$

$$\rightarrow -\frac{\kappa\left\langle\left[\frac{\hat{\underline{k}}_{2}^{B}}{1+\hat{k}}\right]\right\rangle\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)+\left\langle\hat{k}_{1}^{B}\right\rangle\left\langle\hat{k}_{2}\right\rangle}{\left(1-\left(\left\langle\hat{k}\right\rangle+\left(\left\langle\hat{k}_{1}^{B}\right\rangle+\kappa\left\langle\left[\frac{\hat{\underline{k}}_{2}^{B}}{1+\hat{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)\left(1-\left(\left\langle\hat{k}_{1}\right\rangle+\left\langle\hat{k}_{1}^{B}\right\rangle\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)\left\|\hat{\Psi}\right\|^{2}}\frac{\bar{K}}{\left\langle\hat{K}\right\rangle}$$
$$\times\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)$$

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\bar{g}\left(\hat{K}',\hat{X}'\right) \quad (311)$$

$$\simeq \left(\frac{\left(1-\langle\bar{k}\rangle\right)^{2}}{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}\right)^{-1}\left\{\left(\left\langle\hat{\underline{k}}_{1}^{B}\right\rangle+\frac{\left\langle\hat{\underline{k}}_{1}^{B}\right\rangle}{\left(1-\left\langle\hat{k}_{1}\right\rangle\right)}\right)\right.$$
$$\left.\times\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{2}^{B}\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle}\right)\langle A\rangle+\frac{\left(1-\langle\bar{k}\rangle\right)^{2}}{1-\left\langle\bar{k}_{1}\right\rangle}\frac{\left(\bar{k}_{2}\left(\langle\bar{X}\rangle,\bar{X}\right)-\langle\bar{k}_{2}\rangle\right)}{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}\langle\bar{g}\rangle\right\}\frac{\bar{K}}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}$$

$$= \left(\left(\left\langle\hat{\underline{k}}_{1}^{B}\right\rangle+\frac{\left\langle\hat{\underline{k}}_{1}^{B}\right\rangle}{\left(1-\left\langle\hat{k}_{1}\right\rangle\right)}\right)\Delta\left(\hat{k}^{B}\left(\hat{X},\langle\bar{X}\rangle\right)A\right)+\frac{\Delta\bar{k}_{2}\left(\langle\bar{X}\rangle,\bar{X}\right)}{\left(1-\langle\bar{k}_{1}\rangle\right)\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}\langle\bar{g}\rangle\right)\frac{\bar{K}}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle} \quad (312)$$

whr:

$$\langle A\rangle = \left\langle\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\hat{\underline{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_{2}^{B}}{1+\hat{k}}\right]\left(\hat{X}'\right)}\right\rangle$$
$$= \frac{1-\left(\left\langle\hat{\underline{k}}\right\rangle+\left\langle\hat{\underline{k}}_{1}^{B}\right\rangle\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}+\kappa\left\langle\left[\frac{\hat{\underline{k}}_{2}^{B}}{1+\hat{k}}\right]\right\rangle\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)}{\left(1-\left(\left\langle\hat{k}_{1}\right\rangle+\left\langle\hat{k}_{1}^{B}\right\rangle\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}$$
$$\times\left(\left(1-\left\langle\hat{k}\right\rangle\right)\langle\hat{g}\rangle+\left(\left\langle\hat{k}_{1}^{B}\right\rangle+\kappa\frac{\left\langle\hat{k}_{2}^{B}\right\rangle}{1+\left\langle\bar{k}\right\rangle}\left(1-\frac{\langle\bar{k}\rangle}{\left(1+\langle\bar{k}\rangle\right)^{2}}\right)\right)\langle\bar{g}\rangle\right)$$

as a consequence, in averag:

$$\left\langle\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle=0$$

so tht the equation reduces to:

$$0 = \left(\frac{\bar{K}_{1}^{2}\bar{g}^{2}\left(\bar{K}_{1},\bar{X}_{1}\right)}{\sigma_{\hat{K}}^{2}}+\frac{\bar{g}\left(\bar{K}_{1},\bar{X}_{1}\right)}{2}\right)$$
$$+\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}\left(\frac{\hat{K}^{2}\hat{g}\left(\hat{K},\hat{X}\right)}{\sigma_{\hat{K}}^{2}}+\frac{1}{2}\right)\frac{\delta\hat{g}\left(\hat{K},\hat{X}\right)}{\delta\left|\bar{\Psi}\left(\bar{K}_{1},\bar{X}_{1}\right)\right|^{2}}+\frac{1}{\hat{\mu}}\left(\left|\bar{\Psi}\left(\bar{K}_{1},\bar{X}_{1}\right)\right|^{2}-\left|\bar{\Psi}_{0}\left(\bar{X}_{1}\right)\right|^{2}\right)$$

A20.3 Solving saddle point for investors and banks

A20.3.1 Expression for the field

Solving for $\left|\hat{\Psi}\left(\hat{K}_1, \hat{X}_1\right)\right|^2$ and $\left|\bar{\Psi}\left(\bar{K}_1, \bar{X}_1\right)\right|^2$ yields directly:

$$\left|\hat{\Psi}\left(\hat{K}_1, \hat{X}_1\right)\right|^2 = \left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}\left\{\left(\frac{\hat{K}_1^2 \hat{g}^2\left(\hat{X}_1\right)}{2\sigma_{\hat{K}}^2} + \frac{\hat{g}\left(\hat{X}_1\right)}{2}\right) - \left(\frac{\left\langle\hat{K}\right\rangle^2 \langle\hat{g}\rangle}{\sigma_{\hat{K}}^2} + \frac{1}{2}\right)\langle\hat{g}^{ef}\rangle \frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle}\right\} \tag{313}$$

$$\left|\bar{\Psi}\left(\bar{K}_1, \bar{X}_1\right)\right|^2 = \left|\bar{\Psi}_0\left(\bar{X}_1\right)\right|^2 - \hat{\mu}\left\{\left(\frac{\bar{K}_1^2 \bar{g}^2\left(\bar{X}_1\right)}{\sigma_{\hat{K}}^2} + \frac{\bar{g}\left(\bar{X}_1\right)}{2}\right) - \left(\frac{\left\langle\hat{K}\right\rangle^2 \langle\hat{g}\rangle}{\sigma_{\hat{K}}^2} + \frac{1}{2}\right)\langle\hat{g}^{Bef}\rangle \frac{\bar{K}_1}{\left\langle\hat{K}\right\rangle}\right\} \tag{314}$$

and these formula will be used to compute average capital per sector.

A20.3.2 Finding the maximal capital

the maximal value for $\hat{K}_0$ and $\hat{K}_0$ is found by setting $\left|\hat{\Psi}\left(\hat{K}_1, \hat{X}_1\right)\right|^2 = 0$.

$$0 = \left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}\left\{\left(\frac{\hat{K}_0^2 \hat{g}^2\left(\hat{X}_1\right)}{2\sigma_{\hat{K}}^2} + \frac{\hat{g}\left(\hat{X}_1\right)}{2}\right) - \left(\frac{\left\langle\hat{K}\right\rangle^2 \langle\hat{g}\rangle}{\sigma_{\hat{K}}^2} + \frac{1}{2}\right)\langle\hat{g}^{ef}\rangle \frac{\hat{K}_0}{\left\langle\hat{K}\right\rangle}\right\} \tag{315}$$

$$0 = \left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2 - \hat{\mu}\left\{\left(\frac{\bar{K}_0^2 \bar{g}^2\left(\bar{X}_1\right)}{\sigma_{\hat{K}}^2} + \frac{\bar{g}\left(\bar{X}_1\right)}{2}\right) - \left(\frac{\left\langle\hat{K}\right\rangle^2 \langle\hat{g}\rangle}{\sigma_{\hat{K}}^2} + \frac{1}{2}\right)\langle\hat{g}^{Bef}\rangle \frac{\bar{K}_0}{\left\langle\hat{K}\right\rangle}\right\}$$

and we find:

$$\begin{aligned}
\hat{K}_0^2 &\simeq \frac{2\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}} - \left(\frac{\hat{g}\left(\hat{X}_1\right)}{2} + \left(\frac{\left\langle\hat{K}\right\rangle^2 \langle\hat{g}\rangle}{\sigma_{\hat{K}}^2} + \frac{1}{2}\right)\langle\hat{g}^{ef}\rangle \frac{\left\langle\hat{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}\right)\right) \\
&\simeq 2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}} - \left(\frac{\left\langle\hat{K}\right\rangle^2 \langle\hat{g}\rangle}{\sigma_{\hat{K}}^2} + \frac{1}{2}\right)\langle\hat{g}^{ef}\rangle \frac{\left\langle\hat{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}\right)
\end{aligned} \tag{316}$$

and:

$$\bar{K}_0^2 \simeq 2\frac{\sigma_{\hat{K}}^2}{\bar{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2}{\hat{\mu}} - \left(\frac{\left\langle\hat{K}\right\rangle^2 \langle\hat{g}\rangle}{\sigma_{\hat{K}}^2} + \frac{1}{2}\right)\langle\hat{g}^{Bef}\rangle \frac{\langle\bar{K}_0\rangle}{\left\langle\hat{K}\right\rangle}\right)$$

In first approximation, we obtain:

$$\left\langle\hat{K}_0\right\rangle^2 \simeq 2\frac{\sigma_{\hat{K}}^2}{\langle\hat{g}\rangle^2}\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}} - \left(\frac{\left\langle\hat{K}\right\rangle^2 \langle\hat{g}\rangle}{\sigma_{\hat{K}}^2} + \frac{1}{2}\right)\langle\hat{g}^{ef}\rangle \frac{\left\langle\hat{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}\right) \tag{317}$$

or:

$$\left\langle\hat{K}_0\right\rangle^2 \simeq 2\frac{\sigma_{\hat{K}}^2}{\langle\hat{g}\rangle^2}\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\hat{\mu}} - \left(\frac{\left\langle\hat{K}\right\rangle^2 \langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}\right)\langle\hat{g}^{ef}\rangle \frac{\left\langle\hat{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}\right)$$

and:

$$\left\langle \bar{K}_0 \right\rangle^2 \simeq 2 \frac{\sigma_{\hat{K}}^2}{\langle \bar{g} \rangle^2} \left(\frac{\left\| \bar{\Psi}_0 \right\|^2}{\hat{\mu}} - \left(\frac{\left\langle \hat{K} \right\rangle^2 \langle \hat{g} \rangle}{\sigma_{\hat{K}}^2} \right) \left\langle \hat{g}^{Bef} \right\rangle \frac{\left\langle \bar{K}_0 \right\rangle}{\left\langle \hat{K} \right\rangle} \right)$$

A20.4 Fields per sector and average capital per sector

A20.4.1 Fields expression

Integrating the expressions (313) and (314) for the field yields the number of investors in one sector:

$$\begin{aligned} \left\| \hat{\Psi} \left(\hat{X}_1 \right) \right\|^2 &= \hat{K}_0 \left\| \hat{\Psi}_0 \left(\hat{X}_1 \right) \right\|^2 - \hat{\mu} \left\{ \left(\frac{\hat{K}_0^3 \hat{g}^2 \left(\hat{X}_1 \right)}{6 \sigma_{\hat{K}}^2} + \frac{\hat{K}_0 \hat{g} \left(\hat{X}_1 \right)}{2} \right) \right. \\ & \left. - \hat{K}_0 \left(\frac{\left\langle \hat{K} \right\rangle^2 \langle \hat{g} \rangle}{\sigma_{\hat{K}}^2} + \frac{1}{2} \right) \left\langle \hat{g}^{ef} \right\rangle \frac{\hat{K}_0}{2 \left\langle \hat{K} \right\rangle} \right\} \end{aligned} \tag{318}$$

using that $\hat{K}_0^2$ satisfies (246) we obtain:

$$\begin{aligned} \left\| \hat{\Psi} \left(\hat{X}_1 \right) \right\|^2 &= \hat{\mu} \hat{K}_0^3 \frac{\hat{g}^2 \left(\hat{X}_1 \right)}{3 \sigma_{\hat{K}}^2} - \hat{\mu} \frac{\hat{K}_0^2}{2 \left\langle \hat{K} \right\rangle} \left(\frac{\left\langle \hat{K} \right\rangle^2 \langle \hat{g} \rangle}{\sigma_{\hat{K}}^2} + \frac{1}{2} \right) \left\langle \hat{g}^{ef} \right\rangle \\ &\simeq \hat{\mu} \frac{\hat{K}_0^3}{\sigma_{\hat{K}}^2} \left(\frac{\hat{g}^2 \left(\hat{X}_1 \right)}{3} - \frac{\left\langle \hat{K} \right\rangle}{2 \hat{K}_0} \langle \hat{g} \rangle \left\langle \hat{g}^{ef} \right\rangle \right) \end{aligned} \tag{319}$$

Similarly, the value of $\left\| \bar{\Psi} \left(\bar{X}_1 \right) \right\|^2$ is given by

$$\left\| \bar{\Psi} \left(\bar{X}_1 \right) \right\|^2 \simeq \hat{\mu} \frac{\bar{K}_0^3}{\sigma_{\hat{K}}^2} \left(\frac{\bar{g}^2 \left(\bar{X}_1 \right)}{3} - \frac{\left\langle \hat{K} \right\rangle}{2 \bar{K}_0} \langle \hat{g} \rangle \left\langle \hat{g}^{Bef} \right\rangle \right)$$

A20.4.2 Capital per sector

Multiplying (313) and (314) by $\hat{K}$ and $\bar{K}$ respectively and integrating leads:

$$\begin{aligned} \hat{K}_{\hat{X}} \left\| \hat{\Psi} \left(\hat{X}_1 \right) \right\|^2 &= \frac{\hat{K}_0^2}{2} \left\| \hat{\Psi}_0 \left(\hat{X}_1 \right) \right\|^2 - \hat{\mu} \hat{K}_0 \left\{ \left(\frac{\hat{K}_0^3 \hat{g} \left(\hat{X}_1 \right)}{8 \sigma_{\hat{K}}^2} + \frac{\hat{K}_0 \hat{g} \left(\hat{X}_1 \right)}{4} \right) \right. \\ & \left. - \frac{\hat{K}_0^2}{2} \left(\frac{\left\langle \hat{K} \right\rangle^2 \langle \hat{g} \rangle}{\sigma_{\hat{K}}^2} + \frac{1}{2} \right) \left(\frac{2 \hat{K}_0}{3 \left\langle \hat{K} \right\rangle} \left\langle \hat{g}^{ef} \right\rangle \right) \right\} \end{aligned}$$

and since $\hat{K}_0$ satitsfies (246), we find:

$$\begin{aligned} \hat{K}_{\hat{X}} \left\| \hat{\Psi} \left(\hat{X}_1 \right) \right\|^2 &= \hat{\mu} \frac{\hat{K}_0^4 \hat{g}^2 \left(\hat{X}_1 \right)}{8 \sigma_{\hat{K}}^2} - \hat{\mu} \frac{\hat{K}_0^2}{6 \left\langle \hat{K} \right\rangle} \left(\frac{\left\langle \hat{K} \right\rangle^2 \langle \hat{g} \rangle}{\sigma_{\hat{K}}^2} + \frac{1}{2} \right) \left\langle \hat{g}^{ef} \right\rangle \langle \hat{g} \rangle \\ &\simeq \hat{\mu} \frac{\hat{K}_0^4}{2 \sigma_{\hat{K}}^2} \left(\frac{\hat{g}^2 \left(\hat{X}_1 \right)}{4} - \frac{\left\langle \hat{K} \right\rangle \langle \hat{g} \rangle}{3 \left\langle \hat{K}_0 \right\rangle} \left\langle \hat{g}^{ef} \right\rangle \right) \end{aligned} \tag{320}$$

we also have:

$$\bar{K}_{\bar{X}} \left\| \bar{\Psi} \left(\bar{X}_1 \right) \right\|^2 \simeq \hat{\mu} \frac{\bar{K}_0^4}{2\sigma_{\hat{K}}^2} \left(\frac{\bar{g}^2 \left(\hat{X}_1 \right)}{4} - \frac{\left\langle \hat{K} \right\rangle \langle \hat{g} \rangle}{3 \left\langle \bar{K}_0 \right\rangle} \left\langle \hat{g}^{Bef} \right\rangle \right) \tag{321}$$

A20.5 Average field and capital

A20.5.1 Averages and ratio $\frac{\langle \hat{K} \rangle}{\langle \hat{K}_0 \rangle}$

avergs for (248) and (249) become:

$$\left\| \hat{\Psi} \right\|^2 \simeq \hat{\mu} V \frac{\left\langle \hat{K}_0 \right\rangle^3}{\sigma_{\hat{K}}^2} \left(\frac{1}{3} - \frac{\left\langle \hat{K} \right\rangle}{2 \left\langle \hat{K}_0 \right\rangle} \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} \right) \langle \hat{g} \rangle^2 \tag{322}$$

$$\left\langle \hat{K} \right\rangle \left\| \hat{\Psi} \right\|^2 = \hat{\mu} V \frac{\left\langle \hat{K}_0 \right\rangle^4}{2\sigma_{\hat{K}}^2} \left(\frac{1}{4} - \frac{\left\langle \hat{K} \right\rangle}{3 \left\langle \hat{K}_0 \right\rangle} \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} \right) \langle \hat{g} \rangle^2$$

wh V the volume of sectors space. The ratio average capital over maximal capital is given by:

$$\frac{\left\langle \hat{K} \right\rangle}{\left\langle \hat{K}_0 \right\rangle} = \frac{1}{2} \frac{\frac{1}{4} - \frac{\langle \hat{K} \rangle}{3 \langle \hat{K}_0 \rangle} \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle}}{\frac{1}{3} - \frac{\langle \hat{K} \rangle}{2 \langle \hat{K}_0 \rangle} \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle}}$$

with solution

$$\frac{\left\langle \hat{K} \right\rangle}{\left\langle \hat{K}_0 \right\rangle} = \frac{1}{6 \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle}} \left(2 + \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} \right) \left(4 - \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} \right)} \right) \tag{323}$$

Coming back to the equation for $\left\langle \hat{K}_0 \right\rangle^2$

$$\left\langle \hat{K}_0 \right\rangle^2 \simeq 2 \frac{\sigma_{\hat{K}}^2}{\langle \hat{g} \rangle^2} \left(\frac{\left\| \hat{\Psi}_0 \right\|^2}{\hat{\mu}} - \left(\frac{\left\langle \hat{K} \right\rangle^2 \langle \hat{g} \rangle}{\sigma_{\hat{K}}^2} \right) \left\langle \hat{g}^{ef} \right\rangle \frac{\left\langle \hat{K}_0 \right\rangle}{\left\langle \hat{K} \right\rangle} \right) \tag{324}$$

allows to find the exprssion for $\left\langle \hat{K}_0 \right\rangle$. Actually (324) wrts:

$$1 \simeq 2 \frac{\sigma_{\hat{K}}^2}{\langle \hat{g} \rangle^2} \left(\frac{\left\| \hat{\Psi}_0 \right\|^2}{\hat{\mu} \left\langle \hat{K}_0 \right\rangle^2} - \left(\frac{\left\langle \hat{K} \right\rangle \langle \hat{g} \rangle}{\sigma_{\hat{K}}^2 \left\langle \hat{K}_0 \right\rangle} \right) \left\langle \hat{g}^{ef} \right\rangle \right)$$

that becomes:

$$2 \frac{\left\langle \hat{K} \right\rangle \left\langle \hat{g}^{ef} \right\rangle}{\left\langle \hat{K}_0 \right\rangle \langle \hat{g} \rangle} = 2 \frac{\sigma_{\hat{K}}^2}{\langle \hat{g} \rangle^2} \frac{\left\| \hat{\Psi}_0 \right\|^2}{\hat{\mu} \left\langle \hat{K}_0 \right\rangle^2} - 1$$

and yields an expression for $\frac{\langle \hat{K} \rangle}{\langle \hat{K}_0 \rangle}$ that completes (323):

$$\frac{\left\langle \hat{K} \right\rangle}{\left\langle \hat{K}_0 \right\rangle} = \frac{1}{2} \frac{\langle \hat{g} \rangle}{\left\langle \hat{g}^{ef} \right\rangle} \left(2 \frac{\sigma_{\hat{K}}^2}{\langle \hat{g} \rangle^2} \frac{\left\| \hat{\Psi}_0 \right\|^2}{\hat{\mu} \left\langle \hat{K}_0 \right\rangle^2} - 1 \right) \tag{325}$$

Equating (325) and (323) yields:

$$\frac{1}{2}\frac{\langle\hat{g}\rangle}{\langle\hat{g}^{ef}\rangle}\left(2\frac{\sigma_{\hat{K}}^{2}}{\langle\hat{g}\rangle^{2}}\frac{\left\|\hat{\Psi}_{0}\right\|^{2}}{\hat{\mu}\left\langle\hat{K}_{0}\right\rangle^{2}}-1\right)=\frac{1}{6\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}}\left(2+\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\right)$$

leading ultimately to the expressions for $\left\langle\hat{K}_{0}\right\rangle^{2}$ and $\left\langle\hat{K}\right\rangle^{2}$:

$$\left\langle\hat{K}_{0}\right\rangle^{2}=6\frac{\sigma_{\hat{K}}^{2}}{\hat{\mu}}\frac{\left\|\hat{\Psi}_{0}\right\|^{2}}{\langle\hat{g}\rangle^{2}\left(5+\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\right)}\tag{326}$$

$$\begin{aligned}\left\langle\hat{K}\right\rangle^{2}&=\left(\frac{\langle\hat{g}\rangle}{6\langle\hat{g}^{ef}\rangle}\left(2+\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\right)\right)^{2}\left\langle\hat{K}_{0}\right\rangle^{2}\\&\rightarrow\frac{1}{6}\frac{\sigma_{\hat{K}}^{2}}{\hat{\mu}}\frac{\left(\left(2+\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\right)\right)^{2}\left\|\hat{\Psi}_{0}\right\|^{2}}{\langle\hat{g}^{ef}\rangle^{2}\left(5+\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\right)}\end{aligned}\tag{327}$$

A20.5.2 Computation of $\frac{\langle\bar{K}_{0}\rangle}{\langle\hat{K}_{0}\rangle}$

We proceed similarly for fild and average capital for banks:

$$\left\|\bar{\Psi}\right\|^{2}=\hat{\mu}V\frac{\langle\bar{K}_{0}\rangle^{3}}{\sigma_{\hat{K}}^{2}}\left(\frac{1}{3}-\frac{\left\langle\hat{K}\right\rangle}{2\langle\bar{K}_{0}\rangle}\frac{\langle\hat{g}^{Bef}\rangle}{\langle\bar{g}\rangle}\right)\langle\bar{g}\rangle^{2}$$

$$\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^{2}=\hat{\mu}V\frac{\langle\bar{K}_{0}\rangle^{4}}{2\sigma_{\hat{K}}^{2}}\left(\frac{1}{4}-\frac{\left\langle\hat{K}\right\rangle}{3\langle\bar{K}_{0}\rangle}\frac{\langle\hat{g}^{Bef}\rangle}{\langle\bar{g}\rangle}\right)\langle\bar{g}\rangle^{2}$$

along with the value of average maximal capital:

$$\langle\bar{K}_{0}\rangle^{2}\simeq 2\frac{\sigma_{\hat{K}}^{2}}{\langle\bar{g}\rangle^{2}}\left(\frac{\left\|\bar{\Psi}_{0}\right\|^{2}}{\hat{\mu}}-\left(\frac{\left\langle\hat{K}\right\rangle^{2}\langle\hat{g}\rangle}{\sigma_{\hat{K}}^{2}}\right)\langle\hat{g}^{Bef}\rangle\frac{\langle\bar{K}_{0}\rangle}{\left\langle\hat{K}\right\rangle}\right)$$

again this leads to write:

$$1\simeq 2\frac{\sigma_{\hat{K}}^{2}}{\langle\bar{g}\rangle^{2}}\left(\frac{\left\|\bar{\Psi}_{0}\right\|^{2}}{\hat{\mu}\langle\bar{K}_{0}\rangle^{2}}-\frac{\left\langle\hat{K}\right\rangle\langle\hat{g}\rangle}{\langle\bar{K}_{0}\rangle\sigma_{\hat{K}}^{2}}\langle\hat{g}^{Bef}\rangle\right)$$

and the ratio of averages maximal capital is:

$$\frac{\langle\bar{K}_{0}\rangle^{2}}{\left\langle\hat{K}_{0}\right\rangle^{2}}\simeq 2\frac{\sigma_{\hat{K}}^{2}}{\langle\bar{g}\rangle^{2}}\left(\frac{\left\|\bar{\Psi}_{0}\right\|^{2}}{\hat{\mu}\left\langle\hat{K}_{0}\right\rangle^{2}}-\frac{\left\langle\hat{K}\right\rangle\langle\hat{g}\rangle}{\left\langle\hat{K}_{0}\right\rangle\sigma_{\hat{K}}^{2}}\langle\hat{g}^{Bef}\rangle\frac{\langle\bar{K}_{0}\rangle}{\left\langle\hat{K}_{0}\right\rangle}\right)$$

using (323):

$$\frac{\left\langle \hat{K}\right\rangle }{\left\langle \hat{K}_{0}\right\rangle }\simeq\frac{\left\langle \hat{g}\right\rangle }{6\left\langle \hat{g}^{ef}\right\rangle }\left(2+\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }-\sqrt{\left(1-\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }\right)\left(4-\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }\right)}\right)$$

the ratio $\frac{\left\langle \bar{K}_{0}\right\rangle ^{2}}{\left\langle \hat{K}_{0}\right\rangle ^{2}}$ rewrts:

$$\begin{aligned}
\frac{\left\langle \bar{K}_{0}\right\rangle ^{2}}{\left\langle \hat{K}_{0}\right\rangle ^{2}} &\simeq 2\frac{\sigma_{\hat{K}}^{2}}{\left\langle \bar{g}\right\rangle ^{2}}\left(\frac{\left\Vert \bar{\Psi}_{0}\right\Vert ^{2}}{\hat{\mu}\left\langle \hat{K}_{0}\right\rangle ^{2}}-\frac{\left\langle \hat{K}\right\rangle \left\langle \hat{g}\right\rangle }{\left\langle \hat{K}_{0}\right\rangle \sigma_{\hat{K}}^{2}}\left\langle \hat{g}^{Bef}\right\rangle \frac{\left\langle \bar{K}_{0}\right\rangle }{\left\langle \hat{K}_{0}\right\rangle }\right)\\
&\simeq 2\frac{\sigma_{\hat{K}}^{2}}{\left\langle \bar{g}\right\rangle ^{2}}\left(\frac{\left\Vert \bar{\Psi}_{0}\right\Vert ^{2}}{\hat{\mu}\left\langle \hat{K}_{0}\right\rangle ^{2}}-\frac{1}{6}\frac{\left\langle \hat{g}\right\rangle ^{2}}{\sigma_{\hat{K}}^{2}}\frac{\left\langle \hat{g}^{Bef}\right\rangle }{\left\langle \hat{g}^{ef}\right\rangle }\left(2+\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }-\sqrt{\left(1-\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }\right)\left(4-\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }\right)}\right)\frac{\left\langle \bar{K}_{0}\right\rangle }{\left\langle \hat{K}_{0}\right\rangle }\right)
\end{aligned}$$

with solution:

$$\frac{\left\langle \bar{K}_{0}\right\rangle }{\left\langle \hat{K}_{0}\right\rangle }=-\frac{\left\langle \hat{K}\right\rangle \left\langle \hat{g}\right\rangle \left\langle \hat{g}^{Bef}\right\rangle }{\left\langle \bar{g}\right\rangle ^{2}\left\langle \hat{K}_{0}\right\rangle }+\sqrt{\left(\frac{\left\langle \hat{K}\right\rangle \left\langle \hat{g}\right\rangle \left\langle \hat{g}^{Bef}\right\rangle }{\left\langle \bar{g}\right\rangle ^{2}\left\langle \hat{K}_{0}\right\rangle }\right)^{2}+2\frac{\sigma_{\hat{K}}^{2}}{\left\langle \bar{g}\right\rangle ^{2}}\frac{\left\Vert \bar{\Psi}_{0}\right\Vert ^{2}}{\hat{\mu}\left\langle \hat{K}_{0}\right\rangle ^{2}}}$$

or in first approximation:

$$\frac{\left\langle \bar{K}_{0}\right\rangle }{\left\langle \hat{K}_{0}\right\rangle }\rightarrow-\frac{\left\langle \hat{K}\right\rangle \left\langle \hat{g}\right\rangle \left\langle \hat{g}^{Bef}\right\rangle }{\left\langle \bar{g}\right\rangle ^{2}\left\langle \hat{K}_{0}\right\rangle }+\sqrt{2\frac{\sigma_{\hat{K}}^{2}}{\left\langle \bar{g}\right\rangle ^{2}}\frac{\left\Vert \bar{\Psi}_{0}\right\Vert ^{2}}{\hat{\mu}\left\langle \hat{K}_{0}\right\rangle ^{2}}} \tag{328}$$

replacing for $\left\langle \hat{K}_{0}\right\rangle$ (326) and ug (327) leads to the following expression:

$$\begin{aligned}
\frac{\left\langle \bar{K}_{0}\right\rangle }{\left\langle \hat{K}_{0}\right\rangle } &\rightarrow -\frac{\left\langle \hat{K}\right\rangle \left\langle \hat{g}\right\rangle \left\langle \hat{g}^{Bef}\right\rangle }{\left\langle \hat{K}_{0}\right\rangle \left\langle \bar{g}\right\rangle ^{2}}+\sqrt{\left(\frac{\left\langle \hat{K}\right\rangle \left\langle \hat{g}\right\rangle \left\langle \hat{g}^{Bef}\right\rangle }{\left\langle \hat{K}_{0}\right\rangle \left\langle \bar{g}\right\rangle ^{2}}\right)^{2}+2\frac{\sigma_{\hat{K}}^{2}}{\left\langle \bar{g}\right\rangle ^{2}}\frac{\left\Vert \bar{\Psi}_{0}\right\Vert ^{2}}{\hat{\mu}}}\\
&= -\frac{6\left\langle \hat{g}\right\rangle ^{2}\left\langle \hat{g}^{Bef}\right\rangle }{\left\langle \bar{g}\right\rangle ^{2}\left\langle \hat{g}^{ef}\right\rangle }\left(2+\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }-\sqrt{\left(1-\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }\right)\left(4-\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }\right)}\right)\\
&\quad +\sqrt{36\left(\frac{\left\langle \hat{g}\right\rangle ^{2}\left\langle \hat{g}^{Bef}\right\rangle }{\left\langle \bar{g}\right\rangle ^{2}\left\langle \hat{g}^{ef}\right\rangle }\left(2+\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }-\sqrt{\left(1-\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }\right)\left(4-\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }\right)}\right)\right)^{2}+2\frac{\sigma_{\hat{K}}^{2}}{\left\langle \bar{g}\right\rangle ^{2}}\frac{\left\Vert \bar{\Psi}_{0}\right\Vert ^{2}}{\hat{\mu}}}
\end{aligned}$$

**

$$\frac{\left\langle \bar{K}_{0}\right\rangle }{\left\langle \hat{K}_{0}\right\rangle }\rightarrow-\frac{3}{8}\frac{\left\langle \hat{g}\right\rangle \left\langle \hat{g}^{Bef}\right\rangle }{\left\langle \bar{g}\right\rangle ^{2}}+\sqrt{\frac{\left\langle \hat{g}\right\rangle ^{2}}{3\left\langle \bar{g}\right\rangle ^{2}}\frac{\left\Vert \bar{\Psi}_{0}\right\Vert ^{2}\left(5+\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }-\sqrt{\left(1-\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }\right)\left(4-\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }\right)}\right)}{\left\Vert \hat{\Psi}_{0}\right\Vert ^{2}}}$$

$$\rightarrow -\frac{3}{8}\frac{\left\langle \hat{g}\right\rangle \left\langle \hat{g}^{Bef}\right\rangle }{\left\langle \bar{g}\right\rangle ^{2}}+\sqrt{\frac{\left\langle \hat{g}\right\rangle ^{2}}{\left\langle \bar{g}\right\rangle ^{2}}\frac{\left\Vert \bar{\Psi}_{0}\right\Vert ^{2}\left(1+\frac{3}{4}\frac{\left\langle \hat{g}^{ef}\right\rangle }{\left\langle \hat{g}\right\rangle }\right)}{\left\Vert \hat{\Psi}_{0}\right\Vert ^{2}}}$$

A20.5.3 Computation of $\frac{\langle \bar{K}_0 \rangle}{\langle \hat{K} \rangle}$

Multiply (328) by $\frac{\langle \hat{K}_0 \rangle}{\langle \hat{K} \rangle}$:

$$\frac{\langle \bar{K}_0 \rangle}{\left\langle \hat{K} \right\rangle} = -\frac{\langle \hat{g} \rangle \langle \hat{g}^{Bef} \rangle}{\langle \bar{g} \rangle^2} + \sqrt{\left(\frac{\langle \hat{g} \rangle \langle \hat{g}^{Bef} \rangle}{\langle \bar{g} \rangle^2} \right)^2 + 2 \frac{\sigma_{\hat{K}}^2}{\langle \bar{g} \rangle^2} \frac{\left\| \bar{\Psi}_0 \right\|^2}{\hat{\mu} \left\langle \hat{K} \right\rangle^2}}$$

and replacing $\left\langle \hat{K} \right\rangle^2$ by its expressn (327) we find:

$$\frac{\langle \bar{K}_0 \rangle}{\left\langle \hat{K} \right\rangle} = -\frac{\langle \hat{g} \rangle \langle \hat{g}^{Bef} \rangle}{\langle \bar{g} \rangle^2} + \sqrt{\left(\frac{\langle \hat{g} \rangle \langle \hat{g}^{Bef} \rangle}{\langle \bar{g} \rangle^2} \right)^2 + 2 \frac{\sigma_{\hat{K}}^2}{\langle \bar{g} \rangle^2} \frac{\left\| \bar{\Psi}_0 \right\|^2}{\hat{\mu} \left\langle \hat{K} \right\rangle^2}} \simeq -\frac{\langle \hat{g} \rangle \langle \hat{g}^{Bef} \rangle}{\langle \bar{g} \rangle^2} + \sqrt{2 \frac{\sigma_{\hat{K}}^2}{\langle \bar{g} \rangle^2} \frac{\left\| \bar{\Psi}_0 \right\|^2}{\hat{\mu} \left\langle \hat{K} \right\rangle^2}} \tag{329}$$

and (328) bcms:

A20.5.4 Computation of $\frac{\langle \bar{K} \rangle \|\bar{\Psi}\|^2}{\langle \hat{K} \rangle \|\hat{\Psi}\|^2}$

We define the ratio:

$$Z = \frac{\langle \bar{K} \rangle \left\| \bar{\Psi} \right\|^2}{\left\langle \hat{K} \right\rangle \left\| \hat{\Psi} \right\|^2}$$

which satisfies:

$$Z = \frac{\langle \bar{K}_0 \rangle^4 \left(\frac{1}{4} - \frac{\langle \hat{K} \rangle}{3 \langle \bar{K}_0 \rangle} \frac{\langle \hat{g}^{Bef} \rangle}{\langle \bar{g} \rangle} \right) \langle \bar{g} \rangle^2}{\left\langle \hat{K}_0 \right\rangle^4 \left(\frac{1}{4} - \frac{\langle \hat{K} \rangle}{3 \langle \hat{K}_0 \rangle} \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right) \langle \hat{g} \rangle^2}$$

$$\frac{\left\langle \hat{K} \right\rangle}{\left\langle \hat{K}_0 \right\rangle} = \frac{1}{6 \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle}} \left(2 + \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right) \left(4 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right)} \right)$$

Leading to:

$$\frac{\left\langle \hat{K} \right\rangle}{3 \left\langle \hat{K}_0 \right\rangle} \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} = \frac{1}{18} \left(2 + \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right) \left(4 - \frac{\langle \hat{g}^{ef} \rangle}{\langle \hat{g} \rangle} \right)} \right)$$

