# Statistical Field Theory and Neural Structures Dynamics IV: Field-Theoretic Formalism for Interacting Collective States

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November 2023

**Abstract**

Building upon the findings presented in the first three papers of this series, we formulate an effective field theory for interacting collective states. These states consist of a large number of interconnected neurons and are distinguished by their intrinsic activity. The field theory encompasses an infinite set of fields, each of which characterizes the dynamics of a specific type of collective state. Interaction terms within the theory drive transitions between various collective states, allowing us to describe processes such as activation, association, and deactivation of these states.

## 1 Introduction

In this series of papers, we present a field-theoretic approach to the dynamics of connectivities within a set of interacting spiking neurons. In ([6]), we introduced a two-field model that characterizes both the dynamics of neural activity and the connectivity between points along the thread. The first field describes the assembly of neurons and is akin to the one introduced in ([5]). The second field, on the other hand, captures the dynamics of connectivity between cells. Both fields interact with themselves, representing interactions within the thread, and also interact with each other, encapsulating the mutual influences between neural activities and connectivities. This field-based framework encompasses both collective and individual aspects of the system. The system composed of these two fields is described by a field action functional, which encapsulates the interactions within the system at the microscopic level. This action functional provides insights into the dynamics of the entire system.

In ([6]), by focusing on the field for connectivity functions, our framework enabled the derivation of background fields for both neural interactions and connectivities, which minimize the action functional. These background fields encompass the collective configurations of the system and determine the potential static equilibria for neural activities and connectivities. These equilibria serve as the foundational framework for conditioning and shaping fluctuations and signal propagation within the system. Their existence is contingent on internal system parameters and external stimuli. We demonstrated the feasibility of several background states, each associated with specific connectivities, with the thread primarily organized into groups of interconnected points.

In ([7]), we demonstrated how repeated activations at specific points can propagate throughout the thread, progressively altering the connectivity functions. We discussed interference phenomena

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and their influence on connectivities. At points of constructive interference, the background state for connectivities and average connectivities is modified, leading to the emergence of states with enhanced connectivities between certain points. These states gradually fade over time but can be reactivated by external perturbations. Furthermore, the association of these emerging states is possible when their activation occurs at closed times. Activating any of these states rekindles their combination. These enhanced connectivity states exhibit characteristics typical of interacting partial neuronal assemblies.

In Article ([8]), we expanded this approach by focusing on the system of connectivities itself. Expressing the individual cells background field as an effective quantity depending on the connectivity field, we described the effective dynamics of the connectivity fields. The ntegration of the cell field degrees of freedom led to self-interactions for the connectivity field. This introduced internal dynamics that could modify the static background state we initially started with at some points of the thread. The self-interactions, possibly induced by specific perturbations, initiated internal patterns of connections between cells. Depending on various internal parameters, permanent shifts in the background states of connectivity in specific areas of the thread arise while others remain unaffected. This effective theory also allowed us to elucidate the mechanisms behind the reinforcement of connectivities between multiple cells and the emergence of groups with altered connectivities. These collective shifts can be seen as additional structures that emerge above the background field, but whose possibility of emergence depends on these background fields.

The emergence of such states brings about a shift in perspective. In our previous works, collective states emerged from interactions within the entire self-interacting system of neurons and consisted of a large number of interacting individual states. Here, conversely, the dynamics of individual neurons take on a secondary role as they contribute to one or several connected states. Cells may exhibit varying activation patterns depending on their involvement in collective patterns. The collective states become prominent and determine the individual neurons'activity.

Further investigation of the implications of our formalism should prompt us to explicitly consider families of collective states, including the possibility for multiple activations or deactivations of such states. In other words, we should contemplate an effective field theory for emerging and interacting states in its own right. This constitutes the focus of the present study.

Building upon our prior research, this paper establishes a formalism for interacting collective sets of connectivities. By directly considering these emerging states as our starting point, we introduce an effective field formalism to describe their dynamics and interactions. This process unfolds in two stages. Initially, assuming the existence of such states, we elucidate their principal attributes as assemblies of individual states. These structures are characterized by various parameters, including their spatial extent, the average connectivities between their constituent elements, and the activities of each element. The activities exhibit potential dynamic oscillations whose frequencies play a role in interactions with other structures. Importantly, these parameters are not unique. In fact, the same structure can be described by a discrete, and potentially infinite, set of parameters, each corresponding to different possible average connectivities and activity frequencies. As a result, a structure may exist in different states and may undergo transitions between these states.

To account for the existence of multiple states and transitions between them, we then develop a field theoretic model of collective states. We introduce a field for each possible structure, so that the states of this field representing activations or multiple activations of the structure. The action functional describes not only the primary characteristics of these states but also their dynamic interactions. The field formalism enables the activation or deactivation of these structures, as well as transitions of the same structure between states induced by interactions or external perturbations. Consequently, a single structure may experience activation or deactivation, which, in turn, may activate other structures. In this context, state transitions of a given structure and the associated modifications in terms of connectivities and activity frequencies can facilitate or inhibit interactions with other collective states, resulting in synchronization or desynchronization with other structures.

The interactions between collective states also involve mechanisms that describe the assembly of some states to create a larger set. An effective formalism, based on the integration of intermediate degrees of freedom, characterizes the binding of several structures through indirect interactions mediated by intermediate collective states. This effective formalism is presented in two alternative ways: a field formalism in which several structures are projected in their background state, depending on their interactions with others, and a perturbation formalism that enables the extraction of the interaction content of the bound structures.

This work is organized as follows. The first part describes the emergence of collective states from the formalism developed in our previous works and their properties. Starting from the field formalism for neurons and connectivities in Section 2, we recall the concept of a background field for the system. We provide conditions for the emergence of enhanced activation between large sets of cells and describe the characteristics of such states. Section 3 reviews the definition and the principles of deriving the background field, as presented in ([6]). Section 4 outlines the effective formalism for emerging collective states, which was developed in ([7]) and ([8]). In Section 5, we derive the characteristics of these states, including enhanced connectivities and multiple activity frequencies. Section 6 anticipates the formalism of the second part by describing the characteristics of composed collective states involving different types of cell fields. This allows in Section 7 to consider transitions of several collective states to form a composite one. This description serves as a guide for the next part, which significantly expands upon this approach through the cell states formalism.

The second part of this article develops the field formalism for a large number of interacting collective states. Section 8 elaborates on this field formalism, beginning with collective states as basic elements and then constructing the associated field approach. Section 9 incorporates interactions between structures.

The third part of this work describes the different mechanisms of dynamical transitions implied by the formalism, providing several examples of these mechanisms. In Section 10, we present three approaches to transitions between structures. The first approach is based on a perturbative expansion, allowing for a straightforward description of transition mechanisms induced by interactions but misses non-perturbative effects. The second approach is an effective formalism that helps understand how some structures, considered as backgrounds, can initiate activations of other structures. Two examples are presented. Then, an operator formalism that elucidates how indirect interactions can bind several structures is developed. Section 11 presents the application of the effective and perturbative formalism to a system with three structures. Section 12 extends the formalism by introducing an extension considering non-localized structures. Section 13 is a conclusion.

# I Field theory, emergence and description of collective states

In this first part, we start by revisiting the foundational model developed in ([6]), ([7]) and ([8]), along with the pertinent results relevant to our present study. In ([8]), we derived the possibility of emerging collective states above some given background. We conducted an examination of the associated modification to the background state, interpreting these modifications as distinct states themselves. While we partially described their interactions, a comprehensive elucidation of their properties in terms of activities or connectivity frequencies remained incomplete. The primary objective of this first part, therefore, is to further elucidate the characteristics of these emerging states. Ultimately, by qualitatively exploring the transitions between various states, we will pave the way for the field formalism in the subsequent part of this paper.

# 2 Field theoretic description of the system

In the three next sections, we sum-up some results of the previous papers of this series. Then, we focus on the effective action for connectivity field.

## 2.1 Fields and action functional

Based on [1][2][3][4], we gave in (([6]), resume in ([7])) a statistical field formalism to describe both cells and connectivities dynamics. This decription relies on two fields, $\Psi$ for cells, and $\Gamma$ for connectivities. The field action for the system is:

$$\begin{aligned} S_{full} &= -\frac{1}{2}\Psi^{\dagger}(\theta,Z,\omega)\nabla\left(\frac{\sigma_{\theta}^{2}}{2}\nabla-\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)\right)\Psi(\theta,Z)+V(\Psi) \\ &\quad +\frac{1}{2\eta^{2}}\left(S_{\Gamma}^{(0)}+S_{\Gamma}^{(1)}+S_{\Gamma}^{(2)}+S_{\Gamma}^{(3)}+S_{\Gamma}^{(4)}\right)+U\left(\left\{|\Gamma(\theta,Z,Z',C,D)|^{2}\right\}\right) \end{aligned} \tag{1}$$

where the activities $\omega\left(J,\theta,Z,|\Psi|^{2}\right)$ satisfy:

$$\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)=G\left(J(\theta)+\frac{\kappa}{N}\int T(Z,Z_{1},\theta)\frac{\omega\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1}\right)}{\omega(\theta,Z)}\left|\Psi\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1}\right)\right|^{2}dZ_{1}\right) \tag{2}$$

and $S_{\Gamma}^{(1)}$, $S_{\Gamma}^{(2)}$, $S_{\Gamma}^{(3)}$, $S_{\Gamma}^{(4)}$ are given in appendix 1. in (1), we added a potential:

$$U\left(\left\{|\Gamma(\theta,Z,Z',C,D)|^{2}\right\}\right)=U\left(\int T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^{2}dTd\hat{T}\right)$$

that models the constraint about the number of active connections in the system.

## 2.2 Interpretation of the various field

The action functional depends on two distinct fields: $\Psi(\theta,Z)$ and $\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)$. These abstract quantities play a crucial role in deriving the comprehensive dynamics of the entire system and subsequently, in the analysis of transitions between distinct states. However, the squared modulus of these two functions can be interpreted in terms of statistical distribution, depending on the chosen framework. If we consider a system comprised of simple cells distributed along a thread, the function $|\Psi(\theta,Z)|^{2}$ measures at time $\theta$, the density of active cells at point $Z$. In the context of complex cells with multiple axons and dendrites, we can regard each cell as residing at point $Z$, and $|\Psi(\theta,Z)|^{2}$ measures the density of axons for that particular cell. A similar interpretation can be applied to $\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)$. Within the framework of a system composed of simple cells 'accumulated' in the vicinity of $Z$, $\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^{2}$ quantifies the density of connections with the value $T$ (and auxiliary variables $\hat{T}$, $C,D$ between the set of cells plying at points $Z$ and $Z'$ respectivly. In the context of complex cells, it characterizes the density of connections with strength $T$ between sets of axons and dendrites of the cells.

## 3 Derivation of background fields of the model

In the previous parts, we derived the background fields of the system described above. We considered first that the evolution for activity is at a faster pace than the dynamics of connectivits described by $\sum S_{\Gamma}^{(i)}$.

Starting by saddle point equations for:

$$-\frac{1}{2}\Psi^{\dagger}(\theta,Z)\nabla\left(\frac{\sigma_{\theta}^{2}}{2}\nabla-\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)\right)\Psi(\theta,Z)+V(\Psi)$$

allowed to describ the background states for $\Psi(\theta,Z)$ and activity $\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)$ as a function of connectvity $\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)$ (see appendix 1) for more details.

We then minimized the action for connectivties:

$$\frac{\delta\sum S_{\Gamma}^{(i)}}{\delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)}=\frac{\delta\sum S_{\Gamma}^{(i)}}{\delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)}=0$$

and obtained the background field for this field along with the associated average values of connectivities and activities. The relevant formulas are given in Appendix 1. It is important to note that various background fields are conceivable, contingent upon specific external conditions. This, in turn, results in several potential configurations of average connectivities. When subjected to external perturbations, we observed that configurations featuring localized clusters of heightened connectivities were favored, with these clusters partially connected[1]. In other words perturbations around the background state induce emergence of collective states, but this emergence is conditioned by the background states.

## 4 Effective formalism and emerging connectivity collective states

### 4.1 Effective action for variation around the background field.

Pertubations and internal dynamics in a given background obtained by expansion of action around this background. We where led to define:

$$\begin{aligned}\Gamma\left(T,\hat{T},\theta,Z,Z'\right)&=\Gamma_{0}\left(T,\hat{T},\theta,Z,Z'\right)+\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)&=\Gamma_{0}^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)+\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\end{aligned}$$

where $\Gamma_0$ and $\Gamma_0^{\dagger}$ are the background fields.

A second order expansion around the background fld $\Gamma_0$ and $\Gamma_0^{\dagger}$ which minimize $\sum S_{\Gamma}^{(i)}$ led to the effective action for the connectivity field. Formally, this effective action is given by:

$$S\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right)=\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\frac{\delta^{2}\sum S_{\Gamma}^{(i)}}{\delta\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\delta\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)}\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$$

This effective action is the starting point of a formalism of collective states. The expanded form

[1] Formulas for the averages are detailed in the appendix.

for this effective action above the background field is (see appendix 1 for derivation):

$$\begin{aligned}
& S\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right) && (3)\\
= & -\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\nabla_{T}\left(\nabla_{T}+\frac{\Delta T-\lambda\Delta\hat{T}}{\tau\omega_{0}(Z)}\right)\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\
& -\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\nabla_{\hat{T}}\left(\nabla_{\hat{T}}+\left|\bar{\Psi}_{0}(Z,Z')\right|^{2}\Delta\hat{T}\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)-\hat{V}\left(\Delta\Gamma,\Delta\Gamma^{\dagger}\right)\\
& +U_{\Delta\Gamma}\left(\left\|\Delta\Gamma(Z,Z')\right\|^{2}\right)
\end{aligned}$$

with:

$$\left|\bar{\Psi}_{0}(Z,Z')\right|^{2}=\frac{\rho\left(C(\theta)\left|\Psi_{0}(Z)\right|^{2}\omega_{0}(Z)+D(\theta)\hat{T}\left|\Psi_{0}(Z')\right|^{2}\omega_{0}(Z')\right)}{\omega_{0}(Z)}$$

$$\begin{aligned}
& \hat{V}\left(\Delta\Gamma,\Delta\Gamma^{\dagger}\right)=\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right) && (4)\\
& \times\left(\nabla_{\hat{T}}\left(\frac{\rho D(\theta)\left\langle\hat{T}\right\rangle\left|\Psi_{0}(Z')\right|^{2}}{\omega_{0}(Z)}\check{T}\left(1-\left(1+\left\langle\left|\Psi_{\Gamma}\right|^{2}\right\rangle\right)\check{T}\right)^{-1}\left[O\frac{\Delta T\left|\Delta\Gamma(\theta_{1},Z_{1},Z_{1}')\right|^{2}}{T\Lambda^{2}}\right]\right)\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)
\end{aligned}$$

and:

$$O(Z,Z',Z_{1})=-\frac{|Z-Z'|}{c}\nabla_{\theta_{1}}+\frac{(Z'-Z)^{2}}{2}\left(\frac{\nabla_{Z_{1}}^{2}}{2}+\frac{\nabla_{\theta_{1}}^{2}}{2c^{2}}-\frac{\nabla_{Z}^{2}\omega_{0}(Z)}{2}\right) \tag{5}$$

$$\begin{aligned}
\Delta T &= T-\langle T\rangle\\
\Delta\hat{T} &= \hat{T}-\left\langle\hat{T}\right\rangle
\end{aligned}$$

with $\langle T\rangle$ and $\left\langle\hat{T}\right\rangle$ are averages in the background field. The potential

$$U_{\Delta\Gamma}\left(\left\|\Delta\Gamma(Z,Z')\right\|^{2}\right)$$

is the second order expansion of $U\left(\left\{\left|\Gamma(\theta,Z,Z',C,D)\right|^{2}\right\}\right)$ around $\Gamma_{0}$ and $\Gamma_{0}^{\dagger}$.

## 4.2 Static collective states

In ([8]), we derived the conditions of existence for collective states of the effective action (3). Such state are described as a collection of shifted states. The values of these shifts depend both on the characteristics of the background field and the potential $U_{\Delta\Gamma}$. For shifted states, the average connectivities are modified: $\left\langle\hat{T}\right\rangle\rightarrow\left\langle\hat{T}\right\rangle+\underline{\Delta\left\langle\hat{T}\right\rangle}$ and $\langle T\rangle\rightarrow\langle T\rangle+\underline{\Delta\langle T\rangle}$, where the values of the shift $\underline{\Delta\left\langle\hat{T}\right\rangle}$ and $\underline{\Delta\langle T\rangle}$ are computed in ([8]). For later purposes, in appendix 1 we recall the derivation of these states and the associated average shifts along with the description of collective states.

### 4.2.1 Formula for shifted states

The static collective states are described by a set $W$ of doublet, such that if $(Z,Z')\in W$ The solutions to the saddle point equation of (3) at these points become:

$$\begin{aligned}&\Delta\Gamma_{\delta}\left(T,\hat{T},\theta,Z,Z'\right) \quad (6)\\ =\;& \exp\left(-\frac{1}{2}\left(\mathbf{\Delta T}-\mathbf{\Delta\bar{T}}\right)^{t}\hat{U}\left(\mathbf{\Delta T}-\mathbf{\Delta\bar{T}}\right)\right)\\ &\times H_{p}\left(\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\frac{\sigma_{T}\lambda_{+}}{2\sqrt{2}}\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\right)H_{p-\delta}\left(\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\frac{\sigma_{\hat{T}}\lambda_{-}}{2\sqrt{2}}\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\right)\end{aligned}$$

and:

$$\begin{aligned}&\Delta\Gamma_{\delta}^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right) \quad (7)\\ =\;& H_{p}\left(\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\frac{\sigma_{T}\lambda_{+}}{2\sqrt{2}}\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\right)H_{p-\delta}\left(\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\frac{\sigma_{\hat{T}}\lambda_{-}}{2\sqrt{2}}\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\right)\end{aligned}$$

where $H_p$ and $H_{p-\delta}$ are Hermite polynomials.

The variables involved are:

$$\begin{aligned}\mathbf{\Delta T}-\mathbf{\Delta\bar{T}} &= \begin{pmatrix}\Delta T-\langle\Delta T\rangle\\ \Delta\hat{T}-\left\langle\Delta\hat{T}\right\rangle\end{pmatrix} \quad (8)\\ \mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}' &= P^{-1}\left(\mathbf{\Delta T}-\mathbf{\Delta\bar{T}}\right)\end{aligned}$$

The matrices and parameters involved are provided in the appendices, along with the formula for $\langle\Delta T\rangle$, $\left\langle\Delta\hat{T}\right\rangle$.

The density of connectivities between $Z$ and $Z'$ is given by $\left|\Delta\Gamma_{\delta}\left(T,\hat{T},\theta,Z,Z'\right)\right|^{2}$ and can be understood as follows: regardless of how the system is interpreted, whether as a set of groups of simple cells or single complex cells at each point, the stable backgrounds are not defined with a specific connectivity value. On the contrary, the background states are described by a distribution around some average value. In other words, the cells or groups of axons/dendrites are connected with srength of connectivities that are distributed around this average.

### 4.2.2 Average shifts

When a shifted state exists, the average shifts are given by:

$$\langle\Delta T\rangle\simeq\frac{\omega_{0}(Z)\langle T\rangle}{\rho D(\theta)\left\langle\hat{T}\right\rangle\left|\Psi_{0}(Z')\right|^{2}k\underline{A_{1}}\left\|\Delta\Gamma\right\|^{6}}\left\langle\rho\frac{\left|\bar{\Psi}_{0}(Z,Z')\right|^{2}}{A}\right\rangle^{2}\langle T\rangle \quad (9)$$

and:

$$\left\langle\Delta\hat{T}\right\rangle=\hat{A}\langle\Delta T\rangle \quad (10)$$

with:

$$A_{1}(Z,Z')=\frac{\rho D(\theta)\left\langle\hat{T}\right\rangle\left|\Psi_{0}(Z')\right|^{2}}{\omega_{0}(Z)}\underline{A_{1}}(Z,Z')$$

and:

$$\underline{A_{1}}(Z,Z')=\left\langle\left[F(Z_{2},Z_{2}')\left[\check{T}\left(1-\left\langle\left|\Psi_{0}\right|^{2}\right\rangle\frac{\langle\Delta T\rangle}{T}\left\|\Delta\Gamma\right\|^{2}\right)^{-1}O\right]\right]_{(T,\hat{T},\theta,Z,Z')}\right\rangle$$

$$\hat{A}\simeq-\frac{1}{v}A\frac{\left\|\Delta\Gamma\right\|^{2}}{\langle T\rangle} \quad (11)$$

### 4.2.3 Conditions for shifted state

Shifted connectivities emerge at certain points under stability conditions. Depending on the potential and the activity of cells at these points, connectivity may either be enhanced or reduced. We established the conditions for the existence of a shift in a previous work ([8]). The existence conditions for such a set depend on the background $|\Psi_0(Z)|^2$ and the potential $U''_{\Delta\Gamma}$:

$$|(u+v) - \langle u+v\rangle| < \sqrt{-8U_{\Delta\Gamma}\left(\|\Delta\Gamma(Z,Z')\|^2_{\min}\right)U''_{\Delta\Gamma}\left(\|\Delta\Gamma(Z,Z')\|^2_{\min}\right)} \tag{12}$$

where $\|\Delta\Gamma(Z,Z')\|^2_{\min}$ is the minimum of the potential $U_{\Delta\Gamma}$ at $(Z,Z')$ and:

$$\begin{aligned}
u &= \frac{|\Psi_0(Z)|^2}{\tau\omega_0(Z)} \\
v &= \rho C\frac{|\Psi_0(Z)|^2 h_C(\omega_0(Z))}{\omega_0(Z)} + \rho D\frac{|\Psi_0(Z')|^2 h_D(\omega_0(Z'))}{\omega_0(Z)} \\
s &= -\frac{\lambda|\Psi_0(Z)|^2}{\omega_0(Z)}
\end{aligned}$$

The bracket $\langle u+v\rangle$ represents the average of $u+v$ over the entire space. Therefore, the condition (12) for a shift is relative. The main factor for allowing the emergence of modified connectivities is the relative level of activity and frequencies with respect to the entire system.

This condition means that, in first approximation, $|\Psi_0(Z)|^2$ must be below a threshold provided by the right-hand-side of (12) for a state with enhanced connectivities to exist.

## 5 Description of dynamic emergent collective states

Our earlier results were obtained by averaging over the entire system to identify the elements that would become activated when connectivity was modified. However, the resulting states themselves were not extensively studied, especially in terms of their group interactions or the interactions between different potential collective states. Additionally, there was no mention of the activities associated with these possible states.

In the current context, when a group of states experiences a shift in connectivity, these states are expected to interact collectively due to these changes. Furthermore, several group of states should interact with each other. These interactions are contingent upon the activities of each constituent element within the group. The primary objective is to describe the characteristic activity patterns for such independent groups of interconnected cells and investigate the role of these characteristics in their interactions.

To achieve this, we need to introduce dynamic aspects into the description of collective states and expand the formalism to encompass interacting groups of collective states. This begins with a reworking of (3) to incorporate dynamic aspects of interactions between elements within the activated group. In ([8]), the effective action (3) was initially derived to ascertain conditions for the emergence of states, and it was sufficient to work with the neurons background activity. Consequently, we will assume that neuronal activity is given by:

$$\omega_0(Z) + \Delta\omega\left(\theta, Z, |\Psi|^2\right)$$

where $\Delta\omega\left(\theta, Z, |\Psi|^2\right)$ is the internal activity of the group. We will find the formula for the possible values $\Delta\omega\left(\theta, Z, |\Psi|^2\right)$ and this will describe the set of possible activities for the emerging states. In fact we will look specifically for stable oscillating forms for $\Delta\omega\left(\theta,Z,|\Psi|^2\right)$. This choice is justified in detail in appendices 2 and 3 based on ([5]), ([6]), ([7]). The reason relies on a field-theoretic perturbative argument in ([5]): a perturbation in activities modifies the background state $\Psi$ which compensate any dampening or enhancement in activity oscillations, resulting in stable patterns.

## 5.1 Effective action

We assume the existence of states with finite set $S^2=\{(Z,Z')\}$, with $\left\|\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right\|^2\neq 0$[2]. Introducing the averages:

$$\left(\underline{\Delta\langle T\rangle},\underline{\Delta\left\langle\hat{T}\right\rangle}\right)$$

we can rewrite the effective action (3) for the elements of the group of activated states as:

$$\begin{aligned}
&S\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right) \qquad (13)\\
=\;&-\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\nabla_T\left(\nabla_T+\frac{\left(\Delta T-\underline{\Delta\langle T\rangle}\right)-\lambda\left(\Delta\hat{T}-\underline{\Delta\left\langle\hat{T}\right\rangle}\right)}{\tau\omega_0(Z)}\right)\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\
&-\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\nabla_{\hat{T}}\left(\nabla_{\hat{T}}+\left|\bar{\Psi}_0(Z,Z')\right|^2\left(\Delta\hat{T}-\underline{\Delta\left\langle\hat{T}\right\rangle}\right)\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)+U_{\Delta\Gamma}\left(\left\|\Delta\Gamma(Z,Z')\right\|^2\right)
\end{aligned}$$

Where the sum over $Z\in S$ and $Z'\in S$ is implicit. However, this formula relies on the averages background activities $\omega_0(Z)$, since it was designed to find average conditions for emergence of modified states. Once a set of cells are activated, their interaction implies some additional activity frequency $\Delta\omega\left(\theta,Z,|\Psi|^2\right)$ inducing a modification of action that becomes:

$$\hat{S}\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right)=S\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right)-\Delta V\left(\Delta\Gamma,\Delta\Gamma^{\dagger}\right) \qquad (14)$$

where:

$$\begin{aligned}
&\Delta V\left(\Delta\Gamma,\Delta\Gamma^{\dagger}\right)=\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right) \qquad (15)\\
&\times\left(\nabla_{\hat{T}}\left(\frac{\rho}{\omega_0(Z)}\left(D(\theta)\left\langle\hat{T}\right\rangle\left|\Psi_0(Z')\right|^2\left(\left((Z-Z')\left(\nabla_Z+\nabla_Z\omega_0(Z)\right)+\frac{|Z-Z'|}{c}\right)\Delta\omega\left(\theta,Z,|\Psi|^2\right)\right)\right)\right)\right)\\
&\times\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)
\end{aligned}$$

This additional term is similar to the potential (4) but accounts for an additional interaction between the activated elements as implied by the terms $\Delta\omega\left(\theta,Z,|\Psi|^2\right)$. Eventhough similar to (4), this term was not present in (3) since this action describes the possibility of emerging groups, not the internal dynamics of such states.