The formula for $\langle \bar{K}_0 \rangle^2$:

$$\langle \bar{K}_0 \rangle^2 \simeq 2 \frac{\sigma_{\hat{K}}^2}{\langle \bar{g} \rangle^2} \left(\frac{\left\| \bar{\Psi}_0 \right\|^2}{\hat{\mu}} - \left(\frac{\left\langle \hat{K} \right\rangle^2 \langle \hat{g} \rangle}{\sigma_{\hat{K}}^2} \right) \langle \hat{g}^{Bef} \rangle \frac{\langle \bar{K}_0 \rangle}{\left\langle \hat{K} \right\rangle} \right)$$

allows to find $2 \frac{\sigma_{\hat{K}}^2}{\langle \bar{g} \rangle^2} \frac{\|\bar{\Psi}_0\|^2}{\hat{\mu} \langle \hat{K} \rangle^2}$:

$$2\frac{\sigma_{\hat{K}}^{2}}{\langle\bar{g}\rangle^{2}}\frac{\left\|\bar{\Psi}_{0}\right\|^{2}}{\hat{\mu}\left\langle\hat{K}\right\rangle^{2}}$$

$$\rightarrow \quad \frac{12\left\langle\hat{g}^{ef}\right\rangle^{2}\left(5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)\left\|\bar{\Psi}_{0}\right\|^{2}}{\langle\bar{g}\rangle^{2}\left(\left(2+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)\right)^{2}\left\|\hat{\Psi}_{0}\right\|^{2}} \simeq \frac{196\left\|\bar{\Psi}_{0}\right\|^{2}\langle\hat{g}\rangle^{2}}{27\left\|\hat{\Psi}_{0}\right\|^{2}\langle\bar{g}\rangle^{2}}$$

and rewrite (329) as:

$$\begin{aligned}\frac{\left\langle\bar{K}_{0}\right\rangle}{\left\langle\hat{K}\right\rangle} &= -\frac{\langle\hat{g}\rangle\left\langle\hat{g}^{Bef}\right\rangle}{\langle\bar{g}\rangle^{2}}+\sqrt{\left(\frac{\langle\hat{g}\rangle\left\langle\hat{g}^{Bef}\right\rangle}{\langle\bar{g}\rangle^{2}}\right)^{2}+2\frac{\sigma_{\hat{K}}^{2}}{\langle\bar{g}\rangle^{2}}\frac{\left\|\bar{\Psi}_{0}\right\|^{2}}{\hat{\mu}\left\langle\hat{K}\right\rangle^{2}}} \\ &\rightarrow -\frac{\langle\hat{g}\rangle\left\langle\hat{g}^{Bef}\right\rangle}{\langle\bar{g}\rangle^{2}}+\sqrt{\left(\frac{\langle\hat{g}\rangle\left\langle\hat{g}^{Bef}\right\rangle}{\langle\bar{g}\rangle^{2}}\right)^{2}+2\frac{\sigma_{\hat{K}}^{2}}{\langle\bar{g}\rangle^{2}}\frac{\left\|\bar{\Psi}_{0}\right\|^{2}}{\hat{\mu}\left\langle\hat{K}\right\rangle^{2}}} \simeq \sqrt{\frac{196\left\|\bar{\Psi}_{0}\right\|^{2}}{27\left\|\hat{\Psi}_{0}\right\|^{2}}}\frac{\langle\hat{g}\rangle}{\langle\bar{g}\rangle}-\frac{\langle\hat{g}\rangle\left\langle\hat{g}^{Bef}\right\rangle}{\langle\bar{g}\rangle^{2}}\end{aligned}$$

$$\begin{aligned}\frac{\left\langle\bar{K}_{0}\right\rangle}{\left\langle\hat{K}_{0}\right\rangle} &= -\frac{3}{8}\frac{\langle\hat{g}\rangle\left\langle\hat{g}^{Bef}\right\rangle}{\langle\bar{g}\rangle^{2}}+\sqrt{\frac{\langle\hat{g}\rangle^{2}}{3\langle\bar{g}\rangle^{2}}\frac{\left\|\bar{\Psi}_{0}\right\|^{2}\left(5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)}{\left\|\hat{\Psi}_{0}\right\|^{2}}} \\ &\simeq -\frac{3}{8}\frac{\langle\hat{g}\rangle\left\langle\hat{g}^{Bef}\right\rangle}{\langle\bar{g}\rangle^{2}}+\sqrt{\frac{\langle\hat{g}\rangle^{2}}{\langle\bar{g}\rangle^{2}}\frac{\left\|\bar{\Psi}_{0}\right\|^{2}\left(1+\frac{3}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}{\left\|\hat{\Psi}_{0}\right\|^{2}}}\end{aligned}$$

and the ratio Z becomes:

$$\begin{aligned}Z &= \frac{\left\langle\bar{K}_{0}\right\rangle^{4}\left(\frac{1}{4}-\frac{\left\langle\hat{K}\right\rangle}{3\left\langle\bar{K}_{0}\right\rangle}\frac{\left\langle\hat{g}^{Bef}\right\rangle}{\langle\bar{g}\rangle}\right)\langle\bar{g}\rangle^{2}}{\left\langle\hat{K}_{0}\right\rangle^{4}\left(\frac{1}{4}-\frac{\left\langle\hat{K}\right\rangle}{3\left\langle\hat{K}_{0}\right\rangle}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\langle\hat{g}\rangle^{2}} \\ &\rightarrow \frac{\left\langle\bar{K}_{0}\right\rangle^{4}\left(\frac{1}{4}-\frac{1}{3\left(\sqrt{\frac{196\left\|\bar{\Psi}_{0}\right\|^{2}}{27\left\|\hat{\Psi}_{0}\right\|^{2}}}\frac{\langle\hat{g}\rangle}{\langle\bar{g}\rangle}-\frac{\langle\hat{g}\rangle\left\langle\hat{g}^{Bef}\right\rangle}{\langle\bar{g}\rangle^{2}}\right)}\frac{\left\langle\hat{g}^{Bef}\right\rangle}{\langle\bar{g}\rangle}\right)\langle\bar{g}\rangle^{2}}{\left\langle\hat{K}_{0}\right\rangle^{4}\left(\frac{1}{4}-\frac{1}{18}\left(2+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)\right)\langle\hat{g}\rangle^{2}} \simeq \frac{\left\langle\bar{K}_{0}\right\rangle^{4}\left(\frac{1}{4}-\sqrt{\frac{3\left\|\hat{\Psi}_{0}\right\|^{2}}{196\left\|\bar{\Psi}_{0}\right\|^{2}}}\frac{\left\langle\hat{g}^{Bef}\right\rangle}{\langle\hat{g}\rangle}\right)\langle\bar{g}\rangle^{2}}{\left\langle\hat{K}_{0}\right\rangle^{4}\left(\frac{1}{4}-\frac{1}{18}\frac{5}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\langle\hat{g}\rangle^{2}}\end{aligned}$$

that is:

$$Z \simeq \frac{\left\langle\bar{K}_{0}\right\rangle^{4}\left(1-\frac{1}{2}\sqrt{\frac{\left\|\hat{\Psi}_{0}\right\|^{2}}{\left\|\bar{\Psi}_{0}\right\|^{2}}}\frac{\left\langle\hat{g}^{Bef}\right\rangle}{\langle\hat{g}\rangle}\right)\langle\bar{g}\rangle^{2}}{\left\langle\hat{K}_{0}\right\rangle^{4}\langle\hat{g}\rangle^{2}}$$

with expanded form:

$$Z \simeq \left(-\frac{3}{8}\frac{\left\langle \hat{g}^{Bef}\right\rangle}{\langle \bar{g}\rangle} + \sqrt{\frac{\left\|\bar{\Psi}_0\right\|^2 \left(1+\frac{3}{4}\frac{\left\langle \hat{g}^{ef}\right\rangle}{\langle \hat{g}\rangle}\right)}{\left\|\hat{\Psi}_0\right\|^2}}\right)^2 \left(1-\frac{1}{2}\sqrt{\frac{\left\|\hat{\Psi}_0\right\|^2}{\left\|\bar{\Psi}_0\right\|^2}}\frac{\left\langle \hat{g}^{Bef}\right\rangle}{\langle \hat{g}\rangle}\right)$$

$$Z = \left(1+\frac{3}{4}\frac{\left\langle \hat{g}^{ef}\right\rangle}{\langle \hat{g}\rangle}\right)\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2} - \left(\frac{3}{4}\frac{\left\langle \hat{g}^{Bef}\right\rangle}{\langle \bar{g}\rangle} + \frac{1}{2}\frac{\left\langle \hat{g}^{Bef}\right\rangle}{\langle \hat{g}\rangle}\right)\sqrt{\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}}$$

where:

$$\left\langle \hat{g}^{ef}\right\rangle = -\frac{\left(\kappa\left\langle\left[\frac{\hat{k}_2^B}{1+\underline{k}}\right]\right\rangle\left(1-\left\langle\hat{\underline{k}}\right\rangle\right)+\left\langle\hat{k}_1^B\right\rangle\left\langle\hat{k}_2\right\rangle\right)\left(\langle\hat{g}\rangle+\frac{1}{1-\left\langle\hat{\underline{k}}\right\rangle}\bar{N}\langle\bar{g}\rangle\right)\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}}{\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}\right)\right)\left(1-\left(\left\langle\hat{\underline{k}}\right\rangle+\left(\left\langle\hat{\underline{k}}_1^B\right\rangle+\kappa\left\langle\left[\frac{\hat{k}_2^B}{1+\underline{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}\right)\right)}$$

and:

$$\begin{aligned}\left\langle \hat{g}^{Bef}\right\rangle &= -\frac{\kappa\left\langle\left[\frac{\hat{k}_2^B}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\hat{k}\right\rangle\right)+\left\langle\hat{k}_1^B\right\rangle\left\langle\hat{k}_2\right\rangle}{\left(1-\left(\left\langle\hat{k}\right\rangle+\left(\left\langle\hat{k}_1^B\right\rangle+\kappa\left\langle\left[\frac{\hat{k}_2^B}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}\right)\right)\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}\right)\right)}\\ &\quad\times\left(\langle\hat{g}\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\langle\bar{g}\rangle\right)\end{aligned}$$

We thus obtain:

$$\left\langle\bar{K}_0\right\rangle^2 \simeq 2\frac{\sigma_{\hat{K}}^2}{\langle\bar{g}\rangle^2}\left(\frac{\left\|\bar{\Psi}_0\right\|^2}{\hat{\mu}} - \left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}\right)\left\langle\hat{g}^{Bef}\right\rangle\frac{\left\langle\bar{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}\right)$$

and:

$$\begin{aligned}\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2 &= \hat{\mu}V\frac{\left\langle\hat{K}_0\right\rangle^4}{2\sigma_{\hat{K}}^2}\left(\frac{1}{4}-\frac{\left\langle\hat{K}\right\rangle}{3\left\langle\hat{K}_0\right\rangle}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\langle\hat{g}\rangle^2\\ &= \frac{18\sigma_{\hat{K}}^2}{\hat{\mu}}V\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\langle\hat{g}\rangle^2\left(5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)}\right)^2\\ &\quad\times\left(\frac{1}{4}-\frac{1}{18}\left(2+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)\right)\langle\hat{g}\rangle^2\end{aligned}$$

A20.6 Expression for field and capital per sector

We use the formula (318) and (319):

$$\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2 \simeq \hat{\mu}\frac{\hat{K}_0^3}{\sigma_{\hat{K}}^2}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{3}-\frac{\left\langle\hat{K}\right\rangle}{2\hat{K}_0}\langle\hat{g}\rangle\left\langle\hat{g}^{ef}\right\rangle\right) \tag{330}$$

and:

$$\left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2 \simeq \hat{\mu}\frac{\bar{K}_0^3}{\sigma_{\hat{K}}^2}\left(\frac{\bar{g}^2\left(\bar{X}_1\right)}{3}-\frac{\left\langle\hat{K}\right\rangle}{2\bar{K}_0}\left\langle\hat{g}\right\rangle\left\langle\hat{g}^{Bef}\right\rangle\right)$$

We also use our previous formula for capital (320) and (321):

$$\hat{K}_{\hat{X}}\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2 \simeq \hat{\mu}\frac{\hat{K}_0^4}{2\sigma_{\hat{K}}^2}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\left\langle\hat{K}\right\rangle\left\langle\hat{g}\right\rangle}{3\left\langle\hat{K}_0\right\rangle}\left\langle\hat{g}^{ef}\right\rangle\right) \tag{331}$$

and:

$$\bar{K}_{\bar{X}}\left\|\bar{\Psi}\left(\bar{X}_1\right)\right\|^2 \simeq \hat{\mu}\frac{\bar{K}_0^4}{2\sigma_{\hat{K}}^2}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\left\langle\hat{K}\right\rangle\left\langle\hat{g}\right\rangle}{3\left\langle\bar{K}_0\right\rangle}\left\langle\hat{g}^{Bef}\right\rangle\right) \tag{332}$$

A20.7 Formula for $\hat{K}_0^2$ and $\bar{K}_0^2$

$$\hat{K}_0^2 \simeq 2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}}-\left(\frac{\left\langle\hat{K}\right\rangle^2\left\langle\hat{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{ef}\right\rangle\frac{\left\langle\hat{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}\right) \tag{333}$$

$$\bar{K}_0^2 \simeq 2\frac{\sigma_{\hat{K}}^2}{\bar{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2}{\hat{\mu}}-\left(\frac{\left\langle\hat{K}\right\rangle^2\left\langle\hat{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{Bef}\right\rangle\frac{\left\langle\bar{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}\right)$$

A20.8 Including deviation from averages in functional derivative

Using the previous results for averages (326) and ug (327) and including the deviation to averages for the functional derivatives:

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)=\Delta\left(\hat{k}^B\left(\hat{X},\left\langle\bar{X}\right\rangle\right)A\right)\frac{\hat{K}}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}$$

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)=\left(\left\langle\underline{\hat{k}}^B\right\rangle^{ef}\Delta\left(\hat{k}^B\left(\hat{X},\left\langle\bar{X}\right\rangle\right)A\right)+\frac{\Delta\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}\right)}{\left(1-\left\langle\bar{k}_1\right\rangle\right)\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\left\langle\bar{g}\right\rangle\right)\frac{\bar{K}}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}$$

and using, including normalizations:

$$\left\langle\underline{\hat{k}}^B\right\rangle^{ef}=\left\langle\underline{\hat{k}}_1^B\right\rangle+\frac{\left\langle\underline{\hat{k}}_1^B\right\rangle}{\left(1-\left\langle\hat{k}_1\right\rangle\right)}$$

A20.9 Expressions for fields

The solutions for the fields including the deviations becomes:

$$\begin{aligned}\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2 \quad = \quad & \left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}\left\{\left(\frac{\hat{K}_1^2\hat{g}^2\left(\hat{X}_1\right)}{2\sigma_{\hat{K}}^2}+\frac{\hat{g}\left(\hat{X}_1\right)}{2}\right)\right.\\ & \left.+\left(\frac{\left\langle\bar{K}\right\rangle^2\left\langle\bar{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)\frac{\left\|\bar{\Psi}\right\|^2\hat{K}_1}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}-\left(\frac{\left\langle\hat{K}\right\rangle^2\left\langle\hat{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{ef}\right\rangle\frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle}\right\}\end{aligned}$$

$$\left|\bar{\Psi}\left(\bar{K}_1,\bar{X}_1\right)\right|^2 = \left|\bar{\Psi}_0\left(\bar{X}_1\right)\right|^2-\hat{\mu}\left\{\left(\frac{\bar{K}_1^2\bar{g}^2\left(\bar{X}_1\right)}{\sigma_{\hat{K}}^2}+\frac{\bar{g}\left(\bar{X}_1\right)}{2}\right)\right.$$

$$+\left(\frac{\left\langle\bar{K}\right\rangle^2\left\langle\bar{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left(\left\langle\underline{\hat{k}}^B\right\rangle^{ef}\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)+\frac{\Delta\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}\right)}{\left(1-\left\langle\bar{k}_1\right\rangle\right)\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\left\langle\bar{g}\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\bar{K}_1}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}$$

$$\left.-\left(\frac{\left\langle\hat{K}\right\rangle^2\left\langle\hat{g}\right\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{Bef}\right\rangle\frac{\bar{K}_1}{\left\langle\hat{K}\right\rangle}\right\}$$

and integratng:

$$\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2 \simeq \hat{\mu}\frac{\hat{K}_0^3}{\sigma_{\hat{K}}^2}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{3}+\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)Z\frac{\left\langle\bar{K}\right\rangle}{2\hat{K}_0}\left\langle\bar{g}\right\rangle-\frac{\left\langle\hat{K}\right\rangle}{2\hat{K}_0}\left\langle\hat{g}\right\rangle\left\langle\hat{g}^{ef}\right\rangle\right) \tag{334}$$

$$\simeq \hat{\mu}\frac{\hat{K}_0^3}{\sigma_{\hat{K}}^2}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{3}-\frac{\left\langle\hat{K}\right\rangle}{2\hat{K}_0}\left\langle\hat{g}\right\rangle\left\langle\hat{g}^{ef}\right\rangle\right)$$

similarly:

$$\left\|\bar{\Psi}\left(\bar{X}_1\right)\right\|^2$$

$$\simeq \hat{\mu}\frac{\bar{K}_0^3}{\sigma_{\hat{K}}^2}\left(\frac{\bar{g}^2\left(\bar{X}_1\right)}{3}+\left(\left\langle\underline{\hat{k}}^B\right\rangle^{ef}\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)+\frac{\Delta\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}_1\right)}{\left(1-\left\langle\bar{k}_1\right\rangle\right)\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\left\langle\bar{g}\right\rangle\right)\frac{\left\langle\bar{K}\right\rangle}{2\bar{K}_0}\left\langle\bar{g}\right\rangle-\frac{\left\langle\hat{K}\right\rangle}{2\bar{K}_0}\left\langle\hat{g}\right\rangle\left\langle\hat{g}^{Bef}\right\rangle\right)$$

$$\simeq \hat{\mu}\frac{\bar{K}_0^3}{\sigma_{\hat{K}}^2}\left(\frac{\bar{g}^2\left(\bar{X}_1\right)}{3}-\frac{\left\langle\hat{K}\right\rangle}{2\bar{K}_0}\left\langle\hat{g}\right\rangle\hat{g}^{Bef}\left(\bar{X}_1\right)\right)$$

A20.10 Exprssion for capital per sector

multiply $\left|\bar{\Psi}\left(\bar{K}_1,\bar{X}_1\right)\right|^2$ by $\bar{K}_1$ and integrate:

$$\hat{K}_{\hat{X}}\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2 \simeq \hat{\mu}\frac{\hat{K}_0^4}{2\sigma_{\hat{K}}^2}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}+\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)Z\frac{\left\langle\bar{K}\right\rangle}{3\hat{K}_0}\left\langle\bar{g}\right\rangle-\frac{\left\langle\hat{K}\right\rangle\left\langle\hat{g}\right\rangle}{3\left\langle\hat{K}_0\right\rangle}\left\langle\hat{g}^{ef}\right\rangle\right) \tag{335}$$

$$\simeq \hat{\mu}\frac{\hat{K}_0^4}{2\sigma_{\hat{K}}^2}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\left\langle\hat{K}\right\rangle}{3\bar{K}_0}\left\langle\hat{g}\right\rangle\hat{g}^{Bef}\left(\hat{X}_1\right)\right)$$

we also have:

$$\bar{K}_{\bar{X}}\left\|\bar{\Psi}\left(\bar{X}_1\right)\right\|^2 \tag{336}$$

$$\simeq \hat{\mu}\frac{\bar{K}_0^4}{2\sigma_{\hat{K}}^2}\left(\frac{\bar{g}^2\left(\hat{X}_1\right)}{4}+\left(\left\langle\underline{\hat{k}}^B\right\rangle^{ef}\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)+\frac{\Delta\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}_1\right)}{\left(1-\left\langle\bar{k}_1\right\rangle\right)\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\left\langle\bar{g}\right\rangle\right)\frac{\left\langle\bar{K}\right\rangle}{3\bar{K}_0}\left\langle\bar{g}\right\rangle-\frac{\left\langle\hat{K}\right\rangle\left\langle\hat{g}\right\rangle}{3\left\langle\bar{K}_0\right\rangle}\left\langle\hat{g}^{Bef}\right\rangle\right)$$

$$\simeq \hat{\mu}\frac{\bar{K}_0^4}{2\sigma_{\hat{K}}^2}\left(\frac{\bar{g}^2\left(\hat{X}_1\right)}{4}-\frac{\left\langle\hat{K}\right\rangle\left\langle\hat{g}\right\rangle}{3\left\langle\bar{K}_0\right\rangle}\hat{g}^{Bef}\left(\bar{X}_1\right)\right)$$

where:

$$\hat{K}_0^2 \simeq 2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}}+\left(\frac{\left\langle\bar{K}\right\rangle^2\langle\bar{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)Z\langle\bar{g}\rangle-\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{ef}\right\rangle\frac{\left\langle\hat{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}\right) \tag{337}$$

and:

$$\begin{aligned}\bar{K}_0^2 \quad\simeq\quad & 2\frac{\sigma_{\hat{K}}^2}{\bar{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2}{\hat{\mu}}+\left(\frac{\left\langle\bar{K}\right\rangle^2\langle\bar{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left(\left\langle\underline{\hat{k}}^B\right\rangle^{ef}\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)+\frac{\Delta\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}_1\right)}{\left(1-\left\langle\bar{k}_1\right\rangle\right)\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\langle\bar{g}\rangle\right)Z\langle\bar{g}\rangle\right.\\ & \left.-\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{Bef}\right\rangle\frac{\left\langle\bar{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}\right)\end{aligned}$$

These formula can be written depending on some parameters:

$$\begin{aligned}\hat{K}\left[\hat{X}_1\right] &= \hat{K}_{\hat{X}}\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2 \simeq \hat{\mu}\frac{\hat{K}_0^4}{2\sigma_{\hat{K}}^2}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}+\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)Z\frac{\left\langle\bar{K}\right\rangle}{3\hat{K}_0}\langle\bar{g}\rangle-\frac{\left\langle\hat{K}\right\rangle\langle\hat{g}\rangle}{3\hat{K}_0}\left\langle\hat{g}^{ef}\right\rangle\right)\\ &= \frac{\hat{\mu}}{2\sigma_{\hat{K}}^2}\left(2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}}-\hat{D}\left(\hat{X}_1\right)\right)\right)^2\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\langle\hat{g}\rangle\,\hat{g}^{ef}\left(\hat{X}_1\right)}{3}\right)\end{aligned}$$

and $\hat{g}^{ef}\left(\hat{X}_1\right)$ and $\hat{D}\left(\hat{X}_1\right)$ defined b:

$$\hat{g}^{ef}\left(\hat{X}_1\right)=4\left(\frac{\left\langle\hat{K}\right\rangle}{3\left\langle\hat{K}_0\right\rangle}\left\langle\hat{g}^{ef}\right\rangle-\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)Z\frac{\left\langle\bar{K}\right\rangle}{3\hat{K}_0}\frac{\langle\bar{g}\rangle}{\langle\hat{g}\rangle}\right)$$

$$\hat{D}\left(\hat{X}_1\right)=\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\frac{\hat{K}_0}{\left\langle\hat{K}\right\rangle}\left\langle\hat{g}^{ef}\right\rangle-\left(\frac{\left\langle\bar{K}\right\rangle^2\langle\bar{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)Z\langle\bar{g}\rangle$$

Similarly:

$$\begin{aligned}& \bar{K}_{\bar{X}_1}\left\|\bar{\Psi}\left(\bar{X}_1\right)\right\|^2\\ \simeq\quad & \hat{\mu}\frac{\bar{K}_0^4}{2\sigma_{\hat{K}}^2}\left(\frac{\bar{g}^2\left(\hat{X}_1\right)}{4}+\left(\left\langle\underline{\hat{k}}^B\right\rangle^{ef}\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)+\frac{\Delta\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}_1\right)}{\left(1-\left\langle\bar{k}_1\right\rangle\right)\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\langle\bar{g}\rangle\right)\frac{\left\langle\bar{K}\right\rangle}{3\bar{K}_0}\langle\bar{g}\rangle-\frac{\left\langle\hat{K}\right\rangle\langle\hat{g}\rangle}{3\bar{K}_0}\left\langle\hat{g}^{Bef}\right\rangle\right)\\ =\quad & \frac{\hat{\mu}}{2\sigma_{\hat{K}}^2}\left(2\frac{\sigma_{\hat{K}}^2}{\bar{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2}{\hat{\mu}}-\bar{D}\left(\bar{X}_1\right)\right)\right)^2\\ & \times\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\left\langle\hat{K}\right\rangle\langle\hat{g}\rangle}{3\bar{K}_0}\left\langle\hat{g}^{Bef}\right\rangle+\left(\left\langle\underline{\hat{k}}^B\right\rangle^{ef}\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)+\frac{\Delta\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}_1\right)}{\left(1-\left\langle\bar{k}_1\right\rangle\right)\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\langle\bar{g}\rangle\right)\frac{\left\langle\bar{K}\right\rangle}{3\bar{K}_0}\langle\bar{g}\rangle\right)\end{aligned}$$

where:

$$\begin{aligned}\bar{D}\left(\bar{X}_1\right) \quad=\quad & \left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left\langle\hat{g}^{Bef}\right\rangle\frac{\bar{K}_0}{\left\langle\hat{K}\right\rangle}\\ & -\left(\frac{\left\langle\bar{K}\right\rangle^2\langle\bar{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\left(\left\langle\underline{\hat{k}}^B\right\rangle^{ef}\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)+\frac{\Delta\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}_1\right)}{\left(1-\left\langle\bar{k}_1\right\rangle\right)\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\langle\bar{g}\rangle\right)Z\langle\bar{g}\rangle\end{aligned}$$

A20.11 Compact expressions for field and capital

A20.11.1 Investors

We use the previous formula for $\hat{K}_0^2$, $\left\langle \hat{K} \right\rangle^2$ to compute $\hat{K}\left[\hat{X}_1\right]$ the amount of capital in sector $\hat{X}_1$. We found:

$$\hat{K}_0^2 = 6\frac{\sigma_{\hat{K}}^2}{\hat{\mu}} \frac{\left\| \hat{\Psi}_0 \left(\hat{X}_1 \right) \right\|^2}{\hat{g}^2\left(\bar{X}_1\right)\left(5 + \frac{\left\langle \hat{g}^{ef} \right\rangle}{\hat{g}\left(\bar{X}_1\right)} - \sqrt{\left(1 - \frac{\hat{g}^{ef}\left(\bar{X}_1\right)}{\hat{g}\left(\bar{X}_1\right)}\right)\left(4 - \frac{\hat{g}^{ef}\left(\bar{X}_1\right)}{\hat{g}\left(\bar{X}_1\right)}\right)}\right)}$$

$$\begin{aligned}
\left\langle \hat{K} \right\rangle^2 &= \left(\frac{\langle \hat{g} \rangle}{6\left\langle \hat{g}^{ef} \right\rangle} \left(2 + \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)\left(4 - \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)} \right) \right)^2 \left\langle \hat{K}_0 \right\rangle^2 \\
&\rightarrow \frac{1}{6}\frac{\sigma_{\hat{K}}^2}{\hat{\mu}} \frac{\left(\left(2 + \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)\left(4 - \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)} \right) \right)^2 \left\| \hat{\Psi}_0 \right\|^2}{\left\langle \hat{g}^{ef} \right\rangle^2 \left(5 + \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)\left(4 - \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)} \right)}
\end{aligned}$$

so that the formula for $\hat{K}\left[\hat{X}_1\right]$ becomes:

$$\hat{K}\left[\hat{X}_1\right] = \hat{K}_{\hat{X}} \left\| \hat{\Psi}\left(\hat{X}_1\right) \right\|^2 \simeq \hat{\mu} \frac{\hat{K}_0^4}{2\sigma_{\hat{K}}^2} \left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4} + \Delta\left(\hat{k}^B\left(\hat{X}_1, \langle \bar{X} \rangle\right) A\right) Z \frac{\langle \bar{K} \rangle}{3\hat{K}_0} \langle \bar{g} \rangle - \frac{\left\langle \hat{K} \right\rangle \langle \hat{g} \rangle}{3\hat{K}_0} \left\langle \hat{g}^{ef} \right\rangle \right) \tag{338}$$

$$\begin{aligned}
&= \frac{\hat{\mu}}{2\sigma_{\hat{K}}^2} \left(6\frac{\sigma_{\hat{K}}^2}{\hat{\mu}} \frac{\left\| \hat{\Psi}_0 \left(\hat{X}_1 \right) \right\|^2}{\hat{g}^2\left(\bar{X}_1\right)\left(5 + \frac{\hat{g}^{ef}\left(\bar{X}_1\right)}{\hat{g}\left(\bar{X}_1\right)} - \sqrt{\left(1 - \frac{\hat{g}^{ef}\left(\bar{X}_1\right)}{\hat{g}\left(\bar{X}_1\right)}\right)\left(4 - \frac{\hat{g}^{ef}\left(\bar{X}_1\right)}{\hat{g}\left(\bar{X}_1\right)}\right)}\right)} \right)^2 \left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4} - \frac{\langle \hat{g} \rangle \hat{g}^{ef}\left(\hat{X}_1\right)}{3} \right) \\
&\simeq \frac{\hat{\mu}}{2\sigma_{\hat{K}}^2} \left(6\frac{\sigma_{\hat{K}}^2}{\hat{\mu}} \frac{\left\| \hat{\Psi}_0 \left(\hat{X}_1 \right) \right\|^2}{\left(5 + \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)\left(4 - \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)}\right)} \right)^2 \left(\frac{1}{4\hat{g}^2\left(\bar{X}_1\right)} - \frac{\langle \hat{g} \rangle \hat{g}^{ef}\left(\hat{X}_1\right)}{3\hat{g}^4\left(\bar{X}_1\right)} \right)
\end{aligned} \tag{339}$$

and the associated field is:

$$\left\| \hat{\Psi}\left(\hat{X}_1\right) \right\|^2 = \frac{\hat{\mu}}{2\sigma_{\hat{K}}^2} \left(6\frac{\sigma_{\hat{K}}^2}{\hat{\mu}} \frac{\left\| \hat{\Psi}_0 \left(\hat{X}_1 \right) \right\|^2}{\left(5 + \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)\left(4 - \frac{\left\langle \hat{g}^{ef} \right\rangle}{\langle \hat{g} \rangle}\right)}\right)} \right)^{\frac{3}{2}} \left(\frac{1}{3\hat{g}^2\left(\bar{X}_1\right)} - \frac{\langle \hat{g} \rangle \hat{g}^{ef}\left(\hat{X}_1\right)}{2\hat{g}^4\left(\bar{X}_1\right)} \right)$$

with:

$$\begin{aligned}
\hat{g}^{ef}\left(\hat{X}_1\right) &= \frac{\left\langle \hat{K} \right\rangle}{\left\langle \hat{K}_0 \right\rangle} \hat{g}^{ef}\left(\hat{X}_1\right) \\
&= \frac{\langle \hat{g} \rangle}{6} \left(2 + \frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle \hat{g} \rangle} - \sqrt{\left(1 - \frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle \hat{g} \rangle}\right)\left(4 - \frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle \hat{g} \rangle}\right)} \right) \simeq \frac{3}{8}\hat{g}^{ef}\left(\hat{X}_1\right)
\end{aligned}$$

Then, replacing:

$$\frac{\hat{\mu}}{18\sigma_{\hat{K}}^2}\frac{\hat{K}\left[\hat{X}_1\right]}{\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\left(5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)}\right)^2}=\left(\frac{1}{4\hat{g}^2\left(\bar{X}_1\right)}-\frac{\langle\hat{g}\rangle\,\hat{g}^{ef}\left(\hat{X}_1\right)}{3\hat{g}^4\left(\bar{X}_1\right)}\right)$$

leads to:

$$\hat{g}\left(\hat{X}_1\right)\simeq\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\left(5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)\sqrt{\frac{2\hat{\mu}}{9\sigma_{\hat{K}}^2}\hat{K}\left[\hat{X}_1\right]+\frac{4\hat{g}^{ef}\left(\hat{X}_1\right)}{3\langle\hat{g}\rangle^3}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\left(5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)}\right)^2}}\tag{340}$$

A20.11.2 Banks

Similarly the value of $\bar{K}\left[\bar{X}_1\right]$:

$$\bar{K}\left[\bar{X}_1\right]=\bar{K}_{\bar{X}_1}\left\|\bar{\Psi}\left(\bar{X}_1\right)\right\|^2\simeq\hat{\mu}\frac{\bar{K}_0^4}{2\sigma_{\hat{K}}^2}\left(\frac{\bar{g}^2\left(\bar{X}_1\right)}{4}-\frac{\left\langle\hat{K}\right\rangle\langle\hat{g}\rangle}{3\left\langle\bar{K}_0\right\rangle}\hat{g}^{Bef}\left(\bar{X}_1\right)\right)$$

is writen by taking into account:

$$\begin{aligned}\frac{\bar{K}_0}{\left\langle\hat{K}\right\rangle}&=-\frac{\langle\hat{g}\rangle\,\hat{g}^{Bef}\left(\bar{X}_1\right)}{\bar{g}^2\left(\bar{X}_1\right)}+\sqrt{\left(\frac{\langle\hat{g}\rangle\,\hat{g}^{Bef}\left(\bar{X}_1\right)}{\bar{g}^2\left(\bar{X}_1\right)}\right)^2+2\frac{\sigma_{\hat{K}}^2}{\bar{g}^2\left(\bar{X}_1\right)}\frac{\left\|\bar{\Psi}_0\right\|^2}{\hat{\mu}\left\langle\hat{K}\right\rangle^2}}\\&\rightarrow-\frac{\langle\hat{g}\rangle\,\hat{g}^{Bef}\left(\bar{X}_1\right)}{\bar{g}^2\left(\bar{X}_1\right)}+\sqrt{\left(\frac{\langle\hat{g}\rangle\,\hat{g}^{Bef}\left(\bar{X}_1\right)}{\bar{g}^2\left(\bar{X}_1\right)}\right)^2+2\frac{\sigma_{\hat{K}}^2}{\bar{g}^2\left(\bar{X}_1\right)}\frac{\left\|\bar{\Psi}_0\right\|^2}{\hat{\mu}\left\langle\hat{K}\right\rangle^2}}\simeq\sqrt{\frac{196\left\|\bar{\Psi}_0\right\|^2}{27\left\|\hat{\Psi}_0\right\|^2}\frac{\langle\hat{g}\rangle}{\bar{g}^2\left(\bar{X}_1\right)}}-\frac{\langle\hat{g}\rangle\,\hat{g}^{Bef}\left(\bar{X}_1\right)}{\bar{g}^2\left(\bar{X}_1\right)}\end{aligned}$$

and:

$$\hat{g}^{Bef}\left(\bar{X}_1\right)\rightarrow\frac{\left\langle\hat{K}\right\rangle}{\left\langle\bar{K}_0\right\rangle}\hat{g}^{Bef}\left(\bar{X}_1\right)=\frac{\hat{g}^{Bef}\left(\bar{X}_1\right)}{\sqrt{\frac{196\left\|\bar{\Psi}_0\right\|^2}{27\left\|\hat{\Psi}_0\right\|^2}\frac{\langle\hat{g}\rangle}{\bar{g}^2\left(\bar{X}_1\right)}}-\frac{\langle\hat{g}\rangle\hat{g}^{Bef}\left(\bar{X}_1\right)}{\bar{g}^2\left(\bar{X}_1\right)}}$$

with:

$$\begin{aligned}\frac{\bar{K}_0}{\left\langle\hat{K}_0\right\rangle}&\rightarrow-\frac{3}{8}\frac{\langle\hat{g}\rangle\,\hat{g}^{Bef}\left(\bar{X}_1\right)}{\bar{g}^2\left(\bar{X}_1\right)}+\sqrt{\frac{\langle\hat{g}\rangle^2}{3\bar{g}^2\left(\bar{X}_1\right)}\frac{\left\|\bar{\Psi}_0\right\|^2\left(5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)}{\left\|\hat{\Psi}_0\right\|^2}}\\&\rightarrow-\frac{3}{8}\frac{\langle\hat{g}\rangle\,\hat{g}^{Bef}\left(\bar{X}_1\right)}{\bar{g}^2\left(\bar{X}_1\right)}+\sqrt{\frac{\langle\hat{g}\rangle^2}{\bar{g}^2\left(\bar{X}_1\right)}\frac{\left\|\bar{\Psi}_0\right\|^2\left(1+\frac{3}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}{\left\|\hat{\Psi}_0\right\|^2}}\end{aligned}$$

Consequently, we find ultimately:

$$\bar{K}_{\bar{X}_1}\left\|\bar{\Psi}\left(\bar{X}_1\right)\right\|^2 \simeq \frac{\hat{\mu}}{2\sigma_{\hat{K}}^2}\left(\sqrt{\frac{\langle\hat{g}\rangle^2}{\bar{g}^2\left(\bar{X}_1\right)}\frac{\left\|\bar{\Psi}_0\left(\hat{X}_1\right)\right\|^2\left(1+\frac{3}{4}\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}{\left\|\hat{\Psi}_0\left(\bar{X}_1\right)\right\|^2}}-\frac{3}{8}\frac{\langle\hat{g}\rangle\,\hat{g}^{Bef}\left(\bar{X}_1\right)}{\bar{g}^2\left(\bar{X}_1\right)}\right)^4 \tag{341}$$

$$\times\left(6\frac{\sigma_{\hat{K}}^2}{\hat{\mu}}\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\left(5+\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\right)}\right)^2\left(\frac{\bar{g}^2\left(\bar{X}_1\right)}{4}-\frac{\langle\hat{g}\rangle}{3}\hat{g}^{Bef}\left(\bar{X}_1\right)\right)$$

$$= 18\frac{\sigma_{\hat{K}}^2}{\hat{\mu}}\left(\frac{\sqrt{\langle\hat{g}\rangle^2\left(1+\frac{3}{4}\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\left\|\bar{\Psi}_0\left(\hat{X}_1\right)\right\|-\frac{3}{8}\langle\hat{g}\rangle\frac{\hat{g}^{Bef}\left(\bar{X}_1\right)}{\bar{g}\left(\bar{X}_1\right)}\left\|\hat{\Psi}_0\left(\bar{X}_1\right)\right\|}{\sqrt{5+\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}}}\right)^4$$

$$\times\left(\frac{1}{4\bar{g}^2\left(\bar{X}_1\right)}-\frac{\langle\hat{g}\rangle\,\hat{g}^{Bef}\left(\bar{X}_1\right)}{3\bar{g}^4\left(\bar{X}_1\right)}\right)$$

and the field:

$$\left\|\bar{\Psi}\left(\bar{X}_1\right)\right\|^2 \simeq 18\frac{\sigma_{\hat{K}}^2}{\hat{\mu}}\left(\frac{\sqrt{\langle\hat{g}\rangle^2\left(1+\frac{3}{4}\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\left\|\bar{\Psi}_0\left(\hat{X}_1\right)\right\|-\frac{3}{8}\langle\hat{g}\rangle\frac{\hat{g}^{Bef}\left(\bar{X}_1\right)}{\bar{g}\left(\bar{X}_1\right)}\left\|\hat{\Psi}_0\left(\bar{X}_1\right)\right\|}{\sqrt{5+\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}}}\right)^4\left(\frac{1}{3\bar{g}^2\left(\bar{X}_1\right)}-\frac{\langle\hat{g}\rangle\,\hat{g}^{Bef}\left(\bar{X}_1\right)}{2\bar{g}^4\left(\bar{X}_1\right)}\right)$$

allowing to express the return:

$$\bar{g}\left(\bar{X}_1\right) \simeq \frac{\left(\sqrt{\langle\hat{g}\rangle^2\left(1+\frac{3}{4}\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\left\|\bar{\Psi}_0\left(\hat{X}_1\right)\right\|-\frac{3}{8}\langle\hat{g}\rangle\frac{\hat{g}^{Bef}\left(\bar{X}_1\right)}{\langle\hat{g}\rangle}\left\|\hat{\Psi}_0\left(\bar{X}_1\right)\right\|\right)^2}{\left(5+\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\right)^2\sqrt{\frac{2\hat{\mu}}{9\sigma_{\hat{K}}^2}\hat{K}\left[\hat{X}_1\right]+\frac{4\langle\hat{g}\rangle\hat{g}^{ef}\left(\bar{X}_1\right)}{3\langle\hat{g}\rangle^4}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\left(5+\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\right)}\right)^2}} \tag{342}$$

Finally, we also need the ratio $Z\left(\bar{X}_1\right)$:

$$Z\left(\bar{X}_1\right)=\frac{\bar{K}_{\bar{X}}\left\|\bar{\Psi}\left(\bar{X}_1\right)\right\|^2}{\hat{K}_{\hat{X}}\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2}\simeq\frac{\left(\frac{1}{\bar{g}^2\left(\bar{X}_1\right)}\left(\frac{\left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2}{\hat{\mu}}-\bar{D}\left(\bar{X}_1\right)\right)\right)^2\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\langle\hat{K}\rangle\langle\hat{g}\rangle}{3\langle\hat{K}_0\rangle}\langle\hat{g}^{Bef}\rangle\right)}{\left(\frac{1}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}}-\hat{D}\left(\hat{X}_1\right)\right)\right)^2\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\langle\hat{g}\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{3}\right)}$$

that can be detailed using again:

$$\frac{\bar{K}_0}{\hat{K}_0}$$

$$\rightarrow -\frac{3}{8}\frac{\hat{g}\left(\hat{X}_1\right)\left\langle\hat{g}^{Bef}\right\rangle}{\bar{g}^2\left(\bar{X}_1\right)}+\sqrt{\frac{\hat{g}^2\left(\hat{X}_1\right)}{3\bar{g}^2\left(\bar{X}_1\right)}\frac{\left\|\bar{\Psi}_0\right\|^2\left(5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}\right)}\right)}{\left\|\hat{\Psi}_0\right\|^2}}$$

$$\rightarrow -\frac{3}{8}\frac{\hat{g}\left(\hat{X}_1\right)\left\langle\hat{g}^{Bef}\right\rangle}{\bar{g}^2\left(\bar{X}_1\right)}+\sqrt{\frac{\hat{g}^2\left(\hat{X}_1\right)}{\bar{g}^2\left(\bar{X}_1\right)}\frac{\left\|\bar{\Psi}_0\right\|^2\left(1+\frac{3}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}\right)}{\left\|\hat{\Psi}_0\right\|^2}}$$

$$Z = \frac{\left\langle\bar{K}_0\right\rangle^4\left(\frac{1}{4}-\frac{\left\langle\hat{K}\right\rangle}{3\bar{K}_0}\frac{\left\langle\hat{g}^{Bef}\right\rangle}{\bar{g}\left(\bar{X}_1\right)}\right)\left\langle\bar{g}\right\rangle^2}{\left\langle\hat{K}_0\right\rangle^4\left(\frac{1}{4}-\frac{\left\langle\hat{K}\right\rangle}{3\hat{K}_0}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}\right)\left\langle\hat{g}\right\rangle^2}$$

$$\rightarrow \frac{\bar{K}_0^4\left(\frac{1}{4}-\frac{1}{3\left(\sqrt{\frac{196\left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2}{27\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}\frac{\hat{g}\left(\hat{X}_1\right)}{\bar{g}\left(\bar{X}_1\right)}}-\frac{\hat{g}\left(\hat{X}_1\right)\left\langle\hat{g}^{Bef}\right\rangle}{\bar{g}^2\left(\bar{X}_1\right)}\right)}\frac{\left\langle\hat{g}^{Bef}\right\rangle}{\bar{g}\left(\bar{X}_1\right)}\right)\bar{g}^2\left(\bar{X}_1\right)}{\hat{K}_0^4\left(\frac{1}{4}-\frac{1}{18}\left(2+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}\right)}\right)\right)\hat{g}^2\left(\hat{X}_1\right)} \simeq \frac{\bar{K}_0^4\left(\frac{1}{4}-\sqrt{\frac{3\left\|\hat{\Psi}_0\right\|^2}{196\left\|\bar{\Psi}_0\right\|^2}}\frac{\left\langle\hat{g}^{Bef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}\right)\bar{g}^2\left(\bar{X}_1\right)}{\hat{K}_0^4\left(\frac{1}{4}-\frac{1}{18}\frac{5}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}\right)\hat{g}^2\left(\hat{X}_1\right)}$$