To consider the possible dynamics of the group as a whole, we first consider its possible dynamic activities given its shape and connectivity magnitude.

[2] The condition of existence for such collective state in a dynamic context is discussed below.

## 5.2 Dynamic activities $\Delta\omega\left(\theta, Z, |\Psi|^2\right)$ of collective stats

In a state with $S^2 = \{(Z, Z')\}$, we aim at finding the possible activity frequencies $\omega\left(\theta, Z, |\Psi|^2\right)$ associated to the state $\Delta T$. We consider the state $\{\Delta T(Z, Z')\}$ as a system with its own associated activity $\Delta\omega\left(\theta, Z, |\Psi|^2\right)$ in the given background field $|\Psi|^2$.

To find $\omega\left(\theta, Z, |\Psi|^2\right)$, we start with the defining equation:

$$\Delta\omega^{-1}\left(\theta, Z, |\Psi|^2\right) = G\left(\int \frac{\kappa}{N} \frac{\Delta\omega\left(J, \theta - \frac{|Z-Z_1|}{c}, Z_1, \Psi\right)\Delta T\left(Z, \theta, Z_1, \theta - \frac{|Z-Z_1|}{c}\right)}{\Delta\omega\left(J, \theta, Z, |\Psi|^2\right)} \left|\Psi\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)\right|^2 dZ_1\right)$$

We show in appendix 3 that $\Delta\omega^{-1}\left(\theta, Z, |\Psi|^2\right)$ decomposes into a static part and a variable part.

### 5.2.1 Static part of the activities

The static part satisfies:

$$\overline{\Delta\omega}^{-1}\left(Z, |\Psi|^2\right) = G\left(\int \frac{\kappa}{N} \frac{\Delta\omega(Z_1, \Psi)\Delta T(Z, Z_1)}{\Delta\omega\left(Z, |\Psi|^2\right)} |\Psi(Z_1)|^2 dZ_1\right)$$

Using that the modifications states is a group over a bounded domain and involve some finite number of points we find the equation for the modification modification at each point $Z_i$ of this group:

$$\overline{\Delta\omega}^{-1}\left(Z_i, |\Psi|^2\right) = G\left(\sum_j \frac{\kappa}{N} \frac{\overline{\Delta\omega}(Z_j, \Psi)\Delta T(Z_i, Z_j)}{\overline{\Delta\omega}\left(Z_i, |\Psi|^2\right)} |\Psi(Z_j)|^2\right)$$

with solution:

$$\overline{\Delta\omega}\left(\mathbf{Z}, \mathbf{T}, |\Psi|^2\right)$$

where $\omega\left(\mathbf{Z}, \mathbf{T}, |\Psi|^2\right)$ is the vector with coordinates $\omega\left(Z_i, |\Psi|^2\right)$.

### 5.2.2 Variable oscillatory part of the activities

The variable part is a first order variation:

$$\Delta\omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)$$

In fact this variable part should induce some fluctuations in $|\Psi|^2$. We show in appendix 2 that we can equivalently consider $|\Psi|^2$ to remain constant while the variable part exhibits stable oscillations. Therefore, in a first approximation, we can investigate stable oscillations for $\Delta\omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)$.

Using the fact that the modified states form a group over a bounded domain and involve a finite number of points we find in appendix 2 the equation for this modification at each point $Z_i$ of this group:

$$\Delta\omega\left(\theta, Z_i, |\Psi|^2\right) = \sum_j \hat{T}(Z_i, Z_j)\Delta\omega\left(\theta - \frac{|Z_i - Z_j|}{c}, Z_j, \Psi\right) \tag{16}$$

where:

$$\hat{T}(Z_i, Z_j) = \frac{\kappa}{N} \frac{T(Z_i, Z_j)\overline{\Delta\omega}(J, Z_j, \Psi)|\Psi_0(Z_j)|^2}{G^{-1}\left(\overline{\Delta\omega}^{-1}\left(J, Z_i, |\Psi|^2\right)\right) - \overline{\Delta\omega}^{-1}\left(J, Z_i, |\Psi|^2\right)}$$

Appendix 2 shows the existence of stable oscillatory solutions:

$$\Delta\omega\left(\theta,\mathbf{Z},|\Psi|^{2}\right)=A\left(Z_{1}\right)\left(1,\left(1-\hat{\mathbf{T}}\exp\left(-i\Upsilon_{p}\frac{|\mathbf{\Delta Z}|}{c}\right)\right)^{-1}\hat{T}_{1}\left(\mathbf{Z}\right)\exp\left(-i\Upsilon_{p}\frac{|\mathbf{\Delta Z}_{1}|}{c}\right)\right)^{t}\exp\left(i\Upsilon_{p}\left(\hat{\mathbf{T}}\right)\theta\right) \tag{17}$$

where $\overline{\Delta\omega}\left(\theta,\mathbf{Z},|\Psi|^{2}\right)$ and $\hat{T}_{1}\left(\mathbf{Z}\right)$ are vectors with coordinates $\overline{\Delta\omega}\left(J,\theta,Z_{i},|\Psi|^{2}\right)$ and $\hat{T}\left(Z_{1},Z_{j}\right)$ respectively. The point $Z_{1}$ is an arbitrary point chosen in the grp nd $A\left(Z_{1}\right)$ is the amplitude of $\Delta\omega$ at some gvn $Z_{1}$.

The matrix $\hat{\mathbf{T}}$ has elements $\hat{T}\left(Z_{i},Z_{j}\right)$ and the frequencies $\Upsilon_{p}\left(\mathbf{T}\right)$ belong to a discrete set satisfying the equation:

$$\det\left(1-\hat{T}\left(Z_{i},Z_{j}\right)\exp\left(-i\Upsilon_{p}\frac{|Z_{i}-Z_{j}|}{c}\right)\right)=0 \tag{18}$$

The possible osillatory activities associated to the assembly is thus given by the sets:

$$\left\{\left\{A\left(Z_{i}\right)\right\}_{i=1,\dots n},\Upsilon_{p}\left(\left\{\hat{T}\left(Z_{i},Z_{j}\right)\right\}\right)\right\}_{p}$$

where $p$ refers to the frequencies $\Upsilon$, solutions of (16):

$$A\left(Z_{i}\right)=\sum_{j\neq i}A\left(Z_{j}\right)\hat{T}\left(Z_{i},Z_{j}\right)\exp\left(-i\Upsilon_{p}\left(\left\{\hat{T}\left(Z_{i},Z_{j}\right)\right\}\right)\frac{|Z_{i}-Z_{j}|}{c}\right)$$

the $\Upsilon_{p}\left(\hat{T}\left(Z_{i},Z_{j}\right)\right)$ are solutions of (18).

### 5.2.3 Overall activity of the collective state

Gathering (16) and (17), the solution for the specific frequencies of the group is:

$$\begin{aligned}&\Delta\omega\left(\theta,Z_{i},\mathbf{T},|\Psi|^{2}\right)\\=\;&\overline{\Delta\omega}\left(\mathbf{Z},\mathbf{T},|\Psi|^{2}\right)\\&+A\left(Z_{1}\right)\left(1,\left(1-\hat{\mathbf{T}}\exp\left(-i\Upsilon_{p}\left(\hat{\mathbf{T}}\right)\frac{|\mathbf{\Delta Z}|}{c}\right)\right)^{-1}\hat{T}_{1}\left(\mathbf{Z}\right)\exp\left(-i\Upsilon_{p}\left(\hat{\mathbf{T}}\right)\frac{|\mathbf{\Delta Z}_{1}|}{c}\right)\right)^{t}\exp\left(i\Upsilon_{p}\left(\hat{\mathbf{T}}\right)\theta\right)\end{aligned}$$

where:

$$\begin{aligned}\left(\hat{T}_{1}\left(\mathbf{Z}\right)\right)_{i}&=\hat{T}\left(Z_{i},Z_{1}\right)\\\Upsilon_{p}\left(\hat{\mathbf{T}}\right)&=\Upsilon_{p}\left(\left\{\hat{T}\left(Z_{i},Z_{j}\right)\right\}\right)\end{aligned}$$

## 5.3 averages computations for activated stts

Now, considering (14) for the group of shifted states, we rewrite the action taking into account their particular interactions. Having obtained the frequencies, we can come back to the action minimization and compute the connectivity states along with average connectivities. To do so, we replace in (15) (see ([8])):

$$\begin{aligned}\left(\left(\left(Z-Z'\right)\left(\nabla_{Z}+\nabla_{Z}\omega_{0}\left(Z\right)\right)+\frac{|Z-Z'|}{c}\right)\Delta\omega\left(\theta,Z,|\Psi|^{2}\right)\right)&\rightarrow\frac{|Z-Z'|}{c}\Delta\omega\left(\theta,Z,|\Psi|^{2}\right)\\&\rightarrow g\left|Z-Z'\right|\Delta\omega\left(\theta,Z,|\Psi|^{2}\right)\end{aligned} \tag{19}$$

At the connectivity time scale we replace $\Delta\omega\left(\theta, Z, |\Psi|^2\right)$ by its static part $\overline{\Delta\omega}\left(Z, |\Psi|^2\right)$. Replacing this formula in (15), using (14) and (13), leads to the following action:

$$\begin{aligned}
&\hat{S}\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right) \qquad (20)\\
=\ & -\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\nabla_T\left(\nabla_T+\frac{\left(\Delta T-\underline{\Delta\langle T\rangle}\right)-\lambda\left(\Delta\hat{T}-\underline{\Delta\left\langle\hat{T}\right\rangle}\right)}{\tau\omega_0(Z)}\left|\Psi(\theta,Z)\right|^2\right)\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\
&-\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\nabla_{\hat{T}}\left(\nabla_{\hat{T}}+\left|\bar{\Psi}_0(Z,Z')\right|^2\left(\Delta\hat{T}-\left(\underline{\Delta\left\langle\hat{T}\right\rangle}+\Delta^{\omega}\left\langle\hat{T}\right\rangle\right)\right)\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\
&+U_{\Delta\Gamma}\left(\left\|\Delta\Gamma(Z,Z')\right\|^2\right)
\end{aligned}$$

with:

$$\Delta^{\omega}\hat{T}=\frac{\left(D(\theta)\left\langle\hat{T}\right\rangle\left|\Psi_0(Z')\right|^2|Z-Z'|\,g\overline{\Delta\omega}\left(Z,\mathbf{T},|\Psi|^2\right)\right)}{\left(C(\theta)\left|\Psi_0(Z)\right|^2\omega_0(Z)+D(\theta)\left\langle\hat{T}\right\rangle\left|\Psi_0(Z')\right|^2\omega_0(Z')\right)} \qquad (21)$$

Then we define $\Delta\langle T\rangle$ and $\Delta\left\langle\hat{T}\right\rangle$ as solutions of:

$$\begin{aligned}
\Delta\langle T\rangle &= \underline{\Delta\langle T\rangle}+\lambda\left\langle\Delta^{\omega}\hat{T}\right\rangle\\
\Delta\left\langle\hat{T}\right\rangle &= \underline{\Delta\left\langle\hat{T}\right\rangle}+\left\langle\Delta^{\omega}\hat{T}\right\rangle
\end{aligned}$$

The first equation defines the set $\langle\mathbf{\Delta T}\rangle$ and the second one yields $\Delta\left\langle\hat{\mathbf{T}}\right\rangle$. There are several solutions:

$$\left(\langle\mathbf{\Delta T}\rangle^{\alpha},\left\langle\mathbf{\Delta}\hat{\mathbf{T}}\right\rangle^{\alpha}\right) \qquad (22)$$

For each of these solutions, a sequence of frequencies $(\Upsilon_p^{\alpha})$ satisfying (18) are compatible. The variable contribution of activities is given by:

$$\Delta\omega_p^{\alpha}(\theta,Z,\mathbf{\Delta T})=\overline{\Delta\omega}\left(Z,\mathbf{T},|\Psi|^2\right)+\left(\mathbf{N}_p^{\alpha}\right)^{-1}\Delta\boldsymbol{\omega}_0\exp\left(-i\Upsilon_p^{\alpha}\frac{|\Delta\mathbf{Z}_i|}{c}\right)$$

where:

$$\left[\mathbf{N}_p^{\alpha}\right]_{(Z_i,Z_j)}=\left(\delta_{ij}-[\mathbf{\Delta T}]_{(Z_i,Z_j)}\exp\left(-i\Upsilon_p\frac{|Z_i-Z_j|}{c}\right)\right)^{-1}$$

## 5.4 Form of the activated state

We will now derive the formula for $\Delta\Gamma$ and its conjugate $\Delta\Gamma^{\dagger}$ in the activated state. In appendix 1, we show that after a change of variables, as (20), the effective action rewrites :

$$\begin{aligned}
&\hat{S}\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right) \qquad (23)\\
=\ & -\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\nabla_T^2+\nabla_{\hat{T}}^2-\frac{1}{2}\left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^{\alpha}\right)^t\mathbf{A}_p^{\alpha}\left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^{\alpha}\right)\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\
&+C(Z,Z')\left\|\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right\|^2
\end{aligned}$$

and that the minimization equations leads to the activated state:

$$\Delta\Gamma=\prod_{Z,Z'}\left|\Delta T(Z,Z'),\Delta\hat{T}(Z,Z'),\alpha(Z,Z'),p(Z,Z')\right\rangle\equiv\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle \qquad (24)$$

with $S^2 = \{(Z, Z')\}$ where the states are activated. In a developped form, this state is given by formula similar to (6) and (7):

$$\begin{aligned} & \left|\boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle \qquad (25) \\ = & \exp\left(-\frac{1}{2}\left(\boldsymbol{\Delta}\mathbf{T} - \left\langle\boldsymbol{\Delta}\mathbf{T}\right\rangle_p^{\alpha}\right)^t \mathbf{A}_p^{\alpha}\left(\boldsymbol{\Delta}\mathbf{T} - \left\langle\boldsymbol{\Delta}\mathbf{T}\right\rangle_p^{\alpha}\right)\right) H_p\left(\frac{1}{2}\left(\boldsymbol{\Delta}\mathbf{T} - \left\langle\boldsymbol{\Delta}\mathbf{T}\right\rangle_p^{\alpha}\right)^t \mathbf{A}_p^{\alpha}\left(\boldsymbol{\Delta}\mathbf{T} - \left\langle\boldsymbol{\Delta}\mathbf{T}\right\rangle_p^{\alpha}\right)\right) \\ & \times H_p\left(\left(\boldsymbol{\Delta}\mathbf{T}' - \left\langle\boldsymbol{\Delta}\mathbf{T}\right\rangle_p^{\alpha}\right)_1^t \left(\mathbf{D}_p^{\alpha}\right)_1 \left(\boldsymbol{\Delta}\mathbf{T}' - \left\langle\boldsymbol{\Delta}\mathbf{T}\right\rangle_p^{\alpha}\right)_1^t\right) H_{p-\delta}\left(\left(\boldsymbol{\Delta}\mathbf{T}' - \left\langle\boldsymbol{\Delta}\mathbf{T}\right\rangle_p^{\alpha}\right)_2^t \left(\mathbf{D}_p^{\alpha}\right)_2 \left(\boldsymbol{\Delta}\mathbf{T}' - \left\langle\boldsymbol{\Delta}\mathbf{T}\right\rangle_p^{\alpha}\right)_2^t\right) \end{aligned}$$

with the conjugate:

$$\begin{aligned} \Delta\Gamma^{\dagger} &= \left\langle\boldsymbol{\alpha}, \mathbf{p}, S^2\right| \qquad (26) \\ &= H_p\left(\left(\boldsymbol{\Delta}\mathbf{T}' - \left\langle\boldsymbol{\Delta}\mathbf{T}\right\rangle_p^{\alpha}\right)_1^t \left(\mathbf{D}_p^{\alpha}\right)_1 \left(\boldsymbol{\Delta}\mathbf{T}' - \left\langle\boldsymbol{\Delta}\mathbf{T}\right\rangle_p^{\alpha}\right)_1^t\right) H_{p-\delta}\left(\left(\boldsymbol{\Delta}\mathbf{T}' - \left\langle\boldsymbol{\Delta}\mathbf{T}\right\rangle_p^{\alpha}\right)_2^t \left(\mathbf{D}_p^{\alpha}\right)_2 \left(\boldsymbol{\Delta}\mathbf{T}' - \left\langle\boldsymbol{\Delta}\mathbf{T}\right\rangle_p^{\alpha}\right)_2^t\right) \end{aligned}$$

where $H_p$ and $H_{p-\delta}$ are Hermite polynomials. The coordinates $\left(\boldsymbol{\Delta}\mathbf{T}' - \left\langle\boldsymbol{\Delta}\mathbf{T}\right\rangle_p^{\alpha}\right)_1^t$ as well as elements $\left(\mathbf{D}_p^{\alpha}\right)_i$ are obtained through the diagonalization of $\mathbf{A}_p^{\alpha}$.

# 6 Collective state for $n$ interacting fields

In this section, we explore collective states that emerge from the interactions of various types of fields. This section follows a similar structure to the previous one. We begin by revisiting the findings from ([8]) and then delve into the dynamic collective states that result from the interactions within the system. This generalization will be used in the following section when we discuss the transition from a system with multiple collective states (one for each field type) to a state that combines each type of cells. Consequently, introducing the extended field formalism for collective states developed in ([7]) and ([8]) becomes useful. Describing the formalism for collective states involving $n$ different fields will allow us to compare multiple collective states, each associated with a specific field, to a single state resulting from interactions and mergers. This, in turn, will provide insights for developing a formalism for transitions between collective states.

## 6.1 General set up

We saw in ([6]) how to describe $n$ interacting types of cells, with arbitrary interactions. Each type of cells is caracterized by its activity $i = 1, ..., n$, and interacts either positively or negatively with each other. Each type is defined by a field $\Psi_i$ and activities $\omega_i(\theta, Z)$. The system is described by a sum of terms similar to (1):

$$\begin{aligned} S_{full} &= -\frac{1}{2}\Psi_i^{\dagger}(\theta, Z)\nabla\left(\frac{\sigma_{\theta}^2}{2}\nabla - \omega_i^{-1}\left(J, \theta, Z, \left(\left|\Psi_k\right|^2\right)_{k\leqslant n}\right)\right)\Psi_i(\theta, Z) + V(\Psi_i) \qquad (27) \\ & + \frac{1}{2\eta^2}\left(S_{\Gamma_{ij}}^{(0)} + S_{\Gamma_{ij}}^{(1)} + S_{\Gamma_{ij}}^{(2)} + S_{\Gamma_{ij}}^{(3)} + S_{\Gamma_{ij}}^{(4)}\right) + U\left(\left(\left|\Gamma_{ij}(\theta, Z, Z', C, D)\right|^2\right)_{i\leqslant n, j\leqslant n}\right) \end{aligned}$$

The field $\Gamma_{ij}$ describes the connectivities between types $i$ and $j$. The functionals $S_{\Gamma_{ij}}^{(c)}$ involve $\Gamma_{ij}$ and $\Psi_i(\theta, Z)$ and $\Psi_j(\theta, Z)$. The formula are given in appendix 4. As in the one field case, we replace $\Psi_i(\theta, Z)$ and $\omega_i^{-1}$ by functionals of $\Gamma_{ij}(\theta, Z, Z', C, D)$.

Then the background fields are derived by minimizing:

$$\frac{\delta\sum S_{\Gamma_{ij}}^{(a)}}{\delta\Delta\Gamma_{ij}^{\dagger}\left(T, \hat{T}, \theta, Z, Z'\right)} = \frac{\delta\sum S_{\Gamma_{ij}}^{(a)}}{\delta\Delta\Gamma_{ij}\left(T, \hat{T}, \theta, Z, Z'\right)} = 0$$

We obtain the background for this field along with the associated average values of connectivities and activities. The average connectivity functions $T_{ij}$ is given by:

$$T_{ij}(Z,Z_1)=\int T_{ij}\left|\Gamma_{ij}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z',C_{ij},D_{ij}\right)\right|^2$$

and the equilibrium static activities is derived as a function of the non interacting activities, under the assumption:

$$G^{ij}T_{ij}<<T_{ii}\text{ for }i\neq j$$

that is in limit of weak interaction between different fields[3]. The formula are recalled in appendix.

## 6.2 Effective action for collective stts of several type of interacting strctrs

### 6.2.1 Effective action without collective internal dynamics

As in ([8]), the effective action involving $n$ field is obtained directly by a second order expansion around the background field minimizing $\sum S_{\Gamma}^{(i)}$. This leads to the effective action for connectivities. Formally it is given by:

$$S\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right)=\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\frac{\delta^2\sum S_{\Gamma}^{(i)}}{\delta\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\delta\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)}\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$$

Modifying (3) and (4) to the case of several fields leads to:

$$\begin{aligned}
&S\left(\Delta\Gamma_{ij}\left(T,\hat{T},\theta,Z,Z'\right)\right) &(28)\\
=\;&\Delta\Gamma_{ij}^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\nabla_T\left(\nabla_T+\frac{\left(T_{ij}-\langle T_{ij}\rangle\right)}{\tau\omega_i(Z)}\left|\Psi(\theta,Z)\right|^2\right)\right)\Delta\Gamma_{ij}\left(T,\hat{T},\theta,Z,Z'\right)\\
&+\Delta\Gamma_{ij}^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\nabla_{\hat{T}}\left(\nabla_{\hat{T}}+\left|\bar{\Psi}_{0ij}(Z,Z')\right|^2\left(\hat{T}_{ij}-\left\langle\hat{T}_{ij}\right\rangle\right)\right)\Delta\Gamma_{ij}\left(T,\hat{T},\theta,Z,Z'\right)+\hat{V}\left(\Delta\Gamma_{ij},\Delta\Gamma_{ij}^{\dagger}\right)\\
&+U_{\Delta\Gamma_{ij}}\left(\left(\left|\Delta\Gamma_{ij}\left(\theta,Z,Z',C,D\right)\right|^2\right)_{i\leqslant n,j\leqslant n}\right)
\end{aligned}$$

with:

$$\left|\bar{\Psi}_{0ij}(Z,Z')\right|^2=\frac{\rho\left(C(\theta)\left|\Psi_{0i}(Z)\right|^2\omega_i(Z)+D(\theta)\hat{T}_{ij}\left|\Psi_{0j}(Z')\right|^2\omega_j(Z')\right)}{\omega_i(Z)}$$

$$\begin{aligned}
&\hat{V}\left(\Delta\Gamma_{ij},\Delta\Gamma_{ij}^{\dagger}\right)=\Delta\Gamma_{ij}^{\dagger}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z'\right) &(29)\\
&\times\left(\nabla_{\hat{T}}\left(\frac{\rho D(\theta)\left\langle\hat{T}_{ij}\right\rangle\left|\Psi_{0j}(Z')\right|^2}{\omega_{0i}(Z)}\check{T}_{ij}\left(1-\left(1+\left\langle\left|\Psi_{\Gamma_{ij}}\right|^2\right\rangle\right)\check{T}_{ij}\right)^{-1}\left[O_i\frac{\Delta T_{ij}\left|\Delta\Gamma_{ij}\left(\theta_1,Z_1,Z_1'\right)\right|^2}{T_{ij}\Lambda^2}\right]\right)\right)\\
&\times\Delta\Gamma_{ij}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z'\right)
\end{aligned}$$

and:

$$O_i(Z,Z',Z_1)=-\frac{|Z-Z'|}{c}\nabla_{\theta_1}+\frac{(Z'-Z)^2}{2}\left(\frac{\nabla_{Z_1}^2}{2}+\frac{\nabla_{\theta_1}^2}{2c^2}-\frac{\nabla_Z^2\omega_{0i}(Z)}{2}\right)\tag{30}$$

$$\begin{aligned}
\Delta T_{ij}&=T_{ij}-\langle T_{ij}\rangle\\
\Delta\hat{T}&=\hat{T}_{ij}-\left\langle\hat{T}\right\rangle_{ij}
\end{aligned}$$

[3] The matrix $G^{ij}$ describes the interaction between these different types of fields (see appendix 4).

with $\langle T_{ij}\rangle$ and $\left\langle \hat{T}_{ij}\right\rangle$ are the connectivities averages in the background field. The potential:

$$U_{\Delta\Gamma_{ij}}\left(\left(\left|\Delta\Gamma_{ij}\left(\theta,Z,Z',C,D\right)\right|^{2}\right)_{i\leqslant n,j\leqslant n}\right)$$

is the second order expansion of $U\left(\left(\left|\Gamma_{ij}\left(\theta,Z,Z',C,D\right)\right|^{2}\right)_{i\leqslant n,j\leqslant n}\right)$ around the background field.

### 6.2.2 Effective action including internal dynamics

Introducing the average modifications $\left(\underline{\Delta\langle T_{ij}\rangle},\underline{\Delta\left\langle \hat{T}_{ij}\right\rangle}\right)$ of $\left(\Delta T_{ij},\Delta\hat{T}_{ij}\right)$ and including internal dynamics leads to consider the generalization of (13) and (14):

$$\hat{S}\left(\Delta\Gamma_{ij}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z'\right)\right)=S\left(\Delta\Gamma_{ij}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z'\right)\right)-\Delta V\left(\Delta\Gamma_{ij},\Delta\Gamma_{ij}^{\dagger}\right)\tag{31}$$

wth:

$$\begin{aligned}&S\left(\Delta\Gamma_{ij}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z'\right)\right)\\=\;&-\Delta\Gamma_{ij}^{\dagger}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z'\right)\left(\nabla_{T_{ij}}\left(\nabla_{T_{ij}}+\frac{\left(\Delta T_{ij}-\underline{\Delta\langle T_{ij}\rangle}\right)-\lambda\left(\Delta\hat{T}_{ij}-\underline{\Delta\left\langle\hat{T}_{ij}\right\rangle}\right)}{\tau\omega_{0i}\left(Z\right)}\right)\right)\Delta\Gamma_{ij}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z'\right)\\&-\Delta\Gamma_{ij}^{\dagger}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z'\right)\nabla_{\hat{T}_{ij}}\left(\nabla\hat{T}_{ij}+\left|\bar{\Psi}_{0ij}\left(Z,Z'\right)\right|^{2}\left(\Delta\hat{T}_{ij}-\underline{\Delta\left\langle\hat{T}_{ij}\right\rangle}\right)\right)\Delta\Gamma_{ij}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z'\right)\\&+U_{\Delta\Gamma_{ij}}\left(\left(\left|\Delta\Gamma_{ij}\left(\theta,Z,Z',C,D\right)\right|^{2}\right)_{i\leqslant n,j\leqslant n}\right)\end{aligned}\tag{32}$$

As before $\Delta\omega_{i}\left(\theta,Z,\left|\Psi_{i}\right|^{2}\right)$ is some additional activity inducing a modification of action. The potential $V\left(\Delta\Gamma_{ij},\Delta\Gamma_{ij}^{\dagger}\right)$ is:

$$\begin{aligned}&V\left(\Delta\Gamma_{ij},\Delta\Gamma_{ij}^{\dagger}\right)=-\Delta\Gamma_{ij}^{\dagger}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z'\right)\\&\times\nabla_{\hat{T}}\left(\frac{\rho\left(D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_{0j}\left(Z'\right)\right|^{2}\left(\omega_{i}\left(Z\right)\Delta\omega_{j}\left(\theta-\frac{\left|Z-Z'\right|}{c},Z',\left|\Psi_{0j}\right|^{2}\right)-\omega_{j}\left(Z'\right)\Delta\omega_{i}\left(\theta,Z,\left|\Psi_{0i}\right|^{2}\right)\right)\right)}{\omega_{i}^{2}\left(Z\right)}\right)\\&\times\Delta\Gamma_{ij}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z'\right)\end{aligned}\tag{33}$$

Given our assumptions, the interaction $V$ is of relatively small magnitude between two different type of fields. Thus, describing the full collective state relies on mixing non interacting states.

## 6.3 Activities for collective several types of fields

The previous results generalize for a state composed of several type of fields. We consider both the static and variable components of activity. In this section indices $i$ and $j$ denote the distinct types of fields. We consider several groups of points $\{a_i\}$, $\{b_j\}$,... Summing over the entire group will be denoted by $\{\{b_j\}\}$. We define $|\Psi|^2=\left\{|\Psi_i|^2\right\}$, $\Psi=\{\Psi_i\}$. The derivation is similar to the case of a single type of field.

### 6.3.1 Static part of activity

The static part of activity for the $i$-th group at $Z_{a_i}$ is denoted $\omega_i^{-1}\left(Z_{a_i}, |\Psi|^2\right)$ and solves:

$$\omega_i^{-1}\left(Z_{a_i}, |\Psi|^2\right) = G\left(\sum_{\{\{b_j\}\}} \frac{\kappa}{N} G_{ij} \frac{\Delta\omega\left(Z_j, \Psi_j\right) \Delta T\left(Z_i, Z_j\right)}{\Delta\omega\left(Z_{a_i}, |\Psi_i|^2\right)} \left|\Psi_j\left(Z_{b_j}\right)\right|^2\right) \tag{34}$$

### 6.3.2 Variable oscillarory part of activity

The oscillator part satisfies:

$$\Delta\omega_i\left(\theta, Z_{a_i}, |\Psi|^2\right) = \sum_{\{\{b_j\}\}} \hat{T}_{ij}\left(Z_{a_i}, Z_{b_j}\right) \Delta\omega_{b_j}\left(J, \theta - \frac{\left|Z_{a_i} - Z_{b_j}\right|}{c}, Z_{b_j}, \Psi\right) \tag{35}$$

where:

$$\hat{T}_{ij}\left(Z_{a_i}, Z_{b_j}\right) = G_{a_i b_j} \frac{\kappa}{N} \frac{T_{ij}\left(Z_{a_i}, Z_{b_j}\right) \omega_{0b_j}\left(J, Z_{b_j}, \Psi\right) \left|\Psi_{0j}\left(Z_{b_j}\right)\right|^2}{G^{-1}\left(\omega_i^{-1}\left(\theta, Z_{a_i}, |\Psi|^2\right)\right) - \omega_j^{-1}\left(J, Z_{b_j}, |\Psi|^2\right)}$$

The solution of (35) was given before. Defining:

$$A_i\left(Z_{a_i}\right) = \sum_{\{\{b_j\}\}} A_j\left(Z_{b_j}\right) \hat{T}_{ij}\left(Z_{a_i}, Z_{b_j}\right) \exp\left(-i\Upsilon_p\left(\left\{\hat{T}_{ij}\left(Z_{a_i}, Z_{b_j}\right)\right\}\right) \frac{\left|Z_{a_i} - Z_{b_j}\right|}{c}\right)$$

the solution is:

$$\begin{aligned} & \Delta\omega\left(\theta, Z_{a_i}, \mathbf{T}, |\Psi|^2\right) \\ = & \Delta\omega\left(Z_{a_i}, \mathbf{T}\right) \\ & + \sum_{\{\{b_j\}\}} A\left(Z_{b_j}\right)\left(1, \left(1 - \hat{\mathbf{T}} \exp\left(-i\Upsilon_p\left(\hat{\mathbf{T}}\right) \frac{|\mathbf{\Delta Z}|}{c}\right)\right)^{-1} \hat{T}_1\left(\mathbf{Z}\right) \exp\left(-i\Upsilon_p\left(\hat{\mathbf{T}}\right) \frac{|\mathbf{\Delta Z}_1|}{c}\right)\right)^t \exp\left(i\Upsilon_p\left(\hat{\mathbf{T}}\right)\theta\right) \end{aligned}$$

where we defined:

$$\hat{\mathbf{T}} = \left\{\hat{T}_{ij}\left(Z_{a_i}, Z_{b_j}\right)\right\}$$

However, to describe interactions between different collective states and transition between them, it is useful to describe the equilibrium frequencies of oscillations in the activity of a mixed state as a function of the frequencies of collective states of different types.

## 6.4 Frequencies of activities for interacting collective states as function of non interacting states

We formulate the activity equations for a collective state by considering separately the different type of collectives states involved in this state. The activity equations for each type are influenced by the activities of other types. The equations for activities of the $i$-th group as:

$$\begin{aligned} & \Delta\omega_i^{-1}\left(Z_{a_i}, \theta\right) \\ = & G\left(\sum_j \sum_{\{b_j\}} \frac{\kappa}{N} g^{ij} \frac{\Delta\omega_j\left(\theta - \frac{\left|Z_{a_i} - Z_{b_j}\right|}{c}, Z_{b_j}\right) \Delta T_{ij}\left(Z_{a_i}, Z_{b_j}, \theta - \frac{\left|Z_{a_i} - Z_{b_j}\right|}{c}\right)}{\Delta\omega_i\left(Z_{a_i}, \theta\right)} \left|\Psi_j\left(\theta - \frac{\left|Z_{a_i} - Z_{b_j}\right|}{c}, Z_{b_j}\right)\right|^2\right) \end{aligned} \tag{36}$$

Said differently, this is the equation for the activities of the $i$-th component of the whole state.

Assuming a weak interactions between the components:

$$G^{ij}\Delta T_{ij} << \Delta T_{ii}$$

we expand this equation (36) to the first order in $G^{ij}\Delta T_{ij}$. It leads to writing the static and dynamic parts of activities as functions of the non-interacting ones. Details are provided in appendix 5. This description will be useful for depicting transitions between non-interacting and interacting states.

### 6.4.1 Static part

Equilibrium static activities satify:

$$\begin{aligned}
&\overline{\Delta\omega}_i\left(Z_{a_i}\right) \\
=& \sum_{j,b_j}\left(\delta_{(i,a_i)(j,b_j)} - G'\left(G^{-1}\left(\overline{\Delta\omega}_{0i}\left(Z\right)\right)\right)\right. \\
&\times\left.\left(\frac{\kappa}{N}\frac{G^{ij}\overline{\Delta\omega}_{0j}\left(Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\overline{\Delta\omega}_{0i}\left(Z_{a_i}\right)}\left(\mathcal{G}_{j0}+\left|\Psi_j\left(Z_{b_j}\right)\right|^2\right)\right)_{j\neq i}\right)_{ij}^{-1}\overline{\Delta\omega}_{0j}\left(Z_{b_j}\right)
\end{aligned} \tag{37}$$

where the $\omega_{0j}\left(Z_{a_i}\right)$ are the activities in absence of interactions, which satisfy an equation similar to (34):

$$\overline{\Delta\omega}_{i0}^{-1}\left(Z_{a_i}\right) = G\left(\sum_{\beta_i}\frac{\kappa}{N}\frac{\overline{\Delta\omega}_{i0}\left(Z_{\beta_i}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\overline{\Delta\omega}_{i0}\left(Z_{a_i},\theta\right)}\left|\Psi_j\left(Z_{b_j}\right)\right|^2\right) \tag{38}$$

where we assume $G^{ii}=1$. Formula for $\overline{\Delta\omega}_i\left(Z_{a_i}\right)$ are derived in appendix 5 at the first order in $G^{ij}$:

$$\begin{aligned}
&\overline{\Delta\omega}_i\left(Z_{a_i}\right) \\
=& \sum_{j,b_j}\left(\delta_{(i,a_i)(j,b_j)} - G'\left(G^{-1}\left(\overline{\Delta\omega}_{0i}\left(Z\right)\right)\right)\left(\frac{\kappa}{N}\frac{G^{ij}\overline{\Delta\omega}_{0j}\left(Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\overline{\Delta\omega}_{0i}\left(Z_{a_i}\right)}\left(\mathcal{G}_{j0}+\left|\Psi_j\left(Z_{b_j}\right)\right|^2\right)\right)_{j\neq i}\right)_{ij}^{-1} \\
&\times\overline{\Delta\omega}_{0j}\left(Z_{b_j}\right)
\end{aligned} \tag{39}$$

### 6.4.2 Non-static part

**Frequencies without interactions** As before, the non static part of $n$ non-interacting activities are solutions of:

$$\Delta\omega_{i0}^{-1}\left(Z_{a_i},\theta\right) = \sum_{a_j}\hat{T}_{ii}\left(Z_{a_i},Z_{a_j}\right)\Delta\omega_{i0}\left(\theta-\frac{\left|Z_{\alpha_i}-Z_{a_j}\right|}{c},Z_{a_j}\right)$$

where:

$$\hat{T}_{ii}\left(Z,Z_1\right) = \frac{\kappa}{N}\frac{T_{ii}\left(Z,Z_1\right)\omega_{0i}\left(J,Z_1,\Psi\right)\left|\Psi_{0i}\left(Z_1\right)\right|^2 dZ_1}{G'\left(G^{-1}\left(\omega_{0i}^{-1}\left(J,Z,|\Psi|^2\right)\right)\right)-\omega_0^{-1}\left(J,Z,|\Psi|^2\right)}$$

Rewriting the solutions as a vector:

$$\left(\Delta\omega_{0i}^{-1}\left(Z_{a_i},\theta\right)\right)_{a_i} \equiv \Delta\boldsymbol{\omega}_0^{-1}\left(Z_{\mathbf{a}},\theta\right)$$

and looking for oscillatory solutions:

$$\Delta\boldsymbol{\omega}_0^{-1}\left(Z_{\mathbf{a}},\theta\right)=\Delta\boldsymbol{\omega}^{-1}\left(Z_{\mathbf{a}}\right)\exp\left(i\Upsilon\theta\right)$$

yields the following formula, similar to the single type of cells case:

$$\Delta\omega_{0i}\left(\theta,\mathbf{Z},|\Psi|^2\right)=A_i\left(\left(Z_1\right)_i\right)\left(1,\left(1-\hat{\mathbf{T}}_{ii}\exp\left(-i\Upsilon_{ip}\frac{|\mathbf{\Delta Z}|}{c}\right)\right)^{-1}\hat{T}_{1ii}\left(\mathbf{Z}\right)\exp\left(-i\Upsilon_{ip}\frac{|\mathbf{\Delta Z}_1|}{c}\right)\right)^t\exp\left(i\Upsilon_{ip}\left(\mathbf{T}_{ii}\right)\theta\right)$$

As in the 1-field case, $A_i\left(\left(Z_1\right)_i\right)$ is the amplitude of activity at one given point $\left(Z_1\right)_i$ of the group $i$ and $\hat{T}_{1ii}\left(\mathbf{Z}\right)$ is a vector with components $\left(\hat{T}_{1ii}\left(\mathbf{Z}\right)\right)_{a_i}=\hat{T}_{ii}\left(Z_{a_i},\left(Z_1\right)_i\right)$.

The $\Upsilon_{ip}\left(\mathbf{T}_{ii}\right)$ are equilbrium frequencies. They are solutions of:

$$\det\left(1-\hat{T}_{ii}\left(Z,Z_1\right)\exp\left(-i\Upsilon_p\frac{|Z-Z_j|}{c}\right)\right)=0 \tag{40}$$

We write the solutions $\Upsilon_p\left(\mathbf{T}_{ii}\right)$. By diagonalization of the matrix involved in (40), these frequencies satisfy:

$$\prod_k\left(1-f_{i,k}\left(\Upsilon_p\right)\right)=0$$

for some functions $f_{i,k}$. Thus, the solutions form a set:

$$\Upsilon_p^i=\left\{\gamma_{i,k}\right\}_k$$

**Frequencies with interactions** To find the corrections due to interactions, we expand (36) around (37) in appendix 5. We find:

$$\begin{aligned}&\Delta\omega_i^{-1}\left(Z_{a_i},|\Psi|^2\right)\\=\;&\Delta G\left(\sum_{\{\{b_j\}\}}\frac{\kappa}{N}g^{ij}\frac{\omega_j\left(\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c},Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j},\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}\right)}{\omega_i\left(Z_{a_i},\theta\right)}\left|\Psi_j\left(\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c},Z_{b_j}\right)\right|^2\right)\end{aligned}$$

As for one field case, we can neglect:

$$\Delta\left|\Psi_j\left(\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c},Z_{b_j}\right)\right|^2$$

and we obtain the following relation:

$$\Delta\omega_i^{-1}\left(Z_{a_i},\theta\right)=\sum_{\{\{b_j\}\}}\hat{T}_{ij}\left(Z_{a_i},Z_{b_j}\right)\Delta\omega_j\left(\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c},Z_{b_j}\right)$$

with:

$$\hat{T}_{ij}\left(Z_{a_i},Z_{b_j}\right)=G'\left(G^{-1}\left(\omega^{-1}\left(Z_{a_i},|\Psi|^2\right)\right)\right)\frac{\kappa}{N}g^{ij}\frac{\omega_j\left(Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{G'\left(G^{-1}\left(\omega^{-1}\left(Z_{a_i},|\Psi|^2\right)\right)\right)-\omega_i^{-1}\left(Z_{a_i}\right)}\left|\Psi_j\left(Z_{b_j}\right)\right|^2$$

Rewriting the solutions as a vector:

$$\left(\Delta\omega_i^{-1}\left(Z_{\alpha_i},\theta\right)\right)_i \equiv \Delta\boldsymbol{\omega}^{-1}\left(Z_{\boldsymbol{\alpha}},\theta\right)$$

we look for oscillatory solutions. For a composed state to exist, we have to consider non-destructive interactions and this imply that we have to look for similar frequencies between the various groups. As a consequence, we assume that the solutions have the following form:

$$\Delta\boldsymbol{\omega}^{-1}\left(Z_{\mathbf{a}},\theta\right) = \Delta\boldsymbol{\omega}^{-1}\left(Z_{\mathbf{a}}\right)\exp\left(i\Upsilon\theta\right)$$

which implies a solution for :

$$\Delta\boldsymbol{\omega}^{-1}\left(Z_{\mathbf{a}}\right) = M\Delta\boldsymbol{\omega}^{-1}\left(Z_{\mathbf{a}}\right)$$

with:

$$M_{(ia_i),(jb_j)} = \hat{T}_{ij}\left(Z_{a_i},Z_{b_j}\right)\exp\left(-i\Upsilon\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}\right)$$

$$M = \left(M_{(ia_i),(jb_j)}\right) + \left(M_{(ia_i),(jb_j)}\right)_{i\neq j}$$

Without interactions, the frequencies are in some state $\gamma_{i,l_i}$ satisfying:

$$f_{i,l_i}\left(\gamma_{i,l_i}\right) = 1$$

and the solution for the vector $\Upsilon$ depends on these quantities:

$$\Upsilon_{(i,l_i)} = \sum_{i,j\neq i}[[M]]_{i,j}\frac{\gamma_{i,k_i}+\gamma_{j,l_j}}{2} \pm \sqrt{\left(\sum_{i,j\neq i}\frac{[[M]]_{i,j}}{\sum_{i,j\neq i}[[M]]_{i,j}}\frac{\gamma_{i,k_i}-\gamma_{j,l_j}}{2}\right)^2 + \sum_{i,j\neq i}[[M]]_{i,j}} \tag{41}$$

The notation $\Upsilon_{(i,l_i)}$ encapsulates that the resulting frequency for the new structure depends on the states $\left(\gamma_{i,l_i}\right)$ of initial ones. The expression $[[M]]_{i,j}$:

$$[[M]]_{i,j} = \frac{\left[\hat{M}_{j,i}\right]_{l_j,k_i}\left[\hat{M}_{i,j}\right]_{k_i,l_j}}{\left(\frac{\partial}{\partial\gamma}f_{j,l_j}\left(\gamma\right)\right)_{\gamma_{j,l_j}}\left(\frac{\partial}{\partial\gamma}f_{i,k_i}\left(\gamma\right)\right)_{\gamma_{i,k_i}}}$$

depends on the matrix $\hat{M}_{j,i}$ obtained from $M_{ij}$ by a change of basis (see appendix 4).

### 6.4.3 Full activity for composed collective states

Ultimately, the possible activities for a composed state are obtained by gathering static and non-static parts. The modified activity for component $i$

$$\overline{\Delta\omega}_{i0}^{-1}\left(Z_{a_i}\right) + \Delta\omega_i^{-1}\left(Z_{a_i}\right)\exp\left(i\Upsilon_{\{i,l_i\}}\theta\right)$$

wher $\Upsilon_{(i,l_i)}$ is defined in (41).

# 7 Interactions and transitions between collective states

We can now explore the possibilities for several collective states to potentially undergo transitions into other collective states. We have given the descriptions of individual and mixed collective states, defined by connectivities and their associated activities. We may, therefore, consider transitions in which two states can merge to form a third state with its own activities. This transition should depend on the characteristics of each state. Dynamically, the interaction modifies the activities of the initial states, inducing a dynamic process that may converge towards the merged state. The initial states are minima of some action $S_i$, meaning the collection of initial states minimizes $\sum S_i$ without including interactions. The final states minimize the action, including certain interactions $\sum S_i + \sum S_{i,j}$.

This section develops the mechanism of transitions and provides some elements that we will incorporate into a more general field formalism where transitions will result from interaction terms between fields of structures.

## 7.1 Dynamic evolution of connectivity

We want to describe the transition between states wth $S_i$, $T_{ij}$, towards states with $T_{ij}$. We first describe the dynamic of the connectivity $T_{ij}$ of a state. Assuming that starting with $\Delta T_{ij}(Z,Z') = \Delta\hat{T}_{ij}(Z,Z') = 0$, some connectivity above the threshold arises. The term inside the gradient in (33) yields the evolution of a single connectivity. This corresponds to a modification:

$$\begin{aligned}&\frac{d}{dt}\Delta\hat{T}_{ij}(Z,Z')\\=\;&\frac{\rho}{\omega_{0i}^2(Z)}\frac{D(\theta)\left\langle\Delta\hat{T}_{ij}(Z,Z')\right\rangle|\Psi_{0j}(Z')|^2\left(\omega_{0i}(Z)\Delta\omega_j\left(\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)-\omega_{0j}(Z')\Delta\omega_i\left(\theta,Z,|\Psi|^2\right)\right)}{C(\theta)|\Psi_{0i}(Z)|^2\omega_{0i}(Z)+D(\theta)\left\langle\Delta\hat{T}_{ij}\right\rangle|\Psi_{0j}(Z')|^2\omega_{0j}(Z')}\end{aligned}$$

Given that the connectivities fluctuate arounfd their average value, we can focus on the dynamics of:

$$\left\langle\Delta\hat{T}_{ij}(Z,Z')\right\rangle$$

which becomes:

$$\begin{aligned}&\frac{d}{dt}\left\langle\Delta\hat{T}_{ij}(Z,Z')\right\rangle\\=\;&\frac{\rho}{\omega_{0i}^2(Z)}\frac{D(\theta)\left\langle\Delta\hat{T}_{ij}(Z,Z')\right\rangle|\Psi_{0j}(Z')|^2\left(\omega_{0i}(Z)\Delta\omega_j\left(\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)-\omega_{0j}(Z')\Delta\omega_i\left(\theta,Z,|\Psi|^2\right)\right)}{C(\theta)|\Psi_{0i}(Z)|^2\omega_{0i}(Z)+D(\theta)\left\langle\Delta\hat{T}_{ij}\right\rangle|\Psi_{0j}(Z')|^2\omega_{0j}(Z')}\end{aligned}$$

using:

$$\left\langle\Delta\hat{T}_{ij}(Z,Z')\right\rangle\simeq\frac{\langle\Delta T_{ij}(Z,Z')\rangle}{\lambda}$$

this becomes a dynamics for:

$$\begin{aligned}&\frac{d}{dt}\langle\Delta T_{ij}(Z,Z')\rangle\\=\;&\frac{\lambda\rho}{\omega_{0i}^2(Z)}\frac{D(\theta)\langle\Delta T_{ij}(Z,Z')\rangle|\Psi_{0j}(Z')|^2\left(\omega_{0i}(Z)\Delta\omega_j\left(\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)-\omega_{0j}(Z')\Delta\omega_i\left(\theta,Z,|\Psi|^2\right)\right)}{\lambda\left(C(\theta)|\Psi_{0i}(Z)|^2\omega_{0i}(Z)+D(\theta)\langle\Delta T_{ij}(Z,Z')\rangle|\Psi_{0j}(Z')|^2\omega_{0j}(Z')\right)}\end{aligned}$$

This is associated to the dynamics for activities, obtained through mdfctn (196) at the lowest order for $Z$ and its closest point $Z'$ (assuming that $Z$ is also the closest point of $Z'$):

$$\frac{d}{dt}\omega_i(Z)=\sum_{j,b_j}G'\left(G^{-1}\left(\omega_{0i}(Z)\right)\right)\left(\frac{\kappa}{N}\frac{G^{ij}\omega_{0j}(Z')\,\Delta T_{ij}(Z,Z')}{\omega_{0i}(Z)}\left(\mathcal{G}_{j0}+\left|\Psi_j(Z')\right|^2\right)\right)_{j\neq i}\omega_{0j}(Z')$$

This yields system with four dynamic variables:

$$\langle\Delta T_{ij}(Z,Z')\rangle,\langle\Delta T_{ji}(Z,Z')\rangle,\omega_i(Z),\omega_j(Z')$$

whose matrix writes:

$$A=\begin{pmatrix}0&0&A_{i,j}&B_{i,j}\\0&0&B_{j,i}&A_{j,i}\\C_{i,j}&0&0&0\\0&C_{j,i}&0&0\end{pmatrix}\tag{42}$$

the eigenvalues of (42) are:

$$\pm\frac{1}{\sqrt{2}}\sqrt{A_{i,j}C_{i,j}+A_{j,i}C_{j,i}+\sqrt{\left(A_{i,j}C_{i,j}-A_{j,i}C_{j,i}\right)^2+4C_{i,j}C_{j,i}B_{j,i}B_{i,j}}}$$

and:

$$\pm\frac{1}{\sqrt{2}}\sqrt{A_{i,j}C_{i,j}+A_{j,i}C_{j,i}-\sqrt{\left(A_{i,j}C_{i,j}-A_{j,i}C_{j,i}\right)^2+4C_{i,j}C_{j,i}B_{j,i}B_{i,j}}}$$

In the case of enhancing interaction, the dominant eigenvalue:

$$\frac{1}{\sqrt{2}}\sqrt{A_{i,j}C_{i,j}+A_{j,i}C_{j,i}+\sqrt{\left(A_{i,j}C_{i,j}-A_{j,i}C_{j,i}\right)^2+4C_{i,j}C_{j,i}B_{j,i}B_{i,j}}}$$

is real. As a consequence, the system depart from its initial value:

$$\langle\Delta T_{ij}(Z,Z')\rangle=0,\langle\Delta T_{ji}(Z,Z')\rangle=0,\omega_{i0}(Z),\omega_{j0}(Z')$$

and move towards an other equilibrium, given by the previous derivations. In the case of inhibitory interactions, the system oscillates, but starts from:

$$\langle\Delta T_{ij}(Z,Z')\rangle=0,\langle\Delta T_{ji}(Z,Z')\rangle=0$$

and grows slowly toward a new equilibrium if the amplitude of oscillations is large enough.

## 7.2 Dynamic transition for activities

While the $\Delta T_{ij}(Z,Z')$ are evolving towards their new values, the activities depending on these values evolve also towards their new equilibrium values. Temporarily, the term:

$$\omega_{0i}(Z)\,\Delta\omega_j\left(\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)-\omega_{0j}(Z')\,\Delta\omega_i\left(\theta,Z,|\Psi|^2\right)$$

presents some interferences, but these ones may initiate the dynamics for $\Delta T_{ij}(Z,Z')$ by allowing $\Delta\hat{T}_{ij}$ to overcome the threshold for the connectivity dynamics when $T_{ij}(Z,Z')<\delta$:

$$\begin{aligned}&\frac{d}{dt}\Delta\hat{T}_{ij}(Z,Z')\\=\;&\frac{\rho}{\omega_{0i}^2(Z)}\frac{D(\theta)\left\langle\Delta\hat{T}_{ij}(Z,Z')\right\rangle\left|\Psi_{0j}(Z')\right|^2\left(\omega_{0i}(Z)\,\Delta\omega_j\left(\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)-\omega_{0j}(Z')\,\Delta\omega_i\left(\theta,Z,|\Psi|^2\right)\right)}{C(\theta)\left|\Psi_{0i}(Z)\right|^2\omega_{0i}(Z)+D(\theta)\left\langle\Delta\hat{T}_{ij}\right\rangle\left|\Psi_{0j}(Z')\right|^2\omega_{0j}(Z')}\\&-\eta H\left(\delta-T_{ij}(Z,Z')\right)\end{aligned}$$

The convergence towards the new equilibrium should be ensured by the fact that the system's backgrnd minimize an action functional.