This simplifies in frst apprpximation:

$$Z\left(\bar{X}_1\right)\simeq\left(1+\frac{3}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}\right)\frac{\left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2}{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}-\left(\frac{3}{4}\frac{\left\langle\hat{g}^{Bef}\right\rangle}{\bar{g}\left(\bar{X}_1\right)}+\frac{1}{2}\frac{\left\langle\hat{g}^{Bef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}\right)\sqrt{\frac{\left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}}$$

A20.11.3 Alternate expression for field and capital

We present the forms that will be useful to derive the capital equation:

$$\begin{aligned}\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2 &\simeq \hat{\mu}\frac{\hat{K}_0^3}{\sigma_{\hat{K}}^2}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{3}+\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)Z\frac{\left\langle\bar{K}\right\rangle}{2\hat{K}_0}\left\langle\bar{g}\right\rangle-\frac{\left\langle\hat{K}\right\rangle}{2\hat{K}_0}\left\langle\hat{g}\right\rangle\left\langle\hat{g}^{ef}\right\rangle\right) \\ &= \frac{\hat{\mu}}{\sigma_{\hat{K}}^2}\left(2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}}-D\left(\hat{X}_1\right)\right)\right)^{\frac{3}{2}}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\left\langle\hat{g}\right\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{4}\right)\end{aligned} \tag{343}$$

$$\begin{aligned}\hat{K}\left[\hat{X}_1\right] &= \hat{K}_{\hat{X}}\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2\simeq\hat{\mu}\frac{\hat{K}_0^4}{2\sigma_{\hat{K}}^2}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\left\langle\hat{g}\right\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{4}\right) \\ &= \frac{\hat{\mu}}{2\sigma_{\hat{K}}^2}\left(2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}}-D\left(\hat{X}_1\right)\right)\right)^2\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\left\langle\hat{g}\right\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{4}\right)\end{aligned} \tag{344}$$

$$\hat{K}_{\hat{X}_1} = \frac{\hat{K}\left[\hat{X}_1\right]}{\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2} = \frac{\sqrt{2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}} - D\left(\hat{X}_1\right)\right)}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4} - \frac{\langle\hat{g}\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{4}\right)}{2\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4} - \frac{3\langle\hat{g}\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{8}\right)} \tag{345}$$

A20.12 Computation of $\hat{g}$

While deriving the capital expression we found the return (340) and (342). We present an alternative formulation that will be usefl. Gvn (344):

$$\hat{K}\left[\hat{X}_1\right] = \frac{2\sigma_{\hat{K}}^2}{\hat{\mu}}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}D\left(\hat{X}_1\right)}{\hat{g}^2\left(\hat{X}_1\right)}\right)^2\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4} - \frac{\langle\hat{g}\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{4}\right) \tag{346}$$

$$D\left(\hat{X}_1\right) = \left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle^2}{\sigma_{\hat{K}}^2} + \frac{\langle\hat{g}\rangle}{2}\right)\left(\frac{\hat{\underline{k}}^{ef}\left(\hat{X}_1\right)}{\hat{\underline{k}}_2^{Bef}\left(\left\langle\hat{X}\right\rangle\right)} - \frac{6\hat{\underline{k}}_2^{Bef}}{2+\hat{\underline{k}}_2^{Bef} - \sqrt{\left(2+\hat{\underline{k}}\right)^2 - \hat{\underline{k}}_2^{Bef}}}\right)\hat{\underline{k}}_2^{Bef}$$

we can write the followng equation for $\hat{g}\left(\hat{X}_1\right)$:

$$0 = \frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{2\sigma_{\hat{K}}^2}\left(\hat{g}^2\left(\hat{X}_1\right)\right)^2 - \left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}D\left(\hat{X}_1\right)\right)^2\frac{\hat{g}^2\left(\hat{X}_1\right)}{4} + \left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}D\left(\hat{X}_1\right)\right)^2\frac{\langle\hat{g}\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{4}$$

with solution:

$$\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}D\left(\hat{X}_1\right)\right)^2 - \sqrt{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}D\left(\hat{X}_1\right)\right)^4 - \frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{2\sigma_{\hat{K}}^2}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}D\left(\hat{X}_1\right)\right)^2\langle\hat{g}\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}}{\frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{\sigma_{\hat{K}}^2}}$$

and in first approximation:

$$\begin{aligned}\hat{K}\left[\hat{X}_1\right] &= \frac{2\sigma_{\hat{K}}^2}{\hat{\mu}}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}D\left(\hat{X}_1\right)}{\hat{g}^2\left(\hat{X}_1\right)}\right)^2\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4} - \frac{\langle\hat{g}\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{4}\right) \\ &\simeq \frac{\sigma_{\hat{K}}^2}{\hat{\mu}2\hat{g}^2\left(\hat{X}_1\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}D\left(\hat{X}_1\right)\right)^2\left(1 - \frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle\hat{g}\rangle}\right)\end{aligned}$$

equivalently:

$$\hat{g}\left(\hat{X}_1\right) \simeq \frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{1 - \frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle\hat{g}\rangle}}}{\sqrt{\frac{2\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{\sigma_{\hat{K}}^2}}} \tag{347}$$

Appendix 21 Solving for Investors' returns

Recall the formula for investors returns equation:

$$\hat{f}\left(\hat{X}'\right) = \left(1+\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)^{-1}$$

$$\times\frac{k_1\left(\hat{X}',X\right)\hat{K}'}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}\frac{f_1'\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}$$

and its equivalent in term of $\hat{g}\left(\hat{X}\right)$:

$$\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\frac{1}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\left(1-\hat{M}\right)\hat{g}\left(\hat{X}'\right)$$

$$= \frac{k_1\left(\hat{X}',X\right)\hat{K}'}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}\frac{f_1'\left(\hat{K},\hat{X},\Psi,\hat{\Psi}\right)}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}$$

A21.1 Intermediate quantities

28.0.6 A21.1.1 Ratios

We need to define the various ratios:

$$\delta=\frac{k\left(X\right)\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\left(k_1^B\left(X\right)+\frac{\kappa k_2^B\left(X\right)}{1+\bar{k}\left(\bar{X}\right)\frac{\|\bar{\Psi}\|^2}{|\bar{\Psi}_0(\bar{X})|^2}}\right)\bar{K}_X\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}=\frac{k\left(X\right)\hat{K}\left[\hat{X}\right]}{\left(k_1^B\left(X\right)+\frac{\kappa k_2^B\left(X\right)}{1+\bar{k}\frac{\|\bar{\Psi}\|^2}{\left|\bar{\Psi}_0\left(\hat{X}\right)\right|^2}}\right)\bar{K}_X\left[\bar{X}\right]} \tag{348}$$

is the ratio of investors invested capital versus banks invested capital in one sector.

We estimatr the following coefficients:

$$\underline{k}\left(X\right)=\int\frac{k\left(X\right)}{\langle K\rangle}\frac{\hat{K}_{X'}\left|\hat{\Psi}\left(\hat{X}'\right)\right|^2}{\|\Psi_0\|^2}\simeq\frac{k\left(X\right)}{\langle K\rangle}\hat{K}_X\frac{\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\left|\Psi_0\left(X\right)\right|^2}$$

$$\underline{k}_1^{(B)}\left(X'\right)=\int\frac{k_1^B\left(X'\right)}{\langle K\rangle}\bar{K}_{\bar{X}}\frac{\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\left|\Psi_0\left(X'\right)\right|^2}\simeq\frac{k_1^B\left(X'\right)}{\langle K\rangle}\bar{K}_{X'}\frac{\left|\bar{\Psi}\left(X'\right)\right|^2}{\left|\Psi_0\left(X\right)\right|^2}$$

$$\kappa\underline{k}_2^{(B)}\left(X'\right)=\int\kappa\frac{k_2^B\left(X'\right)}{\langle K\rangle}\bar{K}_{\bar{X}}\frac{\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\left|\Psi_0\left(X'\right)\right|^2}\simeq\kappa k_2^B\left(X'\right)\bar{K}_{X'}\frac{\left|\bar{\Psi}\left(X'\right)\right|^2}{\left|\Psi_0\left(X\right)\right|^2}$$

$$\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\simeq\frac{\kappa k_2^B\left(X'\right)\bar{K}_{X'}\frac{\left|\bar{\Psi}\left(X'\right)\right|^2}{\left|\Psi_0\left(X\right)\right|^2}}{1+\frac{\bar{k}}{\langle\bar{K}\rangle}\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^2}{\|\bar{\Psi}\|^2}}\simeq\frac{\kappa k_2^B\left(X'\right)\bar{K}_{X'}\frac{\left|\bar{\Psi}\left(X'\right)\right|^2}{\left|\Psi_0\left(X\right)\right|^2}}{1+\underline{\bar{k}}}$$

We also define the averages

$$\underline{\bar{k}}=\left\langle\bar{k}\left(\bar{X}\right)\right\rangle$$

and the coefficients $k(X)$ by:

$$\underline{k}(X) = k(X)\frac{\hat{K}\left[\hat{X}\right]}{\langle K\rangle \left|\Psi_0(X)\right|^2}$$

so that we can replace $\hat{K}\left[\hat{X}\right]$ as a function of $\underline{k}(X)$

$$\hat{K}\left[\hat{X}\right] = \underline{k}(X)\frac{\langle K\rangle \left|\Psi_0(X)\right|^2}{k(X)}$$

with the factor:

$$k = \frac{\langle K\rangle \left|\Psi_0(X)\right|^2}{k(X)}$$

So that

$$\hat{K}\left[\hat{X}\right] = \frac{\underline{k}(X)}{k(X)}\langle K\rangle \left|\Psi_0(X)\right|^2 \rightarrow \frac{\underline{k}(X)}{k}$$

Moreover, $k(X)$ can be expressed as a function of the averages capital: Note that given the formula (348)

$$\underline{k}(X) = \delta \underline{k}^B(X)$$

we have:

$$\begin{aligned}
\underline{k}(X) &= \frac{k(X)}{\langle K\rangle}\hat{K}_X\frac{\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\left|\Psi_0(X)\right|^2} \\
&= \delta\left(\frac{k_1^B(X)}{\langle K\rangle}\bar{K}_X\frac{\left|\bar{\Psi}\left(\hat{X}\right)\right|^2}{\left|\Psi_0(X)\right|^2} + \frac{\kappa\frac{k_2^B(X)}{\langle K\rangle}\bar{K}_X\frac{\left|\bar{\Psi}(\hat{X})\right|^2}{\left|\Psi_0(X)\right|^2}}{1+\bar{k}\left(\bar{X}\right)\frac{\left\|\bar{\Psi}\right\|^2}{\left|\bar{\Psi}(\bar{X})\right|^2}}\right) \\
&= \delta\left(k_1^B(X) + \frac{\kappa k_2^B(X)}{1+\bar{k}\left(\bar{X}\right)\frac{\left\|\bar{\Psi}\right\|^2}{\left|\bar{\Psi}(\bar{X})\right|^2}}\right)\frac{\bar{K}_X}{\langle K\rangle}\frac{\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\left|\Psi_0(X)\right|^2} \\
&\simeq \delta\left(k_1^B(X) + \frac{\kappa k_2^B(X)}{1+\bar{k}\left(\bar{X}\right)\frac{\left\|\bar{\Psi}\right\|^2}{\left|\bar{\Psi}_0(\bar{X})\right|^2}}\right)\frac{\bar{K}_X}{\langle K\rangle}\frac{\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\left|\Psi_0(X)\right|^2}
\end{aligned}$$

Ultimately, we can write a relation between δ and $Z\left(\bar{X}\right) = \frac{\bar{K}\left[\bar{X}\right]}{\hat{K}\left[\hat{X}\right]}$:

$$\begin{aligned}
\delta &= \frac{k_1^B(X) + \frac{\kappa k_2^B(X)}{1+\bar{k}\left(\bar{X}\right)\frac{\left\|\bar{\Psi}\right\|^2}{\left|\bar{\Psi}_0(\bar{X})\right|^2}}}{k(X)}\frac{\bar{K}_X}{\hat{K}_X}\frac{\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}{\left|\hat{\Psi}\left(\hat{X}\right)\right|^2} \\
&= \frac{k_1^B(X) + \frac{\kappa k_2^B(X)}{1+\bar{k}\left(\bar{X}\right)\frac{\left\|\bar{\Psi}\right\|^2}{\left|\bar{\Psi}_0(\bar{X})\right|^2}}}{k(X)}\frac{\bar{K}\left[\bar{X}\right]}{\hat{K}\left[\hat{X}\right]} \\
&= \frac{k_1^B(X) + \frac{\kappa k_2^B(X)}{1+\bar{k}\left(\bar{X}\right)\frac{\left\|\bar{\Psi}\right\|^2}{\left|\bar{\Psi}_0(\bar{X})\right|^2}}}{k(X)}Z\left(\bar{X}\right)
\end{aligned}$$

with averges:

$$\langle\delta\rangle=\frac{\left\langle k_1^B\right\rangle+\frac{\kappa\left\langle k_2^B\right\rangle}{1+\langle\bar{k}\rangle}}{\langle k\rangle}\frac{\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}$$

where $Z\left(\bar{X}_1\right)$ was computed befr:

$$Z\left(\bar{X}_1\right)\simeq\left(1+\frac{3}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}\right)\frac{\left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2}{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}-\left(\frac{3}{4}\frac{\left\langle\hat{g}^{Bef}\right\rangle}{\bar{g}\left(\bar{X}_1\right)}+\frac{1}{2}\frac{\left\langle\hat{g}^{Bef}\right\rangle}{\hat{g}\left(\hat{X}_1\right)}\right)\sqrt{\frac{\left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}}$$

or, in first approximation

$$Z\left(\bar{X}\right)\simeq\left(1+\frac{3}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\frac{\left\|\bar{\Psi}_0\left(\bar{X}\right)\right\|^2}{\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2}$$

$$\underline{k}^B\left(X\right)=\frac{k^B\left(X\right)}{\left\langle\bar{K}_X\right\rangle}\bar{K}_X\left[\bar{X}\right]$$

A21.1.2 Dedinition and estimation of several coefficints

We first define:

$$\frac{1}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\rightarrow\frac{1}{1+\left[\underline{\hat{k}}_2\left(\hat{X}'\right)\right]_\kappa}$$

Given the normalizatn of coefficients, this quantity rewrits:

$$\begin{aligned}&\frac{1}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\\
\simeq&\frac{1-\left(\left\langle\hat{k}\right\rangle+\left(\left\langle\hat{k}_1^B\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)}{\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left(1+\frac{\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)}\right)}\\
\simeq&\frac{1-\left\langle\hat{K}\right\rangle}{\left(1-\left\langle\hat{K}_1\right\rangle\right)}\frac{1}{1+\frac{\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1-\left\langle\hat{K}_1\right\rangle}}\end{aligned}$$

We also define average connnectivity for banks plus firms:

$$\begin{aligned}\left\langle\hat{k}^\Sigma\right\rangle&=\left(\left\langle\hat{k}\right\rangle+\left(\left\langle\hat{k}_1^B\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\\
\left\langle\hat{k}_1^\Sigma\right\rangle&=\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\end{aligned}$$

so that we can compute:

$$\begin{aligned}
& \frac{1}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)} \\
\simeq & \frac{1-\left\langle\hat{k}^{\Sigma}\right\rangle}{1-\left\langle\hat{k}_1^{\Sigma}\right\rangle}\frac{1}{1+\frac{\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1-\left\langle\hat{K}_1\right\rangle}} \\
= & \frac{1-\left\langle\hat{k}^{\Sigma}\right\rangle}{1-\left\langle\hat{k}_1^{\Sigma}\right\rangle+\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)} \simeq \frac{1-\left\langle\hat{k}^{\Sigma}\right\rangle}{1-\left\langle\hat{k}^{\Sigma}\right\rangle+\hat{k}_2\left(\hat{X}',\left\langle\hat{X}\right\rangle\right)+\kappa\left[\frac{\hat{k}_2^B\left(\hat{X}',\left\langle\bar{X}\right\rangle\right)}{1+\bar{k}}\right]\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}} \\
= & \frac{1}{1+\frac{\hat{k}_2\left(\hat{X}',\left\langle\hat{X}\right\rangle\right)+\kappa\left[\frac{\hat{k}_2^B\left(\hat{X}',\left\langle\bar{X}\right\rangle\right)}{1+\bar{k}}\right]\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}}{1-\left\langle\hat{k}^{\Sigma}\right\rangle}} \simeq \frac{1}{1+\frac{\hat{k}_2\left(\hat{X}',\left\langle\hat{X}\right\rangle\right)+\kappa\left[\frac{\hat{k}_2^B\left(\hat{X}',\left\langle\bar{X}\right\rangle\right)}{1+\bar{k}}\right]\left(\left(1+\frac{3}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\left\langle\hat{g}\right\rangle}\right)\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}\right)}{1-\left\langle\hat{k}^{\Sigma}\right\rangle}} \\
\to & \frac{1}{1+\hat{k}_2^{(n)}\left(\hat{X}',\left\langle\hat{X}\right\rangle\right)+\kappa\left[\frac{\underline{\hat{k}}_2^{B(n)}\left(\hat{X}',\left\langle\bar{X}\right\rangle\right)}{1+\bar{k}}\right]\left(\left(1+\frac{3}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\left\langle\hat{g}\right\rangle}\right)\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}\right)} \\
\equiv & \frac{1}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_{\kappa}}
\end{aligned}$$

A21.2 Development of (147)

A21.2.1 Computation of the left hand side The left hand side:

$$\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\frac{1}{1+\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\left(1-\hat{M}\right) \tag{349}$$

writes:

$$\begin{aligned}
& \frac{1}{1+\left[\underline{\hat{k}}_2\left(\hat{X}\right)\right]_{\kappa}}\left(1-\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}_X}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2\right) \\
& -\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\frac{1}{1+\left[\underline{\hat{k}}_2\left(\hat{X}'\right)\right]_{\kappa}} \\
& +\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\frac{1}{1+\left[\underline{\hat{k}}_2\left(\hat{X}'\right)\right]_{\kappa}}\frac{k\left(X',X''\right)K_{X'}}{1+k\left(X',X''\right)K_{X''}\left\|\hat{\Psi}\left(\hat{X}''\right)\right\|^2}\left\|\hat{\Psi}\left(\hat{X}''\right)\right\|^2
\end{aligned}$$

with the notation:

$$\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\to\left[\underline{\hat{k}}\left(\hat{X}'\right)\right]_{\kappa}$$

we can replace matrices elments with their averages:

$$\frac{1}{1+\left[\hat{\underline{k}}_2\left(\hat{X}\right)\right]_\kappa}\left(1-\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}_X}{1+\left[\hat{\underline{k}}\left(\hat{X}'\right)\right]_\kappa}\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2\right)$$
$$-\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)\hat{K}_X\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2}{1+\left[\hat{\underline{k}}\left(\hat{X}\right)\right]_\kappa}\frac{1}{1+\left[\hat{\underline{k}}_2\left(\hat{X}'\right)\right]_\kappa}$$
$$+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)\hat{K}_X\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2}{1+\left[\hat{\underline{k}}_2\left(\hat{X}'\right)\right]_\kappa}\frac{1}{1+\left[\hat{\underline{k}}_2\left(\hat{X}'\right)\right]_\kappa}\frac{k\left(X',X''\right)K_{X'}}{1+\left[\hat{\underline{k}}\left(\hat{X}'\right)\right]_\kappa}\left\Vert\hat{\Psi}\left(\hat{X}''\right)\right\Vert^2$$

so that the terms:

$$\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)\hat{K}_X\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2}{1+\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}_{X'}}\frac{1}{1+\hat{\underline{k}}_2\left(\hat{X}'\right)}\frac{k\left(X',X''\right)K_{X'}}{1+\hat{\underline{k}}\left(\hat{X}'\right)}\left\Vert\hat{\Psi}\left(\hat{X}''\right)\right\Vert^2$$
$$+\frac{\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\hat{K}_X\left\Vert\hat{\Psi}\left(\left\langle\hat{X}\right\rangle\right)\right\Vert^2}{1+\hat{\underline{k}}\left(\hat{X}\right)}\frac{1}{1+\hat{\underline{k}}_2\left(\langle X\rangle\right)}\frac{k\left(\langle X\rangle,X'\right)\langle K\rangle}{1+\hat{\underline{k}}\left(\left\langle\hat{X}\right\rangle\right)}\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2$$

yield the contribution:

$$-\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)}{1+\left[\hat{\underline{k}}_2\left(\hat{X}\right)\right]_\kappa}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\left[\hat{\underline{k}}\left(\hat{X}'\right)\right]_\kappa}\right)\frac{\hat{K}_X\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2}{1+\left[\hat{\underline{k}}\left(\hat{X}\right)\right]_\kappa}$$
$$+\frac{\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\hat{K}_X\left\Vert\hat{\Psi}\left(\left\langle\hat{X}\right\rangle\right)\right\Vert^2}{1+\left[\hat{\underline{k}}\left(\hat{X}\right)\right]_\kappa}\frac{1}{1+\left[\hat{\underline{k}}_2\left(\left\langle\hat{X}\right\rangle\right)\right]_\kappa}\frac{k\left(\langle X\rangle,X'\right)\langle K\rangle}{1+\left[\hat{\underline{k}}\left(\left\langle\hat{X}\right\rangle\right)\right]_\kappa}\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2$$
$$=-\left(\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)}{1+\left[\hat{\underline{k}}_2\left(\hat{X}\right)\right]_\kappa}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\left[\hat{\underline{k}}_2\left(\hat{X}'\right)\right]_\kappa}\right)-\frac{\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\left\Vert\hat{\Psi}\left(\left\langle\hat{X}\right\rangle\right)\right\Vert^2}{1+\left[\hat{\underline{k}}\left(\left\langle\hat{X}\right\rangle\right)\right]_\kappa}\frac{k\left(\langle X\rangle,X'\right)\langle K\rangle}{1+\left[\hat{\underline{k}}_2\left(\left\langle\hat{X}\right\rangle\right)\right]_\kappa}\right)\frac{\hat{K}_X\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2}{1+\left[\hat{\underline{k}}\left(\hat{X}'\right)\right]_\kappa}$$
$$=-\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\frac{\hat{\underline{k}}_1\left(\langle X\rangle,\hat{X}\right)\left\Vert\hat{\Psi}\left(\left\langle\hat{X}\right\rangle\right)\right\Vert^2}{1+\left[\hat{\underline{k}}\left(\left\langle\hat{X}\right\rangle\right)\right]_\kappa}k\left(\langle X\rangle,X'\right)}{1+\left[\hat{\underline{k}}_2\left(\hat{X}\right)\right]_\kappa}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\left[\hat{\underline{k}}_2\left(\hat{X}'\right)\right]_\kappa}\right)\frac{\hat{K}_X\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2}{1+\left[\hat{\underline{k}}\left(\hat{X}\right)\right]_\kappa}$$

and we can approximate:

$$-\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\frac{\hat{\underline{k}}_1\left(\langle X\rangle,\hat{X}\right)\left\Vert\hat{\Psi}\left(\left\langle\hat{X}\right\rangle\right)\right\Vert^2}{1+\left[\hat{\underline{k}}\left(\left\langle\hat{X}\right\rangle\right)\right]_\kappa}k\left(\langle X\rangle,X'\right)}{1+\left[\hat{\underline{k}}_2^{(n)}\left(\hat{X}\right)\right]_\kappa}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\left[\hat{\underline{k}}_2^{(n)}\left(\hat{X}'\right)\right]_\kappa}\right)\frac{\hat{K}_X\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2}{1+\left[\hat{\underline{k}}\left(\hat{X}\right)\right]_\kappa}$$
$$\simeq-\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\hat{\underline{k}}_1\left(\langle X\rangle,\hat{X}\right)k\left(\langle X\rangle,X'\right)}{1+\left[\hat{\underline{k}}_2^{(n)}\left(\hat{X}\right)\right]_\kappa}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\left[\hat{\underline{k}}_2^{(n)}\left(\hat{X}'\right)\right]_\kappa}\right)\frac{\hat{K}_X\left\Vert\hat{\Psi}\left(\hat{X}'\right)\right\Vert^2}{1+\left[\hat{\underline{k}}\left(\hat{X}\right)\right]_\kappa}$$

where:

$$\hat{\underline{k}}_1\left(\langle X\rangle,\hat{X}\right)=\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\left\Vert\hat{\Psi}\left(\left\langle\hat{X}\right\rangle\right)\right\Vert^2\langle K\rangle$$

Ultimately, we write (349) as:

$$\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\left[\underline{\hat{k}}_2\left(\hat{X}\right)\right]_{\kappa}}-\hat{S}_1^E\left(\hat{X}',\hat{X}\right) \tag{350}$$

A21.2.2 Computation of the right hand side

We introduce the notations:

$$\begin{aligned}\underline{k}_2\left(\hat{X}\right) &= \beta\underline{k}\left(\hat{X}\right)\\ \underline{k}_2\left(\hat{X}\right) &= (1-\beta)\,\underline{k}\left(\hat{X}\right)\end{aligned}$$

$$C^{(e)}=\frac{1+\underline{k}_2(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{\underline{k}}}\right](X)}{1+\underline{k}(X)+\underline{k}_1^B(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{\underline{k}}}\right](X)}C$$

$$\kappa\left[\frac{\underline{k}_2^B}{1+\bar{\underline{k}}}\right](X)>>1$$

$$\kappa\left[\frac{\underline{k}_2^B}{1+\bar{\underline{k}}}\right](X)=\beta^B\left(\underline{k}_1^B(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{\underline{k}}}\right](X)\right)=\beta^B\underline{k}^B(X)$$

note that:

$$1-\beta^B<<1$$

$$\underline{k}(X)=\delta\underline{k}^B(X)$$

which allows to write:

$$\frac{\underline{k}_2(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{\underline{k}}}\right](X)}{\underline{k}(X)+\underline{k}^B(X)}C=\frac{\beta\delta+\beta^B}{1+\delta}$$

$$\frac{\underline{k}_1\left(X',\hat{X}\right)}{\left(1+\underline{k}^B(X')\right)}\to(1-\beta)\,\delta$$

and we compute the return as:

$$\begin{aligned}f_1^{(e)}(X) &= \left(1+\underline{k}_2(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{\underline{k}}}\right](X)\right)f_1(X)-\left(\underline{k}_2(X)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{\underline{k}}}\right](X)\right)\bar{r}\\ &= f_1(X)+\left(\beta\delta+\beta^B\right)\underline{k}^B(X)\left(f_1(X)-R\right)\end{aligned}$$

We use formula (153) for returns

$$\hat{g}\left(\hat{X}_1\right)\simeq\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{\frac{\sigma_{\hat{K}}^2\left(1-\frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle\hat{g}\rangle}\right)}{2\hat{\mu}\hat{K}\left[\hat{X}_1\right]}}$$

altrnately, we can use the full expanded forms (340) and (342), leading to same results.

A21.2.3 Rewrting (80)

The previous expression allow to obtain a detailed form of (80):

$$\left(\frac{1}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}-\hat{S}_1^E\left(\hat{X}',\hat{X}\right)\right)\left(\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{\frac{\sigma_{\hat{K}}^2\left(1-\frac{\hat{g}^{ef}\left(\hat{X}'\right)}{\langle\hat{g}\rangle}\right)}{2\hat{\mu}\hat{K}\left[\hat{X}'\right]}}-\bar{r}'\right) \tag{351}$$

$$= \left(\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{\frac{\sigma_{\hat{K}}^2\left(1-\frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle\hat{g}\rangle}\right)}{2\hat{\mu}\hat{K}\left[\hat{X}_1\right]}}-\bar{r}'}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}\right)$$

$$-\int\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\left(\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{\frac{\sigma_{\hat{K}}^2\left(1-\frac{\hat{g}^{ef}\left(\hat{X}'\right)}{\langle\hat{g}\rangle}\right)}{2\hat{\mu}\hat{K}\left[\hat{X}'\right]}}-\bar{r}'\right)$$

which leads to:

$$\frac{(1-\beta)\delta}{3(1+\delta)}\left(\frac{\left(X+C\frac{\beta\delta+\beta^B}{1+\delta}\right)^2\epsilon\left(\frac{\left(R+\Delta F_\tau\left(\bar{R}(K,X)\right)\right)\left(3X-\frac{\beta\delta+\beta^B}{1+\delta}C\right)\left(\frac{\beta\delta+\beta^B}{1+\delta}C+X\right)}{4\left(f_1(X)+\left(\beta\delta+\beta^B\right)\underline{k}^B(X)R\right)}-\frac{C\left(2X-C\frac{\beta\delta+\beta^B}{1+\delta}\right)}{1+(1+\delta)\underline{k}^B(X)}\right)}{\sigma_{\hat{K}}^2\left(f_1(X)+\left(\beta\delta+\beta^B\right)\underline{k}^B(X)R\right)}\right) \tag{352}$$

$$= \frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{\frac{\sigma_{\hat{K}}^2\left(1-\frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle\hat{g}\rangle}\right)}{2\hat{\mu}\hat{K}\left[\hat{X}_1\right]}}-\bar{r}'}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}$$

$$-\int\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\left(\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{\frac{\sigma_{\hat{K}}^2\left(1-\frac{\hat{g}^{ef}\left(\hat{X}'\right)}{\langle\hat{g}\rangle}\right)}{2\hat{\mu}\hat{K}\left[\hat{X}'\right]}}-\bar{r}'\right)$$

where:

$$\underline{\hat{k}}_2\left(\hat{X}_1\right)=\hat{\beta}\underline{\hat{k}}\left(\hat{X}\right)$$

$$\underline{\hat{k}}^B\left(\hat{X}_1\right)=\underline{\hat{k}}_1^B\left(\hat{X}_1\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}_1\right)$$

$$\underline{\hat{k}}^B\rightarrow\frac{\left(\hat{k}_1^B(X)+\frac{\kappa\hat{k}_2^B(X)}{1+\bar{k}\left(\bar{X}\right)\frac{\|\bar{\Psi}\|^2}{\left|\bar{\Psi}_0\left(\bar{X}\right)\right|^2}}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\bar{K}_X\left|\bar{\Psi}\left(\bar{X}\right)\right|^2$$

$$\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]=\hat{\beta}^B\underline{\hat{k}}^B$$

$$\hat{\delta} = \frac{\hat{k}(X)\hat{K}_X\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{\left(\hat{k}_1^B(X) + \frac{\kappa\hat{k}_2^B(X)}{1+\bar{k}\left(\bar{X}\right)\frac{\|\bar{\Psi}\|^2}{|\bar{\Psi}_0(\bar{X})|^2}}\right)\bar{K}_X\left|\bar{\Psi}\left(\bar{X}\right)\right|^2}$$

$$= \frac{\hat{k}(X)\hat{K}\left[\hat{X}\right]}{\left(\hat{k}_1^B(X) + \frac{\kappa\hat{k}_2^B(X)}{1+\bar{k}\left(\bar{X}\right)\frac{\|\bar{\Psi}\|^2}{|\bar{\Psi}_0(\bar{X})|^2}}\right)\bar{K}_X\left(\bar{X}\right)} \rightarrow \frac{\hat{k}(X)\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}{\left(\hat{k}_1^B(X) + \frac{\kappa\hat{k}_2^B(X)}{1+\bar{k}\left(\bar{X}\right)\frac{\|\bar{\Psi}\|^2}{|\bar{\Psi}_0(\bar{X})|^2}}\right)\left\langle\hat{K}\right\rangle\left\|\bar{\Psi}\right\|^2}$$

so that the following relation holds

$$\underline{\hat{k}}\left(\hat{X}_1\right) = \hat{\delta}\underline{\hat{k}}^B\left(\hat{X}_1\right)$$

A21.3 Solving (352)

Recall that:

$$R = f_1(X) - \bar{r}$$

$$\underline{k}(X) = \frac{k(X)}{\langle K\rangle}\hat{K}_X\frac{\left|\hat{\Psi}\left(\hat{X}\right)\right|^2}{|\Psi_0(X)|^2}$$

$$\hat{K}\left[\hat{X}\right] = \frac{\underline{k}(X)}{k(X)}\langle K\rangle|\Psi_0(X)|^2 \rightarrow \frac{\underline{k}(X)}{k}$$

Define:

$$H = \int \hat{S}_1^E\left(\hat{X}', \hat{X}_1\right)\left(\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2 - \hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{\frac{\sigma_{\hat{K}}^2\left(1 - \frac{\hat{g}^{ef}\left(\hat{X}'\right)}{\langle\hat{g}\rangle}\right)}{2\hat{\mu}\hat{K}\left[\hat{X}'\right]}} - \bar{r}'\right)$$

and (352) becomes:

$$-H - \frac{\bar{r}'}{\left(1 + \left(\hat{\beta}\hat{\delta} + \hat{\beta}^B\right)\underline{\hat{k}}^B\left(\hat{X}_1\right)\right)}$$

$$= \frac{1}{3}\frac{(1-\beta)\delta}{1+\delta}\left(\frac{\left(X + C\frac{\beta\delta+\beta^B}{1+\delta}\right)^2\epsilon\left(\frac{\left(R+\Delta F_\tau\left(\bar{R}(K,X)\right)\right)\left(3X - \frac{\beta\delta+\beta^B}{1+\delta}C\right)\left(\frac{\beta\delta+\beta^B}{1+\delta}C+X\right)}{4\left(f_1(X)+\left(\beta\delta+\beta^B\right)\underline{k}^B(X)R\right)} - \frac{C\left(2X - C\frac{\beta\delta+\beta^B}{1+\delta}\right)}{1+(1+\delta)\underline{k}^B(X)}\right)}{\sigma_{\hat{K}}^2\left(f_1(X) + \left(\beta\delta+\beta^B\right)\underline{k}^B(X)R\right)}\right)$$

$$-\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2 - \hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{\frac{\sigma_{\hat{K}}^2\left(1-\frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle\hat{g}\rangle}\right)}{2\hat{\mu}\hat{K}\left[\hat{X}_1\right]}} - \bar{r}'}{1 + \left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}$$

$$\frac{(1-\beta)\delta}{3(1+\delta)}\frac{\left(X+C\frac{\beta\delta+\beta^B}{1+\delta}\right)^2\epsilon}{\sigma_{\hat{K}}^2\left(f_1(X)+\left(\beta\delta+\beta^B\right)\underline{k}^B(X)R\right)}\frac{C\left(2X-C\frac{\beta\delta+\beta^B}{1+\delta}\right)}{1+(1+\delta)\underline{k}^B(X)}$$

$$+\left(\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle\hat{g}\rangle}\right)}}{\left(1+\left[\hat{\underline{k}}_2^n\left(\hat{X}\right)\right]_\kappa\right)\sqrt{\underline{k}\left(\hat{X}_1\right)}}\right)$$

$$= H+\frac{\bar{r}'}{1+\left[\hat{\underline{k}}_2^n\left(\hat{X}\right)\right]_\kappa}+\frac{1}{3}\frac{(1-\beta)\delta}{1+\delta}\left(\frac{\left(X+C\frac{\beta\delta+\beta^B}{1+\delta}\right)^2\epsilon}{\sigma_{\hat{K}}^2 f_1(X)}\frac{R\left(3X-\frac{\beta\delta+\beta^B}{1+\delta}C\right)\left(\frac{\beta\delta+\beta^B}{1+\delta}C+X\right)}{4f_1(X)}+\Delta F_\tau\left(\bar{R}(K,X)\right)\right)$$

a first order expansion in R leads to:

$$\frac{1}{1+(1+\delta)\underline{k}^B(X)}\left(1-\frac{\left(\beta\delta+\beta^B\right)\underline{k}^B(X)\left(R+\Delta F_\tau\left(\bar{R}(K,X)\right)\right)}{f_1^2(X)}\right)$$

$$+\frac{3(1+\delta)\sigma_{\hat{K}}^2 f_1(X)}{(1-\beta)\delta C\left(X+C\frac{\beta\delta+\beta^B}{1+\delta}\right)^2\left(2X-C\frac{\beta\delta+\beta^B}{1+\delta}\right)\epsilon}\left(\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{k\sigma_{\hat{K}}^2\left(1-\frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle\hat{g}\rangle}\right)}}{\left(1+\left[\hat{\underline{k}}_2^n\left(\hat{X}\right)\right]_\kappa\right)\sqrt{\underline{k}\left(\hat{X}_1\right)}}\right)$$

$$= \frac{3(1+\delta)\sigma_{\hat{K}}^2 f_1(X)}{(1-\beta)\delta C\left(X+C\frac{\beta\delta+\beta^B}{1+\delta}\right)^2\left(2X-C\frac{\beta\delta+\beta^B}{1+\delta}\right)\epsilon}\left(H+\frac{\bar{r}'}{1+\left\langle\hat{K}_2\right\rangle}\right)$$

$$+\frac{\left(3X-\frac{\beta\delta+\beta^B}{1+\delta}C\right)\left(\frac{\beta\delta+\beta^B}{1+\delta}C+X\right)\left(R+\Delta F_\tau\left(\bar{R}(K,X)\right)\right)}{4f_1(X)C\left(2X-C\frac{\beta\delta+\beta^B}{1+\delta}\right)}$$

which becomes:

$$\frac{1}{1+(1+\delta)\underline{k}^B(X)}\left(1+\frac{\left(\beta\delta+\beta^B\right)\left(R+\Delta F_\tau\left(\bar{R}(K,X)\right)\right)}{(1+\delta)f_1^2(X)}\right)$$

$$+\frac{3(1+\delta)\sigma_{\hat{K}}^2 f_1(X)}{(1-\beta)\delta C\left(X+C\frac{\beta\delta+\beta^B}{1+\delta}\right)^2\left(2X-C\frac{\beta\delta+\beta^B}{1+\delta}\right)\epsilon}\left(\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{k\sigma_{\hat{K}}^2\left(1-\frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle\hat{g}\rangle}\right)}}{\left(1+\left[\hat{\underline{k}}_2^n\left(\hat{X}\right)\right]_\kappa\right)\sqrt{\underline{k}\left(\hat{X}_1\right)}}\right)$$

$$= \frac{3(1+\delta)\sigma_{\hat{K}}^2 f_1(X)}{(1-\beta)\delta C\left(X+C\frac{\beta\delta+\beta^B}{1+\delta}\right)^2\left(2X-C\frac{\beta\delta+\beta^B}{1+\delta}\right)\epsilon}\left(H+\frac{\bar{r}'}{\left(1+\left(\hat{\beta}\hat{\delta}+\hat{\beta}^B\right)\hat{\underline{k}}^B\left(\hat{X}_1\right)\right)}\right)$$

$$+\frac{R\left(3X-\frac{\beta\delta+\beta^B}{1+\delta}C\right)\left(\frac{\beta\delta+\beta^B}{1+\delta}C+X\right)}{4f_1(X)C\left(2X-C\frac{\beta\delta+\beta^B}{1+\delta}\right)}+\frac{\left(\beta\delta+\beta^B\right)\left(R+\Delta F_\tau\left(\bar{R}(K,X)\right)\right)}{(1+\delta)f_1^2(X)}$$

This is an equation for $\underline{k}(X)$. Defining:

$$a=\frac{3(1+\delta)\sigma_{\hat{K}}^{2}f_{1}(X)}{(1-\beta)\delta C\left(X+C\frac{\beta\delta+\beta^{B}}{1+\delta}\right)^{2}\left(2X-C\frac{\beta\delta+\beta^{B}}{1+\delta}\right)\epsilon}\left(\frac{\left(\left\Vert\hat{\Psi}_{0}\left(\hat{X}_{1}\right)\right\Vert^{2}-\hat{\mu}D\left(\hat{X}_{1}\right)\right)\sqrt{k\sigma_{\hat{K}}^{2}\left(1-\frac{\hat{g}^{ef}\left(\hat{X}_{1}\right)}{\langle\hat{g}\rangle}\right)}}{1+\left[\underline{\hat{k}}_{2}^{n}\left(\hat{X}\right)\right]_{\kappa}}\right)$$

and:

$$\begin{aligned}c &= \frac{3(1+\delta)\sigma_{\hat{K}}^{2}f_{1}(X)}{(1-\beta)\delta C\left(X+C\frac{\beta\delta+\beta^{B}}{1+\delta}\right)^{2}\left(2X-C\frac{\beta\delta+\beta^{B}}{1+\delta}\right)\epsilon}\left(H+\frac{\bar{r}'}{1+\left[\underline{\hat{k}}_{2}^{n}\left(\hat{X}\right)\right]_{\kappa}}\right)\\ &\quad+\frac{R\left(3X-\frac{\beta\delta+\beta^{B}}{1+\delta}C\right)\left(\frac{\beta\delta+\beta^{B}}{1+\delta}C+X\right)}{4f_{1}(X)C\left(2X-C\frac{\beta\delta+\beta^{B}}{1+\delta}\right)}+\frac{\left(\beta\delta+\beta^{B}\right)R}{(1+\delta)f_{1}^{2}(X)}+\Delta F_{\tau}\left(\bar{R}(K,X)\right)\end{aligned}$$

fr $\underline{k}(X)>>1$ nd $\epsilon<\sigma_{\hat{K}}^{2}f_{1}(X)$ equation simplifies as:

$$\frac{a}{\sqrt{\underline{k}(X)}}-c=0$$

with solution:

$$\sqrt{\underline{k}(X)}=\frac{a}{c}$$

$$\underline{k}(X)=\left(\frac{\left(\left\Vert\hat{\Psi}_{0}\left(\hat{X}_{1}\right)\right\Vert^{2}-\hat{\mu}D\left(\hat{X}_{1}\right)\right)\sqrt{k\sigma_{\hat{K}}^{2}\left(1-\frac{\hat{g}^{ef}\left(\hat{X}_{1}\right)}{\langle\hat{g}\rangle}\right)}}{\left(1+\left[\underline{\hat{k}}_{2}^{n}\left(\hat{X}\right)\right]_{\kappa}\right)\left[H+\frac{\bar{r}'}{\left(1+\left[\underline{\hat{k}}_{2}^{n}\left(\hat{X}_{1}\right)\right]_{\kappa}\right)}+\left(\frac{A(X)}{f_{1}^{2}(X)}+\frac{B(X)}{f_{1}^{3}(X)}\right)\left(R+\Delta F_{\tau}\left(\bar{R}(K,X)\right)\right)\right]}-\sqrt{\frac{\sigma_{\hat{K}}^{2}\hat{\mu}k}{2}}D\left(\hat{X}_{1}\right)\right)^{2}$$

with:

$$\begin{aligned}A(X) &= \frac{\epsilon(1-\beta)\delta\left(X+C\frac{\beta\delta+\beta^{B}}{1+\delta}\right)^{2}}{3(1+\delta)\sigma_{\hat{K}}^{2}}\frac{\left(3X-\frac{\beta\delta+\beta^{B}}{1+\delta}C\right)\left(\frac{\beta\delta+\beta^{B}}{1+\delta}C+X\right)}{4C\left(2X-C\frac{\beta\delta+\beta^{B}}{1+\delta}\right)}\\ B(X) &= \frac{\epsilon(1-\beta)\delta\left(X+C\frac{\beta\delta+\beta^{B}}{1+\delta}\right)^{2}}{3(1+\delta)\sigma_{\hat{K}}^{2}}\frac{\left(\beta\delta+\beta^{B}\right)C\left(2X-C\frac{\beta\delta+\beta^{B}}{1+\delta}\right)}{(1+\delta)}\end{aligned}$$

and the equation for return writes:

$$\begin{aligned}\frac{\hat{g}\left(\hat{X}_{1}\right)}{1+\left[\underline{\hat{k}}_{2}^{n}\left(\hat{X}_{1}\right)\right]_{\kappa}} &= \frac{\left(\left\Vert\hat{\Psi}_{0}\left(\hat{X}_{1}\right)\right\Vert^{2}-\hat{\mu}D\left(\hat{X}_{1}\right)\right)\sqrt{k\sigma_{\hat{K}}^{2}\left(1-\frac{\hat{g}^{ef}\left(\hat{X}_{1}\right)}{\langle\hat{g}\rangle}\right)}}{\left(1+\left[\underline{\hat{k}}_{2}^{n}\left(\hat{X}\right)\right]_{\kappa}\right)\sqrt{\underline{k}\left(\hat{X}_{1}\right)}}\\ &\simeq H+\frac{\bar{r}'}{1+\left[\underline{\hat{k}}_{2}^{n}\left(\hat{X}\right)\right]_{\kappa}}+\frac{\frac{R\left(3X-\frac{\beta\delta+\beta^{B}}{1+\delta}C\right)\left(\frac{\beta\delta+\beta^{B}}{1+\delta}C+X\right)}{4f_{1}(X)C\left(2X-C\frac{\beta\delta+\beta^{B}}{1+\delta}\right)}+\frac{\left(\beta\delta+\beta^{B}\right)R}{(1+\delta)f_{1}^{2}(X)}}{\frac{3(1+\delta)\sigma_{\hat{K}}^{2}f_{1}(X)}{(1-\beta)\delta C\left(X+C\frac{\beta\delta+\beta^{B}}{1+\delta}\right)^{2}\left(2X-C\frac{\beta\delta+\beta^{B}}{1+\delta}\right)\epsilon}}\\ &= H+\frac{\bar{r}'}{1+\underline{\hat{k}}_{2}\left(\hat{X}_{1}\right)}+\left(\frac{A(X)}{f_{1}^{2}(X)}+\frac{B(X)}{f_{1}^{3}(X)}\right)\left(R+\Delta F_{\tau}\left(\bar{R}(K,X)\right)\right)\end{aligned}$$

$$\hat{g}\left(\hat{X}_1\right)-\bar{r}' = \left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa\right)\int \hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\left(\frac{\sqrt{k\sigma_{\hat{K}}^2\left(1-\frac{\hat{g}^{ef}\left(\hat{X}_1\right)}{\langle\hat{g}\rangle}\right)\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)}}{\sqrt{\underline{k}\left(\hat{X}'\right)}}-\bar{r}'\right)$$
$$+\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa\right)\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,\hat{X}'\right)\right)\right)$$

with solution:

$$\hat{g}\left(\hat{X}_1\right)-\bar{r}' = \int\left(1-\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1}$$
$$\times\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,\hat{X}'\right)\right)\right)$$

Remark that given:

$$\hat{g}\left(\hat{X}_1\right)\left(\hat{X}_1\right)-\bar{r}'=(1-M)^{-1}\left(\hat{f}\left(\hat{X}_1\right)-\bar{r}\right)$$

the solution for $\left(\hat{f}\left(\hat{X}_1\right)-\bar{r}\right)$ is obtained directly by multiplying by $(1-M)$.

A21.3 Estimation of $\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$

We estimate:

$$\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$$
$$\rightarrow \frac{\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\frac{\hat{k}_1\left(\langle X\rangle,\hat{X}\right)\left\|\hat{\Psi}\left(\left\langle\hat{X}\right\rangle\right)\right\|^2}{1+\left[\underline{k}\left(\left\langle\hat{X}\right\rangle\right)\right]_\kappa}k\left(\langle X\rangle,X'\right)}{1+\left[\underline{\hat{k}}_2\left(\hat{X}\right)\right]_\kappa}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\left[\underline{\hat{k}}_2\left(\hat{X}'\right)\right]_\kappa}\right)\frac{\hat{K}_X\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2}{1+\left[\underline{\hat{k}}\left(\hat{X}\right)\right]_\kappa}$$

We use (247), (249) and (345). and replace at the lowest order $\hat{g}\left(\left\langle\hat{X}\right\rangle\right)\rightarrow\bar{r}'$

$$\hat{D}\left(\hat{X}_1\right)=\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\frac{\left\langle\hat{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}\left\langle\hat{g}^{ef}\right\rangle-\left(\frac{\left\langle\bar{K}\right\rangle^2\langle\bar{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\Delta\left(\hat{k}^B\left(\hat{X}_1,\left\langle\bar{X}\right\rangle\right)A\right)Z\langle\bar{g}\rangle$$

$$\hat{K}\left[\hat{X}_1\right] = \hat{K}_{\hat{X}}\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2\simeq\hat{\mu}\frac{\hat{K}_0^4}{2\sigma_{\hat{K}}^2}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\langle\hat{g}\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{3}\right) \tag{353}$$
$$= \frac{\hat{\mu}}{2\sigma_{\hat{K}}^2}\left(2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}}-D\left(\hat{X}_1\right)\right)\right)^2\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\langle\hat{g}\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{3}\right)$$

$$\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2 \simeq \hat{\mu}\frac{\hat{K}_0^3}{\sigma_{\hat{K}}^2}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{3}-\frac{\left\langle\hat{K}\right\rangle}{2\hat{K}_0}\langle\hat{g}\rangle\left\langle\hat{g}^{ef}\right\rangle\right) \tag{354}$$

$$= \frac{\hat{\mu}}{\sigma_{\hat{K}}^2}\left(2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}}-D\left(\hat{X}_1\right)\right)\right)^{\frac{3}{2}}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{3}-\frac{\langle\hat{g}\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{2}\right)$$

$$\hat{K}_{\hat{X}_1}=\frac{\hat{K}\left[\hat{X}_1\right]}{\left\|\hat{\Psi}\left(\hat{X}_1\right)\right\|^2}=\frac{\sqrt{2\frac{\sigma_{\hat{K}}^2}{\hat{g}^2\left(\hat{X}_1\right)}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2}{\hat{\mu}}-D\left(\hat{X}_1\right)\right)}\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{4}-\frac{\langle\hat{g}\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{3}\right)}{2\left(\frac{\hat{g}^2\left(\hat{X}_1\right)}{3}-\frac{\langle\hat{g}\rangle\hat{g}^{ef}\left(\hat{X}_1\right)}{2}\right)} \tag{355}$$

$$\hat{g}\left(\hat{X}_1\right)\rightarrow\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^2-\hat{\mu}D\left(\hat{X}_1\right)\right)\sqrt{\left(1-\frac{\hat{g}^{ef}\left(\hat{X}'\right)}{\langle\hat{g}\rangle}\right)}}{\sqrt{\frac{\hat{\mu}\hat{K}\left[\hat{X}_1\right]}{2\sigma_{\hat{K}}^2}}}\rightarrow\bar{r}' \tag{356}$$

$$\hat{K}_{\hat{X}}=\frac{\sqrt{\frac{\sigma_{\hat{K}}^2}{2}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2}{\hat{\mu}}-D\left(\hat{X}\right)\right)}\left(\frac{\left(\bar{r}'\right)^2}{4}-\frac{\bar{r}'\hat{g}^{ef}}{3}\right)}{\left(\frac{\left(\bar{r}'\right)^2}{3}-\frac{\bar{r}'\hat{g}^{ef}}{2}\right)\bar{r}'} \tag{357}$$

$$\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2\simeq\frac{2\sqrt{2\sigma_{\hat{K}}^2}\hat{\mu}}{\bar{r}'}\left(\frac{\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2}{\hat{\mu}}-D\left(\hat{X}'\right)\right)^{\frac{3}{2}}\left(\frac{1}{3}-\frac{\left\langle\hat{K}\right\rangle}{2\left\langle\hat{K}_0\right\rangle}\frac{\hat{g}^{ef}}{\bar{r}'}\right) \tag{358}$$

$$D\left(\hat{X}_1\right)\rightarrow\left(\frac{\left\langle\hat{K}\right\rangle^2\langle\hat{g}\rangle}{\sigma_{\hat{K}}^2}+\frac{1}{2}\right)\frac{\left\langle\hat{K}_0\right\rangle}{\left\langle\hat{K}\right\rangle}\left\langle\hat{g}^{ef}\right\rangle$$

$$\hat{K}_{\hat{X}}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2\rightarrow\frac{2\sqrt{2}\sigma_{\hat{K}}^2}{\hat{\mu}\bar{r}'^2}\sqrt{\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2-\hat{\mu}D\left(\hat{X}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)^{\frac{3}{2}}$$

$$\frac{\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\frac{\underline{\hat{k}}_1\left(\langle X\rangle,\hat{X}\right)}{1+\left[\underline{\hat{k}}\left(\langle\hat{X}\rangle\right)\right]_\kappa}k\left(\langle X\rangle,X'\right)}{1+\left[\underline{\hat{k}}_2\left(\hat{X}\right)\right]_\kappa}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\left[\underline{\hat{k}}_2\left(\hat{X}'\right)\right]_\kappa}\right)\frac{\hat{K}_X\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2}{1+\left[\underline{\hat{k}}\left(\hat{X}\right)\right]_\kappa}$$

$$\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)$$

$$\rightarrow \frac{\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)}{1+\left[\underline{\hat{k}}\left(\hat{X}_1\right)\right]_\kappa}\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\frac{\underline{\hat{k}}_1\left(\langle X\rangle,\hat{X}\right)}{1+\left[\underline{\hat{k}}\left(\langle\hat{X}\rangle\right)\right]_\kappa}k\left(\langle X\rangle,X'\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}'\right)\right]_\kappa}\right)\frac{2\sqrt{2}\sigma_{\hat{K}}^2}{\hat{\mu}\bar{r}'^2}$$
$$\times\frac{\sqrt{\left\|\hat{\Psi}_0\left(\hat{X}\right)\right\|^2-\hat{\mu}D\left(\hat{X}\right)}\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)^{\frac{3}{2}}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}$$

$$\rightarrow \frac{\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)}{1+\left[\underline{\hat{k}}\left(\hat{X}_1\right)\right]_\kappa}\left(\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\underline{\hat{k}}_1\left(\langle X\rangle,\hat{X}\right)k\left(\langle X\rangle,X'\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa}+\frac{\hat{k}_1\left(\hat{X}',\hat{X}\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}'\right)\right]_\kappa}\right)\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)^2}{\left(\left\|\hat{\Psi}_0\right\|^2-\hat{\mu}\langle D\rangle\right)^2}$$

and usng vr frml fr:

$$\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)^2}{\left(\left\|\hat{\Psi}_0\right\|^2-\hat{\mu}\langle D\rangle\right)^2}$$

lds t:

$$\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)^2}{\left(\left\|\hat{\Psi}_0\right\|^2-\hat{\mu}\langle D\rangle\right)^2}$$

$$\simeq\left(\frac{\sqrt{\langle\hat{g}\rangle^2\left(1+\frac{3}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\left\|\bar{\Psi}_0\left(\hat{X}_1\right)\right\|-\frac{3}{8}\langle\hat{g}\rangle\frac{\hat{g}^{Bef}\left(\bar{X}_1\right)}{\bar{g}\left(\bar{X}_1\right)}\left\|\hat{\Psi}_0\left(\bar{X}_1\right)\right\|}{\sqrt{\langle\hat{g}\rangle^2\left(1+\frac{3}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\left\|\bar{\Psi}_0\left(\hat{X}_1\right)\right\|-\frac{3}{8}\langle\hat{g}\rangle\frac{\left\langle\hat{g}^{Bef}\right\rangle}{\langle\bar{g}\rangle}\left\|\hat{\Psi}_0\left(\bar{X}_1\right)\right\|}\right)^4\frac{\frac{1}{4\bar{g}^2\left(\bar{X}_1\right)}-\frac{\langle\hat{g}\rangle\hat{g}^{Bef}\left(\bar{X}_1\right)}{3\bar{g}^4\left(\bar{X}_1\right)}}{\frac{1}{4\langle\bar{g}\rangle^2}-\frac{\langle\hat{g}\rangle\left\langle\hat{g}^{Bef}\right\rangle}{3\langle\bar{g}\rangle^2}}$$

Appendix 22 Banks returns

The equation for bank returns expressed in terms of $\bar{g}$ is:

$$\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\hat{k}}^B\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)} \tag{359}$$
$$-\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\hat{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\frac{\underline{\hat{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}\left(\bar{X}\right)}}$$
$$=\frac{\underline{k}_1^{(B)}\left(X',\bar{X}\right)}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}\frac{\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}$$

A22.1 Use of normalizations and definition of coefficients

Using the normalizations, we replace:

$$\begin{aligned}
& 1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\frac{\hat{\underline{k}}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}} \\
\rightarrow\quad & 1+\left(\hat{k}\left(\hat{X}',\left\langle\hat{X}\right\rangle\right)-\left\langle\hat{k}\right\rangle\right)+\left(\hat{k}_1^B\left(\bar{X}',\left\langle\bar{X}\right\rangle\right)+\kappa\frac{\hat{k}_2^{(B)}\left(\bar{X}',\left\langle\bar{X}\right\rangle\right)}{1+\underline{\bar{k}}}-\left(\left\langle\hat{k}_1^B\right\rangle+\kappa\frac{\left\langle\hat{k}_2^{(B)}\right\rangle}{1+\underline{\bar{k}}}\right)\right)Z \\
=\quad & 1+\left[\hat{\underline{k}}\left(\hat{X}'\right)\right]_\kappa
\end{aligned}$$

and:

$$\begin{aligned}
1+\underline{\bar{k}}_2\left(\bar{X}_1\right) &= \frac{\left(1-\left\langle\bar{k}_1\right\rangle\right)\left(1+\frac{\bar{k}_2\left(\bar{X}_1\right)}{1-\left\langle\bar{k}_1\right\rangle}\right)}{1-\left\langle\bar{k}\right\rangle} \\
&= \frac{1-\left\langle\bar{k}\right\rangle+\bar{k}_2\left(\bar{X}_1,\left\langle\bar{X}\right\rangle\right)-\left\langle\bar{k}_2\right\rangle}{1-\left\langle\bar{k}\right\rangle} \\
&= 1+\frac{\bar{k}_2\left(\bar{X}_1,\left\langle\bar{X}\right\rangle\right)}{1-\left\langle\bar{k}\right\rangle}=1+\bar{k}_2^n\left(\bar{X}_1\right)
\end{aligned}$$

We will write:

$$\frac{1}{1+\underline{\bar{k}}_2\left(\bar{X}_1\right)}\rightarrow\frac{1}{1+\bar{k}_2^n\left(\bar{X}_1\right)}$$

and:

$$1+\underline{\bar{k}}\left(\bar{X}'\right)\simeq 1+\left(\bar{k}\left(\bar{X}',\left\langle\bar{X}\right\rangle\right)-\left\langle\bar{k}\right\rangle\right)$$

A22.2 Definition of diffusion functions

We consider the terms of (359) proportional to $\bar{g}\left(\bar{K}',\bar{X}'\right)$, this defines the functns $\bar{S}_1^E\left(\hat{X}',\hat{X}_1\right)$and $\bar{S}_1^B\left(\hat{X}',\hat{X}_1\right)$:

$$\begin{aligned}
& \left(1-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}_1\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)\bar{g}\left(\bar{K}',\bar{X}'\right)}{1+\bar{k}_2^n\left(\bar{X}'\right)}-\frac{\bar{K}'\hat{\underline{k}}_1^B\left(\hat{X}',\bar{X}_1\right)}{1+\left[\hat{\underline{k}}^n\left(\hat{X}'\right)\right]_\kappa}\frac{\bar{N}\bar{g}\left(\bar{K}',\bar{X}'\right)}{1+\left[\hat{\underline{k}}_2^n\left(\hat{X}'\right)\right]_\kappa} \\
\rightarrow\quad & 1-\bar{S}_1^E\left(\bar{X}',\bar{X}_1\right)-\bar{S}_1^B\left(\hat{X}',\bar{X}_1\right)
\end{aligned}$$

with:

$$\bar{S}_1^E\left(\bar{X}',\bar{X}_1\right)\rightarrow\left(\frac{\bar{k}\left(\bar{X}',\bar{X}_1\right)-\frac{\underline{\bar{k}}_1\left(\left\langle X\right\rangle,\bar{X}_1\right)\bar{k}\left(\left\langle X\right\rangle,\bar{X}_1\right)}{1+\underline{\bar{k}}\left(\left\langle\bar{X}\right\rangle\right)}}{1+\bar{k}_2^n\left(\bar{X}'\right)}+\frac{\bar{k}_1\left(\bar{X}',\bar{X}_1\right)}{1+\bar{k}_2^n\left(\bar{X}'\right)}\right)\frac{1}{1+\underline{\bar{k}}\left(\bar{X}'\right)}$$

and:

$$\begin{aligned}
\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right) &\rightarrow \frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}_1\right)}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\frac{\bar{N}\left(\bar{X}',\bar{X}_1\right)}{1+\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\frac{\hat{\underline{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}\left(\bar{X}\right)}} \\
&= \frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}_1\right)}{1+\left[\hat{\underline{k}}^n\left(\hat{X}\right)\right]_\kappa}\frac{\bar{N}\left(\bar{X}',\bar{X}_1\right)}{1+\left[\hat{\underline{k}}_2^n\left(\hat{X}'\right)\right]_\kappa}
\end{aligned}$$

The coefficient $\bar{N}\left(\bar{X}', \bar{X}_1\right)$ is defined by:

$$\begin{aligned}
&\bar{N}\left(\bar{X}', \bar{X}_1\right) \\
= &\frac{\left(\hat{k}_1^B\left(\hat{X}, \bar{X}'\right) + \kappa \frac{\hat{k}_2^B\left(\hat{X}, \bar{X}'\right)}{1+\int \bar{\bar{k}}\left(\bar{X}', \bar{X}''\right)\bar{K}_0''\left|\bar{\Psi}\left(\bar{K}_0'', \bar{X}''\right)\right|^2} - \kappa \int \frac{\hat{k}_2^B\left(\bar{X}, \bar{X}''\right)\bar{K}_0''\bar{\bar{k}}\left(\bar{X}'', \bar{X}'\right)}{\left(1+\int \bar{\bar{k}}\left(\bar{X}'', \bar{Y}\right)\bar{K}_0^Y\left|\bar{\Psi}\left(\bar{K}_0^Y, \bar{Y}\right)\right|^2\right)^2}\right)\hat{K}}{1+\int \hat{k}\left(\hat{X}, \hat{X}'\right)\left|\hat{\Psi}\left(\hat{K}', \hat{X}'\right)\right|^2 + \int \hat{k}_1^B\left(\hat{X}, \bar{X}'\right)\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0', \bar{X}'\right)\right|^2 + \kappa \int \hat{k}_2^B\left(\hat{X}, \bar{X}'\right)\frac{\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0', \bar{X}'\right)\right|^2}{1+\int \bar{\bar{k}}\left(\bar{X}', \bar{X}''\right)\bar{K}_0''\left|\bar{\Psi}\left(\bar{K}_0'', \bar{X}''\right)\right|^2}} \\
\rightarrow &\frac{\left(\hat{k}_1^B\left(\hat{X}, \bar{X}'\right) + \kappa \frac{\hat{\underline{k}}_2^B\left(\hat{X}\right)}{1+\underline{\bar{\bar{k}}}}\left(1 - \frac{\kappa}{1+\underline{\bar{\bar{k}}}}\right)\right)\hat{K}}{1+\left[\hat{\underline{k}}_2^n\left(\hat{X}\right)\right]_\kappa} \rightarrow \frac{\left(\hat{k}_1^B\left(\hat{X}\right) + \kappa \frac{\hat{\underline{k}}_2^B\left(\hat{X}\right)}{1+\underline{\bar{\bar{k}}}}\left(1 - \frac{\kappa}{1+\underline{\bar{\bar{k}}}}\right)\right)}{1+\left[\hat{\underline{k}}^n\left(\hat{X}\right)\right]_\kappa}
\end{aligned}$$

The terms proportional to $\hat{g}\left(\hat{K}', \hat{X}'\right)$ define the diffusion function $\hat{S}_1^B\left(\bar{X}', \bar{X}_1\right)$:

$$\begin{aligned}
&\frac{\hat{K}'\hat{\underline{k}}_1^B\left(\hat{X}', \bar{X}\right)}{1+\hat{\underline{k}}\left(\hat{X}'\right) + \hat{\underline{k}}_1^B\left(\bar{X}'\right) + \kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)} \frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}', \hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\bar{X}'\right) + \kappa \frac{\hat{\underline{k}}_2^B\left(\bar{X}'\right)}{1+\hat{k}\left(\bar{X}\right)}} \\
\rightarrow \quad &\frac{\hat{K}'\hat{\underline{k}}_1^B\left(\hat{X}', \bar{X}\right)}{1+\left[\hat{\underline{k}}\left(\hat{X}'\right)\right]_\kappa} \frac{1}{1+\left[\hat{\underline{k}}_2\left(\hat{X}\right)\right]_\kappa}\left(1 - \frac{\hat{k}\left(\hat{X}, \hat{X}'\right)\hat{K}_X}{1+\left[\hat{\underline{k}}\left(\hat{X}'\right)\right]_\kappa}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2\right)\hat{g}\left(\hat{K}', \hat{X}'\right) \\
= \quad &\hat{S}_1^B\left(\bar{X}', \bar{X}_1\right)\hat{g}\left(\hat{K}', \hat{X}'\right)
\end{aligned}$$

and we write:

$$\hat{S}_1^B\left(\bar{X}', \bar{X}_1\right) \simeq \frac{\hat{K}'\hat{\underline{k}}_1^B\left(\hat{X}', \bar{X}\right)}{1+\left[\hat{\underline{k}}\left(\hat{X}'\right)\right]_\kappa} \frac{1}{1+\left[\hat{\underline{k}}_2\left(\hat{X}\right)\right]_\kappa}\left(1 - \frac{\hat{k}\left(\hat{X}, \hat{X}'\right)\hat{K}_X}{1+\left[\hat{\underline{k}}\left(\hat{X}'\right)\right]_\kappa}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2\right)$$

Ultimately, we write:

$$\frac{\underline{k}_1^{(B)}\left(X', \bar{X}\right)}{1+\underline{k}\left(\hat{X}'\right) + \underline{k}_1^{(B)}\left(\bar{X}'\right) + \kappa \frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{k}}} \rightarrow \frac{1-\beta^B}{1+\delta}$$

A22.3 Writing (359)

$$\frac{1}{3}\frac{1-\beta^B}{1+\delta}\left(\left(\frac{A\left(\bar{X}_1\right)}{f_1^2\left(\bar{X}_1\right)}+\frac{B\left(\bar{X}_1\right)}{f_1^3\left(\bar{X}_1\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,\bar{X}_1\right)\right)\right)\right) \tag{360}$$

$$-\left(\frac{\left(\left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2-\hat{\mu}D\left(\bar{X}_1\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{Bef}\left(\bar{X}_1\right)}{\langle\bar{g}\rangle}\right)}}{\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\sqrt{\underline{k}\left(\bar{X}_1\right)}}-\frac{\bar{r}'}{1+\bar{k}_2^n\left(\bar{X}_1\right)}\right)$$

$$= -\left(\bar{S}_1^E\left(\bar{X}',\hat{X}_1\right)+\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right)\left(\frac{\left(\left\|\bar{\Psi}_0\left(\bar{X}'\right)\right\|^2-\hat{\mu}D\left(\bar{X}'\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{Bef}\left(\bar{X}'\right)}{\langle\bar{g}\rangle}\right)}}{\sqrt{\underline{k}\left(\bar{X}'\right)}}-\bar{r}'\right)$$

$$-\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)\left(\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{ef}\left(\hat{X}'\right)}{\langle\hat{g}\rangle}\right)}}{\sqrt{\underline{k}\left(\hat{X}'\right)}}-\bar{r}'\right)$$

Using (147):

$$\frac{1}{3}\frac{1-\beta^B}{1+\delta}\left(\left(\frac{A\left(\bar{X}_1\right)}{f_1^2\left(\bar{X}_1\right)}+\frac{B\left(\bar{X}_1\right)}{f_1^3\left(\bar{X}_1\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,\bar{X}_1\right)\right)\right)\right)$$

$$= \frac{1-\beta^B}{(1-\beta)\delta}\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}-\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)\left(\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{\frac{\sigma_{\hat{K}}^2\left(1-\frac{\hat{g}^{ef}\left(\hat{X}'\right)}{\langle\hat{g}\rangle}\right)}{2\hat{\mu}\hat{K}\left[\hat{X}'\right]}}-\bar{r}'\right)$$

we rewrite (360):where:

$$\frac{1-\beta^B}{(1-\beta)\delta}\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}-\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)\left(\frac{\left(\left\|\bar{\Psi}_0\left(\bar{X}'\right)\right\|^2-\hat{\mu}D\left(\bar{X}'\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{Bef}\left(\bar{X}'\right)}{\langle\bar{g}\rangle}\right)}}{\sqrt{\underline{k}\left(\bar{X}'\right)}}-\bar{r}'\right)$$

$$-\left(\frac{\left(\left\|\bar{\Psi}_0\left(\bar{X}_1\right)\right\|^2-\hat{\mu}D\left(\bar{X}_1\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{Bef}\left(\bar{X}_1\right)}{\langle\bar{g}\rangle}\right)}}{\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\sqrt{\underline{k}\left(\bar{X}_1\right)}}-\frac{\bar{r}'}{1+\bar{k}_2^n\left(\bar{X}_1\right)}\right)$$

$$= -\left(\bar{S}_1^E\left(\bar{X}',\hat{X}_1\right)+\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right)\left(\frac{\left(\left\|\bar{\Psi}_0\left(\bar{X}'\right)\right\|^2-\hat{\mu}D\left(\bar{X}'\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{Bef}\left(\bar{X}'\right)}{\langle\bar{g}\rangle}\right)}}{\sqrt{\underline{k}\left(\bar{X}'\right)}}-\bar{r}'\right)$$

$$-\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)\left(\frac{\left(\left\|\hat{\Psi}_0\left(\hat{X}'\right)\right\|^2-\hat{\mu}D\left(\hat{X}'\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{ef}\left(\hat{X}'\right)}{\langle\hat{g}\rangle}\right)}}{\sqrt{\underline{k}\left(\hat{X}'\right)}}-\bar{r}'\right)$$

and we find the followng equation:

$$\begin{aligned}
&\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\frac{1-\beta^B}{(1-\beta)\delta}\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}-\hat{S}_1^E\left(\hat{X}',\bar{X}_1\right)\right)+\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)\right)\\
&\times\left(\frac{\left(\left\|\bar{\Psi}_0\left(\bar{X}'\right)\right\|^2-\hat{\mu}D\left(\bar{X}'\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{Bef}\left(\bar{X}'\right)}{\langle\bar{g}\rangle}\right)}}{\sqrt{\underline{k}\left(\bar{X}'\right)}}-\bar{r}'\right)\\
=&\left(1-\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\bar{S}_1^E\left(\bar{X}',\hat{X}_1\right)+\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right)\right)\left(\frac{\left(\left\|\bar{\Psi}_0\left(\bar{X}'\right)\right\|^2-\hat{\mu}D\left(\bar{X}'\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{Bef}\left(\bar{X}'\right)}{\langle\bar{g}\rangle}\right)}}{\sqrt{\underline{k}\left(\bar{X}'\right)}}-\bar{r}'\right)
\end{aligned}$$

and we find the following expression:

$$\begin{aligned}
&\frac{\left(\left\|\bar{\Psi}_0\left(\bar{X}'\right)\right\|^2-\hat{\mu}D\left(\bar{X}'\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{Bef}\left(\bar{X}'\right)}{\langle\bar{g}\rangle}\right)}}{\sqrt{\underline{k}\left(\bar{X}'\right)}}-\bar{r}'\\
=&\left(1-\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\bar{S}_1^E\left(\bar{X}',\hat{X}_1\right)+\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right)\right)^{-1}\\
&\times\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\frac{1-\beta^B}{(1-\beta)\delta}\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}-\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)+\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)\right)\\
&\times\left(\frac{\left(\left\|\bar{\Psi}_0\left(\bar{X}'\right)\right\|^2-\hat{\mu}D\left(\bar{X}'\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{Bef}\left(\bar{X}'\right)}{\langle\bar{g}\rangle}\right)}}{\sqrt{\underline{k}\left(\bar{X}'\right)}}-\bar{r}'\right)
\end{aligned}$$

or the equivalent formulation:

$$\begin{aligned}
&\frac{\left(\left\|\bar{\Psi}_0\left(\bar{X}'\right)\right\|^2-\hat{\mu}D\left(\bar{X}'\right)\right)\sqrt{k\frac{\sigma_{\hat{K}}^2}{2\hat{\mu}}\left(1-\frac{\hat{g}^{Bef}\left(\bar{X}'\right)}{\langle\bar{g}\rangle}\right)}}{\sqrt{\underline{k}\left(\bar{X}'\right)}}-\bar{r}'\\
=&\left(1-\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\bar{S}_1^E\left(\bar{X}',\hat{X}_1\right)+\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right)\right)^{-1}\\
&\times\left(1+\bar{k}_2^n\left(\bar{X}_1\right)\right)\left(\frac{1-\beta^B}{(1-\beta)\delta}\left(\frac{\Delta\left(\hat{X},\hat{X}'\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}-\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)+\hat{S}_1^B\left(\hat{X}',\bar{X}_1\right)\right)\\
&\times\left(1-\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1}\\
&\times\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\left(\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,\hat{X}'\right)\right)\right)\right)
\end{aligned}$$

where:

$$\delta \rightarrow \frac{\frac{k\hat{K}[\hat{X}_1]}{\sigma^2_{\hat{K}}} + k^B D(\bar{X}_1) A + \sqrt{\left(\frac{k\hat{K}[\hat{X}_1]}{\sigma^2_{\hat{K}}} + k^B D(\bar{X}_1) A\right)^2 + 4\left(\left\|\bar{\Psi}_0(\bar{X}_1)\right\|^4 - k^B D(\bar{X}_1)\bar{r}'\right)\frac{k\hat{K}[\hat{X}_1]}{\sigma^2_{\hat{K}}} A}}{2\left(\left\|\bar{\Psi}_0(\bar{X}_1)\right\|^4 - k^B D(\bar{X}_1)\bar{r}'\right)}$$

$$A = \frac{\left(1-\beta^B\right)\left(1+\bar{\beta}\bar{k}(\bar{X}_1)\right)\left(\left\|\hat{\Psi}_0\left(\hat{X}_1\right)\right\|^4 - \bar{r}'\left(\frac{\hat{K}[\hat{X}_1]}{\sigma^2_{\hat{K}}} + D\left(\hat{X}_1\right)\right)\right)}{\left(\frac{\hat{K}[\hat{X}_1]}{\sigma^2_{\hat{K}}} + D\left(\hat{X}_1\right)\right)\left(\frac{\hat{K}[\hat{X}_1]}{\sigma^2_{\hat{K}}} + D\left(\hat{X}_1\right)\right)(1-\beta)\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)}$$

$$\bar{K}\left[\bar{X}\right] = \frac{\underline{k}^B\left(\bar{X}\right)}{k^B\left(\bar{X}\right)} = \frac{k(X)\hat{K}\left[\hat{X}\right]}{\delta k^B\left(\bar{X}\right)}$$

$$\delta\left(\hat{X}_1\right) = \delta\left(\left\langle\hat{X}\right\rangle, \hat{X}_1\right)$$

Appendix 23 Averages quantities, investors and bank returns

We compute the remaining average quantities: firm capital and return, investors and banks average returns.

28.1 A23.1 Firms average capital

28.1.1 A23.1.1 Average coefficients

We define the following averages coefficients arising in the formula for average capital.