### 7.3 Transitions betweenn states

In dynmics perspective, we can formally describe the previous transition as a states transition between several states of different types:

$$\prod_i \left|\mathbf{a}_i, \mathbf{p}_i, S_i^2\right\rangle$$

where, as before, the states $\left|\mathbf{a}_i, \mathbf{p}_i, S_i^2\right\rangle$ are defined by the state of each of its components:

$$\left|\mathbf{a}_i, \mathbf{p}_i, S_i^2\right\rangle = \prod_{Z_{a_i}, Z_{a'_i}} \left|\Delta T_{ii}\left(Z_{a_i}, Z_{a'_i}\right), \Delta\hat{T}_{ii}\left(Z_{a_i}, Z_{a'_i}\right), \alpha\left(Z_{a_i}, Z_{a'_i}\right), p\left(Z_{a_i}, Z_{a'_i}\right)\right\rangle$$

and a composed state:

$$\begin{aligned}&\left|\mathbf{a}_{i'}, \mathbf{p}_{i'}, (\cup S_i) \times (\cup S_i)\right\rangle \\ =& \prod_{a_i, b_j} \prod_{Z_{a_i}, Z_{b_j}} \left|\Delta T_{ij}\left(Z_{a_i}, Z_{b_j}\right), \Delta\hat{T}_{ij}\left(Z_{a_i}, Z_{b_j}\right), \alpha\left(Z_{a_i}, Z_{b_j}\right), p\left(Z_{a_i}, Z_{b_j}\right)\right\rangle\end{aligned}$$

where indices $i' \in (\cup S_i) \times (\cup S_i)$. The previous sections show that has to happen so that (41) linking activities between non interacting stats and composed one is satisfied. We thus may expect some transitions of the form:

$$\prod_i \left|\mathbf{a}_i, \mathbf{p}_i, S_i^2\right\rangle \rightarrow f\left(\left(\Upsilon_{\mathbf{p}_i}^{\mathbf{a}_i}\right), \Upsilon_{\mathbf{p}'_i}^{\mathbf{a}'_i}\right) \left|\mathbf{a}_{i'}, \mathbf{p}_{i'}, (\cup S_i) \times (\cup S_i)\right\rangle \tag{43}$$

where $f\left(\left(\Upsilon_{\mathbf{p}_i}^{\mathbf{a}_i}\right), \Upsilon_{\mathbf{p}'_i}^{\mathbf{a}'_i}\right)$ is a function close to the Dirac function:

$$\delta\left(\Upsilon_{(i,l_i)} - \left\{\sum_{i,j\neq i} [[M]]_{i,j} \frac{\gamma_{i,k_i} + \gamma_{j,l_j}}{2} \pm \sqrt{\left(\sum_{i,j\neq i} \frac{[[M]]_{i,j}}{\sum_{i,j\neq i} [[M]]_{i,j}} \frac{\gamma_{i,k_i} - \gamma_{j,l_j}}{2}\right)^2 + \sum_{i,j\neq i} [[M]]_{i,j}}\right\}\right)$$

to impose (41) up to some fluctuations. This will be used in the field formalism: transition may occur when the frequencies of the initial states activities and those of the final ones satisfy some consistency conditions.

# II Field formalism for multiple interacting collective states

So far, we have described interactions and activity within a given collective state and considered the transition mechanism between two states and a mixed one resulting from combined signals between the two initial ones. This chapter aims to systematize the interactions and transitions between large sets of collective states.

To do so, we develop a formalism for assemblies of collective states. This formalism extends the previous description developed in the first three parts of this work, as the variables involved in the model will directly represent sets of interacting cells rather than individual cells themselves. The number of degrees of freedom are thus much larger than in our previous field formalism. After rewriting the collective stats in a suitable form for our formalism, we develop the field description of an assembly of collective stat. Interactions are introduced to account for the mechanisms of transition described in the previous section.

# 8 Effective field formalism for assemblies

Developing a field formalism for assemblies of collectiv states $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ amounts to consider arbitrary numbers of such states along with the possibilities of transitions between them. Thus, we first need to consider products of such states. To explore the dynamic transitions between these states, we also need to consider any "intermediate states" in the transition between stable states. In other words, we must account for any possible collective states characterized by activated connectivity between an arbitrary set of cells. This is performed in several steps.

First of all, as demonstrated in equations (25) and (26), the states $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ are non-local states characterized by probability density functions for the connectivities of the activated state. Therefore, we first define states $\left|\Delta\mathbf{T}_p^{\alpha},\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ with given values of connectivities $\Delta\mathbf{T}_p^{\alpha}$. These states serve as a basis for defining arbitrary collective states through linear combinations:

$$\left|\underline{\gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\right\rangle=\int\underline{\gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\left|\Delta\mathbf{T}_p^{\alpha},\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle d\mathbf{T} \tag{44}$$

These states encompass all possible collective states, including unstable ones. However, each of these states describes a single collective state. To transition to a representation where multiple states are involved and interact, we replace the components $\underline{\gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$ with fields $\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$ that account for the possibility of multiple states.

These fields encompass all possible realizations of the components $\underline{\gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$. The dynamic aspects of the states are governed by the field action functional $S\left(\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)$. This functional is derived by initially formulating the action for the individual states $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$, and then extending it to encompass their combinations (44), leading to an action $S\left(\underline{\gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)$ for the components. By replacing these components $\underline{\gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$.with the field of which they are realizations, we obtain a field action functional $S\left(\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)$ which accounts for the possibility of multiple states. Note that doing so, the system as a whole is described by an infinite number of fields $\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$, each of them characerized by a set $\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)$ describing average connectivities, activations and their frequencies, along with its action.

This substitution of components by fields $\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$, which are random variables, is equivalent to describing the system's dynamics through a partition function for the fields $\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$. This partition function is determined by the exponentiated sum of action functionals, with integration over the infinite number of fields $\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$. In this context, interactions between states are modeled through modifications of this action functional. These modifications involve various different fields, and dynamically induce transitions between states associated with these fields. The amplitudes of these transitions are derived by computing Green functions between such states.

Alternatively, the field description of the system is equivalent to an operator-based perspective. In this approach, the fields $\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$ can be interpreted as operators:

$$\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)=\underline{\Gamma}^{+}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)+\underline{\Gamma}^{-}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right) \tag{45}$$

The field $\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$ is decomposed into a sum of creation operator $\underline{\Gamma}^{+}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$ and destruction operator $\underline{\Gamma}^{-}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$ which act on an abstract multiple-state space, constructed by iteratively applying creation operators on a vacuum state, representing the absence of any activated state. This extension of the formalism allows us to consider tensor products of individual states through the action of field operators on the vacuum state. In this approach, acting on such products, operator ,$\underline{\Gamma}^{+}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)$ creates an additional structures with characteristics $\left(\boldsymbol{\alpha},\mathbf{p},S^{2}\right)$ while $\underline{\Gamma}^{-}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)$ deactivates states with such characteristics, enabling transitions between different states.

The same description can be obtained directly through a dual representation, which involves considering the creation and destruction operators $\mathbf{A}^{+}\left(\alpha,p,S^{2}\right)$ and $\mathbf{A}^{-}\left(\alpha,p,S^{2}\right)$ associated with non-local states $\left|\boldsymbol{\alpha},\mathbf{p},S^{2}\right\rangle$. These operators are directly derived from the saddle point equations governing the states $\left|\boldsymbol{\alpha},\mathbf{p},S^{2}\right\rangle$. Furthermore, the field operators $\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)$ (45) are themselve derived from $\mathbf{A}^{+}\left(\alpha,p,S^{2}\right)$ and $\mathbf{A}^{-}\left(\alpha,p,S^{2}\right)$. The space of multiple states is generated by the successive application of this family of operators on the vacuum state. For instance, a specific state with given connectivity values $\left|\Delta\mathbf{T}_{p}^{\alpha},\boldsymbol{\alpha},\mathbf{p},S^{2}\right\rangle$ emerges as an eigenstate of a position operator $\Delta\mathbf{T}_{p}^{\alpha}$[4], which combines the creation and annihilation operators.

In this approach, the interaction terms in the action functional depend on the $\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)$ and are expressed as series of products of creation and destruction operators Thus, interactions can modify collective states or replace one state with another. As a consequence, they can induce transitions between states

At this point, it's important to note that the field description utilized here operates within a much broader space compared to the formalism presented in the first three parts of this work. In those previous articles, the field defining the connectivity functions $\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$ described the state of connectivity between two cells at locations $(Z,Z')$. However, in the current approach, there exists a separate field for each type of potential collective states.

Each of these fields depends on a substantial number of variables, as the collective states involve a multitude of points along with potential values for their associated connectivities, activities frequencies and amplitudes. Our description encompasses an infinite number of these variables, although, due to the action functional under consideration and initial conditions, only certain types of fields may become effective. Nevertheless, this formalism prevents imposing overly restrictive constraints on the emergence of collective states. From this perspective, there shouldn't be a fixed "repertoire" of possible states, even if it may appear that way in practice.

Ultimately, it should be noted that the field formalism considered here implies that several collective states may be activated multiple times or in various different configurations. This pertains to states with enhanced connectivity and their involvement in several interaction processes. This description aligns with our previous works in which individual cells can be perceived as complex entities participating in multiple interactions and assemblies.

## 8.1 Collective states as basic elements

This paragraph summarizes and systematizes the results of the previous chapter. The description of cell states is formulated in a manner that allows for the modeling of multiple states. Each state is characterized by its spatial extent, connectivities, and activities. These variables are encapsulated in several vector parameters.

[4] This operator will be defined in the next section.

### 8.1.1 Description of collective states

Consider assemblies defined as set of points obtained by shifts in background connectivities. To each assembly:

$$S = \{Z_1, ..., Z_n\}$$

we associate the field:

$$\Delta\Gamma\left(\left(T_{ij}, \hat{T}_{ij}, Z_i, Z_j\right)_{Z_i, Z_j \in S}, \theta\right)$$

To account for activation of possible subpattern independently from an initial collective state, we consider a sequence of independent fields:

$$\left\{\Delta\Gamma_\beta\left(\left(T_{ij}, \hat{T}_{ij}, Z_i^\beta, Z_j^\beta\right)_{Z_i^\beta, Z_j^\beta \in S_\beta}, \theta\right)\right\}_{S_\beta \subset S}$$

associated to $\Delta\Gamma$. The link between a pattern and the subpattern will be described by some interactions terms. Our choice implies that both a pattern and several subpatterns may be activated simltaneously, implying the existence, for one neuron, of a multiplicity of connections involving a cell in several patterns.

In the sequel, we will write:

$$S^2 = S \times S$$

with points:

$$Z_{ij} = (Z_i, Z_j)$$

to account for the fact that connectivities in a set $S$ are described by set of doublets $Z_{ij}$.

We will also introduce a change of notation compared to the previous sections. In those sections, collective states were described with an "extra" connectivity vector $\left(\mathbf{\Delta T}, \Delta\hat{\mathbf{T}}\right)$ compared to background field values. To simplify notations and emphasize our focus on the structures themselves, we will use the notation $\mathbf{T}$ and $\hat{\mathbf{T}}$ for the connectivity variables, or $\mathbf{T}_{S^2}$ and $\hat{\mathbf{T}}_{S^2}$ when the spatial extension needs to be specified. The notation $\Delta\mathbf{T}$, with superscripts or subscripts, will be retained to indicate deviations in connectivity from their average values: $\Delta\mathbf{T} = \mathbf{T} - \langle\mathbf{T}\rangle$. Similarly, we rewrite the "extra" activity along the structure $\Delta\omega(Z, \mathbf{\Delta T})$ as $\omega(Z, \mathbf{\Delta T})$.

Given our previous results a state will be parametrized by:

$$\left(S^2, \langle\mathbf{T}(\mathbf{Z})\rangle^\alpha, \Upsilon_p^\alpha, \omega(Z, \mathbf{T})\right) \tag{46}$$

where $\langle\mathbf{T}(\mathbf{Z})\rangle^\alpha$ is one of the possible solution of (22) for the average connectivities, and $\Upsilon_p^\alpha$ one of the associated frequencies for the activities:

$$\omega_p^\alpha(\theta, Z, \mathbf{\Delta T}) = \omega(Z, \mathbf{\Delta T}) + \left(\mathbf{N}_p^\alpha\right)^{-1} \boldsymbol{\omega}_0 \exp\left(-i\Upsilon_p \frac{|\Delta\mathbf{Z}_i|}{c}\right)$$

with:

$$\left[\mathbf{N}_p^\alpha\right]_{(Z_i, Z_j)} = \left(\delta_{ij} - [\mathbf{\Delta T}]_{(Z_i, Z_j)} \exp\left(-i\Upsilon_p \frac{|Z_i - Z_j|}{c}\right)\right)^{-1}$$

The states we are considering have the form:

$$\prod_{Z,Z'} \left|\Delta T(Z, Z'), \Delta\hat{T}(Z, Z'), \alpha(Z, Z'), p(Z, Z')\right\rangle \equiv \left|\boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle \tag{47}$$

The states $\left|\boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle$ and their conjugates $\left\langle\boldsymbol{\alpha}, \mathbf{p}, S^2\right|$ have the form (25) and (26) in absence of interactions.and, or, transitions.

### 8.1.2 Field action functional for states

Having determined the averages for connectivities and activities, we can reformulate the action for the collective states. We consider the same action as (13), but summing only over points $(Z, Z') \in S^2$ belonging to the collective state with spatial extension $S^2$:

$$S(\Delta\Gamma) = \sum_{(Z,Z')\in S^2} S\left(\Delta\Gamma\left(T, \hat{T}, \theta, Z, Z'\right)\right) \tag{48}$$

While considering directly collective states on their own, we saw that the correspondent action for such state is:

$$\begin{aligned} &\hat{S}\left(\left|\boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle\right) \\ =\ &\left\langle \boldsymbol{\alpha}, \mathbf{p}, S^2\right| \left(-\nabla_T^2 - \nabla_{\hat{T}}^2 + \frac{1}{2}\left(\mathbf{\Delta T} - \langle \mathbf{\Delta T}\rangle_p^\alpha\right)^t \mathbf{A}_p^\alpha \left(\mathbf{\Delta T} - \langle \mathbf{\Delta T}\rangle_p^\alpha\right) + \mathbf{C}\right) \left|\boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle \end{aligned} \tag{49}$$

wth:

$$\prod_{Z,Z'} \left|\Delta T(Z,Z'), \Delta\hat{T}(Z,Z'), \alpha(Z,Z'), p(Z,Z')\right\rangle \equiv \left|\mathbf{\Delta T}, \Delta\hat{\mathbf{T}}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle \tag{50}$$

and:

$$\left\langle \boldsymbol{\alpha}, \mathbf{p}, S^2\right| = \left|\boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle^\dagger$$

In (49), we have defined:

$$\mathbf{\Delta T} - \langle \mathbf{\Delta T}\rangle_p^\alpha = \begin{pmatrix} \Delta T - \langle \Delta T\rangle \\ \Delta\hat{T} - \left\langle \Delta\hat{T}\right\rangle \end{pmatrix} \tag{51}$$

and the matrix $\mathbf{A}_p^\alpha$ is derived in appendix 3:

$$\mathbf{A}_p^\alpha = \begin{pmatrix} \left(\frac{1}{\tau\omega_0(Z)}\right)^2 + \left(\mathbf{M}^\alpha(Z,Z')\right)^2 & -\lambda\left(\frac{1}{\tau\omega_0(Z)}\right)^2 + D(Z,Z')\,\mathbf{M}^\alpha(Z,Z') \\ -\lambda\left(\frac{1}{\tau\omega_0(Z)}\right)^2 + D(Z,Z')\,\mathbf{M}^\alpha(Z,Z') & \left(\frac{\lambda}{\tau\omega_0(Z)}\right)^2 + D^2(Z,Z') \end{pmatrix}$$

The matrix $\mathbf{D}$ is diagonal with element:

$$\mathbf{D}(Z,Z') = D\left[\frac{\rho\left(C(\theta)\left|\Psi_0(Z)\right|^2\omega_0(Z) + D(\theta)\hat{T}\left|\Psi_0(Z')\right|^2\omega_0(Z')\right)}{\omega_0(Z)}\right]$$

and:

$$\mathbf{M}^\alpha(Z,Z') = \frac{\rho}{\omega_0(Z)}\left(D(\theta)\left\langle\hat{T}\right\rangle\left|\Psi_0(Z')\right|^2 A\left|Z-Z'\right|\left(\nabla_{\mathbf{\Delta T}_{(Z_1,Z_1')}}\left(\Delta\omega\left(Z, \langle\mathbf{\Delta T}\rangle\right)\right)_{\left(\left\langle \mathbf{\Delta T}_{(Z_1,Z_1')}\right\rangle^\alpha\right)}\right)\right)$$

The vector $\mathbf{C}(Z,Z')$ is defined by:

$$\mathbf{C}(Z,Z') = \frac{\tau\omega_0(Z)}{2} + \frac{\rho\left(C(\theta)\left|\Psi_0(Z)\right|^2\omega_0(Z) + D(\theta)\left|\Psi_0(Z')\right|^2\omega_0(Z')\right)}{2\omega_0(Z)} \tag{52}$$

Ultimately, given our hypothesis that:

$$\left\|\Delta\hat{\mathbf{T}} - \left\langle\Delta\hat{\mathbf{T}}\right\rangle\right\| << \left\|\mathbf{\Delta T} - \langle\mathbf{\Delta T}\rangle^\alpha\right\|$$

the field action simplifies:

$$\begin{aligned}&\hat{S}\left(\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\right) \qquad\qquad \text{(FCT)}\\ =\quad& \left\langle\boldsymbol{\alpha},\mathbf{p},S^2\right|\left(-\nabla_T^2+\frac{1}{2}\left(\mathbf{\Delta T}-\left\langle\mathbf{\Delta T}\right\rangle_p^\alpha\right)^t\mathbf{A}_p^\alpha\left(\mathbf{\Delta T}-\left\langle\mathbf{\Delta T}\right\rangle_p^\alpha\right)+\mathbf{C}\right)\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle+U\left(\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\right)\end{aligned}$$

The matrix $\mathbf{A}^\alpha$ is defined by:

$$\mathbf{A}^\alpha=\sqrt{\mathbf{D}^2+\left(\mathbf{M}^\alpha\right)^t\mathbf{M}^\alpha}$$

## 8.2 From states to fields

Starting from the collective states described in the previous paragraph, we describe general collective states, i.e., states that are not inherently stable. Considering such states is necessary to study transitions.

To do so, we rewrite the states $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ as eigenstates of a certain differential operator. This operator is itself built from some creation and annihilation operators. Creation and annihilation operatrs are the fundamental components used to describe the assembly of states. The creation operators, acting on a vacuum state, provide the product of states we are looking for.

Moreover, a combination of these creation and annihilation operators allows to define collective states that are peaked at some fixed values of connectivty function. These localized states are a suitable basis to define general collective states. The state space of the system is thus generated by tensor products of such spaces. The system is then described by a field, a random variable whose realization are the arbitrary states defined before. We derive a field action functionl that encompass the dynamics and transition between states.

### 8.2.1 Collective states and operators

To introduce products of states, we use that the states $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ are eigenstates of some operators. Actually, let us consider the saddle point equation for $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$:

$$0=\left(-\nabla_T^2+\frac{1}{2}\left(\mathbf{\Delta T}-\left\langle\mathbf{\Delta T}\right\rangle_p^\alpha\right)^t\mathbf{A}_p^\alpha\left(\mathbf{\Delta T}-\left\langle\mathbf{\Delta T}\right\rangle_p^\alpha\right)+\mathbf{C}\right)\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle+U'\left(\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\right)\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$$

In first approximation, for a slowly varying $U'\left(\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\right)$, we can replace[5]:

$$\mathbf{C}+U'\left(\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\right)\rightarrow\mathbf{C}$$

so that $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ are eigenvectors of:

$$-\frac{1}{2}\nabla_{(\mathbf{T})_{S^2}}^2+\frac{1}{2}\left(\mathbf{T}_{S^2}-\left\langle\mathbf{T}_p^\alpha\right\rangle_{S^2}\right)^t\mathbf{A}_{S^2}^\alpha\left(\mathbf{T}_{S^2}-\left\langle\mathbf{T}_p^\alpha\right\rangle_{S^2}\right)=-\frac{1}{2}\nabla_{(\mathbf{T})_{S^2}}^2+\frac{1}{2}\left(\Delta\mathbf{T}_p^\alpha\right)_{S^2}^t\mathbf{A}_{S^2}^\alpha\left(\Delta\mathbf{T}_p^\alpha\right) \tag{53}$$

where:

$$\left(\Delta\mathbf{T}_p^\alpha\right)_{S^2}=\mathbf{T}_{S^2}-\left\langle\mathbf{T}_p^\alpha\right\rangle_{S^2}$$

To describe further the collective states, we assume that the operator:

$$-\frac{1}{2}\nabla_{(\mathbf{T})_{S^2}}^2+\frac{1}{2}\left(\Delta\mathbf{T}_p^\alpha\right)_{S^2}^t\mathbf{A}_{S^2}^\alpha\left(\Delta\mathbf{T}_p^\alpha\right)$$

can be diagonalized as:

$$-\frac{1}{2}\nabla_{(\bar{\mathbf{T}})_{S^2}}^2+\frac{1}{2}\left(\Delta\bar{\mathbf{T}}_p^\alpha\right)_{S^2}^t\bar{\mathbf{D}}_{S^2}^\alpha\left(\Delta\bar{\mathbf{T}}_p^\alpha\right)_{S^2} \tag{54}$$

[5] the constant $C$ has been defined in (52).

with:

$$\begin{aligned}\bar{\mathbf{D}}_{S^2}^{\alpha} &= O^{-1}\left(\mathbf{A}_{S^2}^{\alpha}\right)O\\ \left(\Delta\bar{\mathbf{T}}_p^{\alpha}\right)_{S^2} &= O^{-1}\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}\end{aligned}$$

The possible eigenvalues of (54) associated to $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ are determined simulaneously with the norm of $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ (see ([8])). Actually, in first approximation, the norm of $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ is given by the minimum of $U\left(\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\right)$. However, imposing that the norm of $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ is bounded, implies a condition on the eigenvalue (54). These eigenvalues have the form:

$$\sum_i\left(\sqrt{\bar{\mathbf{D}}_{S^2}^{\alpha}}\right)_i\left(l_i+\frac{1}{2}\right)$$

where the $\left(\sqrt{\bar{\mathbf{D}}_{S^2}^{\alpha}}\right)_i$ are the components of $\sqrt{\bar{\mathbf{D}}_{S^2}^{\alpha}}$ and $l_i$ is a sequence of integer. This imposes a condition yielding the norm of $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ (see also below the condition for collective states).

### 8.2.2 Creation and annihilation operators

We can now introduce the creation and annihilation operators relevant to the system. To do so, we use that operator (54) writes also:

$$\begin{aligned}-\frac{1}{2}\nabla_{(\bar{\mathbf{T}})_{S^2}}^2+\frac{1}{2}\left(\Delta\bar{\mathbf{T}}_p^{\alpha}\right)_{S^2}^t\bar{\mathbf{D}}_{S^2}^{\alpha}\left(\Delta\bar{\mathbf{T}}_p^{\alpha}\right)_{S^2} &= \bar{\mathbf{D}}_{S^2}^{\alpha}\left(\mathbf{A}^{+}\left(\alpha,p,S^2\right)\mathbf{A}^{-}\left(\alpha,p,S^2\right)+\frac{1}{2}\right)\\ &= \left(\bar{\mathbf{D}}_{S^2}^{\alpha}\right)_i\left(\mathbf{A}_i^{+}\left(\alpha,p,S^2\right)\mathbf{A}_i^{-}\left(\alpha,p,S^2\right)+\frac{1}{2}\right)\end{aligned}\tag{55}$$

where:

$$\begin{aligned}\mathbf{A}^{-}\left(\alpha,p,S^2\right) &= \frac{1}{2}\left(\sqrt{\bar{\mathbf{D}}_{S^2}^{\alpha}}\Delta\bar{\mathbf{T}}_p^{\alpha}-\frac{1}{\sqrt{\bar{\mathbf{D}}_{S^2}^{\alpha}}}\nabla_{(\bar{\mathbf{T}})_{S^2}}^2\right)\\ \mathbf{A}^{+}\left(\alpha,p,S^2\right) &= \frac{1}{2}\left(\sqrt{\bar{\mathbf{D}}_{S^2}^{\alpha}}\Delta\bar{\mathbf{T}}_p^{\alpha}+\frac{1}{\sqrt{\bar{\mathbf{D}}_{S^2}^{\alpha}}}\nabla_{(\bar{\mathbf{T}})_{S^2}}^2\right)\end{aligned}\tag{56}$$

and $\mathbf{A}_i^{\pm}\left(\alpha,p,S^2\right)$ and $\left(\bar{\mathbf{D}}_{S^2}^{\alpha}\right)_i$ are cmpnnts of $\mathbf{A}^{\pm}\left(\alpha,p,S^2\right)$ and $\bar{\mathbf{D}}_{S^2}^{\alpha}$ respectively.