The coefficient k was defined by:

$$k \simeq k(X)\frac{\hat{K}\left[\hat{X}\right]}{K_X\left\|\Psi_0\right\|^2}$$

so that in average:

$$\langle k\rangle \simeq k\frac{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}{\langle K\rangle\left\|\Psi_0\right\|^2}$$

$$\begin{aligned}
\left\langle\underline{\hat{k}}(X)\right\rangle &= \left\langle\hat{k}(X,X')\frac{\hat{K}_{X'}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}\right\rangle \simeq \hat{k}\left(\langle X\rangle, \langle X\rangle\right) = \left\langle\hat{k}\right\rangle = \hat{k} \\
\left\langle\underline{k}(X)\right\rangle &= \left\langle k(X,X')\frac{\hat{K}_{X'}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2}{\langle K\rangle\left\|\Psi\left(\hat{X}\right)\right\|^2}\right\rangle \\
&\simeq k\left(\langle X\rangle, \left\langle\hat{X}\right\rangle\right)\frac{\left\langle\hat{K}\right\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}\right\|^2}{\left\|\Psi\right\|^2} \simeq \langle k\rangle\frac{\left\langle\hat{K}\right\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\left\|\Psi_0\right\|^2}
\end{aligned}$$

$$\left\langle X^{(e)}\right\rangle \rightarrow \sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle-\frac{1}{2}\left(\beta\underline{k}\left(\langle f_1\rangle-\bar{r}\right)\right)}$$

$$\simeq \sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\frac{\beta\underline{k}\left(\langle f_1\rangle-\bar{r}\right)}{4\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}}=\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\frac{\beta\langle\underline{k}\rangle\frac{\langle\hat{K}\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\left(\langle f_1\rangle-\bar{r}\right)}{4\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}}$$

$$\left\langle C^{(e)}\right\rangle \simeq \frac{\beta\delta+\beta^B}{1+\delta}C$$

The effective productivity becomes in average:

$$\left\langle f_1^{(e)}\right\rangle=\langle f_1\rangle+\beta\underline{k}\left(\langle f_1\rangle-\bar{r}\right)=\langle f_1\rangle+\beta\langle\underline{k}\rangle\frac{\left\langle\hat{K}\right\rangle}{\langle K\rangle}\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\left(\langle f_1\rangle-\bar{r}\right)$$

The ratio of bank capital invested in firms over investors capital invested in firms is:

$$\langle\delta\rangle\simeq\frac{\left\langle k_1^B\right\rangle+\frac{\kappa\left\langle k_2^B\right\rangle}{1+\langle\bar{k}\rangle}}{\langle k\rangle}Z\simeq\left(\frac{\left\langle k_1^B\right\rangle}{\langle k\rangle}+\frac{\kappa\left\langle k_2^B\right\rangle}{\langle k\rangle\left(1+\langle\bar{k}\rangle\right)}\right)\frac{\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}$$

A23.1.2 Formula for average capital

A23.1.2.1 Firms average capital We use that:

$$\langle K\rangle=\frac{1}{4\left\langle f_1^{(e)}\right\rangle}\left\langle\frac{\left(3X^{(e)}-C^{(e)}\right)\left(C^{(e)}+X^{(e)}\right)}{2X^{(e)}-C^{(e)}}\right\rangle$$

To rewrite average capital as:

$$\langle K\rangle \simeq \left(1-\frac{4\left(\beta\langle\underline{k}\rangle\frac{\left\|\hat{\Psi}_0\right\|^2}{\|\Psi_0\|^2}\langle K\rangle\left(\langle f_1\rangle-\bar{r}\right)\right)\left(2\left\langle X^{(e)}\right\rangle-\frac{\beta\delta+\beta^B}{1+\delta}C\right)}{\left(3\left\langle X^{(e)}\right\rangle-\frac{\beta\delta+\beta^B}{1+\delta}C\right)\left(\left\langle X^{(e)}\right\rangle+\frac{\beta\delta+\beta^B}{1+\delta}C\right)}\right)$$
$$\times\left(3\left\langle X^{(e)}\right\rangle-\frac{\beta\delta+\beta^B}{1+\delta}C\right)\left(\left\langle X^{(e)}\right\rangle+\frac{\beta\delta+\beta^B}{1+\delta}C\right)$$
$$\simeq\frac{\left(3\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\frac{\beta\delta+\beta^B}{1+\delta}C\right)\left(\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}+\frac{\beta\delta+\beta^B}{1+\delta}C\right)}{4\left\langle f_1^{(e)}\right\rangle\left(2\sqrt{\frac{|\Psi_0|^2}{\epsilon}-\frac{1}{2}\langle f_1\rangle}-\frac{\beta\delta+\beta^B}{1+\delta}C\right)}$$

A23.1.2.2 Investors average amount of capital

$$\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2 = \hat{\mu}V\frac{\left\langle\hat{K}_0\right\rangle^4}{2\sigma_{\hat{K}}^2}\left(\frac{1}{4}-\frac{\left\langle\hat{K}\right\rangle}{3\left\langle\hat{K}_0\right\rangle}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\langle\hat{g}\rangle^2$$
$$=\frac{18\sigma_{\hat{K}}^2}{\hat{\mu}}V\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\langle\hat{g}\rangle^2\left(5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)}\right)^2$$
$$\times\left(\frac{1}{4}-\frac{1}{18}\left(2+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\right)\right)\langle\hat{g}\rangle^2$$

$$\langle k\rangle \quad \simeq \quad k\frac{\sigma_{\hat{K}}^{2}}{\hat{\mu}}V\left(\frac{\left\|\bar{\Psi}_{0}\right\|^{2}-\frac{\hat{\mu}}{6}\left(\frac{\left\langle\hat{K}\right\rangle^{2}\langle\hat{g}\rangle}{\sigma_{\hat{K}}^{2}}\right)\left\langle\hat{g}^{Bef}\right\rangle\hat{l}}{\langle\bar{g}\rangle^{2}}\right)^{2}\left(\frac{1}{2}-\frac{2+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}}{9}\right)\langle\hat{g}\rangle^{2}$$

$$\times\frac{4\left\langle f_{1}^{(e)}\right\rangle\left(2\sqrt{\frac{|\Psi_{0}|^{2}}{\epsilon}-\frac{1}{2}\langle f_{1}\rangle}-\frac{\beta\delta+\beta^{B}}{1+\delta}C\right)}{\left(3\sqrt{\frac{|\Psi_{0}|^{2}}{\epsilon}-\frac{1}{2}\langle f_{1}\rangle}-\frac{\beta\delta+\beta^{B}}{1+\delta}C\right)\left(\sqrt{\frac{|\Psi_{0}|^{2}}{\epsilon}-\frac{1}{2}\langle f_{1}\rangle}+\frac{\beta\delta+\beta^{B}}{1+\delta}C\right)}$$

where:

$$\left\langle\underline{k}^{B}\right\rangle=\left\langle k_{1}^{B}\right\rangle\frac{\left\langle\bar{K}_{0}\right\rangle}{\langle K\rangle}\frac{\left\|\bar{\Psi}_{0}\right\|^{2}}{\left\|\Psi_{0}\right\|^{2}}+\left\langle k_{2}^{B}\right\rangle\frac{\kappa\left\langle\bar{K}_{0}\right\rangle}{\langle K\rangle}\frac{\left\|\bar{\Psi}_{0}\right\|^{2}}{\left\|\Psi_{0}\right\|^{2}}$$

$$\langle\delta\rangle\simeq\frac{\left\langle k_{1}^{B}\right\rangle+\frac{\kappa\left\langle k_{2}^{B}\right\rangle}{1+\langle\bar{k}\rangle}}{\langle k\rangle}Z\simeq\left(\frac{\left\langle k_{1}^{B}\right\rangle}{\langle k\rangle}+\frac{\kappa\left\langle k_{2}^{B}\right\rangle}{\langle k\rangle\left(1+\langle\bar{k}\rangle\right)}\right)\frac{\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^{2}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}$$

$$\left\langle\hat{g}^{ef}\right\rangle=-\frac{\left(\kappa\left\langle\left[\frac{\hat{\underline{k}}_{2}^{B}}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\hat{\underline{k}}\right\rangle\right)+\left\langle\hat{k}_{1}^{B}\right\rangle\left\langle\hat{k}_{2}\right\rangle\right)\left(\langle\hat{g}\rangle+\frac{1}{1-\left\langle\hat{\underline{k}}\right\rangle}\bar{N}\langle\bar{g}\rangle\right)\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^{2}}{\langle\hat{K}\rangle\|\hat{\Psi}\|^{2}}}{\left(1-\left(\left\langle\hat{k}_{1}\right\rangle+\left\langle\hat{k}_{1}^{B}\right\rangle\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^{2}}{\langle\hat{K}\rangle\|\hat{\Psi}\|^{2}}\right)\right)\left(1-\left(\left\langle\hat{\underline{k}}\right\rangle+\left(\left\langle\hat{\underline{k}}_{1}^{B}\right\rangle+\kappa\left\langle\left[\frac{\hat{\underline{k}}_{2}^{B}}{1+\bar{k}}\right]\right\rangle\right)\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^{2}}{\langle\hat{K}\rangle\|\hat{\Psi}\|^{2}}\right)\right)}$$

and:

$$\left\langle\hat{g}^{Bef}\right\rangle \quad = \quad -\frac{\kappa\left\langle\left[\frac{\hat{\underline{k}}_{2}^{B}}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\hat{k}\right\rangle\right)+\left\langle\hat{k}_{1}^{B}\right\rangle\left\langle\hat{k}_{2}\right\rangle}{\left(1-\left(\left\langle\hat{k}\right\rangle+\left(\left\langle\hat{k}_{1}^{B}\right\rangle+\kappa\left\langle\left[\frac{\hat{\underline{k}}_{2}^{B}}{1+\bar{k}}\right]\right\rangle\right)\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^{2}}{\langle\hat{K}\rangle\|\hat{\Psi}\|^{2}}\right)\right)\left(1-\left(\left\langle\hat{k}_{1}\right\rangle+\left\langle\hat{k}_{1}^{B}\right\rangle\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^{2}}{\langle\hat{K}\rangle\|\hat{\Psi}\|^{2}}\right)\right)}$$
$$\times\left(\langle\hat{g}\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\langle\bar{g}\rangle\right)$$

with the relation:

$$\left\langle\hat{g}^{Bef}\right\rangle\simeq\frac{\left\langle\hat{g}^{ef}\right\rangle}{\frac{\langle\bar{K}\rangle\|\bar{\Psi}\|^{2}}{\langle\hat{K}\rangle\|\hat{\Psi}\|^{2}}}$$

A23.1.2.3 Banks average amount of capital Using (341):

$$\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^{2}\simeq18\frac{\sigma_{\hat{K}}^{2}}{\langle\bar{g}\rangle^{2}\hat{\mu}}\left(\frac{\sqrt{\langle\hat{g}\rangle^{2}\left(1+\frac{3}{4}\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}\left\|\bar{\Psi}_{0}\right\|-\frac{3}{8}\langle\hat{g}\rangle\frac{\left\langle\hat{g}^{Bef}\right\rangle}{\langle\hat{g}\rangle}\left\|\hat{\Psi}_{0}\right\|}{\sqrt{5+\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\left\langle\hat{g}^{ef}\right\rangle}{\langle\hat{g}\rangle}\right)}}}\right)^{4}\left(\frac{1}{4}-\frac{\langle\hat{g}\rangle\left\langle\hat{g}^{Bef}\right\rangle}{3\langle\bar{g}\rangle^{2}}\right) \qquad (361)$$

A23.2 Investors average returns

A23.2.1 Average coefficients

Using normalizations:

$$\begin{aligned}\left\langle \frac{1}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa}\right\rangle &\simeq \frac{1}{1+\left\langle \hat{k}_2^{(n)}\right\rangle +\kappa\left\langle \frac{\underline{\hat{k}}_2^{B(n)}}{1+\bar{k}}\right\rangle \left(\left(1+\frac{3}{4}\frac{\left\langle \hat{g}^{ef}\right\rangle}{\left\langle \hat{g}\right\rangle}\right)\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}\right)}\\ &\simeq \frac{1}{1+\left\langle \hat{k}_2^{(n)}\right\rangle +\kappa\left\langle \frac{\underline{\hat{k}}_2^{B(n)}}{1+\bar{k}}\right\rangle \frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}}\end{aligned}$$

$$\begin{aligned}&1+\left[\underline{\hat{k}}\left(\hat{X}'\right)\right]_\kappa\\ =\ &1+\left(\hat{k}\left(\hat{X}',\left\langle \hat{X}\right\rangle\right)-\left\langle \hat{k}\right\rangle\right)+\left(\hat{k}_1^B\left(\bar{X}',\left\langle \bar{X}\right\rangle\right)+\kappa\frac{\hat{k}_2^{(B)}\left(\bar{X}',\left\langle \bar{X}\right\rangle\right)}{1+\underline{\bar{k}}}-\left(\left\langle \hat{k}_1^B\right\rangle+\kappa\frac{\left\langle \hat{k}_2^{(B)}\right\rangle}{1+\underline{\bar{k}}}\right)\right)Z\end{aligned}$$

in averages:

$$\left\langle 1+\left[\underline{\hat{k}}\left(\hat{X}_1\right)\right]_\kappa\right\rangle \simeq 1$$

$$\left\langle 1+\left[\underline{\hat{k}}_2^n\left(\hat{X}\right)\right]_\kappa\right\rangle =1+\left\langle \hat{k}_2^n\right\rangle =1+\frac{\left\langle \hat{k}_2^{(n)}\right\rangle +\kappa\left\langle \frac{\underline{\hat{k}}_2^{B(n)}}{1+\bar{k}}\right\rangle \frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}}{1-\left\langle \hat{k}^\Sigma\right\rangle}$$

$$\left\langle \frac{\bar{k}_1\left(\bar{X}',\bar{X}_1\right)}{1+\left[\underline{\hat{k}}^n\left(\hat{X}\right)\right]_\kappa}\right\rangle \left\langle \frac{\bar{N}\left(\bar{X}',\bar{X}_1\right)}{1+\left[\underline{\hat{k}}_2^n\left(\hat{X}'\right)\right]_\kappa}\right\rangle \simeq \frac{\left\langle \bar{k}_1\right\rangle}{1+\left\langle \hat{k}_2^n\right\rangle}\left\langle \bar{N}\left(\bar{X}',\bar{X}_1\right)\right\rangle$$

A23.2.2 Average return equation

In return equation for investors:

$$\begin{aligned}\hat{g}\left(\hat{X}_1\right)-\bar{r}' =& \int\left(1-\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1}\\ &\times\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,\hat{X}'\right)\right)\right)\end{aligned}$$

we estimate the average inverse diffusion matrix:

$$\left(1+\left[\underline{\hat{k}}_2^n\left(\hat{X}_1\right)\right]_\kappa\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\simeq\left(1+\left\langle \hat{k}_2^n\right\rangle\right)\left(\left\langle \hat{k}\right\rangle-\left\langle \hat{k}_1\right\rangle\left\langle k\right\rangle+\left\langle \hat{k}_1\right\rangle\right)=\left(1+\left\langle \hat{k}_2^n\right\rangle\right)\left\langle \hat{k}\right\rangle\left(1-\left\langle \hat{k}_1\right\rangle\right)+\left\langle \hat{k}_1\right\rangle$$

Using that in average:

$$\left\langle \Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right\rangle =0$$

we find:

$$\left(1-\left(1+\underline{\hat{k}}_2\left(\hat{X}_1\right)\right)\hat{S}_1^E\left(\hat{X}',\hat{X}_1\right)\right)^{-1}\simeq\left(1-\left(1+\left\langle \hat{k}_2^n\right\rangle\right)\left(\left\langle \hat{k}\right\rangle\left(1-\left\langle \hat{k}_1\right\rangle\right)+\left\langle \hat{k}_1\right\rangle\right)\right)^{-1}$$

$$\langle \hat{g} \rangle = \bar{r}' + \left(1+\left\langle \hat{k}_2^n \right\rangle\right)\left(1-\left(1+\left\langle \hat{k}_2^n \right\rangle\right)\left(\left\langle \hat{k} \right\rangle\left(1-\left\langle \hat{k}_1 \right\rangle\right)+\left\langle \hat{k}_1 \right\rangle\right)\right)^{-1}$$
$$\times\left\langle\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,\hat{X}'\right)\right)\right)\right\rangle$$

that is:

$$\langle \hat{g} \rangle \simeq \bar{r}' + \left(1-\left(1+\left\langle \hat{k}_2^n \right\rangle\right)\left(\left\langle \hat{k} \right\rangle\left(1-\left\langle \hat{k}_1 \right\rangle\right)+\left\langle \hat{k}_1 \right\rangle\right)\right)^{-1}$$
$$\times\left(1+\left\langle \hat{k}_2^n \right\rangle\right)\left\langle\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,\hat{X}'\right)\right)\right)\right\rangle$$

A23.3 Banks average returns

A23.3.1 Average returns

$$\langle \bar{g} \rangle = \bar{r}' + \left(1+\bar{\beta}\left\langle \underline{\bar{k}} \right\rangle\right)\frac{\left(1-\beta^B\right)}{(1-\beta)\langle\delta\rangle}$$
$$\times\left[\left(1-\left(1+\bar{\beta}\left\langle \underline{\bar{k}} \right\rangle\right)\left(\left\langle \bar{S}_1^E \right\rangle+\left\langle \bar{S}_1^B \right\rangle\right)\right)\right]^{-1}\left(\frac{1}{\left(\frac{1}{1+\underline{\bar{k}}_2^n\left(\bar{X}'\right)}\right)\left\langle \hat{k}^B \right\rangle}-\left(\left\langle \hat{S}_1^E \right\rangle-\frac{(1-\beta)\langle\delta\rangle}{1-\beta^B}\left\langle \hat{S}_1^B \right\rangle\right)\right)\left(\langle \hat{g} \rangle-\bar{r}'\right)$$

Using;

$$\left\langle \frac{1}{1+\underline{\bar{k}}\left(\bar{X}'\right)} \right\rangle \simeq 1$$
$$\left\langle \frac{1}{1+\underline{\bar{k}}_2^n\left(\bar{X}'\right)} \right\rangle \simeq \frac{1}{1+\left\langle \underline{\bar{k}}_2^n \right\rangle}=\frac{1}{1+\frac{\left\langle \bar{k}_2 \right\rangle}{1-\left\langle \bar{k} \right\rangle}}$$

$$\langle \bar{g} \rangle = \bar{r}' + \frac{\left(1-\beta^B\right)}{(1-\beta)\langle\delta\rangle}\left[\left(1-\left(\left\langle \bar{S}_1^E \right\rangle+\left\langle \bar{S}_1^B \right\rangle\right)\right)\right]^{-1}\left(\frac{1}{\left\langle \hat{k}^B \right\rangle}-\left(\left\langle \hat{S}_1^E \right\rangle-\frac{(1-\beta)\langle\delta\rangle}{1-\beta^B}\left\langle \hat{S}_1^B \right\rangle\right)\right)\left(\langle \hat{g} \rangle-\bar{r}'\right)$$

The terms arising in this expression can be rewritten:

$$\left\langle \bar{S}_1^E \right\rangle = \left\langle \bar{k} \right\rangle\left(1-\left\langle \bar{k}_1 \right\rangle\right)+\left\langle \bar{k}_1 \right\rangle$$

$$\bar{S}_1^E\left(\bar{X}',\bar{X}_1\right) \rightarrow \left(1-\left\langle \bar{k}_1 \right\rangle\right)\left\langle \bar{k} \right\rangle+\left\langle \bar{k}_1 \right\rangle$$

$$\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right) \rightarrow \frac{\underline{\hat{k}}_1^B\left(\bar{X}',\bar{X}_1\right)}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\frac{\bar{N}\left(\bar{X}',\bar{X}_1\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\frac{\underline{\hat{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}\left(\bar{X}\right)}}$$

Given that;

$$\bar{N}\left(\bar{X}',\bar{X}_1\right) \rightarrow \frac{\left(\hat{k}_1^B\left(\hat{X}\right)+\kappa\frac{\underline{\hat{k}}_2^B\left(\hat{X}\right)}{1+\underline{\bar{\bar{k}}}}\left(1-\frac{\kappa}{1+\underline{\bar{\bar{k}}}}\right)\right)}{1+\underline{\hat{k}}\left(\hat{X}\right)+\underline{\hat{k}_1^B}\left(\hat{X}\right)+\kappa\frac{\underline{\hat{k}}_2^B\left(\hat{X}\right)}{1+\underline{\bar{\bar{k}}}}}$$

the average $\left\langle \bar{N}\left(\bar{X}',\bar{X}_1\right)\right\rangle$ is given by:

$$\begin{aligned}\left\langle \bar{N}\left(\bar{X}',\bar{X}_1\right)\right\rangle &= \left\langle \hat{\underline{k}}_1^B\right\rangle + \kappa\frac{\left\langle \hat{\underline{k}}_2^B\right\rangle}{1+\underline{\bar{k}}}\left(1-\frac{\kappa}{1+\underline{\bar{k}}}\right)\\ &\rightarrow \left\langle \hat{\underline{k}}_1^B\right\rangle + \kappa\left\langle \hat{\underline{k}}_2^B\right\rangle(1-\kappa)\simeq -\kappa^2\left\langle \hat{\underline{k}}_2^B\right\rangle\end{aligned}$$

and:

$$\left\langle \bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right\rangle = \left\langle \hat{\underline{k}}_1^B\right\rangle\left(\left\langle \hat{\underline{k}}_1^B\right\rangle + \kappa\frac{\left\langle \hat{\underline{k}}_2^B\right\rangle}{1+\underline{\bar{k}}}\left(1-\frac{\kappa}{1+\underline{\bar{k}}}\right)\right)$$

$$\left(1+\bar{\beta}\left\langle\underline{\bar{k}}\right\rangle\right)\left\langle \bar{S}_1^E\right\rangle\rightarrow 1$$

so that:

$$\left(1+\bar{\beta}\left\langle\underline{\bar{k}}\right\rangle\right)\bar{S}_1^B\left(\bar{X}',\bar{X}_1\right)\rightarrow\left\langle \hat{\underline{k}}_1^B\right\rangle\left(\left\langle \hat{\underline{k}}_1^B\right\rangle + \kappa\frac{\left\langle \hat{\underline{k}}_2^B\right\rangle}{1+\underline{\bar{k}}}\left(1-\frac{\kappa}{1+\underline{\bar{k}}}\right)\right)$$

To compute $\left\langle \hat{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right\rangle$, we use:

$$\hat{S}_1^B\left(\bar{X}',\bar{X}_1\right)\simeq\frac{\hat{K}'\hat{\underline{k}}_1^B\left(\hat{X}',\bar{X}\right)}{1+\left[\underline{\hat{k}}\left(\hat{X}'\right)\right]_\kappa}\frac{1}{1+\left[\underline{\hat{k}}_2\left(\hat{X}\right)\right]_\kappa}\left(1-\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}_X}{1+\left[\underline{\hat{k}}\left(\hat{X}'\right)\right]_\kappa}\left\|\hat{\Psi}\left(\hat{X}'\right)\right\|^2\right)$$

and in average, this is equal to:

$$\left\langle \hat{S}_1^B\left(\bar{X}',\bar{X}_1\right)\right\rangle\rightarrow\left\langle \hat{k}_1^B\right\rangle\left(1-\left\langle\hat{k}\right\rangle\right)$$

Ultimately, the average return becomes:

$$\begin{aligned}\langle\bar{g}\rangle &= \bar{r}'+\frac{\left(1-\beta^B\right)}{(1-\beta)\langle\delta\rangle}\left[\left(1-\left(\left(\left(1-\langle\bar{k}_1\rangle\right)\langle\bar{k}\rangle+\langle\bar{k}_1\rangle\right)+\langle\bar{S}_1^B\rangle\right)\right)\right]^{-1}\\ &\times\left(\frac{1}{\left\langle\hat{k}^B\right\rangle}-\left(\left(\left\langle\hat{k}\right\rangle\left(1-\left\langle\hat{k}_1\right\rangle\right)+\left\langle\hat{k}_1\right\rangle\right)-\frac{(1-\beta)\langle\delta\rangle}{1-\beta^B}\left(\left\langle\hat{k}_1^B\right\rangle\left(1-\left\langle\hat{k}\right\rangle\right)\right)\right)\right)\left(\langle\hat{g}\rangle-\bar{r}'\right)\end{aligned}$$

Given that$1-\beta^B<<1$, in first approximation we can cnsdr tht $\langle\bar{g}\rangle=\bar{r}'$.

$$\langle\delta\rangle\simeq\frac{\left\langle k_1^B\right\rangle+\frac{\kappa\left\langle k_2^B\right\rangle}{1+\langle\bar{k}\rangle}}{\langle k\rangle}Z\simeq\left(\frac{\left\langle k_1^B\right\rangle}{\langle k\rangle}+\frac{\kappa\left\langle k_2^B\right\rangle}{\langle k\rangle\left(1+\langle\bar{k}\rangle\right)}\right)\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2}$$

A23.4 Approximate solutions for average global capital

A23.4.1 Expressions for coefficients

To find solutions for $\langle\hat{g}\rangle$ and use some approximtns. First, given that $1-\beta^B<<1$, in first approximation we can cnsdr tht

$$\langle\bar{g}\rangle=\bar{r}'$$

Recall tht:

$$\langle\delta\rangle \simeq \frac{\langle k_1^B\rangle + \frac{\kappa\langle k_2^B\rangle}{1+\langle\bar{k}\rangle}}{\langle k\rangle} Z \simeq \left(\frac{\langle k_1^B\rangle}{\langle k\rangle} + \frac{\kappa\langle k_2^B\rangle}{\langle k\rangle\left(1+\langle\bar{k}\rangle\right)}\right)\frac{\left\|\bar{\Psi}_0\right\|^2}{\left\|\hat{\Psi}_0\right\|^2} >> 1$$

for $\kappa >> 1$.

Moreover, in first pprxmtn:

$$\bar{N} = -\kappa^2\left\langle \underline{\hat{k}}_2^B\right\rangle$$

$$\begin{aligned}
\left\langle \hat{g}^{ef}\right\rangle &= -\frac{\left(\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\underline{\hat{k}}\right\rangle\right)+\left\langle\hat{k}_1^B\right\rangle\left\langle\hat{k}_2\right\rangle\right)\left(\langle\hat{g}\rangle+\frac{1}{1-\left\langle\underline{\hat{k}}\right\rangle}\bar{N}\bar{r}'\right)\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}}{\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}\right)\right)\left(1-\left(\left\langle\underline{\hat{k}}\right\rangle+\left(\left\langle\underline{\hat{k}}_1^B\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\right)\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}\right)\right)} \\
&\simeq \frac{\kappa^2\left\langle\underline{\hat{k}}_2^B\right\rangle}{1-\left\langle\underline{\hat{k}}\right\rangle}\frac{\left(\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\underline{\hat{k}}\right\rangle\right)+\left\langle\hat{k}_1^B\right\rangle\left\langle\hat{k}_2\right\rangle\right)\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}}{\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}\right)\right)\left(1-\left(\left\langle\underline{\hat{k}}\right\rangle+\left(\left\langle\underline{\hat{k}}_1^B\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\right)\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}\right)\right)}\bar{r}' \\
&\simeq \frac{\kappa^3\left\langle\underline{\hat{k}}_2^B\right\rangle\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}}{\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}\right)\right)\left(1-\left(\left\langle\underline{\hat{k}}\right\rangle+\left(\left\langle\underline{\hat{k}}_1^B\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\right)\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}\right)\right)}\bar{r}'
\end{aligned}$$

and:

$$\begin{aligned}
\left\langle \hat{g}^{Bef}\right\rangle &= -\frac{\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\hat{k}\right\rangle\right)+\left\langle\hat{k}_1^B\right\rangle\left\langle\hat{k}_2\right\rangle\left(\langle\hat{g}\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\bar{r}'\right)}{\left(1-\left(\left\langle\hat{k}\right\rangle+\left(\left\langle\hat{k}_1^B\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\right)\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}\right)\right)\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}\right)\right)} \\
&\simeq \frac{\kappa^3\left\langle\underline{\hat{k}}_2^B\right\rangle\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle}{\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}\right)\right)\left(1-\left(\left\langle\underline{\hat{k}}\right\rangle+\left(\left\langle\underline{\hat{k}}_1^B\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\right)\frac{\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2}{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}\right)\right)}\bar{r}'
\end{aligned}$$

A23.4.2 Expression for capital

Average amounts of capital are given by:

$$\begin{aligned}
\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2 &= \frac{18\sigma_{\hat{K}}^2}{\hat{\mu}\langle\hat{g}\rangle^2}V\left(\frac{\left\|\hat{\Psi}_0\right\|^2}{\left(5+\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\right)}\right)^2 \\
&\quad\times\left(\frac{1}{4}-\frac{1}{18}\left(2+\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\right)\right) \\
&\simeq \frac{9\sigma_{\hat{K}}^2}{2\hat{\mu}\langle\hat{g}\rangle^2}V\left\|\hat{\Psi}_0\right\|^4
\end{aligned}$$

so that:

$$\langle\hat{g}\rangle \simeq \frac{\sqrt{\frac{9\sigma_{\hat{K}}^2}{2\hat{\mu}}V}\left\|\hat{\Psi}_0\right\|^2}{\sqrt{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}} \tag{362}$$

We have ssmd tht $\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle} << 1$ but correction at the first order can be easily derived.

For banks capital:

$$\begin{aligned}
\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2 &\simeq 18\frac{\sigma_{\hat{K}}^2 V}{\langle\bar{g}\rangle^2\hat{\mu}}\left(\frac{\sqrt{\langle\hat{g}\rangle^2\left(1+\frac{3}{4}\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}\left\|\bar{\Psi}_0\right\|-\frac{3}{8}\langle\hat{g}\rangle\frac{\langle\hat{g}^{Bef}\rangle}{\langle\bar{g}\rangle}\left\|\hat{\Psi}_0\right\|}{\sqrt{5+\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}-\sqrt{\left(1-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)\left(4-\frac{\langle\hat{g}^{ef}\rangle}{\langle\hat{g}\rangle}\right)}}}\right)^4\left(\frac{1}{4}-\frac{\langle\hat{g}\rangle\langle\hat{g}^{Bef}\rangle}{3\langle\bar{g}\rangle^2}\right) \quad (363)\\
&\simeq 18\frac{\sigma_{\hat{K}}^2 V\langle\hat{g}\rangle^4}{\langle\bar{g}\rangle^2\hat{\mu}}\left(\left\|\bar{\Psi}_0\right\|\right)^4
\end{aligned}$$

Using (362) enables to rewrite at the lowest order:

$$\langle\bar{K}\rangle\left\|\bar{\Psi}\right\|^2 \simeq 18\frac{\sigma_{\hat{K}}^2 V\left(\frac{\sqrt{\frac{9\sigma_{\hat{K}}^2}{2\hat{\mu}}V\left\|\hat{\Psi}_0\right\|^2}}{\sqrt{\langle\hat{K}\rangle\left\|\hat{\Psi}\right\|^2}}\right)^4}{\langle\bar{g}\rangle^2\hat{\mu}}\left(\left\|\bar{\Psi}_0\right\|\right)^4 \simeq 4\frac{\sigma_{\hat{K}}^2 V\left(\frac{9\sigma_{\hat{K}}^2}{2\hat{\mu}}V\right)^3\left\|\hat{\Psi}_0\right\|^8\left\|\bar{\Psi}_0\right\|^4}{(\bar{r}')^2\hat{\mu}\left(\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\right)^2\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2} \quad (364)$$

A23.4.3 Equation for average capital

We rewrite the equation for $\langle\hat{g}\rangle$ in terms of our approximations:

$$\begin{aligned}
\langle\hat{g}\rangle &\simeq \bar{r}'+\left(1-\left(1+\left\langle\hat{k}_2^n\right\rangle\right)\left(\left\langle\hat{k}\right\rangle\left(1-\left\langle\hat{k}_1\right\rangle\right)+\left\langle\hat{k}_1\right\rangle\right)\right)^{-1}\\
&\times\left(1+\left\langle\hat{k}_2^n\right\rangle\right)\left\langle\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,\hat{X}'\right)\right)\right)\right\rangle
\end{aligned}$$

Using that:

$$\delta > 1$$

this becoms

$$\begin{aligned}
\langle\hat{g}\rangle &\simeq \bar{r}'+\left(1-\left(1+\left\langle\hat{k}_2^n\right\rangle\right)\left(\left\langle\hat{k}\right\rangle\left(1-\left\langle\hat{k}_1\right\rangle\right)+\left\langle\hat{k}_1\right\rangle\right)\right)^{-1}\\
&\times\left(1+\left\langle\hat{k}_2^n\right\rangle\right)\left\langle\left(\frac{A\left(\hat{X}'\right)}{f_1^2\left(\hat{X}'\right)}+\frac{B\left(\hat{X}'\right)}{f_1^3\left(\hat{X}'\right)}\right)\left(R+\Delta F_\tau\left(\bar{R}\left(K,\hat{X}'\right)\right)\right)\right\rangle
\end{aligned}$$

or reintroducing $\Delta F_\tau\left(\bar{R}(K,X)\right)$:

$$\begin{aligned}
\langle\hat{g}\rangle-\bar{r}' &\simeq \left(1-\left(1+\left\langle\hat{k}_2^n\right\rangle\right)\left(\left\langle\hat{k}\right\rangle\left(1-\left\langle\hat{k}_1\right\rangle\right)+\left\langle\hat{k}_1\right\rangle\right)\right)^{-1}\left(1+\left\langle\hat{k}_2^n\right\rangle\right)\\
&\times\left\langle\left(\frac{A}{\left(f_1\left(X,\hat{K}[X],\bar{K}[X]\right)\right)^2}+\frac{B}{\left(f_1\left(X,\hat{K}[X],\bar{K}[X]\right)\right)^3}\right)\left(R+\Delta F_\tau\left(\bar{R}(K,X)\right)\right)\right\rangle
\end{aligned}$$

Here:

$$f_1\left(X,\hat{K}[X],\bar{K}[X]\right)=\frac{f_1(X)}{\left(1+k(X)\hat{K}[X]+\left(k_1^B(X)+\kappa\left[\frac{k_2^B}{1+k}\right]\right)\bar{K}[X]\right)^r}-C_0$$

and:
$$\left\langle f_1\left(X,\hat{K}[X],\bar{K}[X]\right)\right\rangle=\frac{\langle f_1(X)\rangle}{\left(1+\langle k\rangle\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2+\left(\left\langle k_1^B\right\rangle+\kappa\left[\frac{k_2^B}{1+\bar{k}}\right]\right)\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^2\right)^r}-C_0$$

Using (362) and (364) leads to write:
$$\left\langle f_1\left(X,\hat{K}[X],\bar{K}[X]\right)\right\rangle\simeq\frac{\langle f_1(X)\rangle}{\left(1+\langle k\rangle\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2+\left(\left\langle k_1^B\right\rangle+\kappa\left[\frac{k_2^B}{1+\bar{k}}\right]\right)\frac{4\sigma_{\hat{K}}^2V\left(\frac{9\sigma_{\hat{K}}^2}{2\hat{\mu}}V\right)^3\left\|\hat{\Psi}_0\right\|^8\left\|\bar{\Psi}_0\right\|^4}{(\bar{r}')^2\hat{\mu}\left(\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\right)^2}\right)^r}-C_0 \tag{365}$$

and return equation becomes an equation for $\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2$:

$$\begin{aligned}\frac{\sqrt{\frac{9\sigma_{\hat{K}}^2}{2\hat{\mu}}V}\left\|\hat{\Psi}_0\right\|^2}{\sqrt{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}}-\bar{r}' \quad\simeq\quad &\left(1-\left(1+\left\langle\hat{k}_2^n\right\rangle\right)\left(\left\langle\hat{k}\right\rangle\left(1-\left\langle\hat{k}_1\right\rangle\right)+\left\langle\hat{k}_1\right\rangle\right)\right)^{-1}\left(1+\left\langle\hat{k}_2^n\right\rangle\right)\\&\times\left\langle\left(\frac{A}{\left(f_1\left(X,\hat{K}[X],\bar{K}[X]\right)\right)^2}+\frac{B}{\left(f_1\left(X,\hat{K}[X],\bar{K}[X]\right)\right)^3}\right)R\right\rangle\end{aligned}$$

This equation is similar to the equation without banks, except that the formula (365), includes banks effects in the definition of disposable ncomes of the firms. Numerical studies indicate that this stabilizes the possibilities of multiple states and reduces the number of stats given some average productivt.

Moreover, including the normaltn f cf:
$$\frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle-\left(\left\langle\hat{k}_1^B\left(\hat{X}',\bar{X}\right)\right\rangle+\kappa\frac{\left\langle\hat{k}_2^B\left(\hat{X}',\bar{X}\right)\right\rangle}{1+\left\langle\bar{k}\right\rangle\left\langle\bar{K}\right\rangle}\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)}$$

The coeficients are higher in this case than in case with no banks where:
$$\frac{\hat{k}_\eta\left(\hat{X}',\hat{X}\right)}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)}$$

As a consequence, the diffusion matrix is larger in this case than in part 1. Investors may now borrow from banks, incrreasing their disposable capital, and invreasing their investment in other investors.

Appendix 24 Block interactions

A24.1 Several groups

To model groups dynamics, we use the alternate description for returns:

$$\begin{aligned}
0 &= \int\left(1-\hat{S}_1\left(\hat{X}',\hat{K}',\hat{X}\right)\right)\frac{\hat{f}\left(\hat{X}'\right)-\bar{r}}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\frac{\underline{\hat{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}}}d\hat{X}'d\hat{K}' \\
&-\int\left(\frac{1+\hat{f}\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}H\left(-\frac{1+\hat{f}\left(\hat{X}'\right)}{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\right)\hat{S}_2\left(\hat{X}',\hat{K}',\hat{X}\right)d\hat{X}'d\hat{K}' \\
&-\int\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}H\left(-\frac{1+f_1'\left(X'\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)}\right)S_2\left(X',K',\hat{X}\right)-\int S_1\left(X',K',\hat{X}\right)\left(\hat{f}_1\left(\hat{K},\hat{X}\right)-\bar{r}\right)
\end{aligned}\tag{366}$$

for investors, and:

$$\begin{aligned}
0 &= \left(1-\bar{S}_1\left(\bar{X}',\bar{X}\right)\right)\frac{\bar{f}\left(\bar{X}'\right)-\bar{r}}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}-\hat{S}_1^B\left(\hat{X}',\bar{X}\right)\left(\frac{\hat{f}\left(\hat{X}'\right)-\bar{r}}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\frac{\underline{\hat{k}}_2^B\left(\bar{X}'\right)}{1+\bar{k}\left(\bar{X}\right)}}\right) \\
&-\frac{\left(1+\bar{f}\left(\bar{X}'\right)\right)H\left(-\left(1+\bar{f}\left(\bar{X}'\right)\right)\right)}{\underline{\bar{\bar{k}}}_2\left(\hat{X}'\right)}\bar{S}_2\left(\bar{X}',\bar{X}\right)-\frac{\left(1+\hat{f}\left(\hat{X}'\right)\right)H\left(-\left(1+\hat{f}\left(\hat{X}'\right)\right)\right)}{\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\hat{S}_2^B\left(\hat{X}',\bar{X}\right) \\
&-\frac{\left(1+f_1'\left(X'\right)\right)H\left(1+f_1'\left(K',X'\right)\right)}{\underline{k}_2\left(X'\right)+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{k}}\right]\left(X'\right)}S_1^B\left(\hat{X}',\bar{X}\right)-S_1^B\left(\hat{X}',\bar{X}\right)\left(\frac{\left(f_1'\left(\bar{K},\bar{X}\right)-\bar{r}\right)}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}+\Delta F_\tau\left(\bar{R}\left(K,X\right)\right)\right)
\end{aligned}\tag{367}$$

for banks. The coefficients are defined in Appendix 23.4.

Applying these formula to groups averages, as in the first part, we are led to define average intra and inter coefficiennts.