The operators $\mathbf{A}^{\pm}\left(\alpha,p,S^2\right)$ satisfy the following commutation relation:

$$\left[\mathbf{A}_i^{-}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right),\mathbf{A}_j^{+}\left(\boldsymbol{\alpha}',\mathbf{p}',S'^2\right)\right]=\delta\left(\mathbf{p}-\mathbf{p}'\right)\delta\left(\boldsymbol{\alpha}-\boldsymbol{\alpha}'\right)\delta\left(S^2-S'^2\right)\delta_{i,j}$$

The states $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ are obtained by successive applications of the components $\mathbf{A}_i^{+}\left(\alpha,p,S^2\right)$ of $\mathbf{A}^{+}\left(\alpha,p,S^2\right)$ on the minimum:

$$\prod_i\left(\mathbf{A}_i^{+}\left(\alpha,p,S^2\right)\right)^{l_i}|vac\rangle=\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\tag{57}$$

The eigenvalue of (54), or which is equivalent, of (55) associated to such state is $\sum_i\left(\sqrt{\bar{\mathbf{D}}_{S^2}^{\alpha}}\right)_i\left(l_i+\frac{1}{2}\right)$

For large number of elementsnote that states $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ are mutually orthogonl:

$$\left\langle\boldsymbol{\alpha},\mathbf{p},S^2\right|\left|\left(\boldsymbol{\alpha}',\mathbf{p}',S^2\right)\right\rangle\simeq\exp\left(-\sum\left(\langle\Delta\mathbf{T}\rangle_p^{\alpha}-\langle\Delta\mathbf{T}\rangle_{p'}^{\alpha'}\right)^2\right)\simeq 0$$

### 8.2.3 Connectivity states and operator

Recall that states $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ are characterized by connectivities at each point $(Z,Z')$ being distributed around an average value (see (6)), modeling that connections between axons and dendrites are defined by a range of values.

However, to calculate transitions between states, it is necessary to consider states with specific values $\Delta\mathbf{T}_p^\alpha$ or its diagonalized form $\Delta\bar{\mathbf{T}}_p^\alpha$. These states differ from $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ and are not stable, instead, they represent all possible connectivity states at a given time. These form the basis for transient states that enable the description of dynamic transitions. Indeed, connectivity values fluctuate due to interactions, and these fluctuations drive state changes.

To derive the form of these states, we assume that for a large number of states most of them are in fundamental states and we consider:

$$\left|\Delta\bar{\mathbf{T}}_p^\alpha,\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle=\prod\left|\Delta\bar{T}_p^\alpha,\alpha,p,S^2\right\rangle$$

that are eigenstate of operator:

$$\Delta\bar{\mathbf{T}}_p^\alpha=\sqrt{\bar{\mathbf{D}}_{S^2}}\left(\mathbf{A}^-\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)+\mathbf{A}^+\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)\tag{58}$$

We also define::

$$\left|\bar{\mathbf{T}}_p^\alpha,\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\equiv\left|\Delta\bar{\mathbf{T}}_p^\alpha,\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$$

that are the same states, seen as eigenstate of:

$$\bar{\mathbf{T}}_{\mathbf{p}}^{\boldsymbol{\alpha}}\equiv\sqrt{\bar{\mathbf{D}}_{S^2}}\left(\mathbf{A}^-\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)+\mathbf{A}^+\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)+\left\langle\bar{\mathbf{T}}_p^\alpha\right\rangle\tag{59}$$

States $\left|\Delta\bar{\mathbf{T}}_p^\alpha,\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ also satisfy:

$$\left\langle\left(\Delta\bar{\mathbf{T}}_p^\alpha\right)',\boldsymbol{\alpha},\mathbf{p},S^2\right|\nabla_{\Delta\bar{\mathbf{T}}}\left|\Delta\bar{\mathbf{T}}_p^\alpha,\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle=\nabla_{\Delta\bar{\mathbf{T}}}\delta\left(\left(\Delta\bar{\mathbf{T}}_p^\alpha\right)'-\Delta\bar{\mathbf{T}}_p^\alpha\right)\tag{60}$$

Moreover, coming back to orignal variables:

$$\left|\Delta\mathbf{T}_p^\alpha,\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle=O\left|\Delta\bar{\mathbf{T}}_p^\alpha,\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$$

this state is eigenstate of operator $\Delta\mathbf{T}_p^\alpha$ and satisfy:

$$\left\langle\left(\Delta\mathbf{T}_p^\alpha\right)',\boldsymbol{\alpha},\mathbf{p},S^2\right|\nabla_{\Delta\bar{\mathbf{T}}}\left|\Delta\mathbf{T}_p^\alpha,\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle=\nabla_{\Delta\mathbf{T}}\delta\left(\left(\Delta\mathbf{T}_p^\alpha\right)'-\Delta\mathbf{T}_p^\alpha\right)\tag{61}$$

### 8.2.4 States $\left|\Delta\bar{\mathbf{T}}_p^\alpha,\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ from creation and annihilation operators

As states $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ are obtained by application of operators on the vacuum, the states $\left|\Delta\bar{\mathbf{T}}_p^\alpha,\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ can be obtained through the fundamental states of the structure. Actually, we use (58):

$$\sqrt{\bar{\mathbf{D}}_{S^2}}\left(\mathbf{A}^-\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)+\mathbf{A}^+\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)\left|\Delta\bar{\mathbf{T}}_p^\alpha,\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle=\Delta\bar{\mathbf{T}}_p^\alpha\left|\Delta\bar{\mathbf{T}}_p^\alpha,\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$$

and the solutions of this equation are:

$$\begin{aligned}&\left|\Delta\bar{\mathbf{T}}_p^\alpha,\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle &(62)\\ =\;&\exp\left(-\left(\Delta\mathbf{T}_p^\alpha\bar{\mathbf{D}}_{S^2}\Delta\mathbf{T}_p^\alpha+2\left(\Delta\bar{\mathbf{T}}_p^\alpha\right)^t\sqrt{\bar{\mathbf{D}}_{S^2}}\mathbf{A}^+\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)+\frac{1}{2}\mathbf{A}^+\left(\boldsymbol{\alpha},\mathbf{p},S^2\right).\mathbf{A}^+\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)\left|Vac\right\rangle\end{aligned}$$

Coming back to the original variables, we also write:

$$\begin{aligned}
& \left|\Delta \mathbf{T}_p^\alpha, \boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle \qquad (63)\\
= & \exp\left(-\left(\left(\Delta \mathbf{T}_p^\alpha\right)^t \mathbf{A}_p^\alpha \Delta \mathbf{T}_p^\alpha + 2\left(\Delta \mathbf{T}_p^\alpha\right)^t \sqrt{\mathbf{A}_p^\alpha} \hat{\mathbf{A}}^+\left(\boldsymbol{\alpha}, \mathbf{p}, S^2\right)\right) + \frac{1}{2} \hat{\mathbf{A}}^+\left(\boldsymbol{\alpha}, \mathbf{p}, S^2\right) . \hat{\mathbf{A}}^+\left(\boldsymbol{\alpha}, \mathbf{p}, S^2\right)\right)|Vac\rangle
\end{aligned}$$

$$\hat{\mathbf{A}}^{\pm}\left(\boldsymbol{\alpha}, \mathbf{p}, S^2\right) = O^{-1} \mathbf{A}^{\pm}\left(\boldsymbol{\alpha}, \mathbf{p}, S^2\right)$$

The states $\left|\Delta \mathbf{T}_p^\alpha, \boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle$ are thus combinations of product of states $\left|\boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle$.

### 8.2.5 General form of collective states

Then, we define the most general state of the structure given spatial extension $S^2$ for state $\boldsymbol{\alpha}, \mathbf{p}$:

$$\left|\underline{\gamma}\left(\boldsymbol{\alpha}, \mathbf{p}, S^2\right)\right\rangle = \int \underline{\gamma}\left(\mathbf{T}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right)\left|\Delta \mathbf{T}_p^\alpha, \boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle d\mathbf{T} \qquad (64)$$

or:

$$\left|\underline{\gamma}\left(\boldsymbol{\alpha}, \mathbf{p}, S^2\right)\right\rangle = \int \underline{\gamma}\left(\bar{\mathbf{T}}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right)\left|\Delta \bar{\mathbf{T}}_p^\alpha, \boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle d\bar{\mathbf{T}} \qquad (65)$$

and summing over all possible states:

$$\begin{aligned}
\left|\underline{\gamma}\left(S^2\right)\right\rangle &= \sum_{\alpha, p}\left|\underline{\gamma}\left(\boldsymbol{\alpha}, \mathbf{p}, S^2\right)\right\rangle = \sum_{\alpha, p} \int \underline{\gamma}\left(\mathbf{T}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right)\left|\Delta \bar{\mathbf{T}}_p^\alpha, \boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle d\mathbf{T} \\
\left|\underline{\gamma}\left(S^2\right)\right\rangle &= \sum_{\alpha, p}\left|\underline{\gamma}\left(\boldsymbol{\alpha}, \mathbf{p}, S^2\right)\right\rangle = \sum_{\alpha, p} \int \underline{\gamma}\left(\bar{\mathbf{T}}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right)\left|\Delta \bar{\mathbf{T}}_p^\alpha, \boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle d\bar{\mathbf{T}}
\end{aligned}$$

Such states do not in general satisfy saddle point equation, but are necessary as transitory states. In fact, the components $\underline{\gamma}\left(\bar{\mathbf{T}}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right)$ are the possible realizations of a field $\underline{\Gamma}\left(\bar{\mathbf{T}}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right)$.

### 8.2.6 Action functional for a general state

Once the states $\left|\underline{\gamma}\left(\mathbf{T}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right)\right\rangle$ are defined as combination of the fundamental states, the action of the field becomes a functional $S\left(\underline{\gamma}\left(\mathbf{T}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right)\right)$:

$$\begin{aligned}
& S\left(\underline{\gamma}\left(\mathbf{T}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right)\right) \qquad (66)\\
= & \int\left\langle\left(\Delta \bar{\mathbf{T}}_p^\alpha\right)', \boldsymbol{\alpha}, \mathbf{p}, S^2\right| \underline{\gamma}^{\dagger}\left(\mathbf{T}', \boldsymbol{\alpha}, \mathbf{p}, S^2\right) \\
& \times\left(-\frac{1}{2} \nabla_{(\mathbf{T})_{S^2}}^2 + \frac{1}{2}\left(\Delta \mathbf{T}_p^\alpha\right)_{S^2}^t \mathbf{A}_{S^2}^\alpha\left(\Delta \mathbf{T}_p^\alpha\right)\right) \underline{\gamma}\left(\mathbf{T}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right)\left|\Delta \bar{\mathbf{T}}_p^\alpha, \boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle d\mathbf{T} d\mathbf{T}'
\end{aligned}$$

Given that $\left|\Delta \bar{\mathbf{T}}_p^\alpha, \boldsymbol{\alpha}, \mathbf{p}, S^2\right\rangle$ are eigenstates of $\Delta \bar{\mathbf{T}}_p^\alpha$ and using (61), this reduces to:

$$S\left(\underline{\gamma}\left(\mathbf{T}, \boldsymbol{\alpha}, \mathbf{p}, S^2, \theta\right)\right) = \int \underline{\gamma}^{\dagger}\left(\mathbf{T}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right)\left(-\frac{1}{2} \nabla_{(\mathbf{T})_{S^2}}^2 + \frac{1}{2}\left(\Delta \mathbf{T}_p^\alpha\right)_{S^2}^t \mathbf{A}_{S^2}^\alpha\left(\Delta \mathbf{T}_p^\alpha\right)\right) \underline{\gamma}\left(\mathbf{T}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right) d\mathbf{T}$$

Working rather with the diagonalized form and using (60), this allows to write:

$$S\left(\underline{\gamma}\left(\bar{\mathbf{T}}, \boldsymbol{\alpha}, \mathbf{p}, S^2, \theta\right)\right) = \int \underline{\gamma}^{\dagger}\left(\mathbf{T}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right)\left(-\frac{1}{2} \nabla_{(\bar{\mathbf{T}})_{S^2}}^2 + \frac{1}{2}\left(\Delta \bar{\mathbf{T}}_p^\alpha\right)_{S^2}^t \bar{\mathbf{D}}_{S^2}^\alpha\left(\Delta \bar{\mathbf{T}}_p^\alpha\right)_{S^2}\right) \underline{\gamma}\left(\mathbf{T}, \boldsymbol{\alpha}, \mathbf{p}, S^2\right) d\bar{\mathbf{T}}$$

### 8.2.7 Field action functional

We now replace $\gamma\left(\bar{\mathbf{T}},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)$ by a field $\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)$ which is a field whose $\underline{\gamma}\left(\bar{\mathbf{T}},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)$ are the realizations. To describe the system as a whole we consider that set of multi-component field $\left\{\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\}$. Including the constants $\mathbf{C}$ defined in (52) and the potential $U$ leads:

$$\begin{aligned}
&S\left(\left\{\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\}\right) &(67)\\
=&\sum_{\left\{\boldsymbol{\alpha},\mathbf{p},S^{2}\right\}}\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\left(-\nabla_{\Delta\mathbf{T}}^{2}+\frac{1}{2}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)_{S^{2}}^{t}\mathbf{A}_{S^{2}}^{\alpha}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)+\mathbf{C}\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\\
&+U\left(\left\|\underline{\Gamma}\left(\left(\Delta\mathbf{T}_{p}^{\alpha}\right)_{S^{2}},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}\right)
\end{aligned}$$

where:

$$\left(\Delta\mathbf{T}_{p}^{\alpha}\right)_{S^{2}}=\mathbf{T}_{S^{2}}-\left\langle\mathbf{T}_{p}^{\alpha}\right\rangle_{S^{2}}$$

$$\mathbf{C}=\sum_{(Z,Z')\in S\times S}C\left(Z,Z'\right)$$

Note that $S\left(\underline{\Gamma}\left(\bar{\mathbf{T}},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right)$ is similar to $S\left(\underline{\gamma}\left(\bar{\mathbf{T}},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right)$ but with the difference that $\underline{\Gamma}\left(\bar{\mathbf{T}},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)$ is a randomvrb with realizations $\underline{\gamma}\left(\bar{\mathbf{T}},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)$. The dynamics of the system will thus be encompassed in the partition function:

$$\int\exp\left(-S\left(\underline{\Gamma}\left(\bar{\mathbf{T}},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right)\right)D\left\{\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\}\tag{68}$$

Moreover, since $S\left(\underline{\Gamma}\left(\bar{\mathbf{T}},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right)$ describes the whole set of possible stts, it depends on the collection $\left\{\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\}$. This will allow to include interactions between collective states in (68).

Note also that the action $S\left(\left\{\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\}\right)$ is similar to $\hat{S}\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right)$ where the dynamics for $\hat{T}$ has been neglected but with the replacement:

$$\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\rightarrow\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)$$

where $S$ is the spatial extension of the collective state. This replacement stresses that the fundamental object are now the states made of the set of activated interacting connections and producing the activities $\Delta\omega_{p}^{\alpha}\left(\theta,\mathbf{Z}\right)$.

### 8.2.8 Oprtrs descriptn

Alternatively, the field action $S\left(\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right)$ can be considered as a matrix element of an operator as in (66). Actually, given (55), the free part of the action (67) can be written:

$$\begin{aligned}
&S_{f}\left(\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right) &(69)\\
=&\int\left\langle\left(\Delta\bar{\mathbf{T}}_{p}^{\alpha}\right)',\boldsymbol{\alpha},\mathbf{p},S^{2}\right|\underline{\Gamma}^{\dagger}\left(\mathbf{T}',\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\\
&\times\left(-\frac{1}{2}\nabla_{(\mathbf{T})_{S^{2}}}^{2}+\frac{1}{2}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)_{S^{2}}^{t}\mathbf{A}_{S^{2}}^{\alpha}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)+\mathbf{C}\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\left|\Delta\bar{\mathbf{T}}_{p}^{\alpha},\boldsymbol{\alpha},\mathbf{p},S^{2}\right\rangle d\mathbf{T}d\mathbf{T}'\\
=&\left\langle\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right|\left(\bar{\mathbf{D}}_{S^{2}}^{\alpha}\left(\mathbf{A}^{+}\left(\alpha,p,S^{2}\right)\mathbf{A}^{-}\left(\alpha,p,S^{2}\right)+\frac{1}{2}+\mathbf{C}\right)\right)\left|\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right\rangle
\end{aligned}$$

Equivalently, using the initial variables:

$$S_f\left(\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\right)=\left\langle\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\right|\left(\left(\hat{\mathbf{A}}^{+}\left(\alpha,p,S^2\right)\right)^t\mathbf{A}_{S^2}^{\alpha}\hat{\mathbf{A}}^{-}\left(\alpha,p,S^2\right)+\frac{1}{2}+\mathbf{C}\right)\left|\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\right\rangle$$

with:

$$\hat{\mathbf{A}}^{\pm-}\left(\alpha,p,S^2\right)=O\mathbf{A}^{\pm-}\left(\alpha,p,S^2\right)$$

Thus, integrating over the degrees of freedom for $\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)$ is equivalent to compute transition elements of operators.

Considering the potential $U\left(\left\|\underline{\Gamma}\left(\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\right\|^2\right)$ in action (67), it is a matrix element between tensor products of states. Actually, starting with a series expansion for $U$:

$$\begin{aligned}&U\left(\left\|\underline{\Gamma}\left(\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\right\|^2\right)\\=&\sum_k\int\prod_{l=1}^k\underline{\Gamma}^{\dagger}\left(\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^{(l)},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\hat{U}_k\left(\left(\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^{(l)}\right)_{l=1\ldots k}\right)\prod_{l=1}^k\underline{\Gamma}\left(\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^{(l)},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\end{aligned}\tag{70}$$

this writes;

$$\begin{aligned}&U\left(\left\|\underline{\Gamma}\left(\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\right\|^2\right)\\=&\sum_k\int\prod_{l=1}^k\left\langle\underline{\Gamma}\left(\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^{(l)},\boldsymbol{\alpha},\mathbf{p},S^2\right)\right|\hat{U}_k\left(\left(\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^{(l)}\right)_{l=1\ldots k}\right)\prod_{l=1}^k\left|\underline{\Gamma}\left(\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^{(l)},\boldsymbol{\alpha},\mathbf{p},S^2\right)\right\rangle\end{aligned}$$

so that the potential is the matrix elements of the operator:

$$\hat{U}=\sum_k\int\prod_{l=1}^k\left|\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^{(l)},\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\hat{U}_k\left(\left(\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^{(l)}\right)_{l=1\ldots k}\right)\prod_{l=1}^k\left\langle\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^{(l)},\boldsymbol{\alpha},\mathbf{p},S^2\right|$$

between tensor product of states. This operator can be written in terms of creation annihilation operators by a change of basis. Defining:

$$\begin{aligned}&\bar{U}_{mn}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)=\sum_k\left(\left\langle\boldsymbol{\alpha},\mathbf{p},S^2\right|\right)^{\otimes m}\hat{U}_k\left(\left(\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^{(l)}\right)_{l=1\ldots k}\right)\left(\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\right)^{\otimes n}\\=&\sum_k\int\hat{U}_k\left(\left(\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^{(l)}\right)_{l=1\ldots k}\right)\\&\times\left(\prod_{l=1}^k\left\langle\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^{(l)},\boldsymbol{\alpha},\mathbf{p},S^2\right|\left(\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\right)^{\otimes n}\right)\left(\prod_{l=1}^k\left\langle\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^{(l)},\boldsymbol{\alpha},\mathbf{p},S^2\right|\left(\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\right)^{\otimes m}\right)^{\dagger}\end{aligned}$$

We can thus replace:

$$\hat{U}=\sum_{m,n}\left(\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\right)^{\otimes n}\bar{U}_{mn}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\left(\left\langle\boldsymbol{\alpha},\mathbf{p},S^2\right|\right)^{\otimes m}$$

and the operator becomes:

$$\hat{U}=\sum_{m,n}\bar{U}_{mn}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\left(\hat{\mathbf{A}}^{+}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)^m\left(\hat{\mathbf{A}}^{-}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)^n$$

As a consequence, field action $S\left(\left\{\underline{\gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\right\}\right)$ has the same content as the operator:

$$\begin{aligned}\mathbf{S}=&\left(\hat{\mathbf{A}}^{+}\left(\alpha,p,S^2\right)\right)^t\mathbf{A}_{S^2}^{\alpha}\hat{\mathbf{A}}^{-}\left(\alpha,p,S^2\right)+\frac{1}{2}+\mathbf{C}\\&+\sum_{m,n}\bar{U}_{mn}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\left(\hat{\mathbf{A}}^{+}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)^m\left(\hat{\mathbf{A}}^{-}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)^n\end{aligned}\tag{71}$$

and this operator will compute the same transitions between states as the integration over the field degrees of freedom of the field $\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)$ in the partition function (68).

### 8.3 Condition for collective state

#### 8.3.1 General condition

In the single state formalism, the condition for the existence of a collective stat can be directly studied using the action (48):

$$S\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right)$$

and study the condition that a solution for the saddle point equations:

$$\frac{\delta S\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right)}{\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)}=\frac{\delta S\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right)}{\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)}=0$$

with:

$$\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\neq 0$$

for a finite number of points $(Z,Z')$ satisfy:

$$S\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right)<0$$

Equivalently, this can be done more in the field formalism directly by minimizing:

$$\begin{aligned}
& S\left(\left\{\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\right\}\right) \\
= & \underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\left(-\nabla^2_{\Delta\mathbf{T}}+\frac{1}{2}\left(\Delta\mathbf{T}_p^{\alpha}\right)^t_{S^2}\mathbf{A}^{\alpha}_{S^2}\left(\Delta\mathbf{T}_p^{\alpha}\right)+\mathbf{C}\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right) \\
& +U\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\right\|^2\right)
\end{aligned} \tag{FCT}$$

with equation:

$$\left(-\nabla^2_{\Delta\mathbf{T}}+\frac{1}{2}\left(\Delta\mathbf{T}_p^{\alpha}\right)^t_{S^2}\mathbf{A}^{\alpha}_{S^2}\left(\Delta\mathbf{T}_p^{\alpha}\right)+\mathbf{C}\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)+U'\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\right\|^2\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)=0$$

which is understood as:

$$\left(\left(-\sum_{(Z,Z')}\nabla^2_{\Delta\mathbf{T}(Z,Z')}+\frac{1}{2}\left(\Delta\mathbf{T}_p^{\alpha}\right)^t_{S^2}\mathbf{A}^{\alpha}_{S^2}\left(\Delta\mathbf{T}_p^{\alpha}\right)+\mathbf{C}\right)+U'\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\right\|^2\right)\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)=0 \tag{72}$$

As in ([8]) we have to compare $C$, considered as some threshold, to the potential

If we diagonalize:

$$\mathbf{A}^{\alpha}_{S^2}=U\mathbf{D}U^{-1}$$

using that $Tr\mathbf{D}=Tr\mathbf{A}^{\alpha}_{S^2}$, the lowest eigenvalue of operator:

$$-\sum_{(Z,Z')}\nabla^2_{\Delta\mathbf{T}(Z,Z')}+\frac{1}{2}\left(\Delta\mathbf{T}_p^{\alpha}\right)^t_{S^2}\mathbf{A}^{\alpha}_{S^2}\left(\Delta\mathbf{T}_p^{\alpha}\right)+\mathbf{C}$$

is:

$$Tr\mathbf{A}^{\alpha}_{S^2}+\mathbf{C}$$

We then rewrite (72) for the state with lowest eigenvalue:

$$Tr\mathbf{A}^{\alpha}_{S^2}+\mathbf{C}+U'\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\right\|^2\right)=0 \tag{73}$$

and the existence of such states is the condition for a collective state.

Equation (73) yields the norm of this state:

$$\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}=\left(U'\right)^{-1}\left(-\left(Tr\mathbf{A}_{S^{2}}^{\alpha}+\mathbf{C}\right)\right)$$

and the corresponding action writes:

$$\begin{aligned}
& S\left(\left\|\Gamma\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_{\alpha}^{2}},\theta\right)\right\|^{2}\right)\\
= & \underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\left(-\nabla_{\Delta\mathbf{T}}^{2}+\frac{1}{2}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)_{S^{2}}^{t}\mathbf{A}_{S^{2}}^{\alpha}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)+\mathbf{C}\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\\
& +U\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}\right)\\
= & \underline{\Gamma}^{\dagger}\left(\Delta\bar{\mathbf{T}},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\left(Tr\mathbf{A}_{S^{2}}^{\alpha}+\mathbf{C}\right)\underline{\Gamma}\left(\Delta\bar{\mathbf{T}},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)+U\left(\left\|\underline{\Gamma}\left(\Delta\bar{\mathbf{T}},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}\right)
\end{aligned}$$

with:

$$\Delta\bar{\mathbf{T}}=U^{-1}\Delta\mathbf{T}$$

Given (73) this simplifies as:

$$\begin{aligned}
S\left(\left\|\Gamma\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_{\alpha}^{2}},\theta\right)\right\|^{2}\right) = & U\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}\right)\\
& -U'\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}\right)\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}
\end{aligned} \tag{74}$$

This has to be inferior to 0, for a collective state to exist.

### 8.3.2 Particular form of the potential:

We assume a potential of the form:

$$\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}\right)=\frac{b}{2}\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}\right)^{2}-a\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}$$

with $b>0$ to ensure a potential bounded from below. We assume a potential whose form increases with background activity:

$$a=a\left(\left|\Psi_{0}\left(\mathbf{Z}\right)\right|^{2}\omega_{0}\left(\mathbf{Z}\right)\right)\text{ with }a'>0$$

The equation (73) writes:

$$0=A+b\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}-a \tag{75}$$

with:

$$A=Tr\mathbf{A}_{S^{2}}^{\alpha}+\mathbf{C}$$

Given (74), we have:

$$\begin{aligned}
S\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}\right) = & U\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}\right)\\
& -U'\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}\right)\frac{a-A}{b}
\end{aligned}$$

tht simplifies as:

$$S\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}\right)=-\frac{\left(A-a\right)^{2}}{2b}<0$$

Thus a collective state exists, only if (75) has a solution. Since:

$$\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}>0$$

the condition for a collective state becomes:

$$A=Tr\mathbf{A}_{S^{2}}^{\alpha}+\mathbf{C}<a$$

Remark that $A$ is an increasing function of $\left|S^{2}\right|$, the number of connections involved in the stt. The larger the state, the more unlikely its emergence. Moreover, given our assumption that $a$ is dependent on the background activity, the emergence of a collective states depends on the level of background activity.

# 9 Interactions between collective states

So far, we have examined fields describing independent collective states. In this section, we introduce interaction terms and explore their implications for transitions.

## 9.1 Principle

Previous mechanism translates in term of fields by considering $n$ multicmpnnts fields correspdng to the strctrs:

$$\begin{aligned}
&S\left(\left\{\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S_{i}^{2},\theta\right)\right\}\right)\\
=\;&-\underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S_{i}^{2},\theta\right)\left(-\nabla_{\mathbf{T}}^{2}+\frac{1}{2}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)_{S_{i}^{2}}^{t}\mathbf{A}_{S_{i}^{2}}^{\alpha}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)_{S_{i}^{2}}+\mathbf{C}\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S_{i}^{2},\theta\right)\\
&+U\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S_{i}^{2},\theta\right)\right\|^{2}\right)
\end{aligned}\tag{76}$$

The set $S_{i}^{2}$ characterizes the structure localization along with its possible states. The multicomponents labelled by $\boldsymbol{\alpha},\mathbf{p}$ transcribes the possible averages and frequencies. The structure emerging from interactions is described by the action functionl:

$$\begin{aligned}
&S\left(\left\{\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},\left(\cup S_{i}\right)\times\left(\cup S_{i}\right),\theta\right)\right\}\right)\\
=\;&\underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},\left(\cup S_{i}\right)\times\left(\cup S_{i}\right),\theta\right)\left(-\nabla_{\mathbf{T}}^{2}+\frac{1}{2}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)_{\left(\cup S_{i}\right)^{2}}^{t}\mathbf{A}_{\left(\cup S_{i}\right)^{2}}^{\alpha}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)_{\left(\cup S_{i}\right)^{2}}+\mathbf{C}\right)\\
&\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},\left(\cup S_{i}\right)\times\left(\cup S_{i}\right),\theta\right)\\
&+U\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},\left(\cup S_{i}\right)\times\left(\cup S_{i}\right),\theta\right)\right\|^{2}\right)
\end{aligned}\tag{77}$$

Implicitly, this structure has relatively low average connectivities for elements of:

$$S_{i}\times S_{j}$$

and $j\neq i$. We will relax this assumption in the next section. However, the mechanisms described above allow to understand the dynamical aspects of interactions between structures.

The full action for the system described above should be a sum of individual actions:

$$\sum_{i}S\left(\left\{\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S_{i}^{2},\theta\right)\right\}\right)+S\left(\left\{\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},\left(\cup S_{i}\right)\times\left(\cup S_{i}\right),\theta\right)\right\}\right)$$

with additional interaction terms. These terms allow transition from states over $S_i^2$ to states over $(\cup S_i)\times(\cup S_i)$. This is possible when considering interaction terms of the type:

$$V\left((\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},(\cup S_i)\times(\cup S_i)),\left\{\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S_i^2\right\}\right)\underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},(\cup S_i)\times(\cup S_i),\theta\right)\prod\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S_i^2,\theta\right)\tag{78}$$

where $V$ is a potential. We will detail the formalism in the next section, but given our previous description, we may expect the potential:

$$V\left((\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},(\cup S_i)\times(\cup S_i)),\left\{\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S_i^2\right\}\right)$$

to depend on the states frequencies, to allow to switch from some equilibrium over the $S_i^2$ to equilibrium over the $(\cup S_i)\times(\cup S_i)$. If the $S_i^2$ have frqncs $\Upsilon_{i,l_i}$ we may expect $(\cup S_i)^2$ to be characterized by the set of frequencies $\Upsilon_{(i,l_i)}$. Similarly, $\Delta\mathbf{T}((\cup S_i)\times(\cup S_i))$ on the diagonal should be close to the $\Delta\mathbf{T}((\cup S_i)\times(\cup S_i))$.

## 9.2 Interactions

### 9.2.1 Different structures

We described previously the collective state resulting by "merging" different types of structures. To describe dynamically this transition in terms of fields, we add to the action a term of the form:

$$\sum_{nn'}\sum_{\substack{k=1\dots n\\ l=1,\dots,n'}}\sum_{\{S_k,S_l\}_{\substack{l=1,\dots,n'\\ k=1\dots n}}}\prod_l\underline{\Gamma}^{\dagger}\left(\mathbf{T}'_l,\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right)V\left(\left\{\left|\mathbf{T}'_l,\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right\rangle\right\},\left\{\left|\mathbf{T}_l,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\rangle\right\}\right)\prod_k\underline{\Gamma}\left(\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)$$

where:

$$V\left(\left\{\left|\mathbf{T}'_l,\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right\rangle\right\},\left\{\left|\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\rangle\right\}\right)=V\left(\left\{\mathbf{T}'_l,\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right\},\left\{\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right)$$

and the action for interacting structures becomes:

$$\begin{aligned}S &= \sum_S\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\left(-\frac{1}{2}\nabla^2_{(\hat{\mathbf{T}})_{S^2}}+\frac{1}{2}\left(\Delta\mathbf{T}_p^{\alpha}\right)^t_{S^2}\mathbf{A}^{\alpha}_{S^2}\left(\Delta\mathbf{T}_p^{\alpha}\right)+\mathbf{C}\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\\&\quad+\sum_{nn'}\sum_{\substack{k=1\dots n\\ l=1,\dots,n'}}\sum_{\{S_k,S_l\}_{\substack{l=1,\dots,n'\\ k=1\dots n}}}\prod_l\underline{\Gamma}^{\dagger}\left(\mathbf{T}'_l,\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right)\\&\quad\times V\left(\left\{\left|\mathbf{T}',\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right\rangle\right\},\left\{\left|\mathbf{T},\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\rangle\right\}\right)\prod_k\underline{\Gamma}\left(\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)\end{aligned}\tag{79}$$

allowing for transitions between sets of several collective states. The form of $V$ is conditionned by frequencies of oscillation:

$$V\left(\left\{\mathbf{T}'_l,\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right\},\left\{\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right)=V\left(\Upsilon_l^{\mathbf{p}'_l}\left(\mathbf{T}'_l\right),\Upsilon_k^{\mathbf{p}_k}\left(\mathbf{T}_k\right)\right)$$

and models the results of the first part of this article, transitions depend both on initial states characteritics and that of the merged ones.

### 9.2.2 Substructures

We also consider the possibility of activation by a substructure. This is a particular case of interaction where the activation of some substructure induces the full structure activation. To describe this type of transition, the term:

$$\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\left(-\frac{1}{2}\nabla^2_{(\hat{\mathbf{T}})_{S^2}}+\frac{1}{2}\left(\Delta\mathbf{T}_p^{\alpha}\right)^t_{S^2}\mathbf{A}^{\alpha}_{S^2}\left(\Delta\mathbf{T}_p^{\alpha}\right)+\mathbf{C}\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$$

is generalized by including free action terms for each substructures plus interaction terms between these substructures, including the full one:

$$\sum_{S_1\subseteq S}\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S_1^2\right)\left(-\frac{1}{2}\nabla^2_{\left(\hat{\mathbf{T}}\right)_{S_1^2}}+\frac{1}{2}\left(\Delta\mathbf{T}_p^{\alpha}\right)^t_{S_1^2}\mathbf{A}^{\alpha}_{S_1^2}\left(\Delta\mathbf{T}_p^{\alpha}\right)+\mathbf{C}\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S_1^2\right)$$
$$+\sum_n\sum_{S_1,\ldots,S_n\subseteq S}\sum_{(\alpha_1,p_1),\ldots,(\alpha_n,p_n)}\sum_k V_k\left(\left(\left\{\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i\right\},(S_i)^2\right)\right)\prod_{i\leqslant k}\underline{\Gamma}^{\dagger}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i\right)\prod_{k+1\leqslant i\leqslant n}\underline{\Gamma}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)$$

The potential $V_k\left(\left(\left\{\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i\right\},(S_i)^2\right)\right)$ induces transition from some state with $k$ substructures towards a state with $n-k$ subtructures. It may include the transition from one or several subsets to the full activated structure. This situation is depicted by a potential of the type:

$$V_k\left(\left(\left\{\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,(S_i)^2\right\}\right)\right)=V_k\left(\left(\left\{\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,(S_i)^2\right\}_{i\leqslant k},\left\{\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,(S_i)^2\right\}_{k+1\leqslant i\leqslant n}\right)\right)$$

the group $\mathbf{Z}$ of possible states is defined, at least partly, by initial background.

## 9.3 Operator formalism for interactions

### 9.3.1 General case

We have seen in (71) the operator formulation for the dynamic of one type of structure:

$$\begin{aligned}\mathbf{S} &= \left(\hat{\mathbf{A}}^{+}\left(\alpha,p,S^2\right)\right)^t\mathbf{A}^{\alpha}_{S^2}\hat{\mathbf{A}}^{-}\left(\alpha,p,S^2\right)+\frac{1}{2}+\mathbf{C}\\&\quad+\sum_{m,n}\bar{U}_{mn}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\left(\hat{\mathbf{A}}^{+}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)^m\left(\hat{\mathbf{A}}^{-}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)^n\end{aligned}\tag{80}$$

As for the field version, we can consider interaction potential between different structures. The operator counterpart of the potential term in (79) is:

$$\sum_{n,n'}\sum_{\{S_k,S_l\}_{\substack{l=1,\ldots,n'\\k=1\ldots n}}}\prod_{l=1}^{n'}\underline{\Gamma}^{\dagger}\left(\mathbf{T}'_l,\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right)V_{n,n'}\left(\left\{\mathbf{T}'_l,\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right\}_{l\leqslant n'},\left\{\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}_{l\leqslant n}\right)\prod_{k=1}^{n}\underline{\Gamma}\left(\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)$$

is found by applying the same technique as for individual potential $U$ in (70). We change the basis by defining:

$$\begin{aligned}&V_{n,n'}\left(\left\{\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2,m'_l\right\}_{l\leqslant n'},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2,m_k\right\}_{l\leqslant n}\right)\\=&\int d\left(\Delta\mathbf{T}'^{\alpha_l}_{p_l}\right)_{S_l'^2}d\left(\Delta\mathbf{T}^{\alpha_k}_{p_k}\right)_{S_k^2}V_{n,n'}\left(\left\{\left(\Delta\mathbf{T}'^{\alpha_l}_{p_l}\right)_{S_l'^2},\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right\}_{l\leqslant n'},\left\{\left(\Delta\mathbf{T}^{\alpha_k}_{p_k}\right)_{S_k^2},\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}_{l\leqslant n}\right)\\&\times\prod_{k=1}^{n}\left\langle\left(\Delta\mathbf{T}^{\alpha_k}_{p_k}\right)_{S_k^2},\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right|\left(\left|\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\rangle\right)^{\otimes m_k}\left(\prod_{l=1}^{n'}\left\langle\left(\Delta\mathbf{T}'^{\alpha_l}_{p_l}\right)_{S_l'^2},\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right|\left(\left|\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\rangle\right)^{\otimes m'_l}\right)^{\dagger}\end{aligned}\tag{81}$$

and the interaction operator writes in terms of annihilation and creation operators:

$$\begin{aligned}\hat{V}=&\sum_{n,n'}\sum_{\{S_k,S_l\}_{\substack{l=1,\ldots,n'\\k=1\ldots n}}}\sum_{\{m'_l,m_k\}}\prod_{l=1}^{n'}\left(\hat{\mathbf{A}}^{+}\left(\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right)\right)^{m'_l}\\&\times V_{n,n'}\left(\left\{\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2,m'_l\right\}_{l\leqslant n'},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2,m_k\right\}_{l\leqslant n}\right)\prod_{k=1}^{n}\left(\hat{\mathbf{A}}^{-}\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)\right)^{m_k}\end{aligned}\tag{82}$$

Thus the corresponding operator is to $S\left(\left\{\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\}\right)$ is:

$$\mathbf{S}=\sum_{S\times S}\bar{\mathbf{D}}_{S^{2}}^{\alpha}\left(\mathbf{A}^{+}\left(\alpha,p,S^{2}\right)\mathbf{A}^{-}\left(\alpha,p,S^{2}\right)+\frac{1}{2}\right) \tag{83}$$
$$+\sum_{m,n}\bar{U}_{mn}\left(\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\left(\hat{\mathbf{A}}^{+}\left(\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right)^{m}\left(\hat{\mathbf{A}}^{-}\left(\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right)^{n}+\hat{V}$$

The advantage of this formulation is to directly translate the dynamics in terms of creation and destruction of structures, describing the transition resulting from such operators. It also allows straightforward computations at the lowest order of approximation, presenting a direct interpretation as transitions of structures.

In the sequel we will simplify the notation:

$$V_{n,n'}\left(\left\{\boldsymbol{\alpha}_{l}',\mathbf{p}_{l}',S_{l}'^{2},m_{l}'\right\}_{l\leqslant n'},\left\{\boldsymbol{\alpha}_{k},\mathbf{p}_{k},S_{k}^{2},m_{k}\right\}_{l\leqslant n}\right)\rightarrow V_{n,n'}\left(\left\{\boldsymbol{\alpha}_{l}',\mathbf{p}_{l}',S_{l}'^{2},m_{l}'\right\},\left\{\boldsymbol{\alpha}_{k},\mathbf{p}_{k},S_{k}^{2},m_{k}\right\}\right)$$

### 9.3.2 Internal perturbations for one structure

We can consider the particular case of several activated states for one structure. This corresponds to several processes arising within the same collective state. This case is intermediate between a single type of structure and multiple interacting structures. In this case the pntl in (83) is replaced by:

$$\hat{V} = \sum_{n,n'}\sum_{\left\{m_{l}',m_{k}\right\}}\prod_{l=1}^{n'}\left(\hat{\mathbf{A}}^{+}\left(\boldsymbol{\alpha}_{l}',\mathbf{p}_{l}',S^{2}\right)\right)^{m_{l}'}$$
$$\times V_{n,n'}\left(\left\{\boldsymbol{\alpha}_{l}',\mathbf{p}_{l}',m_{l}'\right\}_{l\leqslant n'},\left\{\boldsymbol{\alpha}_{k},\mathbf{p}_{k},m_{k}\right\}_{l\leqslant n},S^{2}\right)\prod_{k=1}^{n}\left(\hat{\mathbf{A}}^{-}\left(\boldsymbol{\alpha}_{k},\mathbf{p}_{k},S^{2}\right)\right)^{m_{k}}$$

where:

$$V_{n,n'}\left(\left\{\boldsymbol{\alpha}_{l}',\mathbf{p}_{l}',m_{l}'\right\}_{l\leqslant n'},\left\{\boldsymbol{\alpha}_{k},\mathbf{p}_{k},m_{k}\right\}_{l\leqslant n},S^{2}\right)$$

is given by (81) with $S_k=S_l'=S$ for all $l$ and $k$.

## 9.4 External perturbation

In ([6]) and ([7]), we have studied the effect of external sources on the connectivity functions. External signals induce modified activities and as a consequence, modifications in equilibrium states of structures (see appendix 2 for a detailed account). In terms of collective state formalism, this situation can be described through a modification of the effective action or by an operator description.

### 9.4.1 Effective action

In the present context, external perturbations can be modelled by adding extra terms in the action inducing switches in background states, and dynamical transitions between stts. We show in appendix 0 that we can consider a modified action:

$$S\left(\left\{\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\}\right) \tag{84}$$
$$=\sum_{\left\{\boldsymbol{\alpha},\mathbf{p},S^{2}\right\}}\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\left(-\nabla_{\Delta\mathbf{T}}^{2}+\frac{1}{2}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)_{S^{2}}^{t}\mathbf{A}_{S^{2}}^{\alpha}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)+\mathbf{C}\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)$$
$$+U\left(\left\Vert\underline{\Gamma}\left(\left(\Delta\mathbf{T}_{p}^{\alpha}\right)_{S^{2}},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\Vert^{2}\right)+\underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)J\left(\Delta\mathbf{T},S^{2},\theta\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)$$

The source term $J\left(S^2,\theta\right)$ can be considered as a sum of contributions acting at each points of $S$:

$$J\left(S^2,\theta\right)=\sum_{Z\in S}J\left(Z,\theta\right)$$

The introduction of $J\left(S^2,\theta\right)$ modifies the saddle points equations of the system, and may induce some structures to be switched off or on. They may also induce some different structures to combine through effective interaction.

**Terms inducing types transition** The presence of:

$$\underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)J\left(S^2,\theta\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right) \tag{85}$$

in the action accounts for the possibility to turn on or off the structure. More generally, an external source may induce transition between two states of the same strcture, and this is described by:

$$\underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T}',\boldsymbol{\alpha}',\mathbf{p}',S^2,\theta\right)J\left(S^2,\theta\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right) \tag{86}$$

switching the connectivity states from $(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p})$ to $(\Delta\mathbf{T}',\boldsymbol{\alpha}',\mathbf{p}')$.

In an effective formalism, if the signals modify several structures we may assume that switches between states may be modelled directly by current induced interactions. In this case, the interactions (85) or (86) can be replaced in (84) by:

$$\sum_{\left\{\boldsymbol{\alpha},\mathbf{p},S^2\right\},\left\{\boldsymbol{\alpha}',\mathbf{p}',S'^2\right\}}\underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T}',\boldsymbol{\alpha}',\mathbf{p}',S'^2,\theta\right)J\left(S'^2,\theta'\right)J\left(S^2,\theta\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right) \tag{87}$$

and this terms ensures dynamics switching between different states. This corresponds to the integration of some "faster" structure, connecting different states through some perturbations.

**Activation or deactivation terms** This possiility represents activation or deactivation of structures due to the source terms. It is modelled by field linear terms of the form:

$$J\left(S^2,\theta\right)\left(\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)+\underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\right) \tag{88}$$

### 9.4.2 operator formalism

In term of operators the additional terms (85) and (87) are modeled using the operators technique described above.

**Transition type trms** The operator representing (85) is:

$$\sum_{m,m'}\left(\hat{\mathbf{A}}^{+}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)^{m'}J_{m,m'}\left(\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\left(\hat{\mathbf{A}}^{-}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)^{m}$$

with:

$$J_{m,m'}\left(\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)=\left(\left\langle\boldsymbol{\alpha},\mathbf{p},S^2\right|\right)^{\otimes m'}J\left(S^2,\theta\right)\left(\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\right)^{\otimes m}$$

and (87) is translated to:

$$\sum_{m,m'}\left(\hat{\mathbf{A}}^{+}\left(\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2\right)\right)^{m'}V_{m,m'}\left(\boldsymbol{\alpha}',\mathbf{p}',S'^2,\theta',\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\left(\hat{\mathbf{A}}^{-}\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)\right)^{m}$$
$$+\left(\hat{\mathbf{A}}^{+}\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)\right)^{m}V_{m',m}\left(\boldsymbol{\alpha},\mathbf{p},S^2,\theta,\boldsymbol{\alpha}',\mathbf{p}',S'^2,\theta'\right)\left(\hat{\mathbf{A}}^{-}\left(\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2\right)\right)^{m'}$$

with:

$$V_{m,m'}\left(\boldsymbol{\alpha}',\mathbf{p}',S'^2,\theta',\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)=\sum_k J_{m,k}\left(S'^2,\theta\right)J_{k,m'}\left(S^2,\theta\right)$$

**Activation or deactivation terms** The operator equivalent of (88) is:

$$J\left(S^2,\theta\right)\left(\hat{\mathbf{A}}^{+}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)+\hat{\mathbf{A}}^{-}\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)\right)$$

# III Approaches to transitions, examples and extensions

The formalism presented above enables the computation of transition mechanisms between various states. Nevertheless, exact computation of the path integral is not feasible. We introduce three approaches for these computations. Two of them are perturbative in nature (perturbation expansion and operator formalism), while the third one, effective field theory, is not. Each of these methods has its own advantages. We present some illustrative examples of these methods

## 10 Mechanisms of transition

We present three different complementary approaches to the interactions and transitions of states. The perturbation expansion is the most straightforward method for studying transitions. By considering the "free" action obtained by neglecting the interaction terms, we can compute the Green functions of individual structures. These Green functions describe the dynamic fluctuations of a structure in the absence of interactions. Subsequently, the interaction terms are incorporated to compute the transitions of such free states to others. These transitions may involve the activation of some structures and the deactivation of others. The advantage of this approach lies in its clarity, as it directly calculates the probabilities of the considered transitions. However, it falls short in addressing non-perturbative effects, which encompass global effects where some structures act as a background for others. These effects differ from perturbative ones since they correspond to the impact of a permanent landscape in which structures evolve. Nevertheless, since it mainly operates at the background field level, it may not fully capture the precise mechanisms by which some structures constitute effective interactions between others. The third approach, operator formalism, addresses this issue to some extent by integrating certain interactions to establish indirect ones. Consequently, this approach combines elements of the other two methods.

### 10.1 Perturbation expansion

The first approach to transitions of states is the most direct and the most readable due to its direct computation of transition functions based on series expansion.

As explained in the introduction of this section, it begins by calculating the Green functions, which are the transition functions for the free structures. Then, perturbations that enable transitions are introduced in an ordered manner to compute the transitions induced by interactions.

#### 10.1.1 Green Functions

Grn fnctns cmptd wth free part in (79):

$$\underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)\left(-\nabla_{\Delta\mathbf{T}}^2+\frac{1}{2}\left(\Delta\mathbf{T}_p^{\alpha}\right)_{S^2}^t\mathbf{A}_{S^2}^{\alpha}\left(\Delta\mathbf{T}_p^{\alpha}\right)+\mathbf{C}\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right)$$

and the trnstn prtr s gvn b:

$$\left(-\nabla_{\Delta\mathbf{T}}^{2}+\frac{1}{2}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)_{S^{2}}^{t}\mathbf{A}_{S^{2}}^{\alpha}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)+\mathbf{C}\right)^{-1}$$

It is diagonalized as:

$$U\left(-\frac{1}{2}\nabla_{(\bar{\mathbf{T}})_{S^{2}}}^{2}+\frac{1}{2}\left(\bar{\mathbf{D}}_{S^{2}}\bar{\mathbf{T}}_{p}^{\alpha}\right)^{2}+\mathbf{C}\right)^{-1}U^{-1}$$

For lrg nmbr f vrbls, lowest order krnl wth grnd stts:

$$G\left(\bar{\mathbf{T}}_{p}^{\prime\alpha},\bar{\mathbf{T}}_{p}^{\alpha}\right)=\frac{\exp\left(-\left(\bar{\mathbf{T}}_{p}^{\prime\alpha}\right)^{t}\bar{\mathbf{D}}_{S^{2}}\bar{\mathbf{T}}_{p}^{\prime\alpha}-\left(\bar{\mathbf{T}}_{p}^{\alpha}\right)^{t}\bar{\mathbf{D}}_{S^{2}}\bar{\mathbf{T}}_{p}^{\alpha}\right)}{\sqrt{\det\left(\bar{\mathbf{D}}_{S^{2}}\right)}}\tag{89}$$

and coming back to the original varianles:

$$G\left(\mathbf{T}_{p}^{\prime\alpha},\mathbf{T}_{p}^{\alpha}\right)=\frac{\exp\left(-\left(\mathbf{T}_{p}^{\prime\alpha}\right)^{t}\mathbf{A}_{S^{2}}^{\alpha}\mathbf{T}_{p}^{\prime\alpha}-\left(\mathbf{T}_{p}^{\alpha}\right)^{t}\mathbf{A}_{S^{2}}^{\alpha}\mathbf{T}_{p}^{\alpha}\right)}{\sqrt{\det\left(\mathbf{A}_{S^{2}}^{\alpha}\right)}}\tag{90}$$

This computes the probability of a transition for a specific structure defined by its data $\bar{\mathbf{T}}_{p}^{\alpha}$, including thus average connectivities and internal frequencies of activities, to an other state $\bar{\mathbf{T}}_{p}^{\prime\alpha}$. This transition occurs in an average timespan normalized here to 1

When there are no interactions or external signals, the structure remains unchanged. The characteristics represented by the parameters $\alpha$ and $p$ remain unchanged. The structure only undergoes fluctuations around its average values.

However, these fluctuations can, in the presence of interactions, lead to a switch from one state to another, cause the structure to turn off, or enable it to combine with another. In practice, a deviation induced by the interaction may bring the values $\bar{\mathbf{T}}_{p}^{\alpha}$ closer to a new average determined by some parameters $\alpha'$ and $p'$, resulting in a state transition. As a consequence, the quantity (89) (or (90)) will be the main component in all calculations of transition functions. Mathematically, it represents the lowest-order expansion for transitions, and interactions will modify this formula, leading to transition effects.