A24.2 Investor return

A24.2.1 Mixed formulation

The return equation for investors rewrites:

$$\begin{aligned}
0 &= \begin{pmatrix} 1-\underline{\hat{S}}_1^{[ii]} & -\underline{\hat{S}}_1^{[ji]} \\ -\underline{\hat{S}}_1^{[ij]} & 1-\underline{\hat{S}}_1^{[jj]} \end{pmatrix}\begin{pmatrix} \frac{f^{[i]}-\bar{r}}{1+\underline{\hat{k}}_2^{[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}} \\ \frac{f^{[j]}-\bar{r}}{1+\underline{\hat{k}}_2^{[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[j]}} \end{pmatrix}-\begin{pmatrix} \underline{\hat{S}}_2^{[ii]} & \underline{\hat{S}}_2^{[ji]} \\ \underline{\hat{S}}_2^{[ij]} & \underline{\hat{S}}_2^{[jj]} \end{pmatrix}\begin{pmatrix} \frac{\left(1+f^{[i]}\right)H\left(-\left(1+f^{[i]}\right)\right)}{\underline{\hat{k}}_2^{[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}} \\ \frac{\left(1+f^{[j]}\right)H\left(-\left(1+f^{[j]}\right)\right)}{\underline{\hat{k}}_2^{[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[j]}} \end{pmatrix} \\
&-\begin{pmatrix} \underline{S}_2^{[ii]} & \underline{S}_2^{[ji]} \\ \underline{S}_2^{[ij]} & \underline{S}_2^{[jj]} \end{pmatrix}\begin{pmatrix} \frac{\left(1+f_1'^{[i]}\right)H\left(-\left(1+f_1'^{[i]}\right)\right)}{\underline{k}_2^{[i]}+\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]^{[i]}} \\ \frac{\left(1+f_1'^{[j]}\right)H\left(-\left(1+f_1'^{[j]}\right)\right)}{\underline{k}_2^{[j]}+\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]^{[j]}} \end{pmatrix}-\begin{pmatrix} \underline{S}_1^{[ii]} & \underline{S}_1^{[ji]} \\ \underline{S}_1^{[ij]} & \underline{S}_1^{[jj]} \end{pmatrix}\begin{pmatrix} \frac{f_1'^{[i]}-\bar{r}}{1+\underline{k}_2^{[i]}+\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]^{[i]}} \\ \frac{f_1'^{[j]}-\bar{r}}{1+\underline{k}_2^{[j]}+\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]^{[j]}} \end{pmatrix}
\end{aligned}\tag{368}$$

where we defined:

$$\kappa\left(\frac{\underline{\hat{k}}_\epsilon^{B[ii]}}{1+\bar{k}^{[i]}}+\frac{\underline{\hat{k}}_\epsilon^{B[ij]}}{1+\bar{k}^{[j]}}\right)\rightarrow\kappa\left[\frac{\underline{\hat{k}}_\epsilon^{B}}{1+\bar{k}}\right]^{[i]}$$

$$\kappa\left(\frac{\underline{\hat{k}}^{B[ii]}}{1+\bar{k}^{[i]}}+\frac{\underline{\hat{k}}^{B[ij]}}{1+\bar{k}^{[j]}}\right)\rightarrow\kappa\left[\frac{\underline{\hat{k}}^{B}}{1+\bar{k}}\right]^{[i]}$$

$$\kappa\left(\frac{\underline{k}_\epsilon^{B[ii]}}{1+\bar{k}^{[i]}}+\frac{\underline{k}_\epsilon^{B[ij]}}{1+\bar{k}^{[j]}}\right)\rightarrow\kappa\left[\frac{\underline{k}_\epsilon^{B}}{1+\bar{k}}\right]^{[i]}$$

$$\kappa\left(\frac{\underline{k}^{B[ii]}}{1+\bar{k}^{[i]}}+\frac{\underline{k}^{B[ij]}}{1+\bar{k}^{[j]}}\right)\rightarrow\kappa\left[\frac{\underline{k}^{B}}{1+\bar{k}}\right]^{[i]}$$

and the following rates:

$$\underline{\hat{S}}_\eta^{[ii]} = \frac{\underline{\hat{k}}_\eta^{[ii]}}{\left(1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\bar{k}}\right]^{[i]}\right)}$$

$$\underline{\hat{S}}_\eta^{[ij]} = \frac{\underline{\hat{k}}_\eta^{[ij]}}{\left(1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\bar{k}}\right]^{[i]}\right)}$$

$$\underline{\hat{S}}^{[ii]} = \underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_2^{[ii]}$$
$$\underline{\hat{S}}^{[ij]} = \underline{\hat{S}}_1^{[ij]}+\underline{\hat{S}}_2^{[ij]}$$

$$\underline{S}_\eta^{[ii]} = \frac{k_\eta^{[ii]}}{1+\underline{k}^{[i]}+\underline{k}_1^{B[i]}+\kappa\left[\frac{\underline{k}_2^{B}}{1+\bar{k}}\right]^{[i]}}$$

$$\underline{S}_\eta^{[ij]} = \frac{k_\eta^{[ij]}}{1+\underline{k}^{[i]}+\underline{k}_1^{B[i]}+\kappa\left[\frac{\underline{k}_2^{B}}{1+\bar{k}}\right]^{[i]}}$$

$$\underline{S}^{[ii]} = \underline{S}_1^{[ii]}+S_2^{[ii]}$$
$$\underline{S}^{[ij]} = \underline{S}_1^{[ij]}+S_2^{[ij]}$$

A24.2.2 Constraints

The follwing constraints apply on outgoing flux of capital:

$$1 = \frac{\underline{\hat{k}}^{[ji]}}{1+\underline{\hat{k}}^{[j]}+\underline{\hat{k}}_1^{B[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\bar{k}}\right]^{[j]}}+\frac{\underline{\hat{k}}^{[ii]}}{1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\bar{k}}\right]^{[i]}}$$
$$+\frac{\underline{k}^{[ji]}}{1+\underline{k}^{[j]}+\underline{k}_1^{B[j]}+\kappa\left[\frac{\underline{k}_2^{B}}{1+\bar{k}}\right]^{[j]}}+\frac{\underline{k}^{[ii]}}{1+\underline{k}^{[i]}+\underline{k}_1^{B[i]}+\kappa\left[\frac{\underline{k}_2^{B}}{1+\bar{k}}\right]^{[i]}}$$

and:

$$
\begin{aligned}
1 \quad = \quad & \frac{\underline{\hat{k}}_2^{[ii]}}{1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]^{[i]}}+\frac{\underline{\hat{k}}_2^{[ji]}}{1+\underline{\hat{k}}^{[j]}+\underline{\hat{k}}_1^{B[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]^{[j]}} \\
& +\frac{\underline{k}_2^{[ii]}}{1+\underline{k}^{[i]}+\underline{k}_1^{B[i]}+\kappa\left[\frac{\underline{k}_2^B}{1+\underline{k}}\right]^{[i]}}+\frac{\underline{k}_2^{[ji]}}{1+\underline{k}^{[j]}+\underline{k}_1^{B[j]}+\kappa\left[\frac{\underline{k}_2^B}{1+\underline{k}}\right]^{[j]}} \\
& +\frac{\underline{\hat{k}}_1^{[ii]}}{1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]^{[i]}}+\frac{\underline{\hat{k}}_1^{[ji]}}{1+\underline{\hat{k}}^{[j]}+\underline{\hat{k}}_1^{B[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]^{[j]}} \\
& +\frac{\underline{k}_1^{[ii]}}{1+\underline{k}^{[i]}+\underline{k}_1^{B[i]}+\kappa\left[\frac{\underline{k}_2^B}{1+\underline{k}}\right]^{[i]}}+\frac{\underline{k}_1^{[ji]}}{1+\underline{k}^{[j]}+\underline{k}_1^{B[j]}+\kappa\left[\frac{\underline{k}_2^B}{1+\underline{k}}\right]^{[j]}}
\end{aligned}
$$

that are translated in terms of shars:

$$\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ji]}+\underline{S}^{[ii]}+\underline{S}^{[ji]}=1$$

A24.2.3 Full alternate formulation

The previous definitions allow to write all quantities as functions of the new parameters. Actually we have for the coefficients related to investment in firms:

$$\underline{\hat{S}}_\eta^{[i]}=\underline{\hat{S}}_\eta^{[ii]}+\underline{\hat{S}}_\eta^{[ij]}=\frac{\underline{\hat{k}}_\eta^{[i]}}{1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]^{[i]}}$$

$$\underline{\hat{S}}^{[i]}=\underline{\hat{S}}_1^{[i]}+\underline{\hat{S}}_2^{[i]}=\frac{\underline{\hat{k}}^{[i]}}{1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]^{[i]}}$$

$$\underline{S}^{[i]}=\underline{S}_1^{[i]}+\underline{S}_2^{[i]}=\frac{k^{[i]}}{1+\underline{k}^{[i]}+\underline{k}_1^{B[i]}+\kappa\left[\frac{\underline{k}_2^B}{1+\underline{k}}\right]^{[i]}}$$

$$
\begin{aligned}
\underline{\hat{S}}^{[ii]} &= \underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_2^{[ii]} \\
\underline{\hat{S}}^{[ij]} &= \underline{\hat{S}}_1^{[ij]}+\underline{\hat{S}}_2^{[ij]}
\end{aligned}
$$

$$\underline{\hat{S}}^{[i]}=\underline{\hat{S}}_1^{[i]}+\underline{\hat{S}}_2^{[i]}=\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}$$

$$\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ji]}+\underline{S}^{[ii]}+\underline{S}^{[ji]}=1$$

While for the coefficients related to bank investment in investrs:

$$\underline{\hat{S}}_1^{B[ii]} = \frac{\underline{\hat{k}}_1^{B[ii]}}{\left(1 + \underline{\hat{k}}^{[i]} + \underline{\hat{k}}_1^{B[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}\right)}$$

$$\underline{\hat{S}}_1^{B[ij]} = \frac{\underline{\hat{k}}_1^{B[ij]}}{\left(1 + \underline{\hat{k}}^{[i]} + \underline{\hat{k}}_1^{B[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}\right)}$$

$$\underline{\hat{S}}_2^{B[ii]} = \frac{\kappa \frac{\underline{\hat{k}}_2^{B[ii]}}{1+\bar{k}^{[i]}}}{1 + \underline{\hat{k}}^{[i]} + \underline{\hat{k}}_1^{B[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}}$$

$$\underline{\hat{S}}_2^{B[ij]} = \frac{\kappa \frac{\underline{\hat{k}}_2^{B[ij]}}{1+\bar{k}^{[j]}}}{1 + \underline{\hat{k}}^{[i]} + \underline{\hat{k}}_1^{B[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}}$$

$$\underline{\hat{S}}_\eta^{B[i]} = \underline{\hat{S}}_\eta^{B[ii]} + \underline{\hat{S}}_\eta^{B[ij]}$$

$$\underline{\hat{S}}^{B[i]} = \underline{\hat{S}}_1^{B[i]} + \underline{\hat{S}}_2^{B[i]} = \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}$$

$$\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]} = \frac{\underline{\hat{k}}^{[i]} + \underline{\hat{k}}_1^{B[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}}{1 + \underline{\hat{k}}^{[i]} + \underline{\hat{k}}_1^{B[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}}$$

As a consequence, we can write the following coefficients:

$$\frac{1}{1 + \underline{\hat{k}}^{[i]} + \underline{\hat{k}}_1^{B[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}} = 1 - \left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}\right)$$

$$\underline{\hat{k}}_\eta^{[i]} \left(1 - \left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}\right)\right) = \underline{\hat{S}}_\eta^{[ii]} + \underline{\hat{S}}_\eta^{[ij]}$$

and:

$$\underline{\hat{k}}_\eta^{[i]} = \frac{\underline{\hat{S}}_\eta^{[ii]} + \underline{\hat{S}}_\eta^{[ij]}}{1 - \left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}\right)}$$

$$\underline{\hat{k}}_1^{B[i]} = \frac{\underline{\hat{S}}_1^{B[ii]} + \underline{\hat{S}}_1^{B[ij]}}{1 - \left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}\right)}$$

$$\kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]} = \frac{\underline{\hat{S}}_2^{B[ii]} + \underline{\hat{S}}_2^{B[ij]}}{1 - \left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}\right)}$$

$$\underline{\hat{k}}_2^{[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}} \right]^{[i]} = \frac{\underline{\hat{S}}_2^{[ii]} + \underline{\hat{S}}_2^{[ij]}}{1 - \left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]} \right)} + \frac{\underline{\hat{S}}_2^{B[ii]} + \underline{\hat{S}}_2^{B[ij]}}{1 - \left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]} \right)}$$

$$= \frac{\underline{\hat{S}}_2^{[ii]} + \underline{\hat{S}}_2^{[ij]} + \underline{\hat{S}}_2^{B[ii]} + \underline{\hat{S}}_2^{B[ij]}}{1 - \left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]} \right)}$$

$$1 + \underline{\hat{k}}_2^{[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}} \right]^{[i]} = 1 + \frac{\underline{\hat{S}}_2^{[ii]} + \underline{\hat{S}}_2^{[ij]}}{1 - \left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]} \right)} + \frac{\underline{\hat{S}}_2^{B[ii]} + \underline{\hat{S}}_2^{B[ij]}}{1 - \left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]} \right)}$$

$$= \frac{1 - \left(\underline{\hat{S}}_1^{[ii]} + \underline{\hat{S}}_1^{[ij]} + \underline{\hat{S}}_1^{B[ii]} + \underline{\hat{S}}_1^{B[ij]} \right)}{1 - \left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]} \right)}$$

$$\underline{S}_2^{B[ii]} = \frac{\kappa \frac{\underline{k}_2^{(B)[ii]}}{1+\bar{\underline{k}}^{[i]}}}{\left(1 + \underline{k}^{[i]} + \underline{k}_1^{(B)[i]} + \kappa \left[\frac{\underline{k}_2^{(B)}}{1+\bar{\underline{k}}} \right]^{[i]} \right)}$$

$$\underline{S}_2^{B[ij]} = \frac{\kappa \frac{\underline{k}_2^{(B)[ij]}}{1+\bar{\underline{k}}^{[j]}}}{\left(1 + \underline{k}^{[i]} + \underline{k}_1^{(B)[i]} + \kappa \left[\frac{\underline{k}_2^{(B)}}{1+\bar{\underline{k}}} \right]^{[i]} \right)}$$

$$\underline{k}_\eta^{[i]} = \frac{\underline{S}_\eta^{[ii]} + \underline{S}_\eta^{[ij]}}{1 - \left(\underline{S}^{[ii]} + \underline{S}^{[ij]} + \underline{S}^{B[ii]} + \underline{S}^{B[ij]} \right)}$$

$$\underline{k}_1^{(B)[ij]} = \frac{\underline{S}_1^{B[ii]} + \underline{S}_1^{B[ij]}}{1 - \left(\underline{S}^{[ii]} + \underline{S}^{[ij]} + \underline{S}^{B[ii]} + \underline{S}^{B[ij]} \right)}$$

$$\kappa \left[\frac{\underline{k}_2^{(B)}}{1+\bar{\underline{k}}} \right]^{[i]} = \frac{\underline{S}_2^{B[ii]} + \underline{S}_2^{B[ij]}}{1 - \left(\underline{S}^{[ii]} + \underline{S}^{[ij]} + \underline{S}^{B[ii]} + \underline{S}^{B[ij]} \right)}$$

$$\underline{k}_2^{[i]} + \left[\frac{\underline{k}_2^B}{1+\bar{k}} \right]^{[i]} = \frac{\underline{S}_2^{[ii]} + \underline{S}_2^{[ij]} + \underline{S}_2^{B[ii]} + \underline{S}_2^{B[ij]}}{1 - \left(\underline{S}^{[ii]} + \underline{S}^{[ij]} + \underline{S}^{B[ii]} + \underline{S}^{B[ij]} \right)}$$

$$1 + \underline{k}_2^{[i]} + \left[\frac{\underline{k}_2^B}{1+\bar{k}} \right]^{[i]} = \frac{1 - \left(\underline{S}_1^{[ii]} + \underline{S}_1^{[ij]} + \underline{S}_1^{B[ii]} + \underline{S}_1^{B[ij]} \right)}{1 - \left(\underline{S}^{[ii]} + \underline{S}^{[ij]} + \underline{S}^{B[ii]} + \underline{S}^{B[ij]} \right)}$$

$$f_1'^{[i]} = \left(1 + \underline{k}_2^{[i]} + \left[\frac{\underline{k}_2^B}{1+\bar{k}} \right]^{[i]} \right) f_1^{[i]} - \bar{r} \left(\underline{k}_2^{[i]} + \left[\frac{\underline{k}_2^B}{1+\bar{k}} \right]^{[i]} \right)$$

$$
\begin{aligned}
\underline{\hat{S}}_1^{B[ii]} &= \frac{\underline{\hat{k}}_1^{B[ii]}}{\left(1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\underline{\bar{k}}}\right]^{[i]}\right)} \\
\underline{\hat{S}}_1^{B[ij]} &= \frac{\underline{\hat{k}}_1^{B[ij]}}{\left(1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\underline{\bar{k}}}\right]^{[i]}\right)}
\end{aligned}
$$

$$
\begin{aligned}
\underline{\hat{S}}_2^{B[ii]} &= \frac{\kappa\frac{\underline{\hat{k}}_2^{B[ii]}}{1+\underline{\bar{k}}^{[i]}}}{1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\underline{\bar{k}}}\right]^{[i]}} \\
\underline{\hat{S}}_2^{B[ij]} &= \frac{\kappa\frac{\underline{\hat{k}}_2^{B[ij]}}{1+\underline{\bar{k}}^{[j]}}}{1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\underline{\bar{k}}}\right]^{[i]}}
\end{aligned}
$$

$$
\begin{aligned}
\underline{\hat{S}}_\eta^{B[i]} &= \underline{\hat{S}}_\eta^{B[ii]}+\underline{\hat{S}}_\eta^{B[ij]} \\
\underline{\hat{S}}^{B[i]} &= \underline{\hat{S}}_1^{B[i]}+\underline{\hat{S}}_2^{B[i]} = \underline{\hat{S}}^{B[ii]}+\underline{\hat{S}}^{B[ij]}
\end{aligned}
$$

$$
\frac{1}{1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\underline{\bar{k}}}\right]^{[i]}} = 1-\left(\underline{\hat{S}}^{B[ii]}+\underline{\hat{S}}^{B[ij]}\right) = 1-\underline{\hat{S}}^{B[i]}
$$

$$
\underline{\hat{S}}_\eta^{B[i]} = \frac{\underline{\hat{k}}_\eta^{B[i]}}{1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\underline{\bar{k}}}\right]^{[i]}} = \underline{\hat{k}}_\eta^{B[i]}\left(1-\left(\underline{\bar{S}}^{[ii]}+\underline{\bar{S}}^{[ij]}\right)\right)
$$

$$
\underline{\bar{k}}_\eta^{[i]} = \frac{\underline{\bar{S}}_\eta^{[i]}}{1-\underline{\bar{S}}^{[i]}} = \frac{\underline{\bar{S}}_\eta^{[ii]}+\underline{\bar{S}}_\eta^{[ij]}}{1-\left(\underline{\bar{S}}^{[ii]}+\underline{\bar{S}}^{[ij]}\right)}
$$

$$
\begin{aligned}
\underline{S}_2^{B[ii]} &= \frac{\kappa\frac{\underline{k}_2^{(B)[ii]}}{1+\underline{\bar{k}}^{[i]}}}{\left(1+\underline{k}^{[i]}+\underline{k}_1^{(B)[i]}+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{\bar{k}}}\right]^{[i]}\right)} \\
\underline{S}_2^{B[ij]} &= \frac{\kappa\frac{\underline{k}_2^{(B)[ij]}}{1+\underline{\bar{k}}^{[j]}}}{\left(1+\underline{k}^{[i]}+\underline{k}_1^{(B)[i]}+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{\bar{k}}}\right]^{[i]}\right)}
\end{aligned}
$$

$$
\underline{\bar{S}}^{[ii]}+\underline{\bar{S}}^{[ji]}+\underline{\hat{S}}^{B[ii]}+\underline{\hat{S}}^{B[ji]}+\underline{S}^{B[ii]}+\underline{S}^{[ji]} = 1
$$

$$\underline{\hat{k}}_2^{[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]} = \frac{\underline{\hat{S}}_2^{[ii]} + \underline{\hat{S}}_2^{[ij]}}{1-\left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}\right)} + \frac{\underline{\hat{S}}_2^{B[ii]} + \underline{\hat{S}}_2^{B[ij]}}{1-\left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}\right)}$$
$$= \frac{\underline{\hat{S}}_2^{[ii]} + \underline{\hat{S}}_2^{[ij]} + \underline{\hat{S}}_2^{B[ii]} + \underline{\hat{S}}_2^{B[ij]}}{1-\left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}\right)}$$

$$1 + \underline{\hat{k}}_2^{[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]} = 1 + \frac{\underline{\hat{S}}_2^{[ii]} + \underline{\hat{S}}_2^{[ij]}}{1-\left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}\right)} + \frac{\underline{\hat{S}}_2^{B[ii]} + \underline{\hat{S}}_2^{B[ij]}}{1-\left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}\right)}$$
$$= \frac{1-\left(\underline{\hat{S}}_1^{[ii]} + \underline{\hat{S}}_1^{[ij]} + \underline{\hat{S}}_1^{B[ii]} + \underline{\hat{S}}_1^{B[ij]}\right)}{1-\left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}\right)}$$

$$\underline{k}_2^{[i]} + \left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]^{[i]} = \frac{\underline{S}_2^{[ii]} + \underline{S}_2^{[ij]} + \underline{S}_2^{B[ii]} + \underline{S}_2^{B[ij]}}{1-\left(\underline{S}^{[ii]} + \underline{S}^{[ij]} + \underline{S}^{B[ii]} + \underline{S}^{B[ij]}\right)}$$

$$\frac{\underline{\hat{k}}^{[ji]}}{1 + \underline{\hat{k}}^{[j]} + \underline{\hat{k}}_1^{B[j]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[j]}} + \frac{\underline{\hat{k}}^{[ii]}}{1 + \underline{\hat{k}}^{[i]} + \underline{\hat{k}}_1^{B[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}} + \frac{\underline{k}^{[ji]}}{1 + \underline{k}^{[j]} + \underline{k}_1^{B[j]} + \kappa \left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]^{[j]}} + \frac{\underline{k}^{[ii]}}{1 + \underline{k}^{[i]} + \underline{k}_1^{B[i]} + \kappa \left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]^{[i]}} = 1$$

$$\frac{\underline{\hat{k}}_1^{[ii]}}{\left(1 + \underline{\hat{k}}^{[i]} + \underline{\hat{k}}_1^{B[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}\right)\left(1 + \underline{\hat{k}}_2^{[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}\right)} = \frac{\underline{\hat{S}}_1^{[ii]}}{1 + \underline{\hat{k}}_2^{[i]} + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]^{[i]}}$$

Replacing these coeficients in equation (368):

$$0 = \begin{pmatrix} 1-\underline{\hat{S}}_1^{[ii]} & -\underline{\hat{S}}_1^{[ji]} \\ -\underline{\hat{S}}_1^{[ij]} & 1-\underline{\hat{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \left(f^{[i]}-\bar{r}\right)\underline{\hat{s}}_1^{[i]} \\ \left(f^{[j]}-\bar{r}\right)\underline{\hat{s}}_1^{[j]} \end{pmatrix} - \begin{pmatrix} \underline{\hat{S}}_2^{[ii]} & \underline{\hat{S}}_2^{[ji]} \\ \underline{\hat{S}}_2^{[ij]} & \underline{\hat{S}}_2^{[jj]} \end{pmatrix} \begin{pmatrix} \left(1+f^{[i]}\right)\underline{\hat{s}}_2^{[i]} H\left(-\left(1+f^{[i]}\right)\right) \\ \left(1+f^{[j]}\right)\underline{\hat{s}}_2^{[i]} H\left(-\left(1+f^{[j]}\right)\right) \end{pmatrix}$$
$$- \begin{pmatrix} \underline{S}_2^{[ii]} & \underline{S}_2^{[ji]} \\ \underline{S}_2^{[ij]} & \underline{S}_2^{[jj]} \end{pmatrix} \begin{pmatrix} \left(1+f_1'^{[i]}\right)\underline{s}_2^{[i]} H\left(-\left(1+f_1'^{[i]}\right)\right) \\ \left(1+f_1'^{[j]}\right)\underline{s}_2^{[j]} H\left(-\left(1+f_1'^{[j]}\right)\right) \end{pmatrix} - \begin{pmatrix} \underline{S}_1^{[ii]} & \underline{S}_1^{[ji]} \\ \underline{S}_1^{[ij]} & \underline{S}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \left(f_1'^{[i]}-\bar{r}\right)\underline{s}_1^{[i]} \\ \left(f_1'^{[j]}-\bar{r}\right)\underline{s}_1^{[j]} \end{pmatrix}$$

with:

$$\underline{\hat{s}}_1^{[i]} = \frac{1-\left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]} + \underline{\hat{S}}_1^{[ij]} + \underline{\hat{S}}_1^{B[ii]} + \underline{\hat{S}}_1^{B[ij]}\right)}, \ \underline{\hat{s}}_2^{[i]} = \frac{1-\left(\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ij]} + \underline{\hat{S}}^{B[ii]} + \underline{\hat{S}}^{B[ij]}\right)}{\underline{\hat{S}}_2^{[ii]} + \underline{\hat{S}}_2^{[ij]} + \underline{\hat{S}}_2^{B[ii]} + \underline{\hat{S}}_2^{B[ij]}}$$
$$\underline{s}_1^{[i]} = \frac{1-\left(\underline{S}^{[ii]} + \underline{S}^{[ij]} + \underline{S}^{B[ii]} + \underline{S}^{B[ij]}\right)}{1-\left(\underline{S}_1^{[ii]} + \underline{S}_1^{[ij]} + \underline{S}_1^{B[ii]} + \underline{S}_1^{B[ij]}\right)}, \ \underline{s}_2^{[i]} = \frac{1-\left(\underline{S}^{[ii]} + \underline{S}^{[ij]} + \underline{S}^{B[ii]} + \underline{S}^{B[ij]}\right)}{\underline{S}_2^{[ii]} + \underline{S}_2^{[ij]} + \underline{S}_2^{B[ii]} + \underline{S}_2^{B[ij]}}$$

A24.3 Bank return

A24.3.1 Constraints

The constraint for bank investments are:

$$
\begin{aligned}
1 \quad = \quad & \frac{\underline{k}_1^{(B)[ii]}}{\left(1+\underline{k}^{[i]}+\underline{k}_1^{(B)[i]}+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{\bar{k}}}\right]^{[i]}\right)}+\frac{\underline{k}_1^{(B)[ji]}}{\left(1+\underline{k}^{[j]}+\underline{k}_1^{(B)[j]}+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{\bar{k}}}\right]^{[j]}\right)} \\
& +\frac{\underline{\hat{k}}_1^{B[ii]}}{1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{(1+\underline{\bar{k}})}\right]^{[i]}}+\frac{\underline{\hat{k}}_1^{B[ji]}}{1+\underline{\hat{k}}^{[j]}+\underline{\hat{k}}_1^{B[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{(1+\underline{\bar{k}})}\right]^{[j]}}+\frac{\underline{\bar{\bar{k}}}^{[ii]}}{1+\underline{\bar{\bar{k}}}^{[i]}}+\frac{\underline{\bar{\bar{k}}}^{[ji]}}{1+\underline{\bar{\bar{k}}}^{[j]}}
\end{aligned}
$$

which translates in terms of new set of paramters:

$$
\underline{\bar{S}}^{[ii]}+\underline{\bar{S}}^{[ji]}+\underline{\hat{S}}_1^{B[ii]}+\underline{\hat{S}}_1^{B[ji]}+\underline{S}_1^{B[ii]}+\underline{S}_1^{B[ji]}=1
$$

while the constraint for investors:

$$
\begin{aligned}
1 \quad = \quad & \frac{\kappa\frac{\underline{k}_2^{(B)[ii]}}{1+\underline{\bar{k}}^{[i]}}}{\left(1+\underline{k}^{[i]}+\underline{k}_1^{(B)[i]}+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{\bar{k}}}\right]^{[i]}\right)}+\frac{\kappa\frac{\underline{k}_2^{(B)[ji]}}{1+\underline{\bar{k}}^{[i]}}}{\left(1+\underline{k}^{[j]}+\underline{k}_1^{(B)[j]}+\kappa\left[\frac{\underline{k}_2^{(B)}}{1+\underline{\bar{k}}}\right]^{[j]}\right)} \\
& +\frac{\kappa\frac{\underline{\hat{k}}_2^{B[ii]}}{\left(1+\underline{\bar{k}}^{[i]}\right)}}{1+\underline{\hat{k}}^{[i]}+\underline{\hat{k}}_1^{B[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{(1+\underline{\bar{k}})}\right]^{[i]}}+\frac{\kappa\frac{\underline{\hat{k}}_2^{B[ji]}}{\left(1+\underline{\bar{k}}^{[j]}\right)}}{1+\underline{\hat{k}}^{[j]}+\underline{\hat{k}}_1^{B[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{(1+\underline{\bar{k}})}\right]^{[j]}}
\end{aligned}
$$

translates into:

$$
\underline{\hat{S}}_2^{B[ii]}+\underline{\hat{S}}_2^{B[ji]}+\underline{S}_2^{B[ii]}+\underline{S}_2^{B[ji]}=1
$$

A24.3.2 Bank equation

The formula for average returns per group is:

$$
\begin{aligned}
0 \quad = \quad & \begin{pmatrix} 1-\underline{\bar{S}}_1^{[ii]} & -\underline{\bar{S}}_1^{[ji]} \\ -\underline{\bar{S}}_1^{[ij]} & 1-\underline{\bar{S}}_1^{[jj]} \end{pmatrix}\begin{pmatrix} \frac{\bar{f}^{[i]}-\bar{r}}{1+\underline{\bar{k}}^{[i]}} \\ \frac{\bar{f}^{[j]}-\bar{r}}{1+\underline{\bar{k}}^{[j]}} \end{pmatrix}-\begin{pmatrix} \underline{\hat{S}}_2^{[ii]} & \underline{\hat{S}}_\eta^{[ij]} \\ \underline{\hat{S}}_\eta^{[ij]} & \underline{\hat{S}}_2^{[jj]} \end{pmatrix}\begin{pmatrix} \frac{\hat{f}^{[i]}-\bar{r}}{1+\underline{\hat{k}}_2^{[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\bar{k}}\right]^{[i]}} \\ \frac{\hat{f}^{[j]}-\bar{r}}{1+\underline{\hat{k}}_2^{[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\bar{k}}\right]^{[j]}} \end{pmatrix} \\
& -\begin{pmatrix} \underline{\bar{S}}_2^{[ii]} & \underline{\bar{S}}_2^{[ji]} \\ \underline{\bar{S}}_2^{B[ij]} & \underline{\bar{S}}_2^{B[jj]} \end{pmatrix}\begin{pmatrix} \frac{1+\bar{f}^{[i]}}{\underline{\bar{\bar{k}}}_2^{[i]}}H\left(-\left(1+\bar{f}^{[i]}\right)\right) \\ \frac{1+\bar{f}^{[j]}}{\underline{\bar{\bar{k}}}_2^{[j]}}H\left(-\left(1+\bar{f}^{[j]}\right)\right) \end{pmatrix}-\begin{pmatrix} \underline{\hat{S}}_2^{B[ii]} & \underline{\hat{S}}_2^{B[ji]} \\ \underline{\hat{S}}_2^{B[ij]} & \underline{\hat{S}}_2^{B[jj]} \end{pmatrix}\begin{pmatrix} \frac{1+\hat{f}^{[i]}}{\underline{\hat{k}}_2^{[i]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\bar{k}}\right]^{[i]}}H\left(-\left(1+\hat{f}^{[i]}\right)\right) \\ \frac{1+\hat{f}^{[j]}}{\underline{\hat{k}}_2^{[j]}+\kappa\left[\frac{\underline{\hat{k}}_2^{B}}{1+\bar{k}}\right]^{[j]}}H\left(-\left(1+\hat{f}^{[j]}\right)\right) \end{pmatrix} \\
& -\begin{pmatrix} \underline{S}_2^{B[ii]} & \underline{S}_2^{B[ji]} \\ \underline{S}_2^{B[ij]} & \underline{S}_2^{B[jj]} \end{pmatrix}\begin{pmatrix} \frac{1+f_1'^{[i]}}{\underline{k}_2^{[i]}+\left[\frac{\underline{k}_2^{B}}{1+\bar{k}}\right]^{[i]}}H\left(-\left(1+f_1'^{[i]}\right)\right) \\ \frac{1+f_1'^{[j]}}{\underline{k}_2^{[j]}+\left[\frac{\underline{k}_2^{B}}{1+\bar{k}}\right]^{[j]}}H\left(-\left(1+f_1'^{[j]}\right)\right) \end{pmatrix}-\begin{pmatrix} \underline{S}_1^{B[ii]} & \underline{S}_1^{B[ji]} \\ \underline{S}_1^{B[ij]} & \underline{S}_1^{B[jj]} \end{pmatrix}\begin{pmatrix} f_1^{[i]}-\bar{r} \\ f_1^{[i]}-\bar{r} \end{pmatrix}
\end{aligned}
$$

where $f_1'^{[i]}$ is defined by:

$$\frac{f_1'^{[i]} - \bar{r}}{1 + \underline{k}_2^{[i]} + \left[\frac{\underline{k}_2^B}{1+\bar{\underline{k}}}\right]^{[i]}} = f_1^{[i]} - \bar{r}$$

where:

$$\begin{aligned}
\bar{\underline{S}}_\eta^{[ii]} &= \frac{\bar{\underline{k}}_\eta^{[ii]}}{\left(1 + \bar{\underline{k}}^{[i]}\right)} \\
\bar{\underline{S}}_\eta^{[ij]} &= \frac{\bar{\underline{k}}_\eta^{[ij]}}{\left(1 + \bar{\underline{k}}^{[i]}\right)}
\end{aligned}$$

with:

$$\begin{aligned}
\bar{\underline{S}}^{[ii]} &= \bar{\underline{S}}_1^{[ii]} + \bar{\underline{S}}_2^{[ii]} \\
\bar{\underline{S}}^{[ij]} &= \bar{\underline{S}}_1^{[ij]} + \bar{\underline{S}}_2^{[ij]}
\end{aligned}$$

A24.3.3 Full alternate formulation of return equation

From these formula we compute the coefficnts:

$$\bar{\underline{S}}^{[i]} = \bar{\underline{S}}^{[ii]} + \bar{\underline{S}}^{[ij]} = \frac{\bar{\underline{k}}^{[i]}}{\left(1 + \bar{\underline{k}}^{[i]}\right)}$$

$$\frac{1}{1 + \hat{\underline{k}}^{[i]}} = 1 - \left(\bar{\underline{S}}^{[ii]} + \bar{\underline{S}}^{[ij]}\right) = 1 - \bar{\underline{S}}^{[i]}$$

$$\bar{\underline{S}}_\eta^{[i]} = \frac{\bar{\underline{k}}_\eta^{[i]}}{\left(1 + \bar{\underline{k}}^{[i]}\right)} = \bar{\underline{k}}_\eta^{[i]} \left(1 - \left(\bar{\underline{S}}^{[ii]} + \bar{\underline{S}}^{[ij]}\right)\right)$$

$$\bar{\underline{k}}_\eta^{[i]} = \frac{\bar{\underline{S}}_\eta^{[i]}}{1 - \bar{\underline{S}}^{[i]}} = \frac{\bar{\underline{S}}_\eta^{[ii]} + \bar{\underline{S}}_\eta^{[ij]}}{1 - \left(\bar{\underline{S}}^{[ii]} + \bar{\underline{S}}^{[ij]}\right)}$$

$$1 + \bar{\underline{k}}_\eta^{[i]} = \frac{1 - \left(\bar{\underline{S}}_{3-\eta}^{[ii]} + \bar{\underline{S}}_{3-\eta}^{[ij]}\right)}{1 - \left(\bar{\underline{S}}^{[ii]} + \bar{\underline{S}}^{[ij]}\right)}$$

$$\begin{aligned}
\bar{\underline{S}}_\eta^{[ii]} &= \frac{\bar{\underline{k}}_\eta^{[ii]}}{\left(1 + \bar{\underline{k}}^{[i]}\right)} \\
\bar{\underline{S}}_\eta^{[ij]} &= \frac{\bar{\underline{k}}_\eta^{[ij]}}{\left(1 + \bar{\underline{k}}^{[i]}\right)}
\end{aligned}$$

$$\begin{aligned}
\bar{\underline{S}}^{[ii]} &= \bar{\underline{S}}_1^{[ii]} + \bar{\underline{S}}_2^{[ii]} \\
\bar{\underline{S}}^{[ij]} &= \bar{\underline{S}}_1^{[ij]} + \bar{\underline{S}}_2^{[ij]}
\end{aligned}$$

$$\underline{\hat{S}}_{\eta}^{B[ii]} = \frac{\kappa \frac{\underline{\hat{k}}_{\eta}^{B[ii]}}{1+\underline{\bar{k}}^{[i]}}}{\left(1 + \underline{\hat{k}}^{[i]} + \underline{\hat{k}}_{1}^{B[i]} + \kappa \left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\bar{k}}}\right]^{[i]}\right)}$$

$$\underline{\hat{S}}_{\eta}^{B[ij]} = \frac{\kappa \frac{\underline{\hat{k}}_{\eta}^{B[ij]}}{1+\underline{\bar{k}}^{[i]}}}{\left(1 + \underline{\hat{k}}^{[i]} + \underline{\hat{k}}_{1}^{B[i]} + \kappa \left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\bar{k}}}\right]^{[i]}\right)}$$

$$\underline{S}_{\eta}^{B[ii]} = \frac{\kappa \frac{\underline{k}_{\eta}^{(B)[ii]}}{1+\underline{\bar{k}}^{[i]}}}{\left(1 + \underline{k}^{[i]} + \underline{k}_{1}^{(B)[i]} + \kappa \left[\frac{\underline{k}_{2}^{(B)}}{1+\underline{\bar{k}}}\right]^{[i]}\right)}$$

$$\underline{S}_{\eta}^{B[ij]} = \frac{\kappa \frac{\underline{k}_{\eta}^{(B)[ij]}}{1+\underline{\bar{k}}^{[i]}}}{\left(1 + \underline{k}^{[i]} + \underline{k}_{1}^{(B)[i]} + \kappa \left[\frac{\underline{k}_{2}^{(B)}}{1+\underline{\bar{k}}}\right]^{[i]}\right)}$$

Finaly, the formulation with the new set of parameters becomes:

$$\begin{aligned} 0 = & \begin{pmatrix} 1-\underline{\bar{S}}_{1}^{[ii]} & -\underline{\bar{S}}_{1}^{[ji]} \\ -\underline{\bar{S}}_{1}^{[ij]} & 1-\underline{\bar{S}}_{1}^{[jj]} \end{pmatrix} \begin{pmatrix} \frac{\bar{f}^{[i]}-\bar{r}}{1+\underline{\bar{k}}^{[i]}} \\ \frac{\bar{f}^{[j]}-\bar{r}}{1+\underline{\bar{k}}^{[j]}} \end{pmatrix} \\ & - \begin{pmatrix} \underline{\bar{S}}_{2}^{[ii]} & \underline{\bar{S}}_{2}^{[ji]} \\ \underline{\bar{S}}_{2}^{[ij]} & \underline{\bar{S}}_{2}^{[jj]} \end{pmatrix} \begin{pmatrix} \frac{1+\bar{f}^{[i]}}{\underline{\bar{k}}_{2}^{[i]}} H\left(-\left(1+\bar{f}^{[i]}\right)\right) \\ \frac{1+\bar{f}^{[j]}}{\underline{\bar{k}}_{2}^{[j]}} H\left(-\left(1+\bar{f}^{[j]}\right)\right) \end{pmatrix} - \begin{pmatrix} \underline{\hat{S}}_{2}^{B'[ii]} & \underline{\hat{S}}_{2}^{B'[ji]} \\ \underline{\hat{S}}_{2}^{B'[ij]} & \underline{\hat{S}}_{2}^{B'[jj]} \end{pmatrix} \begin{pmatrix} \frac{1+\hat{f}^{[i]}}{\underline{\hat{k}}_{2}^{[i]}+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\bar{k}}}\right]^{[i]}} H\left(-\left(1+\hat{f}^{[i]}\right)\right) \\ \frac{1+\hat{f}^{[j]}}{\underline{\hat{k}}_{2}^{[j]}+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\bar{k}}}\right]^{[j]}} H\left(-\left(1+\hat{f}^{[j]}\right)\right) \end{pmatrix} \\ & - \begin{pmatrix} \underline{S}_{2}^{B'[ii]} & \underline{S}_{2}^{B'[ji]} \\ \underline{S}_{2}^{B'[ij]} & \underline{S}_{2}^{B'[jj]} \end{pmatrix} \begin{pmatrix} \frac{1+f_{1}'^{[i]}}{\underline{k}_{2}^{[i]}+\left[\frac{\underline{k}_{2}^{B}}{1+\underline{\bar{k}}}\right]^{[i]}} H\left(-\left(1+f_{1}'^{[i]}\right)\right) \\ \frac{1+f_{1}'^{[j]}}{\underline{k}_{2}^{[j]}+\left[\frac{\underline{k}_{2}^{B}}{1+\underline{\bar{k}}}\right]^{[j]}} H\left(-\left(1+f_{1}'^{[j]}\right)\right) \end{pmatrix} - \begin{pmatrix} \underline{S}_{1}^{B'[ii]} & \underline{S}_{1}^{B'[ji]} \\ \underline{S}_{1}^{B'[ij]} & \underline{S}_{1}^{B'[jj]} \end{pmatrix} \begin{pmatrix} \frac{f_{1}'^{[i]}-\bar{r}}{1+\underline{k}_{2}^{[i]}+\left[\frac{\underline{k}_{2}^{B}}{1+\underline{\bar{k}}}\right]^{[i]}} \\ \frac{f_{1}^{[i]}-\bar{r}}{\frac{f_{1}'^{[j]}-\bar{r}}{1+\underline{k}_{2}^{[j]}+\left[\frac{\underline{k}_{2}^{B}}{1+\underline{\bar{k}}}\right]^{[j]}}} \end{pmatrix} \end{aligned}$$

with:

$$\begin{pmatrix} \underline{\hat{S}}_{2}^{B'[ii]} & \underline{\hat{S}}_{2}^{B'[ji]} \\ \underline{\hat{S}}_{2}^{B'[ij]} & \underline{\hat{S}}_{2}^{B'[jj]} \end{pmatrix} = \begin{pmatrix} \underline{\hat{S}}_{2}^{B[ii]} & \underline{\hat{S}}_{2}^{B[ji]} \\ \underline{\hat{S}}_{2}^{B[ij]} & \underline{\hat{S}}_{2}^{B[jj]} \end{pmatrix} + \begin{pmatrix} \underline{\hat{S}}_{2}^{[ii]} & \underline{\hat{S}}_{\eta}^{[ij]} \\ \underline{\hat{S}}_{\eta}^{[ij]} & \underline{\hat{S}}_{2}^{[jj]} \end{pmatrix} \begin{pmatrix} 1-\underline{\hat{S}}_{1}^{[ii]} & -\underline{\hat{S}}_{1}^{[ji]} \\ -\underline{\hat{S}}_{1}^{[ij]} & 1-\underline{\hat{S}}_{1}^{[jj]} \end{pmatrix}^{-1} \begin{pmatrix} \underline{\hat{S}}_{2}^{[ii]} & \underline{\hat{S}}_{2}^{[ji]} \\ \underline{\hat{S}}_{2}^{[ij]} & \underline{\hat{S}}_{2}^{[jj]} \end{pmatrix}$$

$$\begin{pmatrix} \underline{S}_{\eta}^{B'[ii]} & \underline{S}_{\eta}^{B'[ji]} \\ \underline{S}_{\eta}^{B'[ij]} & \underline{S}_{\eta}^{B'[jj]} \end{pmatrix} = \begin{pmatrix} \underline{S}_{\eta}^{B[ii]} & \underline{S}_{\eta}^{B[ji]} \\ \underline{S}_{\eta}^{B[ij]} & \underline{S}_{\eta}^{B[jj]} \end{pmatrix} + \begin{pmatrix} \underline{\hat{S}}_{2}^{[ii]} & \underline{\hat{S}}_{\eta}^{[ij]} \\ \underline{\hat{S}}_{\eta}^{[ij]} & \underline{\hat{S}}_{2}^{[jj]} \end{pmatrix} \begin{pmatrix} 1-\underline{\hat{S}}_{1}^{[ii]} & -\underline{\hat{S}}_{1}^{[ji]} \\ -\underline{\hat{S}}_{1}^{[ij]} & 1-\underline{\hat{S}}_{1}^{[jj]} \end{pmatrix}^{-1} \begin{pmatrix} \underline{S}_{\eta}^{[ii]} & \underline{S}_{\eta}^{[ji]} \\ \underline{S}_{\eta}^{[ij]} & \underline{S}_{\eta}^{[jj]} \end{pmatrix}$$

with:

$$\underline{\hat{S}}^{[ii]} + \underline{\hat{S}}^{[ji]} + \underline{S}^{[ii]} + \underline{S}^{[ji]} = 1$$

$$\underline{\bar{S}}^{[ii]} + \underline{\bar{S}}^{[ji]} + \underline{\hat{S}}_{1}^{B[ii]} + \underline{\hat{S}}_{1}^{B[ji]} + \underline{S}_{1}^{B[ii]} + \underline{S}_{1}^{B[ji]} = 1$$

$$\underline{\hat{S}}_{2}^{B[ii]} + \underline{\hat{S}}_{2}^{B[ji]} + \underline{S}_{2}^{B[ii]} + \underline{S}_{2}^{B[ji]} = 1$$

$$\underline{\bar{k}}_2^{[i]} = \frac{\underline{\bar{S}}_2^{[i]}}{1-\underline{\bar{S}}^{[i]}} = \frac{\underline{\bar{S}}_2^{[ii]}+\underline{\bar{S}}_2^{[ij]}}{1-\left(\underline{\bar{S}}^{[ii]}+\underline{\bar{S}}^{[ij]}\right)}$$

$$1+\underline{\bar{k}}_2^{[i]} = \frac{1-\left(\underline{\bar{S}}_1^{[ii]}+\underline{\bar{S}}_1^{[ij]}\right)}{1-\left(\underline{\bar{S}}^{[ii]}+\underline{\bar{S}}^{[ij]}\right)}$$

$$\frac{1}{1+\underline{\bar{k}}^{[i]}} = \frac{1}{1-\left(\underline{\bar{S}}^{[ii]}+\underline{\bar{S}}^{[ij]}\right)}$$

$$\begin{aligned}
0 \quad = \quad & \begin{pmatrix} 1-\underline{\bar{S}}_1^{[ii]} & -\underline{\bar{S}}_1^{[ji]} \\ -\underline{\bar{S}}_1^{[ij]} & 1-\underline{\bar{S}}_1^{[jj]} \end{pmatrix} \begin{pmatrix} \left(\bar{f}^{[i]}-\bar{r}\right)\underline{\bar{s}}_1^{[i]} \\ \left(\bar{f}^{[j]}-\bar{r}\right)\underline{\bar{s}}_1^{[j]} \end{pmatrix} \\
& -\begin{pmatrix} \underline{\bar{S}}_2^{[ii]} & \underline{\bar{S}}_2^{[ji]} \\ \underline{\bar{S}}_2^{[ij]} & \underline{\bar{S}}_2^{[jj]} \end{pmatrix} \begin{pmatrix} \left(1+\bar{f}^{[i]}\right)\underline{\bar{s}}_2^{[i]} H\left(-\left(1+\bar{f}^{[i]}\right)\right) \\ \left(1+\bar{f}^{[j]}\right)\underline{\bar{s}}_2^{[j]} H\left(-\left(1+\bar{f}^{[j]}\right)\right) \end{pmatrix} - \begin{pmatrix} \underline{\hat{S}}_2^{B'[ii]} & \underline{\hat{S}}_2^{B'[ji]} \\ \underline{\hat{S}}_2^{B'[ij]} & \underline{\hat{S}}_2^{B'[jj]} \end{pmatrix} \begin{pmatrix} \left(1+f^{[i]}\right)\underline{\hat{s}}_2^{[i]} H\left(-\left(1+f^{[i]}\right)\right) \\ \left(1+f^{[j]}\right)\underline{\hat{s}}_2^{[j]} H\left(-\left(1+f^{[j]}\right)\right) \end{pmatrix} \\
& -\begin{pmatrix} \underline{S}_2^{B'[ii]} & \underline{S}_2^{B'[ji]} \\ \underline{S}_2^{B'[ij]} & \underline{S}_2^{B'[jj]} \end{pmatrix} \begin{pmatrix} \left(1+f_1'^{[i]}\right)\underline{s}_2^{[i]} H\left(-\left(1+f_1'^{[i]}\right)\right) \\ \left(1+f_1'^{[j]}\right)\underline{s}_2^{[j]} H\left(-\left(1+f_1'^{[j]}\right)\right) \end{pmatrix} - \begin{pmatrix} \underline{S}_1^{B'[ii]} & \underline{S}_1^{B'[ji]} \\ \underline{S}_1^{B'[ij]} & \underline{S}_1^{B'[jj]} \end{pmatrix} \begin{pmatrix} \left(f_1'^{[i]}-\bar{r}\right)\underline{s}_1^{[i]} \\ \left(f_1'^{[j]}-\bar{r}\right)\underline{s}_1^{[j]} \end{pmatrix}
\end{aligned}$$

$$\underline{\bar{s}}_1^{[i]} = \frac{1-\left(\underline{\bar{S}}_1^{[ii]}+\underline{\bar{S}}_1^{[ij]}\right)}{1-\left(\underline{\bar{S}}^{[ii]}+\underline{\bar{S}}^{[ij]}\right)},\ \underline{\bar{s}}_2^{[i]} = \frac{1-\left(\underline{\bar{S}}^{[ii]}+\underline{\bar{S}}^{[ij]}\right)}{\underline{\bar{S}}_2^{[ii]}+\underline{\bar{S}}_2^{[ij]}}$$

$$\begin{aligned}
\underline{\hat{s}}_1^{[i]} \quad &= \quad \frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}+\underline{\hat{S}}^{B[ii]}+\underline{\hat{S}}^{B[ij]}\right)}{1-\left(\underline{\hat{S}}_1^{[ii]}+\underline{\hat{S}}_1^{[ij]}+\underline{\hat{S}}_1^{B[ii]}+\underline{\hat{S}}_1^{B[ij]}\right)},\ \underline{\hat{s}}_2^{[i]} = \frac{1-\left(\underline{\hat{S}}^{[ii]}+\underline{\hat{S}}^{[ij]}+\underline{\hat{S}}^{B[ii]}+\underline{\hat{S}}^{B[ij]}\right)}{\underline{\hat{S}}_2^{[ii]}+\underline{\hat{S}}_2^{[ij]}+\underline{\hat{S}}_2^{B[ii]}+\underline{\hat{S}}_2^{B[ij]}} \\
\underline{s}_1^{[i]} \quad &= \quad \frac{1-\left(\underline{S}^{[ii]}+\underline{S}^{[ij]}+\underline{S}^{B[ii]}+\underline{S}^{B[ij]}\right)}{1-\left(\underline{S}_1^{[ii]}+\underline{S}_1^{[ij]}+\underline{S}_1^{B[ii]}+\underline{S}_1^{B[ij]}\right)},\ \underline{s}_2^{[i]} = \frac{1-\left(\underline{S}^{[ii]}+\underline{S}^{[ij]}+\underline{S}^{B[ii]}+\underline{S}^{B[ij]}\right)}{\underline{S}_2^{[ii]}+\underline{S}_2^{[ij]}+\underline{S}_2^{B[ii]}+\underline{S}_2^{B[ij]}}
\end{aligned}$$

Appendix 25 Computation of the functionl derivatives

We comput the returns derivativ with respect to the various flds.

A25.1 Derivatives with respect to $\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2$

Starting with investors nd banks return equation:

$$\begin{aligned}
&\left(\Delta\left(\hat{X},\hat{X}'\right) - \frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right) \frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\bar{K}',\bar{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)} \\
= \quad & \frac{k_1\left(X',X'\right)\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)\left|\Psi\left(K',X'\right)\right|^2}{\left(1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)\left(1+\underline{k}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)}
\end{aligned}$$

$$\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)} \tag{369}$$

$$-\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)\left|\Psi\left(K',X'\right)\right|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)\right)}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\bar{K}',\bar{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)}$$

$$=\quad\frac{\underline{k}_1^{(B)}\left(X',\bar{X}\right)}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{k}}}\frac{\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)\left|\Psi\left(K',X'\right)\right|^2}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{k}}}$$

and first consder:

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{k_1\left(X',X'\right)\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)\left|\Psi\left(K',X'\right)\right|^2}{\left(1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)\left(1+\underline{k}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)}$$

$$\rightarrow\quad-\frac{k_1\left(X',X'\right)\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)\left|\Psi\left(K',X'\right)\right|^2}{\left(1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)\left(1+\underline{k}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)^2}\frac{\left\langle k_2\left(X',\hat{X}\right)\right\rangle}{\left\langle K\right\rangle\left\|\Psi\right\|^2}$$

$$-\frac{k_1\left(X',X'\right)\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)\left|\Psi\left(K',X'\right)\right|^2}{\left(1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)^2\left(1+\underline{k}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)}\frac{\left\langle k\left(X',\hat{X}\right)\right\rangle}{\left\langle K\right\rangle\left\|\Psi\right\|^2}$$

we use that:

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{k_1\left(X',X'\right)\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)\left|\Psi\left(K',X'\right)\right|^2}{\left(1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)\left(1+\underline{k}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)}<<1$$

similarly, we have:

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{\underline{k}_1^{(B)}\left(X',\bar{X}\right)}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{k}}}\frac{\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)\left|\Psi\left(K',X'\right)\right|^2}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{k}}}<<1$$

The derivativ for return becomes:

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)}\right)A\right)\simeq 0$$

and:

$$\frac{\delta\left\{\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}-\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left(\hat{X}'\right)}A\right\}}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\simeq 0 \tag{370}$$

where:

$$A=\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)}$$

$$B=\frac{\left(1-\bar{M}\right)\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}$$

$$\hat{D}\left(\hat{X}'\right)=1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)$$

and:

$$\bar{N}=\frac{\left(\hat{k}_1^B\left(\hat{X},\bar{X}'\right)+\kappa\frac{\hat{k}_2^B\left(\hat{X},\bar{X}'\right)}{1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)}-\kappa\frac{\hat{k}_2^B\left(\bar{X},\bar{X}''\right)\bar{K}_0''\bar{\bar{k}}\left(\bar{X}'',\bar{X}'\right)}{\left(1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)\right)^2}\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left(\hat{X}'\right)\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}$$

$$\begin{aligned}0\simeq&\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)\\&-\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left\{\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}A\right\}\end{aligned}\tag{371}$$

$$\begin{aligned}&\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)\\\simeq&\left[\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}\right]A\\&+\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}A\end{aligned}\tag{372}$$

so that:

$$\begin{aligned}&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)\\\simeq&\left\{\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}\right\}^{-1}\times\\&\left[\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)}{\hat{D}\left(\hat{X}'\right)}A+\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left(\hat{X}'\right)}\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}A\right]\end{aligned}\tag{373}$$

A25.1.1 Estimation of $\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}A$

Recall that:

$$A=\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

and:

$$\frac{\delta\left(\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]\left(\hat{X}'\right)}\right)\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]\left(\hat{X}'\right)}\right)}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\simeq 0$$

$$\begin{aligned}&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{k}}\right]\left(\hat{X}'\right)} \\ = & \left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left(\hat{X}'\right)}\right)^{-1}\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left(\hat{X}'\right)}A\end{aligned}\tag{374}$$

A25.1.1.1 Estimation of:

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\left(\frac{1}{\hat{D}\left(\hat{X}\right)}\right)$$

given the normalizations:

$$\begin{aligned}\hat{D}\left(\hat{X}\right) = & 1+\int\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2+\int\frac{\hat{k}_1^B\left(\hat{X},\bar{X}'\right)-\left\langle\hat{k}_1^B\left(\hat{X},\bar{X}'\right)\right\rangle}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0',\bar{X}'\right)\right|^2 \\ & +\kappa\int\frac{\hat{k}_2^B\left(\hat{X},\bar{X}'\right)-\left\langle\hat{k}_2^B\left(\hat{X},\bar{X}'\right)\right\rangle}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\frac{\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0',\bar{X}'\right)\right|^2}{1+\int\bar{k}\left(\bar{X}',\bar{X}''\right)\bar{K}_0''\left|\bar{\Psi}\left(\bar{K}_0'',\bar{X}''\right)\right|^2}\end{aligned}$$

$$\hat{k}\left(\hat{X},\hat{X}'\right)\rightarrow\frac{\hat{k}\left(\hat{X},\hat{X}'\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}=\frac{\hat{k}\left(\hat{X},\hat{X}'\right)}{\int\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}$$

$$\hat{k}_\eta^B\left(\hat{X},\bar{X}'\right)\rightarrow\frac{\hat{k}_\eta^B\left(\hat{X},\bar{X}'\right)}{\int\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}$$

and:

$$\frac{\delta\left(\int\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2\right)}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}<<1$$

we find:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\left(\frac{1}{\hat{D}\left(\hat{X}\right)}\right)\\
= \quad & -\frac{\left(\hat{k}\left(\hat{X},\hat{X}_1\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\left(\hat{D}\left(\hat{X}\right)\right)^2}\\
&+\left\{\frac{\int\frac{\left(\hat{k}\left(\hat{X},\hat{X}'\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}+\int\frac{\left(\hat{k}_1^B\left(\hat{X},\bar{X}'\right)-\left\langle\hat{k}_1^B\left(\hat{X},\bar{X}'\right)\right\rangle\right)\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0',\bar{X}'\right)\right|^2}{\int\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}}{\left(\hat{D}\left(\hat{X}\right)\right)^2}\right.\\
&\left.+\frac{\kappa}{\left(\hat{D}\left(\hat{X}\right)\right)^2}\int\frac{\left(\hat{k}_2^B\left(\hat{X},\bar{X}'\right)-\left\langle\hat{k}_2^B\left(\hat{X},\bar{X}'\right)\right\rangle\right)\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0',\bar{X}'\right)\right|^2}{\int\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2\left(1+\int\bar{k}\left(\bar{X}',\bar{X}''\right)\bar{K}_0''\left|\bar{\Psi}\left(\bar{K}_0'',\bar{X}''\right)\right|^2\right)}\right\}\\
&\times\frac{\hat{K}_1}{\left(\int\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|\right)}
\end{aligned}$$

in averages this leads to:

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\left(\frac{1}{\hat{D}\left(\hat{X}\right)}\right)\rightarrow-\frac{\left(\hat{k}\left(\hat{X},\hat{X}_1\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\left(\hat{D}\left(\hat{X}\right)\right)^2}$$

A25.1.1.2 Estimation of

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left(\hat{X}'\right)}A$$

Using the previous result:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\left(\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\underline{\bar{k}}}\right]\left(\hat{X}'\right)\right)}\right)\\
\simeq\quad&\frac{\hat{K}_1\hat{k}_1\left(\hat{X}_1,\hat{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\hat{D}}-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\hat{D}\left(\hat{X}'\right)}\frac{\hat{K}_1}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\\
&-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\hat{D}\left(\hat{X}'\right)}\frac{\left(\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\hat{D}\left(\hat{X}\right)}
\end{aligned}$$

as before, this is can be replaced in matricl elements by:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left(\hat{X}'\right)}A\\
=&\frac{\hat{K}_1\hat{k}_1\left(\hat{X}_1,\hat{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}A-\frac{\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\left(1+\left(\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\langle A\rangle\\
\simeq&\frac{\hat{K}_1\hat{k}_1\left(\hat{X}_1,\hat{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}A-\frac{\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\langle A\rangle
\end{aligned}$$

A25.1.1.3 formula for $\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}A$

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)}\tag{375}$$

$$\begin{aligned}
=&\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left(\hat{X}'\right)}\right)^{-1}\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left(\hat{X}'\right)}A\\
=&\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left(\hat{X}'\right)}\right)^{-1}\left(\frac{\hat{K}_1\hat{k}_1\left(\hat{X}_1,\hat{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}A-\frac{\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\langle A\rangle\right)\\
\simeq&\left(1-\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\right)^{-1}\left(\frac{\hat{K}_1\hat{k}_1\left(\hat{X}_1,\hat{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}A-\frac{\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\langle A\rangle\right)
\end{aligned}$$

A25.1.2 Estimation of $\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)$

We start with formula (373):

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)\tag{376}$$

$$\begin{aligned}
\simeq&\left\{\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}\right\}^{-1}\times\\
&\left[\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\hat{D}\left(\hat{X}'\right)}A+\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\hat{D}\left(\hat{X}'\right)}\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}A\right]
\end{aligned}$$

and compute the various contributions:

A25.1.2.1 Estimation of:

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{\hat{K}'\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)}{\hat{D}\left(\hat{X}'\right)}A$$

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}}\frac{\hat{K}'\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\hat{k}}\right]\left(\hat{X}'\right)\right)}$$

$$\simeq -\frac{\left(\hat{k}\left(\hat{X},\bar{X}\right)-\left\langle\hat{k}\left(\hat{X},\bar{X}\right)\right\rangle\right)\hat{K}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}\left(\hat{D}\left(\hat{X}\right)\right)^{2}}\underline{\hat{k}}_{1}^{B}\left(\left\langle\hat{X}\right\rangle,\bar{X}\right)+\frac{\hat{K}\underline{\hat{k}}_{1}^{B}\left(\hat{X},\bar{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}\hat{D}\left(\hat{X}\right)}-\frac{\hat{K}'\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}\hat{D}\left(\hat{X}\right)}\frac{\hat{K}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}$$

In average, this formula becomes:

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}}\frac{\hat{K}'\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}\hat{D}\left(\hat{X}'\right)}A$$

$$\simeq -\frac{\left(\hat{k}\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\hat{K}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}\left(\hat{D}\left(\hat{X}\right)\right)^{2}}\left(\frac{\hat{K}'\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}\right)A$$

$$+\frac{\hat{K}\underline{\hat{k}}_{1}^{B}\left(\hat{X},\bar{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}\hat{D}\left(\hat{X}\right)}A-\frac{\hat{K}\underline{\hat{k}}_{1}^{B}\left(\left\langle\hat{X}\right\rangle,\bar{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}\hat{D}\left(\hat{X}\right)}\langle A\rangle$$

$$\rightarrow -\frac{\underline{\hat{k}}_{1}^{B}\left(\left\langle\hat{X}\right\rangle,\bar{X}\right)\left(\hat{k}\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\hat{K}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}\langle A\rangle+\frac{\hat{K}\underline{\hat{k}}_{1}^{B}\left(\hat{X},\bar{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}A-\frac{\hat{K}\underline{\hat{k}}_{1}^{B}\left(\left\langle\hat{X}\right\rangle,\bar{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}\langle A\rangle$$

A25.1.2.2 Summing terms

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}}\bar{g}\left(\hat{K}',\hat{X}'\right) \tag{377}$$

$$\simeq \left\{\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_{1}\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^{2}}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)}{1+\underline{\bar{\bar{k}}}_{2}\left(\bar{X}'\right)}\right\}^{-1}$$

$$\times\left[\frac{\underline{\hat{k}}_{1}^{B}\left(\hat{X},\bar{X}\right)\hat{K}A}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}-\frac{\underline{\hat{k}}_{1}^{B}\left(\left\langle\hat{X}\right\rangle,\bar{X}\right)\left(1+\left(\hat{k}\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\right)\hat{K}\langle A\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right.$$

$$+\frac{\hat{K}'\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\hat{D}\left(\hat{X}'\right)}\left(1-\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle\right)^{-1}$$

$$\left.\times\left(\frac{\hat{K}_{1}\hat{k}_{1}\left(\hat{X}_{1},\hat{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}A-\frac{\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle\left(1+\left(\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}_{1}\right)\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}_{1}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}\langle A\rangle\right)\right]$$

Using the normalzations:

$$1+\underline{\bar{k}}_{2}\left(\bar{X}'\right)\rightarrow\frac{1-\left\langle\bar{k}_{1}\left(\bar{X}',\bar{X}\right)\right\rangle}{1-\left\langle\bar{k}\left(\bar{X}',\bar{X}\right)\right\rangle}\left(1+\frac{\underline{\bar{k}}_{2}\left(\hat{X}'\right)}{1-\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle}\right)$$

$$\frac{1}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)} \rightarrow \frac{\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)}{\left(1-\left\langle\bar{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\right)\left(1+\frac{\underline{\bar{k}}_2\left(\hat{X}'\right)}{1-\left\langle\bar{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle}\right)}$$

$$\frac{1}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)} \rightarrow \frac{\left(1-\left\langle\bar{k}\right\rangle\right)}{\left(1-\left\langle\bar{k}_1\right\rangle\right)}$$

The derivative, when inserted in matrices in matrices elements writes:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right) \qquad (378)\\
\simeq\ &\left\{\left(1-\left\langle\bar{k}\left(\bar{X}',\bar{X}\right)\right\rangle\right)^2\right\}^{-1}\\
&\times\left[\frac{\underline{\hat{k}}_1^B\left(\hat{X},\left\langle\bar{X}\right\rangle\right)\hat{K}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}A-\frac{\left\langle\underline{\hat{k}}_1^B\right\rangle\left(1+\left(\hat{k}\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}\right\rangle\right)\right)\hat{K}}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\langle A\rangle\right.\\
&\left.+\left\langle\underline{\hat{k}}_1^B\right\rangle\left\{\left(1-\left\langle\hat{k}_1\right\rangle\right)^{-1}\left(\frac{\hat{K}_1\hat{k}_1\left(\hat{X}_1,\left\langle\hat{X}\right\rangle\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}A-\frac{\left\langle\hat{k}_1\right\rangle\left(1+\left(\hat{k}\left(\hat{X}_1,\hat{X}\right)-\left\langle\hat{k}\right\rangle\right)\right)\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\langle A\rangle\right)\right\}\right]
\end{aligned}$$

which is in averagzs:

$$\rightarrow \frac{\left(\left(\underline{\hat{k}}_1^B\left(\hat{X},\left\langle\bar{X}\right\rangle\right)+\left\langle\underline{\hat{k}}_1^B\right\rangle\frac{\hat{k}_1\left(\hat{X}_1,\left\langle\hat{X}\right\rangle\right)}{1-\left\langle\hat{k}_1\right\rangle}\right)A-\left\langle\underline{\hat{k}}_1^B\left(\hat{X},\left\langle\bar{X}\right\rangle\right)+\left\langle\underline{\hat{k}}_1^B\right\rangle\frac{\hat{k}_1\left(\hat{X}_1,\left\langle\hat{X}\right\rangle\right)}{1-\left\langle\hat{k}_1\right\rangle}\right\rangle\langle A\rangle\right)\hat{K}}{\left(1-\left\langle\bar{k}\left(\bar{X}',\bar{X}\right)\right\rangle\right)^2\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}-$$

A25.1.3 Estimation of $\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\hat{g}\left(\hat{K}',\hat{X}'\right)$

We start with:

$$A=\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

and:

$$\hat{g}\left(\hat{K},\hat{X}\right)=\left(1-\hat{M}\right)^{-1}\left(\left(1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)A-\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)$$

The derivative decomposes as:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\hat{g}\left(\hat{K},\hat{X}\right)\\
=\ &\left(1-\hat{M}\right)^{-1}\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\hat{M}\hat{g}\left(\hat{K},\hat{X}\right)\\
&+\left(1-\hat{M}\right)^{-1}\left(\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)A-\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)\right)
\end{aligned}$$

and in avrage this is:

$$\left(1-\hat{M}\right)^{-1}\left(\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}}\hat{M}\right)\hat{g}\left(\hat{K},\hat{X}\right) \tag{379}$$
$$+\frac{\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}{\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}\left\langle\left(1-\hat{M}\right)^{-1}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)A\right\rangle$$
$$+\left(1-\hat{M}\right)^{-1}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}}A$$
$$-\left(1-\hat{M}\right)^{-1}\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}}\left(\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)$$

The various contributions to (379) are estimatd as:

$$\begin{aligned}
&\frac{\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}{\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}\left\langle\left(1-\hat{M}\right)^{-1}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)A\right\rangle\\
=\;&\frac{\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}{\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)
\end{aligned}$$

$$\begin{aligned}
&\left(1-\hat{M}\right)^{-1}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}}A\\
=\;&\left(1-\left\langle\hat{k}\right\rangle\right)^{-1}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)\left(1-\left\langle\hat{k}_{1}\right\rangle\right)^{-1}\left(\frac{\hat{K}_{1}\hat{k}_{1}\left(\hat{X}_{1},\hat{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}A-\frac{\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle\hat{K}_{1}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}\langle A\rangle\right)\\
\simeq\;&\left(\left(1-\left\langle\hat{k}\right\rangle\right)\left(1-\left\langle\hat{k}_{1}\right\rangle\right)\right)^{-1}\left(\frac{\hat{K}_{1}\hat{k}_{1}\left(\hat{X}_{1},\hat{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}-\frac{\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle\hat{K}_{1}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}\right)\\
&\times\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)
\end{aligned}$$

and gathering the contributions leads to:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\hat{g}\left(\hat{K},\hat{X}\right)\\
=\;&\left(1-\hat{M}\right)^{-1}\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\hat{M}\hat{g}\left(\hat{K},\hat{X}\right)\\
&+\frac{\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}{\left(1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)\\
&+\left(\left(1-\left\langle\hat{k}\right\rangle\right)\left(1-\left\langle\hat{k}_1\right\rangle\right)\right)^{-1}\left(\frac{\hat{K}_1\hat{k}_1\left(\hat{X}_1,\hat{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}-\frac{\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\right)\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)\\
&-\left(1-\hat{M}\right)^{-1}\left(\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)\right)
\end{aligned}$$

The successive contributions are computed below.

A25.1.3.1 Estimtn of

$$\frac{\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}{\left(1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}$$

This term is obtained by developping the contributions involved in its definitn:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)\\
\rightarrow\;&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{\left(1-\left(\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_1^B\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left(1+\frac{\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1-\left(\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_1^B\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)}\right)}{\left(1-\left(\left\langle\underline{\hat{k}}\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B\left(\hat{X}',\bar{X}\right)}{1+\bar{k}}\right]\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)}\\
=\;&-\left(\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B\left(\hat{X}',\bar{X}\right)}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\underline{\hat{k}}_1\left(\hat{X}',\hat{X}\right)\right\rangle\right)+\left\langle\hat{k}_1^B\left(\hat{X}',\hat{X}\right)\right\rangle\left\langle\hat{k}_2\left(\hat{X}',\hat{X}\right)\right\rangle\right)\\
&\times\frac{\left(1+\frac{\underline{\hat{k}}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1-\left(\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_1^B\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)}\right)\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\hat{K}}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle\left(1-\left(\left\langle\underline{\hat{k}}\left(\hat{X}',\hat{X}\right)\right\rangle+\left(\left\langle\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B\left(\hat{X}',\bar{X}\right)}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)^2}\\
&+\frac{\left(\hat{k}_2\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}_2\left(\hat{X}',\hat{X}\right)\right\rangle\right)}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle\left(1-\left(\left\langle\underline{\hat{k}}\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\underline{\hat{k}}_1^B\left(\hat{X}',\bar{X}\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B\left(\hat{X}',\bar{X}\right)}{1+\bar{k}}\right]\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)}
\end{aligned}$$

and thus:

$$\frac{\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}{\left(1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}$$

$$\simeq -\frac{\left(\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\underline{\hat{k}}\right\rangle\right)+\left\langle\hat{k}_1^B\right\rangle\left\langle\hat{k}_2\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\frac{\hat{K}}{\left\langle\hat{K}\right\rangle}}{\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left(1-\left(\left\langle\underline{\hat{k}}\right\rangle+\left\langle\underline{\hat{k}}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left\|\hat{\Psi}\right\|^2}$$
$$+\frac{\left(\hat{k}_2\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}_2\right\rangle\right)\hat{K}}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)}$$

A25.1.3.2 Estimation of

$$\left(\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)$$

Defining:

$$\hat{M}\left(\left(\hat{K},\hat{X}\right),\left(\hat{K}',\hat{X}'\right)\right)=\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left(\hat{X}\right)}$$

Given normalztions:

$$\begin{aligned}\hat{D}\left(\hat{X}\right)&=1+\int\frac{\hat{k}\left(\hat{X},\hat{X}'\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2+\int\frac{\hat{k}_1^B\left(\hat{X},\bar{X}'\right)-\left\langle\hat{k}_1^B\left(\hat{X},\bar{X}'\right)\right\rangle}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0',\bar{X}'\right)\right|^2\\&+\kappa\int\frac{\hat{k}_2^B\left(\hat{X},\bar{X}'\right)-\left\langle\hat{k}_2^B\left(\hat{X},\bar{X}'\right)\right\rangle}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\frac{\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0',\bar{X}'\right)\right|^2}{1+\int\bar{k}\left(\bar{X}',\bar{X}''\right)\bar{K}_0''\left|\bar{\Psi}\left(\bar{K}_0'',\bar{X}''\right)\right|^2}\end{aligned}$$

and:

$$\hat{k}\left(\hat{X},\hat{X}'\right)\rightarrow\frac{\hat{k}\left(\hat{X},\hat{X}'\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}=\frac{\hat{k}\left(\hat{X},\hat{X}'\right)}{\int\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}$$

In average, this leads to:

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}\left(\frac{1}{\hat{D}\left(\hat{X}\right)}\right)\rightarrow-\frac{\left(\hat{k}\left(\hat{X},\hat{X}_1\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\left(\hat{D}\left(\hat{X}\right)\right)^2}$$

$$\begin{aligned}\frac{\delta\hat{M}}{\delta\left|\hat{\Psi}\left(\hat{K}_1,\hat{X}_1\right)\right|^2}&=\frac{\hat{k}\left(\hat{X},\hat{X}_1\right)\hat{K}}{\hat{D}\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}-\int\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\frac{\hat{K}_1}{\left(\int\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|\right)}\\&-\int\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\frac{\left(\hat{k}\left(\hat{X},\hat{X}_1\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\hat{D}\left(\hat{X}\right)}\end{aligned}$$

$$\frac{\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)\left\langle\hat{K}\right\rangle}{\hat{D}\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}-\frac{\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle}{\hat{D}}\frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}$$
$$-\frac{\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle}{\hat{D}}\frac{\left(\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2\hat{D}\left(\hat{X}\right)}$$

Given that in average:

$$\hat{D}\left(\hat{X}\right)\rightarrow\hat{D}\left(\left\langle\hat{X}\right\rangle\right)\simeq 1$$

The contribution for:

$$\left(\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)$$

is given:

$$\left(\frac{\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)}{\left\|\hat{\Psi}\right\|^2}-\frac{\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}-\frac{\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\left(\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\right)$$
$$=\left(\frac{\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)}{\left\|\hat{\Psi}\right\|^2}-\frac{\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\left(1+\left(\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\right)$$

A25.1.3.3 Estimation of:

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)$$

We use that:

$$\bar{N}\rightarrow\frac{\left(\hat{k}_1^B\left(\hat{X},\bar{X}'\right)+\kappa\frac{\hat{k}_2^B\left(\hat{X},\bar{X}'\right)}{1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)}-\kappa\int\frac{\hat{k}_2^B\left(\bar{X},\bar{X}''\right)\bar{K}_0''\bar{\bar{k}}\left(\bar{X}'',\bar{X}'\right)}{\left(1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)\right)^2}\right)\hat{K}}{1+\int\hat{k}\left(\hat{X},\hat{X}'\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2+\int\hat{k}_1^B\left(\hat{X},\bar{X}'\right)\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0',\bar{X}'\right)\right|^2+\kappa\int\hat{k}_2^B\left(\hat{X},\bar{X}'\right)\frac{\bar{K}_0'\left|\bar{\Psi}\left(\bar{K}_0',\bar{X}'\right)\right|^2}{1+\int\bar{\bar{k}}\left(\bar{X}',\bar{X}''\right)\bar{K}_0''\left|\bar{\Psi}\left(\bar{K}_0'',\bar{X}''\right)\right|^2}}$$

Given the normalztn, this reduces to:

$$\bar{N}\rightarrow\left\langle\hat{k}_1^B\left(\hat{X},\bar{X}'\right)\right\rangle+\kappa\frac{\left\langle\hat{k}_2^B\left(\hat{X},\bar{X}'\right)\right\rangle}{1+\left\langle\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)\right\rangle}\left(1-\frac{\left\langle\bar{\bar{k}}\left(\bar{X}'',\bar{X}'\right)\right\rangle}{\left(1+\left\langle\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)\right\rangle\right)^2}\right)$$

so that:

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)$$
$$\rightarrow\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\frac{\left(\hat{k}_1^B\left(\hat{X}',\bar{X}'\right)+\kappa\frac{\hat{k}_2^B\left(\hat{X}',\bar{X}'\right)}{1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)}-\kappa\frac{\hat{k}_2^B\left(\bar{X}',\bar{X}''\right)\bar{K}_0''\bar{\bar{k}}\left(\bar{X}'',\bar{X}'\right)}{\left(1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)\right)^2}\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\hat{D}\left(\hat{X}'\right)}\bar{g}\left(\hat{K}',\hat{X}'\right)$$
$$+\bar{N}\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)$$

and the corresponding contribution writes:

$$\rightarrow \quad -\frac{\left(\hat{k}\left(\hat{X},\hat{X}\right)-\left\langle \hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}_1}{\left\langle \hat{K}\right\rangle\left\Vert\hat{\Psi}\right\Vert^2}\left\langle \bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle$$
$$+\frac{\left(\left(\hat{k}_1^B\left(\hat{X},\bar{X}'\right)-\hat{k}_1^B\left(\left\langle\hat{X}\right\rangle,\bar{X}'\right)\right)+\kappa\frac{\hat{k}_2^B\left(\hat{X},\bar{X}'\right)-\hat{k}_2^B\left(\left\langle\hat{X}\right\rangle,\bar{X}'\right)}{1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)}\left(1-\frac{1}{1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)}\right)\right)\left\langle\bar{K}\right\rangle\left\Vert\bar{\Psi}\right\Vert^2}{\left\langle \hat{K}\right\rangle\left\Vert\hat{\Psi}\right\Vert^2}\left\langle\bar{N}\right\rangle\left\langle \bar{g}\left(\hat{K},\hat{X}\right)\right\rangle$$
$$+\left\langle\bar{N}\right\rangle\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)$$

A25.1.3.4 Estimation of $\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\hat{g}\left(\hat{K},\hat{X}\right)$

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\hat{g}\left(\hat{K},\hat{X}\right)$$

We consider th sum of contributions:

$$\left(1-\hat{M}\right)^{-1}\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\hat{M}\hat{g}\left(\hat{K},\hat{X}\right)$$
$$+\frac{\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}{\left(1+\underline{\hat{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)$$
$$+\left(\left(1-\left\langle\hat{k}\right\rangle\right)\left(1-\left\langle\hat{k}_1\right\rangle\right)\right)^{-1}\left(\frac{\hat{K}_1\hat{k}_1\left(\hat{X}_1,\hat{X}\right)}{\left\langle\hat{K}\right\rangle\left\Vert\hat{\Psi}\right\Vert^2}-\frac{\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\Vert\hat{\Psi}\right\Vert^2}\right)\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)$$
$$-\left(1-\hat{M}\right)^{-1}\left(\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)\right)$$

leadng t:

$$\simeq \quad -\frac{\left(\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\underline{\hat{k}}\right\rangle\right)+\left\langle\hat{k}_1^B\right\rangle\left\langle\hat{k}_2\right\rangle\right)\frac{\left\Vert\bar{\Psi}\right\Vert^2\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle\hat{K}\right\rangle}\frac{\hat{K}}{\left\langle\hat{K}\right\rangle}}{\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^2\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle\hat{K}\right\rangle}\right)\right)\left(1-\left(\left\langle\underline{\hat{k}}\right\rangle+\left\langle\underline{\hat{k}}_1^B\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^2\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle\hat{K}\right\rangle}+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\bar{k}}\right]\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^2\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle\hat{K}\right\rangle}\right)\right)\left\Vert\hat{\Psi}\right\Vert^2}$$
$$+\frac{\left(\hat{k}_2\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}_2\right\rangle\right)\hat{K}}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle\hat{K}\right\rangle\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^2\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^2\left\langle\hat{K}\right\rangle}\right)\right)}$$

which is equal to:

$$\begin{aligned}
&= \left(1-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)^{-1}\left(\frac{\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)}{\left\|\hat{\Psi}\right\|^2}-\frac{\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\left(1+\left(\hat{k}\left(\left\langle\hat{X}\right\rangle,\hat{X}_1\right)\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\right)\hat{g}\left(\hat{K},\hat{X}\right)\\
&+\frac{\left(\hat{k}_2\left(\hat{X}',\hat{X}\right)-\left\langle\hat{k}_2\right\rangle\right)\hat{K}_1\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)}\\
&-\frac{\left(\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\hat{\underline{k}}}\right]\right\rangle\left(1-\left\langle\underline{\hat{k}}\right\rangle\right)+\left\langle\hat{k}_1^B\right\rangle\left\langle\hat{k}_2\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle}\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)}{\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left(1-\left(\left\langle\underline{\hat{k}}\right\rangle+\left\langle\underline{\hat{k}}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}+\kappa\left\langle\left[\frac{\underline{\hat{k}}_2^B}{1+\hat{\underline{k}}}\right]\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left\|\hat{\Psi}\right\|^2}\\
&+\left(\left(1-\left\langle\hat{k}\right\rangle\right)\left(1-\left\langle\hat{k}_1\right\rangle\right)\right)^{-1}\left(\frac{\hat{K}_1\hat{k}_1\left(\hat{X}_1,\hat{X}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}-\frac{\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\right)\\
&\times\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)\\
&-\left(1-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)^{-1}\left(\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)\right)
\end{aligned}$$

Ultimately, we use that:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\left(\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)\\
\rightarrow\;&-\frac{\left(\hat{k}\left(\hat{X},\hat{X}\right)-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{K}_1}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\\
&+\frac{\left(\left(\hat{k}_1^B\left(\hat{X},\bar{X}'\right)-\hat{k}_1^B\left(\left\langle\hat{X}\right\rangle,\bar{X}'\right)\right)+\kappa\frac{\hat{k}_2^B\left(\hat{X},\bar{X}'\right)-\hat{k}_2^B\left(\left\langle\hat{X}\right\rangle,\bar{X}'\right)}{1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)}\left(1-\frac{1}{1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)}\right)\right)\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^2}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}\left\langle\bar{N}\right\rangle\left\langle\bar{g}\left(\hat{K},\hat{X}\right)\right\rangle\\
&+\left\langle\bar{N}\right\rangle\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)
\end{aligned}$$

with:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)\\
\rightarrow\;&\frac{\left(\left(\underline{\hat{k}}_1^B\left(\hat{X},\left\langle\bar{X}\right\rangle\right)+\left\langle\underline{\hat{k}}_1^B\right\rangle\frac{\hat{k}_1\left(\hat{X}_1,\left\langle\hat{X}\right\rangle\right)}{1-\left\langle\hat{k}_1\right\rangle}\right)A-\left\langle\underline{\hat{k}}_1^B\left(\hat{X},\left\langle\bar{X}\right\rangle\right)+\left\langle\underline{\hat{k}}_1^B\right\rangle\frac{\hat{k}_1\left(\hat{X}_1,\left\langle\hat{X}\right\rangle\right)}{1-\left\langle\hat{k}_1\right\rangle}\right\rangle\left\langle A\right\rangle\right)\hat{K}}{\left(1-\left\langle\bar{k}\left(\bar{X}',\bar{X}\right)\right\rangle\right)^2\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^2}
\end{aligned}$$

A25.1.3.5 Dominant contribution Considering all contributions, they are all given by deviation around averages, except:

$$\frac{\delta}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\hat{g}\left(\hat{K},\hat{X}\right)$$
$$\simeq -\frac{\left(\kappa\left\langle\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\hat{\underline{k}}\right\rangle\right)+\left\langle\hat{k}_1^B\right\rangle\left\langle\hat{k}_2\right\rangle\right)\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\frac{1}{1-\left\langle\hat{\underline{k}}\right\rangle}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\frac{\hat{K}_1}{\left\langle\hat{K}\right\rangle}}{\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left(1-\left(\left\langle\hat{\underline{k}}\right\rangle+\left(\left\langle\hat{\underline{k}}_1^B\right\rangle+\kappa\left\langle\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left\|\hat{\Psi}\right\|^2}$$