### 10.1.2 Perturbative contributions

Rewrite the potential in terms of diagonalized variables in (79):

$$V\left(\left\{\mathbf{T}_{l}^{\prime},\boldsymbol{\alpha}_{l}^{\prime},\mathbf{p}_{l}^{\prime},S_{l}^{\prime2}\right\},\left\{\mathbf{T}_{k},\boldsymbol{\alpha}_{k},\mathbf{p}_{k},S_{k}^{2}\right\}\right)=\bar{V}\left(\left\{\bar{\mathbf{T}}_{l}^{\prime},\boldsymbol{\alpha}_{l}^{\prime},\mathbf{p}_{l}^{\prime},S_{l}^{\prime2}\right\},\left\{\bar{\mathbf{T}}_{k},\boldsymbol{\alpha}_{k},\mathbf{p}_{k},S_{k}^{2}\right\}\right)$$

This potential induces corrections to the free Green functions, through the perturbative expansion of the partition function. Actually:

$$\begin{aligned}
&\exp\left(-S\right)\\
=\;&\exp\left(-\sum_{S}\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\left(-\frac{1}{2}\nabla_{(\hat{\mathbf{T}})_{S^{2}}}^{2}+\frac{1}{2}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)_{S^{2}}^{t}\mathbf{A}_{S^{2}}^{\alpha}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)+\mathbf{C}\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right)\\
&\times\sum\frac{1}{r!}\left(\sum_{nn'}\sum_{\substack{k=1\dots n\\ l=1,\dots,n'}}\sum_{\{S_{k},S_{l}\}_{\substack{l\leqslant n'\\ k\leqslant n}}}\prod_{l}\underline{\Gamma}^{\dagger}\left(\mathbf{T}_{l}^{\prime},\boldsymbol{\alpha}_{l}^{\prime},\mathbf{p}_{l}^{\prime},S_{l}^{\prime2}\right)\bar{V}\left(\left\{\bar{\mathbf{T}}_{l}^{\prime},\boldsymbol{\alpha}_{l}^{\prime},\mathbf{p}_{l}^{\prime},S_{l}^{\prime2}\right\},\left\{\bar{\mathbf{T}}_{k},\boldsymbol{\alpha}_{k},\mathbf{p}_{k},S_{k}^{2}\right\}\right)\prod_{k}\underline{\Gamma}\left(\mathbf{T}_{k},\boldsymbol{\alpha}_{k},\mathbf{p}_{k},S_{k}^{2}\right)\right)^{r}
\end{aligned}$$

and the contributions to transition are computed with graphs. The vertices are given by the potential and the legs correspond to the free Green functions (89):

$$\begin{aligned}
&G\left(\left\{\bar{\mathbf{T}}_l,\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right\},\left\{\bar{\mathbf{T}}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right) \qquad (91)\\
=&\int\prod_l\frac{\exp\left(-\left(\bar{\mathbf{T}}_l'\right)^t\bar{\mathbf{D}}_{S_l^2}\bar{\mathbf{T}}_l'-\left(\bar{\mathbf{T}}_l\right)^t\bar{\mathbf{D}}_{S_l^2}\bar{\mathbf{T}}_l\right)}{\sqrt{\det\left(\bar{\mathbf{D}}_{S^2}\right)}}\\
&\times\bar{V}\left(\left\{\bar{\mathbf{T}}_l',\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l'^2\right\},\left\{\bar{\mathbf{T}}_k',\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right)\prod_k\frac{\exp\left(-\left(\bar{\mathbf{T}}_k'\right)^t\bar{\mathbf{D}}_{S_k^2}\bar{\mathbf{T}}_k'-\left(\bar{\mathbf{T}}_k\right)^t\bar{\mathbf{D}}_{S_k^2}\bar{\mathbf{T}}_k\right)}{\sqrt{\det\left(\bar{\mathbf{D}}_{S^2}\right)}}d\Delta\bar{\mathbf{T}}_l'd\Delta\bar{\mathbf{T}}_k'
\end{aligned}$$

Assuming the following form for the potentials:

$$\bar{V}\left(\left\{\bar{\mathbf{T}}_l',\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l'^2\right\},\left\{\bar{\mathbf{T}}_k',\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right)=V\left(\left(\left\|\bar{\mathbf{T}}_l'-\bar{\mathbf{T}}_k'\right\|^2\right)\right)$$

the integrals in (91) can be computed.

Compared to the Green functions (89), the terms (91) introduce a probability of transition between the two structures $S_k^2$ and $S_l'^2$. If $S_k^2=S_l'^2$, it represents a state transition for structure $k$. The same cells are involved and interact, but there is a change in parameters from $(\boldsymbol{\alpha}_k,\mathbf{p}_k)$ to $(\boldsymbol{\alpha}_l,\mathbf{p}_l)$. This change affects the frequencies of activity.

If the two structures are different, potential $V$ inducesa probability of transitioning from one structure to another. The first one can be considered switched off, while the second one switches on. It's worth noting that among other possibilities, this may model spatial transitions of the same structures along the thread. Some information is retained but not necessarily at a fixed location.

### 10.1.3 Exemple of perturbative transition

Here, we study a transition due to sources, and consider (86):

$$\underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T}',\boldsymbol{\alpha}',\mathbf{p}',S^2,\theta\right)J\left(S^2,\theta\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2,\theta\right) \qquad (92)$$

along with the following interaction term between two structures, each of them in a given state $\left\{\mathbf{T}_l',\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right\}$ and $\left\{\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}$ respectively:

$$V\left(\left\{\mathbf{T}_l',\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right\},\left\{\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right)=\bar{V}\left(\left\{\bar{\mathbf{T}}_l',\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l'^2\right\},\left\{\bar{\mathbf{T}}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right)$$

The combination of these two contibutions in the effective action induces the possibility of a transition:

$$\left|\bar{\mathbf{T}}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\rangle\overset{J}{\to}\left|\bar{\mathbf{T}}_k',\boldsymbol{\alpha}_k',\mathbf{p}_k',S_k^2\right\rangle\overset{\bar{V}}{\to}\left|\bar{\mathbf{T}}_l,\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right\rangle$$

The first arrow corresponds to the transition due to the external source, and the second arrow describes the transition due to the structure-structure interaction.

In terms of amplitude, this corresponds to the perturbative expansion of the Green function:

$$\begin{aligned}
&G\left(\left\{\bar{\mathbf{T}}_l,\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right\},\left\{\bar{\mathbf{T}}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right) \\
= &\int\frac{\exp\left(-\left(\bar{\mathbf{T}}_l'\right)^t\bar{\mathbf{D}}_{S_l^2}\bar{\mathbf{T}}_l'-\left(\bar{\mathbf{T}}_l\right)^t\bar{\mathbf{D}}_{S_l^2}\bar{\mathbf{T}}_l\right)}{\sqrt{\det\left(\bar{\mathbf{D}}_{S^2}\right)}} \\
&\times\bar{V}\left(\left\{\bar{\mathbf{T}}_l',\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right\},\left\{\bar{\mathbf{T}}_k',\boldsymbol{\alpha}_k',\mathbf{p}_k',S_k^2\right\}\right)\frac{\exp\left(-\left(\bar{\mathbf{T}}_k'\right)^t\bar{\mathbf{D}}_{S^2}\bar{\mathbf{T}}_k'-\left(\bar{\mathbf{T}}_k''\right)^t\bar{\mathbf{D}}_{S^2}\bar{\mathbf{T}}_k''\right)}{\sqrt{\det\left(\bar{\mathbf{D}}_{S^2}\right)}} \\
&\times J\left(S_k^2,\theta\right)\frac{\exp\left(-\left(\bar{\mathbf{T}}_k\right)^t\bar{\mathbf{D}}_{S^2}\bar{\mathbf{T}}_k-\left(\bar{\mathbf{T}}_k'\right)^t\bar{\mathbf{D}}_{S^2}\bar{\mathbf{T}}_{k'}'\right)}{\sqrt{\det\left(\bar{\mathbf{D}}_{S^2}\right)}}d\Delta\bar{\mathbf{T}}_l'd\Delta\bar{\mathbf{T}}_k'd\Delta\bar{\mathbf{T}}_k''
\end{aligned} \tag{93}$$

Assuming the potential $\bar{V}$ to be proportionl to a Dirac delta, so that the transition between structures occur for states in which the structures' frequencies in activity are similar:

$$\delta\left(\left\{\bar{\mathbf{T}}_l',\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l'^2\right\}-\left\{\bar{\mathbf{T}}_k',\boldsymbol{\alpha}_k',\mathbf{p}_k',S_k^2\right\}\right)$$

the integration over $\Delta\bar{\mathbf{T}}_k'$ and $\Delta\bar{\mathbf{T}}_k''$ reduces the amplitude (93) to:

$$G\left(\left\{\bar{\mathbf{T}}_l,\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right\},\left\{\bar{\mathbf{T}}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right)=AJ\left(S_k^2,\theta\right)\frac{\exp\left(-\left(\bar{\mathbf{T}}_l\right)^t\bar{\mathbf{D}}_{S_l^2}\bar{\mathbf{T}}_l-\left(\bar{\mathbf{T}}_k\right)^t\bar{\mathbf{D}}_{S^2}\bar{\mathbf{T}}_k\right)}{\sqrt{\det\left(\bar{\mathbf{D}}_{S^2}\right)}}$$

with $A$ an integration constant. As a consequence, an apparent transition from one structure to a different one has arisen from the source signal. However, the transition is feasible only through a decay of the initial structure towards a state synchronized with the new emerging state.

It is noteworthy that the same mechanism arises if the extrrnal source activates a substructure and that substructure, in turn, activates the full structure:

$$\left|\mathbf{vac}\right\rangle\overset{J}{\rightarrow}\left|\bar{\mathbf{T}}_l,\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\subset S_k^2\right\rangle\overset{\bar{V}}{\rightarrow}\left|\bar{\mathbf{T}}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\rangle$$

The first arrow represents the activation of the substructure through $J$, while $\bar{V}$ induces the transition to the structure $\left|\bar{\mathbf{T}}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\rangle$. The associated amplitude is:

$$\begin{aligned}
&G\left(\left\{\mathbf{vac}\right\},\left\{\bar{\mathbf{T}}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right) \\
= &\int\frac{\exp\left(-\left(\bar{\mathbf{T}}_k\right)^t\bar{\mathbf{D}}_{S^2}\bar{\mathbf{T}}_k-\left(\bar{\mathbf{T}}_k'\right)^t\bar{\mathbf{D}}_{S^2}\bar{\mathbf{T}}_{k'}'\right)}{\sqrt{\det\left(\bar{\mathbf{D}}_{S^2}\right)}}\bar{V}\left(\left\{\bar{\mathbf{T}}_l',\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right\},\left\{\bar{\mathbf{T}}_k',\boldsymbol{\alpha}_k',\mathbf{p}_k',S_k^2\right\}\right) \\
&\times J\left(S_k^2,\theta\right)\frac{\exp\left(-\left(\bar{\mathbf{T}}_l'\right)^t\bar{\mathbf{D}}_{S_l^2}\bar{\mathbf{T}}_l'-\left(\bar{\mathbf{T}}_l\right)^t\bar{\mathbf{D}}_{S_l^2}\bar{\mathbf{T}}_l\right)}{\sqrt{\det\left(\bar{\mathbf{D}}_{S^2}\right)}}d\Delta\bar{\mathbf{T}}_l'd\Delta\bar{\mathbf{T}}_k'
\end{aligned} \tag{94}$$

## 10.2 Effective field approach

Transition from two structures of different types to a third combined one may be described in several ways. The perturbation expansion enables an understanding of the transition mechanism but may not capture non-perturbative or long-lasting aspects. The effective field approach overcomes this limitation by concentrating on the initial and final background states. This approach allows for the consideration of permanent effects of some background structures on others. By determining the background state for some structure, we can rewrite the effective action for the remaining components.

### 10.2.1 Principle

We start with the partition function describing the full potential system including the entire set of structures, independent and composed n:

$$\int \exp\left(\sum_i S\left(\underline{\Gamma_i}\right) + S\left(\underline{\Gamma_{[1,n]}}\right) + U\left(\left(\underline{\Gamma_i}\right), \underline{\Gamma_{[1,n]}}\right)\right) \prod_i \mathcal{D}\Gamma_i \mathcal{D}\Gamma_{[1,n]} \tag{95}$$

In this context, we have divided the fields into two sets. Our objective is to perform an integration over the degrees of freedom of $\Gamma_i$ to derive the effective action for the fields $\Gamma_{[1,n]}$. Consequently, the interaction between $\underline{\Gamma_i}$ and $\underline{\Gamma_{[1,n]}}$ is integrated out, which in turn modifies the action of the $\underline{\Gamma_{[1,n]}}$. This formulation is based on the assumption that certain potential structures are present, some of which are activated (represented by the $\underline{\Gamma_i}$), while others are not. The interactions among the $\underline{\Gamma_i}$, whose overall impact is encompassed in the solutions of the saddle point equations, create a landscape that facilitates the emergence of the combined structure described by an action $S_e\left(\underline{\Gamma_{[1,n]}}\right)$, so that after integration over $\Gamma_i$, the effective action $S_e\left(\underline{\Gamma_{[1,n]}}\right)$ has an associated partition function:

$$\int \exp\left(S_e\left(\underline{\Gamma_{[1,n]}}\right)\right) \mathcal{D}\Gamma_{[1,n]}$$

Practically, at the lowest order of approximation, we solve the saddle point equations for the $\underline{\Gamma_i}$:

$$\frac{\delta S\left(\underline{\Gamma_i}\right)}{\delta \underline{\Gamma_i}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2, \theta\right)} + \frac{\delta U\left(\left(\underline{\Gamma_i}\right), \underline{\Gamma_{[1,n]}}\right)}{\delta \underline{\Gamma_i}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2, \theta\right)} = 0$$

This set of equation allows to express $\Gamma_i$ as a functional of $\Gamma_{[1,n]}$ and (95) becoms:

$$\int \exp\left(S\left(\underline{\Gamma_{[1,n]}}\right) + \sum_i S\left(\underline{\Gamma_i}\left(\Gamma_{[1,n]}\right)\right) + U\left(\left(\underline{\Gamma_i}\left(\Gamma_{[1,n]}\right)\right), \underline{\Gamma_{[1,n]}}\right)\right) \mathcal{D}\Gamma_{[1,n]}$$

The interaction $U\left(\left(\underline{\Gamma_i}\right), \underline{\Gamma_{[1,n]}}\right)$ between structures $\underline{\Gamma_i}$ and $\Gamma_{[1,n]}$ is replaced by an effective potential:

$$U_e\left(\underline{\Gamma_{[1,n]}}\right) = U\left(\left(\underline{\Gamma_i}\left(\Gamma_{[1,n]}\right)\right), \underline{\Gamma_{[1,n]}}\right)$$

This modified potential may induce a non-trivial stable background for $\Gamma_{[1,n]}$ that is, a set of activated structures.

For a potential such that the interaction depends on some compatibility conditions between the $\left(\underline{\Gamma_i}\right)$ and the $\left(\underline{\Gamma_{[1,n]}}\right)$, we may assume that:

$$U\left(\left(\underline{\Gamma_i}\right), \underline{\Gamma_{[1,n]}}\right)$$

is proportional to:

$$\delta\left(f\left(\left(\boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right), \boldsymbol{\alpha}_{[1,n]}, \mathbf{p}_{[1,n]}\right)\right) \tag{96}$$

the Dirac function $\delta$ implementing the condition between the structures characteristics to interaction.

We can deduce the condition of activation for the $\Gamma_{[1,n]}$. Actually, given the condition (96), the $\left(\underline{\Gamma_i}\right)$ remains neutral, that is unactivated, if the following condition:

$$\frac{\delta U\left(\left(\underline{\Gamma_i}\right), \underline{\Gamma_{[1,n]}}\right)}{\delta \underline{\Gamma_i}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2, \theta\right)} = 0 \tag{97}$$

is not satisfied.

If on the contrry the $\left(\underline{\Gamma_i}\right)$ are activated, their effective action at their minima is negative:

$$\sum_i S\left(\underline{\Gamma_i}\left(\Gamma_{[1,n]}\right)\right)+U\left(\left(\underline{\Gamma_i}\left(\Gamma_{[1,n]}\right)\right),\underline{\Gamma_{[1,n]}}\right)<0 \tag{98}$$

The full effective action for the $\underline{\Gamma_{[1,n]}}$ is

$$S_e\left(\underline{\Gamma_{[1,n]}}\right)=S\left(\underline{\Gamma_{[1,n]}}\right)+\sum_i S\left(\underline{\Gamma_i}\left(\Gamma_{[1,n]}\right)\right)+U\left(\left(\underline{\Gamma_i}\left(\Gamma_{[1,n]}\right)\right),\underline{\Gamma_{[1,n]}}\right)$$

and as a consequence of (98):

$$S_e\left(\underline{\Gamma_{[1,n]}}\right)<S\left(\underline{\Gamma_{[1,n]}}\right)$$

This lower value of the effective action for $\Gamma_{[1,n]}$ leads to a possibility of activation, even if the $\Gamma_{[1,n]}$ were initially neutral.

#### 10.2.2 First example

Assume two initial structures, i.e., $n=2$ and the following action functional for the system:

$$\begin{aligned}
&\sum_{i=1,2}\underline{\Gamma_i}^{\dagger}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\left(-\frac{1}{2}\nabla^2_{\left(\hat{\mathbf{T}}\right)_{S_i^2}}+\frac{1}{2}\left(\left(\mathbf{D}_{S_i^2}+\left(\mathbf{M}^{\boldsymbol{\alpha}_i}_{p_i}\right)_{S_i^2}\right)\Delta\mathbf{T}^{\boldsymbol{\alpha}_i}_{p_i}\right)^2\right)\underline{\Gamma_i}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\\
&+V_i\left(\left|\underline{\Gamma_i}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right|^2\right)\\
&+\underline{\Gamma_{12}}^{\dagger}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\left(-\frac{1}{2}\nabla^2_{\left(\Delta\mathbf{T}^{\alpha\beta}_p\right)}+\frac{1}{2}\left(\left(\mathbf{D}+\left(\mathbf{M}^{\alpha\beta}_p\right)\right)\left(\Delta\mathbf{T}^{\alpha\beta}_p-\langle\Delta\mathbf{T}\rangle^{\alpha\beta}_p\right)\right)^2\right)\underline{\Gamma_{12}}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\\
&+V_{12}\left(\left|\underline{\Gamma_{12}}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\right|^2\right)\\
&+U\left(\prod_{i=1,2}\left|\underline{\Gamma_i}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right|^2 f\left(\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right),\left\{\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right\}_{1,2}\right),\left|\underline{\Gamma_{12}}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\right|^2\right)
\end{aligned} \tag{99}$$

It describes two independent structures $\underline{\Gamma_i}$ along with a potential composite structure $\underline{\Gamma_{12}}$. The potential $U$ transcribes the possible transitions between $\underline{\Gamma_i}$ and $\Gamma_{12}$.

Assume that $V_i$, $V_{12}$ are such that $V_{12}(0)=0$ and the ground states $\underline{\Gamma_i}$ are activated, but $\underline{\Gamma_{12}}$ is not. Thus, the stable state for the composite structure is $\underline{\Gamma_{12}}=0$.

**Ground state without interaction** In absence of interaction, integrating the $\Gamma_i$' degrees of freedom corresponds in first approximation to replace the dields $\underline{\Gamma_i}$ by their saddle point solution $\underline{\Gamma_i}^{(0)}$ where these saddle points equations are independent:

$$\left(-\frac{1}{2}\nabla^2_{\left(\mathbf{T}\right)_{S_i^2}}+\frac{1}{2}\left(\left(\mathbf{D}_{S_i^2}+\left(\mathbf{M}^{\boldsymbol{\alpha}_i}_{p_i}\right)_{S_i^2}\right)\Delta\mathbf{T}^{\boldsymbol{\alpha}_i}_{p_i}\right)^2\right)\underline{\Gamma_i}^{(0)}+\frac{\delta}{\delta\left|\underline{\Gamma_i}\right|^2}V_i\left(\left|\underline{\Gamma_i^{(0)}}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right|^2\right)\underline{\Gamma_i}^{(0)}=0$$

as explained above, in this state, the $\underline{\Gamma_i}$ are activated. They do not interact with the field $\Gamma_{12}$, and the combined structure $\underline{\Gamma_{12}}$ is not activated. Actually, since $V_{12}(0)=0$, the ground state for this structure is $\underline{\Gamma_{12}}=0$.

**Interactions modified background** To describe the interactions we first define:

$$f\left(\{i\},(1,2)\right)=f\left(\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^{2}\right),\left\{\left(\mathbf{T}_{i},\boldsymbol{\alpha}_{i},\mathbf{p}_{i},S_{i}^{2}\right)\right\}_{1,2}\right)$$

Then, including the interactions in the action, the saddle point equation rewrites:

$$\begin{aligned}0 &= \left(-\frac{1}{2}\nabla_{(\mathbf{T})_{S_i^2}}^{2}+\frac{1}{2}\left(\left(\mathbf{D}_{S_i^2}+\left(\mathbf{M}_{p_i}^{\boldsymbol{\alpha}_i}\right)_{S_i^2}\right)\Delta\mathbf{T}_{p_i}^{\boldsymbol{\alpha}_i}\right)^{2}\right)\underline{\Gamma_i}+\frac{\delta}{\delta\left|\underline{\Gamma_i}\right|^{2}}V_i\left(\left|\underline{\Gamma_i}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right|^{2}\right)\underline{\Gamma_i}\\&+\frac{\delta}{\delta\left|\underline{\Gamma_i}\right|^{2}}U\left(\prod_{i=1,2}\left|\underline{\Gamma_i}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right|^{2}f\left(\{i\},(1,2)\right),\left|\underline{\Gamma_{12}}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\right|^{2}\right)\underline{\Gamma_i}\end{aligned}$$

This is solved in first approximation, by decomposing the field $\underline{\Gamma_i}$ as the background plus a fluctuation:

$$\underline{\Gamma_i}\simeq\underline{\Gamma_i}^{(0)}+\Delta\underline{\Gamma_i}$$

and we are led to:

$$\begin{aligned}0 &\simeq \left(\left(-\frac{1}{2}\nabla_{(\mathbf{T})_{S_i^2}}^{2}+\frac{1}{2}\left(\left(\mathbf{D}_{S_i^2}+\left(\mathbf{M}_{p_i}^{\boldsymbol{\alpha}_i}\right)_{S_i^2}\right)\Delta\mathbf{T}_{p_i}^{\boldsymbol{\alpha}_i}\right)^{2}\right)+\frac{\delta V_i\left(\left|\underline{\Gamma_i^{(0)}}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right|^{2}\right)}{\delta\left|\underline{\Gamma_i^{(0)}}\right|^{2}}\right)\Delta\underline{\Gamma_i}\\&+\frac{\delta^2 V_i\left(\left|\underline{\Gamma_i^{(0)}}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right|^{2}\right)}{\delta^2\left(\left|\underline{\Gamma_i^{(0)}}\right|^{2}\right)^{2}}\left|\underline{\Gamma_i^{(0)}}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right|^{2}\Delta\underline{\Gamma_i}\\&+\frac{\delta}{\delta\left|\underline{\Gamma_i^{(0)}}\right|^{2}}U\left(\prod_{i=1,2}\left|\underline{\Gamma_i^{(0)}}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right|^{2}f\left(\{i\},(1,2)\right),\left|\underline{\Gamma_{12}}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\right|^{2}\right)\Gamma_i^{(0)}\end{aligned}$$

with solution:

$$\underline{\Gamma_i}\simeq\underline{\Gamma_i}^{(0)}+O_i^{-1}\frac{\delta}{\delta\Gamma_i^{\dagger(0)}}U\left(\prod_{i=1,2}\left|\underline{\Gamma_i^{(0)\dagger}}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right|^{2}f\left(\{i\},(1,2)\right),\left|\underline{\Gamma_{12}}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\right|^{2}\right)\quad(100)$$

where:

$$O_{\alpha}^{-1}=\left(\left(-\frac{1}{2}\nabla_{(\hat{\mathbf{T}})_{S_i^2}}^{2}+\frac{1}{2}\left(\left(\mathbf{D}_{S_i^2}+\left(\mathbf{M}_{p_i}^{\boldsymbol{\alpha}_i}\right)_{S_i^2}\right)\Delta\mathbf{T}_{p_i}^{\boldsymbol{\alpha}_i}\right)^{2}\right)+\frac{\delta V_i\left(\left|\underline{\Gamma_i}^{(0)}\right|^{2}\right)}{\delta\left|\underline{\Gamma_i}^{(0)}\right|^{2}}+\frac{\delta^2 V_i\left(\left|\underline{\Gamma_i^{(0)}}\right|^{2}\right)}{\delta\left(\left|\underline{\Gamma_i^{(0)}}\right|^{2}\right)^{2}}\left|\underline{\Gamma_i^{(0)}}\right|^{2}\right)^{-1}$$

**Effective action for $\Gamma_{12}$** Once the saddle point solution is known, we can rewrite the effective action for $\Gamma_i$ as a function of $\Gamma_{12}$, thereby integrating the degrees of freedom associated with $\Gamma_i$ to obtain an effective potential for $\Gamma_{12}$:

$$\begin{aligned}
&\underline{\Gamma_i}^{\dagger}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\left(-\frac{1}{2}\nabla^2_{(\hat{\mathbf{T}})_{S_i^2}}+\frac{1}{2}\left(\left(\mathbf{D}_{S_i^2}+\left(\mathbf{M}_{p_i}^{\boldsymbol{\alpha}_i}\right)_{S_i^2}\right)\Delta\mathbf{T}_{p_i}^{\boldsymbol{\alpha}_i}\right)^2\right)\underline{\Gamma_i}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right) \qquad (101)\\
&+V_i\left(\left|\underline{\Gamma_i}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right|^2\right)\\
=\ & V_i\left(\left|\underline{\Gamma_i}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right|^2\right)\\
&-\underline{\Gamma_i}^{\dagger}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\left(\frac{\delta V_i\left(\left|\underline{\Gamma_i}\right|^2\right)}{\delta\left|\underline{\Gamma_i}\right|^2}+\frac{\delta}{\delta\left|\underline{\Gamma_i}\right|^2}U\left(\prod_{i=1,2}\left|\underline{\Gamma_i}\right|^2 f\left(\{i\},(1,2)\right),\left|\underline{\Gamma_{12}}\right|^2\right)\right)\underline{\Gamma_i}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\\
\simeq\ & V_i\left(\left|\underline{\Gamma_i^{(0)}}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right|^2\right)+\frac{1}{2}\frac{\delta^2 V_i\left(\left|\underline{\Gamma_i^{(0)}}\right|^2\right)}{\delta\left(\left|\underline{\Gamma_i^{(0)}}\right|^2\right)^2}\left|\Delta\underline{\Gamma_i}\right|^2\\
&-\underline{\Gamma_i^{(0)}}^{\dagger}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\frac{\delta}{\delta\left|\underline{\Gamma_i^0}\right|^2}U\left(\prod_{i=1,2}\left|\underline{\Gamma_i}\right|^2 f\left(\{i\},(1,2)\right),\left|\underline{\Gamma_{12}}\right|^2\right)\underline{\Gamma_i^{(0)}}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)
\end{aligned}$$

The effective action for $\underline{\Gamma_{12}}$ combines the $\Gamma_{12}$ part of (99) plus (101), so that it writes:

$$\begin{aligned}
&\underline{\Gamma_{12}}^{\dagger}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\left(-\frac{1}{2}\nabla^2_{\left(\Delta\mathbf{T}_p^{\alpha\beta}\right)}+\frac{1}{2}\left(\left(\mathbf{D}+\left(\mathbf{M}_p^{\alpha\beta}\right)\right)\left(\Delta\mathbf{T}_p^{\alpha\beta}-\langle\Delta\mathbf{T}\rangle_p^{\alpha\beta}\right)\right)^2\right)\underline{\Gamma_{12}}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\\
&+V_{12}\left(\left|\underline{\Gamma_{12}}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\right|^2\right)+U\left(\prod_{i=1,2}\left|\underline{\Gamma_i}^{(0)}\right|^2 f\left(\{i\},(1,2)\right),\left|\underline{\Gamma_{12}}\right|^2\right)\\
&+\frac{1}{2}\frac{\delta^2 V_i\left(\left|\underline{\Gamma_i^{(0)}}\right|^2\right)}{\delta\left(\left|\underline{\Gamma_i^{(0)}}\right|^2\right)^2}\left|\Delta\underline{\Gamma_i}\right|^2-\underline{\Gamma_i^{(0)}}^{\dagger}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\frac{\delta U\left(\prod_{i=1,2}\left|\underline{\Gamma_i}\right|^2 f\left(\{i\},(1,2)\right),\left|\underline{\Gamma_{12}}\right|^2\right)}{\delta\left|\underline{\Gamma_i^0}\right|^2}\underline{\Gamma_i^{(0)}}\left(\mathbf{T}_i,\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)
\end{aligned}$$

Assume that $U$ is proportional to:

$$\prod_{i=1,2}\left|\underline{\Gamma_i}^{(0)\dagger}\right|^2$$

Thus,the effective action for $\underline{\Gamma_{12}}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)$ becomes:

$$\begin{aligned}
&\underline{\Gamma_{12}}^{\dagger}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\left(-\frac{1}{2}\nabla^2_{\left(\Delta\mathbf{T}_p^{\alpha\beta}\right)}+\frac{1}{2}\left(\left(\mathbf{D}+\left(\mathbf{M}_p^{\alpha\beta}\right)\right)\left(\Delta\mathbf{T}_p^{\alpha\beta}-\langle\Delta\mathbf{T}\rangle_p^{\alpha\beta}\right)\right)^2\right)\underline{\Gamma_{12}}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\\
&+V_{12}^{(e)}\left(\left|\underline{\Gamma_{12}}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\right|^2\right)
\end{aligned}$$

with the effective potential defined by:

$$\begin{aligned}
&V_{12}^{(e)}\left(\left|\underline{\Gamma_{12}}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\right|^2\right) \qquad (102)\\
=\ & V_{12}\left(\left|\underline{\Gamma_{12}}\left(\mathbf{T}_{1,2},\boldsymbol{\alpha}_{1,2},\mathbf{p}_{1,2},S_{1,2}^2\right)\right|^2\right)+\frac{1}{2}\frac{\delta^2 V_i\left(\left|\underline{\Gamma_i^{(0)}}\right|^2\right)}{\delta\left(\left|\underline{\Gamma_i^{(0)}}\right|^2\right)^2}\left|\Delta\underline{\Gamma_i}\right|^2-U\left(\prod_{i=1,2}\left|\underline{\Gamma_i}^{(0)}\right|^2 f\left(\{i\},(1,2)\right),\left|\underline{\Gamma_{12}}\right|^2\right)
\end{aligned}$$

The first term in (102) is the potential for structure 12. It is not sufficient to allow the emergenc of the structure. The second term:

$$\frac{1}{2}\frac{\delta^2 V_i\left(\left|\underline{\Gamma}_i^{(0)}\right|^2\right)}{\delta\left(\left|\underline{\Gamma}_i^{(0)}\right|^2\right)^2}\left|\Delta\underline{\Gamma}_i\right|^2$$

stabilizes the structures $i$ in its stable equilibrium and prevent the transition of states.