A25.2 Derivatives with respect to $\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2$

We use the followng conditions:

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\frac{k_1\left(X',X'\right)\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)\left|\Psi\left(K',X'\right)\right|^2}{\left(1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)\left(1+\underline{k}_2\left(\hat{X}'\right)+\kappa\left[\frac{\underline{k}_2^B}{1+\bar{k}}\right]\left(X'\right)\right)}<<1$$

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\frac{\underline{k}_1^{(B)}\left(X',\bar{X}\right)}{1+\underline{k}\left(\hat{X}'\right)+\underline{k}_1^{(B)}\left(\bar{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}\frac{\left(f_1'\left(X'\right)K'-\bar{C}\left(X'\right)\right)\left|\Psi\left(K',X'\right)\right|^2}{1+\underline{k}_2\left(\hat{X}'\right)+\kappa\frac{\underline{k}_2^{(B)}\left(\bar{X}'\right)}{1+\underline{\bar{k}}}}<<1$$

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\simeq 0$$

$$\begin{aligned}&\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\left(\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}\right.\\&\left.-\frac{\hat{K}'\hat{\underline{k}}_1^B\left(\hat{X}',\bar{X}\right)}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)\simeq 0\end{aligned}\tag{380}$$

A25.2.1 Computation of $\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)$

$$\begin{aligned}&\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\left(\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}\right)\bar{g}\left(\hat{K}',\hat{X}'\right)\\&+\left(\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}\right)\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)\\=\;&-\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\frac{\hat{K}'\hat{\underline{k}}_1^B\left(\hat{X}',\bar{X}\right)}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\\&-\frac{\hat{K}'\hat{\underline{k}}_1^B\left(\hat{X}',\bar{X}\right)}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\bar{k}}\right]\left(\hat{X}'\right)}\end{aligned}$$

so that:

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\bar{g}\left(\hat{K}',\hat{X}'\right)$$

$$=\left(\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_{1}\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^{2}}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)}{1+\underline{\bar{k}}_{2}\left(\bar{X}'\right)}\right)^{-1}$$

$$\left\{-\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\frac{\hat{K}'\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\hat{k}}}\right]\left(\hat{X}'\right)\right)}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\hat{k}}}\right]\left(\hat{X}'\right)}\right.$$

$$-\frac{\hat{K}'\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\hat{k}}}\right]\left(\hat{X}'\right)\right)}\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\hat{k}}}\right]\left(\hat{X}'\right)}$$

$$\left.-\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_{1}\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^{2}}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)}{1+\underline{\bar{k}}_{2}\left(\bar{X}'\right)}\right)\bar{g}\left(\hat{K}',\hat{X}'\right)\right\}$$

Various contributions are computed below:

A25.2.1.1 Estimation of

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\hat{M}$$

$$\begin{aligned}
\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\hat{M} &\simeq -\frac{\hat{k}\left(\hat{X},\hat{X}'\right)\hat{K}\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\hat{D}\left(\hat{X}\right)}\\
&\times\frac{\left(\hat{k}_{1}^{B}\left(\hat{X},\bar{X}\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}\left(\hat{X},\bar{X}\right)}{1+\underline{\hat{k}}}\right]-\left(\left\langle\hat{k}_{1}^{B}\left(\hat{X},\bar{X}\right)\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}\left(\hat{X},\bar{X}\right)}{1+\underline{\hat{k}}}\right]\right\rangle\right)\right)\bar{K}}{\hat{D}\left(\hat{X}\right)\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\\
&= -\frac{\hat{K}'\hat{k}\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\underline{\hat{k}}}\right]\left(\hat{X}'\right)\right)^{2}}\\
&\times\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}_{1}\right)-\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}_{1}\right)\right\rangle\right)+\kappa\frac{\int\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}_{1}\right)-\left\langle\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}_{1}\right)\right\rangle\right)}{1+\int\bar{k}\left(\bar{X}',\bar{X}''\right)\bar{K}_{0}''\left|\bar{\Psi}\left(\bar{K}_{0}'',\bar{X}''\right)\right|^{2}}\right)\bar{K}
\end{aligned}$$

In average:

$$\hat{D}\left(\hat{X}\right)=1$$

and:

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\hat{M}\tag{381}$$

$$\begin{aligned}
&\rightarrow -\frac{\hat{K}'\hat{k}\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}_{1}\right)-\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}_{1}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}_{1}\right)-\left\langle\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}_{1}\right)\right\rangle\right)}{1+\int\bar{k}\left(\bar{X}',\bar{X}''\right)\bar{K}_{0}''\left|\bar{\Psi}\left(\bar{K}_{0}'',\bar{X}''\right)\right|^{2}}\right)\bar{K}\\
&\simeq -\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}_{1}\right)-\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}_{1}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}_{1}\right)-\left\langle\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}_{1}\right)\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle}\right)\bar{K}
\end{aligned}$$

A25.2.1.2 Estimation of

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\hat{k}}\right]\left(\hat{X}'\right)}\right)$$

$$\begin{aligned}
\rightarrow\quad & \frac{\hat{K}'\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\hat{k}}\right]\left(\hat{X}'\right)\right)^{2}}\\
& \times\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)}{1+\int\bar{k}\left(\bar{X}',\bar{X}''\right)\frac{\bar{K}_{0}''\left|\bar{\Psi}\left(\bar{K}_{0}'',\bar{X}''\right)\right|^{2}}{\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^{2}}}\right)\bar{K}\\
\rightarrow\quad & \left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle}\right)\bar{K}
\end{aligned}$$

A25.2.1.3 Estimation of

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\frac{1}{1+\underline{\bar{\bar{k}}}_{2}\left(\bar{X}'\right)}$$

gv nrml

$$\frac{1}{1+\underline{\bar{\bar{k}}}_{2}\left(\bar{X}'\right)}\rightarrow\frac{\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)}{\left(1-\left\langle\bar{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\left(1+\frac{\underline{\bar{k}}_{2}\left(\hat{X}'\right)}{1-\left\langle\bar{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle}\right)}$$

$$=-\frac{\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)}{\left(\left(1-\left\langle\bar{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\left(1+\frac{\underline{\bar{k}}_{2}\left(\hat{X}'\right)}{1-\left\langle\bar{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle}\right)\right)^{2}}\times\frac{\left(\bar{k}_{2}\left(\left\langle\bar{X}\right\rangle,\bar{X}_{1}\right)-\left\langle\bar{k}_{2}\left(\bar{X},\bar{X}'\right)\right\rangle\right)}{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}\bar{K}$$

In matrices elements, this becomes:

$$-\frac{\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)}{\left(\left(1-\left\langle\bar{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\right)^{2}}\times\frac{\left(\bar{k}_{2}\left(\left\langle\bar{X}\right\rangle,\bar{X}\right)-\left\langle\bar{k}_{2}\left(\bar{X},\bar{X}'\right)\right\rangle\right)}{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}\bar{K}$$

A25.2.1.4 Estimation of

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_{1}\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^{2}}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)}{1+\underline{\bar{\bar{k}}}_{2}\left(\bar{X}'\right)}\right)$$

Assuming contributions:

$$\begin{aligned}
&\left(\left(\hat{k}_1^B\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_1^B\left(\hat{X}',\bar{X}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_2^B\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_2^B\left(\hat{X}',\bar{X}\right)\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle}\right)\bar{K}\frac{\left(1-\bar{M}\right)}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\\
&+\left(1-\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\right)\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\\
&\times\left(\left(\hat{k}_1^B\left(\hat{X}',\bar{X}_1\right)-\left\langle\hat{k}_1^B\left(\hat{X}',\bar{X}_1\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_2^B\left(\hat{X}',\bar{X}_1\right)-\left\langle\hat{k}_2^B\left(\hat{X}',\bar{X}_1\right)\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle}\right)\bar{K}\frac{1}{1+\underline{\bar{\bar{k}}}_2\left(\bar{X}'\right)}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\\
&-\left(1-\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\right)\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\frac{\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)}{\left(\left(1-\left\langle\bar{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\right)\right)^2}\times\frac{\left(\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}\right)-\left\langle\bar{k}_2\left(\bar{X},\bar{X}'\right)\right\rangle\right)}{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\bar{K}
\end{aligned}$$

with:

$$\begin{aligned}
&\left(\left(\hat{k}_1^B\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_1^B\left(\hat{X}',\bar{X}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_2^B\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_2^B\left(\hat{X}',\bar{X}\right)\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle}\right)\bar{K}\frac{\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)^2}{\left(1-\left\langle\bar{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\right)}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\\
&+\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\left(\left(\hat{k}_1^B\left(\hat{X}',\bar{X}_1\right)-\left\langle\hat{k}_1^B\left(\hat{X}',\bar{X}_1\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_2^B\left(\hat{X}',\bar{X}_1\right)-\left\langle\hat{k}_2^B\left(\hat{X}',\bar{X}_1\right)\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle}\right)\\
&\times\bar{K}\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\\
&-\frac{\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)^2}{1-\left\langle\bar{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle}\times\frac{\left(\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}\right)-\left\langle\bar{k}_2\left(\bar{X},\bar{X}'\right)\right\rangle\right)}{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\bar{K}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle
\end{aligned}$$

and:

$$\begin{aligned}
&\left(\frac{\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)^2}{\left(1-\left\langle\bar{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\right)}+\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\right)\\
&\times\left(\left(\hat{k}_1^B\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_1^B\left(\hat{X}',\bar{X}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_2^B\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_2^B\left(\hat{X}',\bar{X}\right)\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle}\right)\bar{K}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\\
&-\frac{\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)^2}{1-\left\langle\bar{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle}\frac{\left(\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}\right)-\left\langle\bar{k}_2\left(\bar{X},\bar{X}'\right)\right\rangle\right)}{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\bar{K}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle
\end{aligned}$$

A25.2.1.5 Estimation of

$$S=-\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\frac{\hat{K}'\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

$$=\frac{\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle}\right)\bar{K}}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)^{2}}$$

$$\times\left\langle\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right\rangle$$

which becomes in average:

$$\frac{\left\langle\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle}\right)\bar{K}}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}$$

$$\times\left\langle\frac{\left(1-\left\langle\hat{k}\left(\hat{X},\hat{X}'\right)\right\rangle\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right\rangle$$

We can write:

$$\frac{1}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

$$=\frac{1-\left(\left\langle\underline{\hat{k}}\left(\hat{X}',\hat{X}\right)\right\rangle+\left(\left\langle\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}\left(\hat{X}',\bar{X}\right)}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\right)}{\left(1-\left(\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\right)\right)\left(1+\frac{\underline{\hat{k}}_{2}\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1-\left(\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\right)}\right)}$$

In average:

$$\frac{1}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}\rightarrow\frac{1-\left(\left\langle\underline{\hat{k}}\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}+\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}\left(\hat{X}',\bar{X}\right)}{1+\bar{k}}\right]\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\right)}{\left(1-\left(\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\right)\right)}$$

$$\simeq\frac{1-\left(\left\langle\underline{\hat{k}}\right\rangle+\left\langle\underline{\hat{k}}_{1}^{B}\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}+\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\right)}{\left(1-\left(\left\langle\hat{k}_{1}\right\rangle+\left\langle\hat{k}_{1}^{B}\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\right)\right)}$$

$$\bar{N}\rightarrow\left\langle\hat{k}_{1}^{B}\left(\hat{X},\bar{X}'\right)\right\rangle+\kappa\frac{\left\langle\hat{k}_{2}^{B}\left(\hat{X},\bar{X}'\right)\right\rangle}{1+\left\langle\bar{k}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)\right\rangle}\left(1-\frac{\left\langle\bar{k}\left(\bar{X}'',\bar{X}'\right)\right\rangle}{\left(1+\left\langle\bar{k}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)\right\rangle\right)^{2}}\right)$$

$$
\begin{aligned}
S \quad \rightarrow \quad & \frac{\left\langle \underline{\hat{k}}_1^B \left(\left\langle \hat{X}' \right\rangle, \bar{X} \right) \right\rangle}{\left\| \hat{\Psi} \right\|^2 \left\langle \hat{K} \right\rangle} \left(\left(\hat{k}_1^B \left(\hat{X}', \bar{X} \right) - \left\langle \hat{k}_1^B \left(\hat{X}', \bar{X} \right) \right\rangle \right) + \kappa \frac{\left(\hat{k}_2^B \left(\hat{X}', \bar{X} \right) - \left\langle \hat{k}_2^B \left(\hat{X}', \bar{X} \right) \right\rangle \right)}{1 + \left\langle \bar{k} \left(\bar{X}', \bar{X}'' \right) \right\rangle} \right) \bar{K} \\
& \times \frac{1 - \left(\left\langle \underline{\hat{k}} \left(\hat{X}', \hat{X} \right) \right\rangle + \left(\left\langle \underline{\hat{k}}_1^B \left(\hat{X}', \bar{X} \right) \right\rangle + \kappa \left\langle \left[\frac{\underline{\hat{k}}_2^B \left(\hat{X}', \bar{X} \right)}{1 + \bar{k}} \right] \right\rangle \right) \frac{\left\| \bar{\Psi} \right\|^2 \left\langle \bar{K} \right\rangle}{\left\| \hat{\Psi} \right\|^2 \left\langle \hat{K} \right\rangle} \right)}{\left(1 - \left(\left\langle \hat{k}_1 \left(\hat{X}', \hat{X} \right) \right\rangle + \left\langle \hat{k}_1^B \left(\hat{X}', \hat{X} \right) \right\rangle \frac{\left\| \bar{\Psi} \right\|^2 \left\langle \bar{K} \right\rangle}{\left\| \hat{\Psi} \right\|^2 \left\langle \hat{K} \right\rangle} \right) \right)} \\
& \times \left(\left(1 - \left\langle \hat{k} \left(\hat{X}, \hat{X}' \right) \right\rangle \right) \left\langle \hat{g} \left(\hat{K}', \hat{X}' \right) \right\rangle \right. \\
& \left. + \left(\left\langle \hat{k}_1^B \left(\hat{X}, \bar{X}' \right) \right\rangle + \kappa \frac{\left\langle \hat{k}_2^B \left(\hat{X}, \bar{X}' \right) \right\rangle}{1 + \left\langle \bar{\bar{k}} \left(\bar{X}', \left\langle \bar{X}'' \right\rangle \right) \right\rangle} \left(1 - \frac{\left\langle \bar{\bar{k}} \left(\bar{X}'', \bar{X}' \right) \right\rangle}{\left(1 + \left\langle \bar{\bar{k}} \left(\bar{X}', \left\langle \bar{X}'' \right\rangle \right) \right\rangle \right)^2} \right) \right) \left\langle \bar{g} \left(\hat{K}', \hat{X}' \right) \right\rangle \right)
\end{aligned}
$$

A25.2.1.6 Estimation of

$$
\frac{\delta}{\delta \left| \bar{\Psi} \left(\bar{K}, \bar{X} \right) \right|^2} \frac{\left(1 - \hat{M} \right) \hat{g} \left(\hat{K}', \hat{X}' \right) + \bar{N} \bar{g} \left(\hat{K}', \hat{X}' \right)}{1 + \underline{\hat{k}}_2 \left(\bar{X}' \right) + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1 + \bar{k}} \right] \left(\hat{X}' \right)}
$$

$$
\begin{aligned}
& \frac{\delta}{\delta \left| \bar{\Psi} \left(\bar{K}, \bar{X} \right) \right|^2} \frac{\left(1 - \hat{M} \right) \hat{g} \left(\hat{K}', \hat{X}' \right) + \bar{N} \bar{g} \left(\hat{K}', \hat{X}' \right)}{1 + \underline{\hat{k}}_2 \left(\bar{X}' \right) + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1 + \bar{k}} \right] \left(\hat{X}' \right)} \\
= \quad & - \left(\Delta \left(\hat{X}, \hat{X}' \right) - \frac{\hat{K}' \hat{k}_1 \left(\hat{X}', \hat{X} \right) \left| \hat{\Psi} \left(\hat{K}', \hat{X}' \right) \right|^2}{1 + \underline{\hat{k}} \left(\hat{X}' \right) + \underline{\hat{k}}_1^B \left(\bar{X}' \right) + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1 + \bar{k}} \right] \left(\hat{X}' \right)} \right)^{-1} \\
& \times \frac{\delta}{\delta \left| \bar{\Psi} \left(\bar{K}, \bar{X} \right) \right|^2} \left(\Delta \left(\hat{X}, \hat{X}' \right) - \frac{\hat{K}' \hat{k}_1 \left(\hat{X}', \hat{X} \right) \left| \hat{\Psi} \left(\hat{K}', \hat{X}' \right) \right|^2}{1 + \underline{\hat{k}} \left(\hat{X}' \right) + \underline{\hat{k}}_1^B \left(\bar{X}' \right) + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1 + \bar{k}} \right] \left(\hat{X}' \right)} \right) \frac{\left(1 - \hat{M} \right) \hat{g} \left(\hat{K}', \hat{X}' \right) + \bar{N} \bar{g} \left(\hat{K}', \hat{X}' \right)}{1 + \underline{\hat{k}}_2 \left(\bar{X}' \right) + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1 + \bar{k}} \right] \left(\hat{X}' \right)}
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{\delta}{\delta \left| \bar{\Psi} \left(\bar{K}, \bar{X} \right) \right|^2} \left(\Delta \left(\hat{X}, \hat{X}' \right) - \frac{\hat{K}' \hat{k}_1 \left(\hat{X}', \hat{X} \right) \left| \hat{\Psi} \left(\hat{K}', \hat{X}' \right) \right|^2}{1 + \underline{\hat{k}} \left(\hat{X}' \right) + \underline{\hat{k}}_1^B \left(\bar{X}' \right) + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1 + \bar{k}} \right] \left(\hat{X}' \right)} \right) \\
\rightarrow \quad & - \frac{\hat{K}' \hat{k}_1 \left(\hat{X}', \hat{X} \right) \left| \hat{\Psi} \left(\hat{K}', \hat{X}' \right) \right|^2 \left(\left(\hat{k}_1^B \left(\hat{X}', \bar{X} \right) - \left\langle \hat{k}_1^B \left(\hat{X}', \bar{X} \right) \right\rangle \right) + \kappa \frac{\left(\hat{k}_2^B \left(\hat{X}', \bar{X} \right) - \left\langle \hat{k}_2^B \left(\hat{X}', \bar{X} \right) \right\rangle \right)}{1 + \int \bar{k} \left(\bar{X}', \bar{X}'' \right) \frac{\bar{K}_0'' \left| \bar{\Psi} \left(\bar{K}_0'', \bar{X}'' \right) \right|^2}{\left\langle \bar{K} \right\rangle \left\| \bar{\Psi} \right\|^2}} \right)}{\left(1 + \underline{\hat{k}} \left(\hat{X}' \right) + \underline{\hat{k}}_1^B \left(\bar{X}' \right) + \kappa \left[\frac{\underline{\hat{k}}_2^B}{1 + \bar{k}} \right] \left(\hat{X}' \right) \right)^2} \bar{K}
\end{aligned}
$$

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

$$\rightarrow -\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right)^{-1}$$

$$\times\frac{\hat{K}'\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)}{1+\int\bar{k}\left(\bar{X}',\bar{X}''\right)\frac{\bar{K}_{0}''\left|\bar{\Psi}\left(\bar{K}_{0}'',\bar{X}''\right)\right|^{2}}{\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^{2}}}\right)}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}\left(1+\underline{\hat{k}}\left(\hat{X}'\right)+\underline{\hat{k}}_{1}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)^{2}}\bar{K}$$

$$\times\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

In matrices element of

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

where:

$$-\left(1-\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle\right)^{-1}\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle\frac{\left\langle\bar{K}\right\rangle\left|\left\|\bar{\Psi}\right\|\right|^{2}}{\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^{2}}}\right)\bar{K}$$

$$\times\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

we have:

$$-\left(1-\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle\right)^{-1}\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle\frac{\left\langle\bar{K}\right\rangle\left|\left\|\bar{\Psi}\right\|\right|^{2}}{\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^{2}}}\right)\bar{K}$$

$$\times\left\langle\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right\rangle \tag{382}$$

A25.2.1.7 Gathering terms Using that in average:

$$\frac{\left(1-\bar{M}\right)}{1+\underline{\bar{k}}_{2}\left(\bar{X}'\right)}$$

$$\rightarrow \frac{\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)^{2}}{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle\left(1-\left\langle\bar{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\left(1+\frac{\underline{\bar{k}}_{2}\left(\hat{X}'\right)}{1-\left\langle\bar{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle}\right)}$$

and:

$$\left(\Delta\left(\hat{X},\hat{X}'\right)-\frac{\hat{K}'\hat{k}_1\left(\hat{X}',\hat{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)}\right)\frac{\left(1-\bar{M}\right)}{1+\bar{\underline{k}}_2\left(\bar{X}'\right)}$$
$$\rightarrow\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)^2\frac{1}{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}$$

We find:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)\\
=&\left(\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)}{1+\bar{\underline{\bar{k}}}_2\left(\bar{X}'\right)}\right)^{-1}\\
&\left\{-\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\frac{\hat{K}'\hat{\underline{k}}_1^B\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle\left(1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)\right)}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)}\right.\\
&-\frac{\hat{K}'\hat{\underline{k}}_1^B\left(\hat{X}',\bar{X}\right)\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^2}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle\left(1+\hat{\underline{k}}\left(\hat{X}'\right)+\hat{\underline{k}}_1^B\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)\right)}\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)}\\
&\left.-\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\left(\left(\Delta\left(\bar{X}',\bar{X}\right)-\frac{\bar{K}'\bar{k}_1\left(\bar{X}',\bar{X}\right)\left|\bar{\Psi}\left(\bar{K}',\bar{X}'\right)\right|^2}{1+\underline{\bar{k}}\left(\bar{X}'\right)}\right)\frac{\left(1-\bar{M}\right)}{1+\bar{\underline{\bar{k}}}_2\left(\bar{X}'\right)}\right)\bar{g}\left(\hat{K}',\hat{X}'\right)\right\}
\end{aligned}$$

that is:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^2}\bar{g}\left(\hat{K}',\hat{X}'\right)\\
\simeq&\left(\frac{\left(1-\left\langle\bar{k}\right\rangle\right)^2}{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\right)^{-1}\left\{\left(\left\langle\hat{\underline{k}}_1^B\left(\left\langle\hat{X}'\right\rangle,\bar{X}\right)\right\rangle+\frac{\left\langle\hat{\underline{k}}_1^B\left(\hat{X}',\bar{X}\right)\right\rangle}{\left(1-\left\langle\hat{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle\right)}\right)\right.\\
&\times\left(\left(\hat{k}_1^B\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_1^B\left(\hat{X}',\bar{X}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_2^B\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_2^B\left(\hat{X}',\bar{X}\right)\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle}\right)\left\langle A\right\rangle\frac{\bar{K}}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\\
&\left.+\frac{\left(1-\left\langle\bar{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)^2}{1-\left\langle\bar{k}_1\left(\hat{X}',\hat{X}\right)\right\rangle}\frac{\left(\bar{k}_2\left(\left\langle\bar{X}\right\rangle,\bar{X}\right)-\left\langle\bar{k}_2\left(\bar{X},\bar{X}'\right)\right\rangle\right)}{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}\bar{K}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right\}
\end{aligned}\tag{383}$$

We can replace the following averages:

$$\left\langle\frac{1}{1+\hat{\underline{k}}_2\left(\bar{X}'\right)+\kappa\left[\frac{\hat{\underline{k}}_2^B}{1+\hat{k}}\right]\left(\hat{X}'\right)}\right\rangle\rightarrow\frac{1-\left(\left\langle\hat{\underline{k}}\right\rangle+\left\langle\hat{\underline{k}}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}+\kappa\left\langle\left[\frac{\hat{\underline{k}}_2^B}{1+\hat{k}}\right]\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)}{\left(1-\left(\left\langle\hat{k}_1\right\rangle+\left\langle\hat{k}_1^B\right\rangle\frac{\left\|\bar{\Psi}\right\|^2\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}\right)\right)\left\|\hat{\Psi}\right\|^2\left\langle\hat{K}\right\rangle}$$

$$\langle A\rangle = \left\langle \frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}\right\rangle$$

$$= \frac{1-\left(\left\langle \underline{\hat{k}}\right\rangle+\left\langle \underline{\hat{k}}_{1}^{B}\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}+\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\right)}{\left(1-\left(\left\langle\hat{k}_{1}\right\rangle+\left\langle\hat{k}_{1}^{B}\right\rangle\frac{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}\right)\right)\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}$$

$$\times\left(\left(1-\left\langle\hat{k}\right\rangle\right)\langle\hat{g}\rangle+\left(\left\langle\hat{k}_{1}^{B}\right\rangle+\kappa\frac{\left\langle\hat{k}_{2}^{B}\right\rangle}{1+\left\langle\bar{k}\right\rangle}\left(1-\frac{\left\langle\bar{k}\right\rangle}{\left(1+\left\langle\bar{k}\right\rangle\right)^{2}}\right)\right)\langle\bar{g}\rangle\right)$$

and:

$$\bar{N}\rightarrow\left\langle\hat{k}_{1}^{B}\right\rangle+\kappa\frac{\left\langle\hat{k}_{2}^{B}\right\rangle}{1+\left\langle\bar{k}\right\rangle}\left(1-\frac{\left\langle\bar{k}\right\rangle}{\left(1+\left\langle\bar{k}\right\rangle\right)^{2}}\right)$$

We are thus led to:

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\bar{g}\left(\hat{K}',\hat{X}'\right) \tag{384}$$

$$\simeq \left(\frac{\left(1-\left\langle\bar{k}\right\rangle\right)^{2}}{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}\right)^{-1}\left\{\left(\left\langle\underline{\hat{k}}_{1}^{B}\right\rangle+\frac{\left\langle\underline{\hat{k}}_{1}^{B}\right\rangle}{\left(1-\left\langle\hat{k}_{1}\right\rangle\right)}\right)\right.$$

$$\times\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{2}^{B}\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle}\right)\langle A\rangle$$

$$\left.+\frac{\left(1-\left\langle\bar{k}\right\rangle\right)^{2}}{1-\left\langle\bar{k}_{1}\right\rangle}\frac{\left(\bar{k}_{2}\left(\left\langle\bar{X}\right\rangle,\bar{X}\right)-\left\langle\bar{k}_{2}\right\rangle\right)}{\left\Vert\bar{\Psi}\right\Vert^{2}\left\langle\bar{K}\right\rangle}\langle\bar{g}\rangle\right\}\frac{\bar{K}}{\left\Vert\hat{\Psi}\right\Vert^{2}\left\langle\hat{K}\right\rangle}$$

A25.2.2 Computation of $\frac{\delta\hat{g}\left(\hat{K}',\hat{X}'\right)}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}$

We define, as before:

$$A=\frac{\left(1-\hat{M}\right)\hat{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)}{1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}$$

and $\hat{g}\left(\hat{K},\hat{X}\right)$ is given by:

$$\hat{g}\left(\hat{K},\hat{X}\right)=\left(1-\hat{M}\right)^{-1}\left(\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)A-\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)$$

As a consequence, the derivative we are seeking for is given by:

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\hat{g}\left(\hat{K},\hat{X}\right)$$

$$= \left(1-\hat{M}\right)^{-1}\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\hat{M}\hat{g}\left(\hat{K},\hat{X}\right)$$

$$+\left(1-\hat{M}\right)^{-1}\left(\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)A-\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)\right)$$

In average, this becomes:

$$\left(1-\hat{M}\right)^{-1}\left(\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\hat{M}\right)\hat{g}\left(\hat{K},\hat{X}\right) \tag{385}$$
$$+\frac{\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}{\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}\left\langle\left(1-\hat{M}\right)^{-1}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)A\right\rangle$$
$$+\left(1-\hat{M}\right)^{-1}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}A$$
$$-\left(1-\hat{M}\right)^{-1}\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)$$

where the terms in (385) are:

$$\frac{\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}{\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}\left\langle\left(1-\hat{M}\right)^{-1}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)A\right\rangle$$
$$=\frac{\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}{\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)$$

and, using (382)

$$\left(1-\hat{M}\right)^{-1}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}A$$
$$=-\left(1-\left\langle\hat{k}\right\rangle\right)^{-1}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)\left(1-\left\langle\hat{k}_{1}\right\rangle\right)^{-1}$$
$$\times\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{2}^{B}\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle\frac{\left\langle\bar{K}\right\rangle\left\|\left|\bar{\Psi}\right|\right\|^{2}}{\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^{2}}}\right)\bar{K}\left\langle A\right\rangle$$
$$\simeq-\left(\left(1-\left\langle\hat{k}\right\rangle\right)\left(1-\left\langle\hat{k}_{1}\right\rangle\right)\right)^{-1}\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{2}^{B}\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle\frac{\left\langle\bar{K}\right\rangle\left\|\left|\bar{\Psi}\right|\right\|^{2}}{\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^{2}}}\right)$$
$$\times\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)$$

Consequently, we have:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\hat{g}\left(\hat{K},\hat{X}\right)\\
=\;&\left(1-\hat{M}\right)^{-1}\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\hat{M}\hat{g}\left(\hat{K},\hat{X}\right)\\
&+\frac{\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}{\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)\\
&-\left(\left(1-\left\langle\hat{k}\right\rangle\right)\left(1-\left\langle\hat{k}_{1}\right\rangle\right)\right)^{-1}\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{2}^{B}\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle\frac{\left\langle\bar{K}\right\rangle\left|\left\|\bar{\Psi}\right\|\right|^{2}}{\left\langle\bar{K}\right\rangle\left\|\bar{\Psi}\right\|^{2}}}\right)\\
&\times\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)-\left(1-\hat{M}\right)^{-1}\left(\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)\right)
\end{aligned}$$

A25.2.2.1 Estimation of:

$$\frac{\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}{\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}$$

The derivative has th xpanded form:

$$
\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(1+\underline{\hat{k}}_{2}^{B}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)
$$

$$
\rightarrow\quad\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\frac{\left(1-\left(\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)\left(1+\frac{\underline{\hat{k}}_{2}\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1-\left(\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)}\right)}{1-\left(\left\langle\underline{\hat{k}}\left(\hat{X}',\hat{X}\right)\right\rangle+\left(\left\langle\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}\left(\hat{X}',\bar{X}\right)}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)}
$$

$$
\rightarrow\quad\frac{1}{\left(1+\frac{\underline{\hat{k}}_{2}\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1-\left(\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)}\right)^{2}\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}
$$

$$
\times\frac{\left(\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}',,\bar{X}\right)-\left\langle\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}',,\bar{X}\right)\right\rangle\right)\bar{K}}{\left(1-\left(\left\langle\underline{\hat{k}}\left(\hat{X}',\hat{X}\right)\right\rangle+\left(\left\langle\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}\left(\hat{X}',\bar{X}\right)}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)}
$$

$$
-\frac{1}{\left(1+\frac{\underline{\hat{k}}_{2}\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1-\left(\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)}\right)\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}
$$

$$
\times\frac{\left(\left(1-\left\langle\underline{\hat{k}}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}\left(\hat{X}',\bar{X}\right)}{1+\bar{k}}\right]\right\rangle+\left\langle\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\left\langle\underline{\hat{k}}_{2}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\bar{K}}{\left(1-\left(\left\langle\underline{\hat{k}}\left(\hat{X}',\hat{X}\right)\right\rangle+\left(\left\langle\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}\left(\hat{X}',\bar{X}\right)}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)^{2}}
$$

and:

$$
-\left(1-\left(\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)^{-1}
$$

$$
\times\left(\frac{\left(\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}',,\bar{X}\right)-\left\langle\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}',,\bar{X}\right)\right\rangle\right)\bar{K}}{\left(1+\frac{\underline{\hat{k}}_{2}\left(\hat{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)}{1-\left(\left\langle\hat{k}_{1}\left(\hat{X}',\hat{X}\right)\right\rangle+\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\hat{X}\right)\right\rangle\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)}\right)\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right.
$$

$$
\left.+\frac{\left(\left(1-\left\langle\underline{\hat{k}}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}\left(\hat{X}',\bar{X}\right)}{1+\bar{k}}\right]\right\rangle\bar{K}+\left\langle\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle\left\langle\underline{\hat{k}}_{2}\left(\hat{X}',\hat{X}\right)\right\rangle\right)\bar{K}}{\left(1-\left(\left\langle\underline{\hat{k}}\left(\hat{X}',\hat{X}\right)\right\rangle+\left(\left\langle\underline{\hat{k}}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}\left(\hat{X}',\bar{X}\right)}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)}\right)
$$

which is, in average:

$$\frac{\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}{\left(1+\underline{\hat{k}}_{2}\left(\bar{X}'\right)+\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}'\right)\right)}$$

$$\rightarrow \frac{\left(\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}',,\bar{X}\right)-\left\langle\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}',,\bar{X}\right)\right\rangle\right)-\frac{\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)+\left\langle\hat{k}_{1}^{B}\right\rangle\left\langle\hat{k}_{2}\right\rangle}{\left(1-\left(\left\langle\hat{k}\right\rangle+\left(\left\langle\hat{k}_{1}^{B}\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)}}{\left(1-\left(\left\langle\hat{k}_{1}\right\rangle+\left\langle\hat{k}_{1}^{B}\right\rangle\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)\left\|\hat{\Psi}\right\|^{2}}\frac{\bar{K}}{\left\langle\hat{K}\right\rangle}$$

A25.2.2.2 Estimation of $\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\hat{g}\left(\hat{K},\hat{X}\right)$ Using (381):

$$\begin{aligned}
&\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\hat{g}\left(\hat{K},\hat{X}\right)\\
=\;& -\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)^{-1}\\
&\times\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}_{1}\right)-\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}_{1}\right)\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}_{1}\right)-\left\langle\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}_{1}\right)\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle}\right)\frac{\bar{K}}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\hat{g}\left(\hat{K},\hat{X}\right)\\
&+\frac{\left(\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}',,\bar{X}\right)-\left\langle\kappa\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}',,\bar{X}\right)\right\rangle\right)-\frac{\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)+\left\langle\hat{k}_{1}^{B}\right\rangle\left\langle\hat{k}_{2}\right\rangle}{\left(1-\left(\left\langle\hat{k}\right\rangle+\left(\left\langle\hat{k}_{1}^{B}\right\rangle+\kappa\left\langle\left[\frac{\underline{\hat{k}}_{2}^{B}}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)}}{\left(1-\left(\left\langle\hat{k}_{1}\right\rangle+\left\langle\hat{k}_{1}^{B}\right\rangle\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)\left\|\hat{\Psi}\right\|^{2}}\frac{\bar{K}}{\left\langle\hat{K}\right\rangle}\\
&\times\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)\\
&-\left(\left(1-\left\langle\hat{k}\right\rangle\right)\left(1-\left\langle\hat{k}_{1}\right\rangle\right)\right)^{-1}\left(\left(\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\right\rangle\right)+\kappa\frac{\left(\hat{k}_{2}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{2}^{B}\right\rangle\right)}{1+\left\langle\bar{k}\left(\bar{X}',\bar{X}''\right)\right\rangle\frac{\left\langle\bar{K}\right\rangle\left|\left\|\bar{\Psi}\right\|\right|^{2}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}}\right)\\
&\times\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)-\left(1-\hat{M}\right)^{-1}\left(\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)\right)
\end{aligned}$$

A25.2.2.3 Estimation of $\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)$

$$\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(\bar{N}\bar{g}\left(\hat{K}',\hat{X}'\right)\right)=\left[\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\bar{N}\right]\bar{g}\left(\hat{K}',\hat{X}'\right)+\bar{N}\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(\bar{g}\left(\hat{K}',\hat{X}'\right)\right)$$

The first term is obtained by wrtng:and this becomes:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\bar{N}\\
=\;&\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\frac{\left(\hat{k}_{1}^{B}\left(\hat{X},\bar{X}'\right)+\kappa\frac{\hat{k}_{2}^{B}\left(\hat{X},\bar{X}'\right)}{1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)}-\kappa\frac{\hat{k}_{2}^{B}\left(\bar{X},\bar{X}''\right)\bar{K}_{0}''\bar{\bar{k}}\left(\bar{X}'',\bar{X}'\right)}{\left(1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)\right)^{2}}\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\hat{D}\left(\hat{X}'\right)\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}\\
\to\;&-\frac{\left(\hat{k}_{1}^{B}\left(\hat{X},\bar{X}'\right)+\kappa\frac{\hat{k}_{2}^{B}\left(\hat{X},\bar{X}'\right)}{1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)}-\kappa\frac{\hat{k}_{2}^{B}\left(\bar{X},\bar{X}''\right)\bar{K}_{0}''\bar{\bar{k}}\left(\bar{X}'',\bar{X}'\right)}{\left(1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)\right)^{2}}\right)\hat{K}'\left|\hat{\Psi}\left(\hat{K}',\hat{X}'\right)\right|^{2}}{\hat{D}\left(\hat{X}'\right)\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}\\
&\times\frac{\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle+\kappa\left(\left[\frac{\hat{k}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}',\bar{X}\right)-\left\langle\left[\frac{\hat{k}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}',\bar{X}\right)\right\rangle\right)\bar{K}}{\hat{D}\left(\hat{X}'\right)\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}\\
\to\;&-\left(\left\langle\hat{k}_{1}^{B}\left(\hat{X},\bar{X}'\right)\right\rangle+\kappa\frac{\left\langle\hat{k}_{2}^{B}\left(\hat{X},\bar{X}'\right)\right\rangle}{1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)}\left(1-\kappa\frac{\left\langle\bar{\bar{k}}\left(\bar{X}'',\bar{X}'\right)\right\rangle}{1+\bar{\bar{k}}\left(\bar{X}',\left\langle\bar{X}''\right\rangle\right)}\right)\right)\\
&\times\frac{\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)-\left\langle\hat{k}_{1}^{B}\left(\hat{X}',\bar{X}\right)\right\rangle+\kappa\left(\left[\frac{\hat{k}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}',\bar{X}\right)-\left\langle\left[\frac{\hat{k}_{2}^{B}}{1+\bar{k}}\right]\left(\hat{X}',\bar{X}\right)\right\rangle\right)\bar{K}}{\left\langle\hat{K}\right\rangle\left\|\hat{\Psi}\right\|^{2}}
\end{aligned}$$

and the second term $\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\left(\bar{g}\left(\hat{K}',\hat{X}'\right)\right)$ is given by (383).

A25.2.2.4 Dominant term All contributions, bt one, to the derivative $\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\hat{g}\left(\hat{K},\hat{X}\right)$ are deviations around the average. As a consequence, in first approximation, we have:

$$\begin{aligned}
&\frac{\delta}{\delta\left|\bar{\Psi}\left(\bar{K},\bar{X}\right)\right|^{2}}\hat{g}\left(\hat{K},\hat{X}\right)\\
=\;&-\frac{\kappa\left\langle\left[\frac{\hat{k}_{2}^{B}}{1+\bar{k}}\right]\right\rangle\left(1-\left\langle\hat{k}\left(\hat{X}',\hat{X}\right)\right\rangle\right)+\left\langle\hat{k}_{1}^{B}\right\rangle\left\langle\hat{k}_{2}\right\rangle}{\left(1-\left(\left\langle\hat{k}\right\rangle+\left(\left\langle\hat{k}_{1}^{B}\right\rangle+\kappa\left\langle\left[\frac{\hat{k}_{2}^{B}}{1+\bar{k}}\right]\right\rangle\right)\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)\left(1-\left(\left\langle\hat{k}_{1}\right\rangle+\left\langle\hat{k}_{1}^{B}\right\rangle\frac{\left\|\bar{\Psi}\right\|^{2}\left\langle\bar{K}\right\rangle}{\left\|\hat{\Psi}\right\|^{2}\left\langle\hat{K}\right\rangle}\right)\right)\left\|\hat{\Psi}\right\|^{2}}\frac{\bar{K}}{\left\langle\hat{K}\right\rangle}\\
&\times\left(\left\langle\hat{g}\left(\hat{K},\hat{X}\right)\right\rangle+\left(1-\hat{M}\right)^{-1}\bar{N}\left\langle\bar{g}\left(\hat{K}',\hat{X}'\right)\right\rangle\right)
\end{aligned}$$