However, the third term:

$$-U\left(\prod_{i=1,2}\left|\underline{\Gamma}_i{}^{(0)}\right|^2 f\left(\{i\},(1,2)\right),\left|\underline{\Gamma}_{12}\right|^2\right)$$

represents a gain of transitioning from 1 and 2 to 12.

As a consequence, if:

$$\frac{1}{2}\frac{\delta^2 V_i\left(\left|\underline{\Gamma}_i^{(0)}\right|^2\right)}{\delta\left(\left|\underline{\Gamma}_i^{(0)}\right|^2\right)^2}\left|\Delta\underline{\Gamma}_i\right|^2-U\left(\prod_{i=1,2}\left|\underline{\Gamma}_i{}^{(0)}\right|^2 f\left(\{i\},(1,2)\right),\left|\underline{\Gamma}_{12}\right|^2\right)<0$$

then $V_{12}^{(e)} < V_{12}$ and there is a possibility for a composed stable state. One condition is that $f\left(\{i\},(1,2)\right) >> 1$.

### 10.2.3 Second example, structures activation by substructures:

We study an example in which the activation of a substructure may activate the full structure. Consider the transitions:

$$\left|\mathbf{T},\boldsymbol{\alpha},\mathbf{p},(S'\subset S)^2\right\rangle\rightarrow\left|\mathbf{T}',\boldsymbol{\alpha}',\mathbf{p}',S^2\right\rangle \tag{103}$$

The effective action for $S_1^2 \subset S^2$ writes:

$$\begin{aligned}
&\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\left(-\frac{1}{2}\nabla^2_{\left(\hat{\mathbf{T}}\right)_{S^2}}+\frac{1}{2}\left(\left(\mathbf{D}_{S^2}+\left(\mathbf{M}_p^{\alpha}\right)_{S^2}\right)\Delta\mathbf{T}_p^{\alpha}\right)^2\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)+V\left(\left|\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\right|^2\right)\\
&+\underline{\Gamma}^{\dagger}\left(\mathbf{T}_1,\boldsymbol{\alpha}_1,\mathbf{p}_1,S_1^2\right)\left(-\frac{1}{2}\nabla^2_{\left(\hat{\mathbf{T}}\right)_{S_1^2}}+\frac{1}{2}\left(\left(\mathbf{D}_{S_1^2}+\left(\mathbf{M}_p^{\alpha}\right)_{S_1^2}\right)\Delta\mathbf{T}_p^{\alpha}\right)^2\right)\underline{\Gamma}\left(\mathbf{T}_1,\boldsymbol{\alpha}_1,\mathbf{p}_1,S_1^2\right)\\
&+V_1\left(\left|\underline{\Gamma}\left(\mathbf{T}_1,\boldsymbol{\alpha}_1,\mathbf{p}_1,S_1^2\right)\right|^2\right)\\
&+I\left(\left(\boldsymbol{\alpha},\mathbf{p}\right),\left(\boldsymbol{\alpha}_i',\mathbf{p}_i'\right),\left(\boldsymbol{\alpha},\mathbf{p}\right),S^2,S_1^2\right)\underline{\Gamma_i}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S_i^2\right)\underline{\Gamma}_1\left(\mathbf{T},\boldsymbol{\alpha}',\mathbf{p}',S_i^2\right)\underline{\Gamma}\left(\mathbf{T}_1,\boldsymbol{\alpha}_1,\mathbf{p}_1S_1^2\right)
\end{aligned}$$

To model the activation (103), we assume that in absence of interaction term, the state describing the substructure is more stable than the state describing the entire structure. Thus, we assume that for $I=0$:

$$S\left(\left|\underline{\Gamma}\left(\mathbf{T}_1,\boldsymbol{\alpha}_1,\mathbf{p}_1,S_1^2\right)\right|^2,0\right)<S\left(\left|\underline{\Gamma}\left(\mathbf{T}_1,\boldsymbol{\alpha}_1,\mathbf{p}_1,S_1^2\right)\right|^2,\left|\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\right|^2\right)$$

so that the background states can be assumed to satisfy:

$$\left|\underline{\Gamma}\left(\mathbf{T}_1,\boldsymbol{\alpha}_1,\mathbf{p}_1,S_1^2\right)\right|^2\neq 0$$

and:

$$\left|\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\right|^2=0$$

However, if interaction $I$ induces activation of $\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)$ then the saddle point equation for both structures:

$$\begin{aligned}\frac{\delta S\left(\left|\underline{\Gamma}_{1}\right|^{2},\left|\underline{\Gamma}\right|^{2}\right)}{\delta\underline{\Gamma}_{1}} &= \left(-\frac{1}{2}\nabla_{\left(\hat{\mathbf{T}}\right)_{S_{1}^{2}}}^{2}+\frac{1}{2}\left(\left(\mathbf{D}_{S_{1}^{2}}+\left(\mathbf{M}_{p}^{\alpha}\right)_{S_{1}^{2}}\right)\Delta\mathbf{T}_{p}^{\alpha}\right)^{2}\right)\underline{\Gamma}\left(\mathbf{T}_{1},\boldsymbol{\alpha}_{1},\mathbf{p}_{1},S_{1}^{2}\right) \\ &+\frac{\delta V_{1}\left(\left|\underline{\Gamma}\left(\mathbf{T}_{1},\boldsymbol{\alpha}_{1},\mathbf{p}_{1},S_{1}^{2}\right)\right|^{2}\right)}{\delta\underline{\Gamma}_{1}^{\dagger}} \\ &+I\left(\left(\boldsymbol{\alpha},\mathbf{p}\right),\left(\boldsymbol{\alpha}_{i}^{\prime},\mathbf{p}_{i}^{\prime}\right),\left(\boldsymbol{\alpha},\mathbf{p}\right),S^{2},S_{1}^{2}\right)\underline{\Gamma}_{1}\left(\mathbf{T},\boldsymbol{\alpha}^{\prime},\mathbf{p}^{\prime},S_{i}^{2}\right)\underline{\Gamma}\left(\mathbf{T}_{1},\boldsymbol{\alpha}_{1},\mathbf{p}_{1}S_{1}^{2}\right)\end{aligned} \tag{104}$$

and:

$$\begin{aligned}\frac{\delta S\left(\left|\underline{\Gamma}_{1}\right|^{2},\left|\underline{\Gamma}\right|^{2}\right)}{\delta\underline{\Gamma}} &= \left(-\frac{1}{2}\nabla_{\left(\hat{\mathbf{T}}\right)_{S^{2}}}^{2}+\frac{1}{2}\left(\left(\mathbf{D}_{S^{2}}+\left(\mathbf{M}_{p}^{\alpha}\right)_{S^{2}}\right)\Delta\mathbf{T}_{p}^{\alpha}\right)^{2}\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right) \\ &+\frac{\delta V\left(\left|\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right|^{2}\right)}{\delta\underline{\Gamma}^{\dagger}} \\ &+I\left(\left(\boldsymbol{\alpha},\mathbf{p}\right),\left(\boldsymbol{\alpha}_{i}^{\prime},\mathbf{p}_{i}^{\prime}\right),\left(\boldsymbol{\alpha},\mathbf{p}\right),S^{2},S_{1}^{2}\right)\left|\underline{\Gamma}_{1}\left(\mathbf{T},\boldsymbol{\alpha}^{\prime},\mathbf{p}^{\prime},S_{1}^{2}\right)\right|^{2}\end{aligned} \tag{105}$$

present a stable minimum with:

$$\begin{aligned}\left|\underline{\Gamma}\left(\mathbf{T}_{1},\boldsymbol{\alpha}_{1},\mathbf{p}_{1},S_{1}^{2}\right)\right|^{2} &\neq 0 \\ \left|\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right|^{2} &\neq 0\end{aligned}$$

To find the condition fo such stable state We first look at (104). Due to interaction, the background substructure:

$$\underline{\Gamma}_{0}\left(\mathbf{T}_{1},\boldsymbol{\alpha}_{1},\mathbf{p}_{1},S_{1}^{2}\right)$$

is shifted by the interaction to:

$$\underline{\Gamma}_{0}\left(\mathbf{T}_{1},\boldsymbol{\alpha}_{1},\mathbf{p}_{1},S_{1}^{2}\right)+\delta\underline{\Gamma}_{0}\left(\mathbf{T}_{1},\boldsymbol{\alpha}_{1},\mathbf{p}_{1},S_{1}^{2}\right)$$

and the saddle point equation (105) for the structure $\Gamma$ becomes in turn:

$$\begin{aligned}\frac{\delta S\left(\left|\underline{\Gamma}_{1}\right|^{2},\left|\underline{\Gamma}\right|^{2}\right)}{\delta\underline{\Gamma}} &= \left(-\frac{1}{2}\nabla_{\left(\hat{\mathbf{T}}\right)_{S^{2}}}^{2}+\frac{1}{2}\left(\left(\mathbf{D}_{S^{2}}+\left(\mathbf{M}_{p}^{\alpha}\right)_{S^{2}}\right)\Delta\mathbf{T}_{p}^{\alpha}\right)^{2}\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right) \\ &+\frac{\delta V\left(\left|\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right|^{2}\right)}{\delta\underline{\Gamma}^{\dagger}} \\ &+I\left(\left(\boldsymbol{\alpha},\mathbf{p}\right),\left(\boldsymbol{\alpha}_{i}^{\prime},\mathbf{p}_{i}^{\prime}\right),\left(\boldsymbol{\alpha},\mathbf{p}\right),S^{2},S_{1}^{2}\right)\left|\underline{\Gamma}_{10}\left(\mathbf{T},\boldsymbol{\alpha}^{\prime},\mathbf{p}^{\prime},S_{1}^{2}\right)\right|^{2}\end{aligned}$$

Given this equation, we can conclude that if:

$$\frac{\delta V\left(\left|\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right|^{2}\right)}{\delta\underline{\Gamma}^{\dagger}}+I\left(\left(\boldsymbol{\alpha},\mathbf{p}\right),\left(\boldsymbol{\alpha}_{i}^{\prime},\mathbf{p}_{i}^{\prime}\right),\left(\boldsymbol{\alpha},\mathbf{p}\right),S^{2},S_{1}^{2}\right)\left|\underline{\Gamma}_{10}\left(\mathbf{T},\boldsymbol{\alpha}^{\prime},\mathbf{p}^{\prime},S_{1}^{2}\right)\right|^{2}<0$$

then:

$$\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\neq 0$$

and the structure may be activated.

## 10.3 Operators formalism and transformations from independent structures to composite ones

By translating the dynamics in terms of creation and destruction of structures, the advantge of operator formalism, is to read directly which terms induce transitions between structures and thus to understand the dynamical mechanisms of transitions. We found in (83):

$$\begin{aligned} & S^{(O)}\left(\left(\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right) \\ = & \sum_{S\times S}\bar{\mathbf{D}}_{S^{2}}^{\alpha}\left(\mathbf{A}^{+}\left(\alpha,p,S^{2}\right)\mathbf{A}^{-}\left(\alpha,p,S^{2}\right)+\frac{1}{2}\right) \\ & +\sum_{m,n}\bar{U}_{mn}\left(\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\left(\hat{\mathbf{A}}^{+}\left(\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right)^{m}\left(\hat{\mathbf{A}}^{-}\left(\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right)^{n}+\hat{V} \end{aligned} \tag{106}$$

This formulation shows the instablity of a state since the form of interaction always allows apriori for transitions. However, some change of basis makes possible to integrate out the overall results of these interactions and to reveal the appearance of resulting stable dressed structures, having included the action of some structures considered as auxiliary in this perspective. This change of basis is similar to the effective action formalism but is more precise in the present approach.

### 10.3.1 Transformation of $S^{(O)}$ and emergence of composed structures

Starting with operators describing transitions between structures, the idea is to perform a transformation that modifies $S^{(O)}\left(\left(\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right)$. The transformation is performed through an operator $\exp(-F)$, with $F$ to be determined in order, at least in first approximation, to diagonalize partially (106) and cancel the interaction terms between two types of structures. This terms will be replaced by an effective interaction terms between a subset of remaining bound structures.

**Interaction terms** Technically, we divide the structures into two sets. The first one labelled by indices $k$ and $l$ describes the strctrs for which we aim at finding an effective description. The second set labelled by indices $c$ and $d$ corresponds to structures that will be integrated out to produce effective interactions in the remaining subset. The interaction between the subsets takes the form:

$$\begin{aligned} & \sum_{n,n'}\sum_{\left\{S_{k/c},S_{l/d}\right\}_{\substack{l/d=1,\ldots,n'\\ k/c=1\ldots n}}}\sum_{\left\{m'_{l/d},m_{k/c}\right\}}\prod_{l/d=1}^{n'}\left(\hat{\mathbf{A}}^{+}\left(\boldsymbol{\alpha}'_{l/d},\mathbf{p}'_{l/d},S'^{2}_{l/d}\right)\right)^{m'_{l/d}} \\ & V_{n,n'}\left(\left\{\boldsymbol{\alpha}'_{l/d},\mathbf{p}'_{l/d},S'^{2}_{l/d},m'_{l/d}\right\},\left\{\boldsymbol{\alpha}_{k/c},\mathbf{p}_{k/c},S^{2}_{k/c},m_{k/c}\right\}\right)\prod_{k/c=1}^{n}\left(\hat{\mathbf{A}}^{-}\left(\boldsymbol{\alpha}_{k/c},\mathbf{p}_{k/c},S^{2}_{k/c}\right)\right)^{m_{k}} \end{aligned}$$

where indices $l/d$ or $k/c$ indicate that the structures can be of either type. Or goal is to integrate the crossed interactions:

$$\prod_{l=1}^{n'}\left(\hat{\mathbf{A}}^{+}\left(\boldsymbol{\alpha}'_{l},\mathbf{p}'_{l},S'^{2}_{l}\right)\right)^{m'_{l}}V_{n,n'}\left(\left\{\boldsymbol{\alpha}'_{l},\mathbf{p}'_{l},S'^{2}_{l},m'_{l}\right\},\left\{\boldsymbol{\alpha}_{c},\mathbf{p}_{c},S^{2}_{c},m_{c}\right\}\right)\prod_{k/c=1}^{n}\left(\hat{\mathbf{A}}^{-}\left(\boldsymbol{\alpha}_{c},\mathbf{p}_{c},S^{2}_{c}\right)\right)^{m_{c}}+((l,c)\leftrightarrow(d,k)) \tag{107}$$

to obtain an effective action for structures $S_{k}^{2}$, $S_{l}'^{2}$.

**Transformation operator** To integrate the crossed interactions and obtain the required effective action, we consider the following transformation:

$$\left(S^{(O)}\left(\left(\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right)\right)^{\prime}=\exp\left(-F\right)S^{(O)}\left(\left(\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right)\exp\left(F\right) \tag{108}$$

where $F$ will be found to cancel the interaction term (107) after transformation. Doing so modifies $S^{(O)}\left(\left(\boldsymbol{\alpha},\mathbf{p},S^{2}\right)\right)$ to a descrption of composed, stable structures.

Disregarding the potential $\bar{U}_{mm'}\left(\boldsymbol{\alpha},\mathbf{p},S^{2}\right)$ which can be considered as slowly varying and expanding (108) at the lowest order in interactions, while canceling the interaction term (107) leads to the relation:

$$[F,S_0]+I=0 \tag{109}$$

with:

$$S_0=\bar{\mathbf{D}}_{S^2}^{\alpha}\left(\mathbf{A}^{+}\left(\alpha,p,S^{2}\right)\mathbf{A}^{-}\left(\alpha,p,S^{2}\right)+\frac{1}{2}\right)$$

so that $S_0+I$ is transformed into:

$$\begin{aligned}\left(S^{(O)}\right)^{\prime} &= S_0+I+[F,S_0]+[F,I]+\frac{1}{2}[F,[F,S_0+I]]\\ &= S_0+I+[F,S_0]+[F,I]+\frac{1}{2}[F,[F,S_0]]\end{aligned}$$

Then, using (109), we find at lowest order:

$$\left(S^{(O)}\right)^{\prime}=S_0+\frac{1}{2}[I,F] \tag{110}$$

To find this effective action, we solve (109) by postulating that $F$ has the same form as $I$:

$$\begin{aligned}F &= \sum_{nn'}\sum_{\substack{k=1\dots n\\ l=1,\dots,n'}}\sum_{\{S_k,S_l\}_{\substack{l=1,\dots,n'\\ k=1\dots n}}}\prod_{l=1}^{n'}\prod_{s=1}^{m_l'}\mathbf{A}^{+}\left(\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2\right)\\ &\quad\times F\left(\left\{\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2,m_l'\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2,m_k\right\}\right)\prod_{k=1}^{n}\prod_{s=1}^{m_k}\mathbf{A}^{-}\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)\end{aligned}$$

and in appendix 4 the solution of (109) writes:

$$F\left(\left\{\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2,m_l'\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2,m_k\right\}\right)=-\frac{V_{n,n'}\left(\left\{\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2,m_l'\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2,m_k\right\}\right)}{\sum_{l=1}^{n'}m_l'\bar{\mathbf{D}}_{S_l'^2}^{\boldsymbol{\alpha}_l'}-\sum_{k=1}^{n}m_k\bar{\mathbf{D}}_{S_k^2}^{\boldsymbol{\alpha}_k}} \tag{111}$$

**Correction (110) to the action** We also obtain the matrix elements of $[I,F]$ that modify (110). Defining:

$$\begin{aligned}\boldsymbol{\Lambda}_k &= \left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)\\ \boldsymbol{\Lambda}_l' &= \left(\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2\right)\end{aligned}$$

and:

$$\begin{aligned}\bar{\boldsymbol{\Lambda}}_{\bar{k}} &= \left(\bar{\boldsymbol{\alpha}}_{\bar{k}},\bar{\mathbf{p}}_{\bar{k}},\bar{S}_{\bar{k}}^2\right)\\ \bar{\boldsymbol{\Lambda}}_{\bar{l}}' &= \left(\bar{\boldsymbol{\alpha}}_{\bar{l}}',\bar{\mathbf{p}}_{\bar{l}}',\bar{S}_{\bar{l}}'^2\right)\end{aligned}$$

we find:

$$
\begin{aligned}
& [I,F]\left(\left\{\mathbf{\Lambda}'_L, M'_L\right\}, \left\{\mathbf{\Lambda}_K, M_K\right\}\right) \qquad (112)\\
= \; & -\sum_{P_K,P_L}\sum_{\left\{\epsilon'_d\right\},\left\{\epsilon_c\right\}}\sum_{\left\{\delta_k\right\},\left\{\delta'_l\right\}}\prod\left(\epsilon'_d!\epsilon_c!\right)^2\prod_{\bar{k},\bar{l},k,l}(-1)^{\delta'_l}C^{\delta'_l}_{m'_l+\delta'_l}C^{\delta_k}_{m_k+\delta_k}\\
& \times\bar{\mathbf{D}}^{\boldsymbol{\alpha}_k}_{S^2_k}\bar{\mathbf{D}}^{\boldsymbol{\alpha}'_l}_{S^2_k}\bar{\mathbf{D}}^{\boldsymbol{\alpha}_c}_{S^2_c}\bar{\mathbf{D}}^{\boldsymbol{\alpha}'_d}_{S^2_d}\delta\left(\mathbf{\Lambda}_k-\bar{\mathbf{\Lambda}}'_{\bar{l}}\right)\delta\left(\mathbf{\Lambda}'_l-\bar{\mathbf{\Lambda}}_{\bar{k}}\right)\delta\left(\mathbf{\Lambda}_c-\bar{\mathbf{\Lambda}}'_{\bar{d}}\right)\delta\left(\mathbf{\Lambda}'_d-\bar{\mathbf{\Lambda}}_{\bar{c}}\right)\\
& \times\frac{V^{(2)}\left(\left\{\left(\mathbf{\Lambda}'_{l\cup d}, m'_l+\delta'_l,\epsilon'_d\right),\left(\mathbf{\Lambda}_{k\cup c}, m_k+\delta_k,\epsilon_c\right),\left(\bar{\mathbf{\Lambda}}'_{\bar{l}\cup\bar{d}},\bar{m}'_{\bar{l}}+\delta_k,\epsilon_d\right),\left(\bar{\mathbf{\Lambda}}_{\bar{k}\cup\bar{c}},\bar{m}_{\bar{k}}+\delta'_l,\epsilon'_c\right)\right\}\right)}{\sum_{l=1}^{n'}\left(m'_l+\delta'_{l'}\right)\bar{\mathbf{D}}^{\boldsymbol{\alpha}'_l}_{S'^2_l}-\sum_{k=1}^{n}\left(m_k+\delta_k\right)\bar{\mathbf{D}}^{\boldsymbol{\alpha}_k}_{S^2_k}+\sum_{d=1}^{p'}\epsilon'_d\bar{\mathbf{D}}^{\boldsymbol{\alpha}'_d}_{S'^2_d}-\sum_{c=1}^{p}\epsilon_c\bar{\mathbf{D}}^{\boldsymbol{\alpha}_c}_{S^2_c}}
\end{aligned}
$$

where:

$$
\begin{aligned}
& V^{(2)}\left(\left\{\left(\mathbf{\Lambda}'_{l\cup d}, m'_l+\delta'_l,\epsilon'_d\right),\left(\mathbf{\Lambda}_{k\cup c}, m_k+\delta_k,\epsilon_c\right),\left(\bar{\mathbf{\Lambda}}'_{\bar{l}\cup\bar{d}},\bar{m}'_{\bar{l}}+\delta_k,\epsilon_d\right),\left(\bar{\mathbf{\Lambda}}_{\bar{k}\cup\bar{c}},\bar{m}_{\bar{k}}+\delta'_l,\epsilon'_c\right)\right\}\right)\\
= \; & V\left(\left\{\mathbf{\Lambda}'_l, m'_l+\delta'_l\right\}\cup\left\{\mathbf{\Lambda}'_d,\epsilon'_d\right\},\left\{\mathbf{\Lambda}_k, m_k+\delta_k\right\}\cup\left\{\mathbf{\Lambda}_c,\epsilon_c\right\}\right)\\
& \times V\left(\left\{\bar{\mathbf{\Lambda}}'_{\bar{l}},\bar{m}'_{\bar{l}}+\delta_k\right\}\cup\left\{\bar{\mathbf{\Lambda}}'_{\bar{d}},\epsilon_c\right\},\left\{\bar{\mathbf{\Lambda}}_{\bar{k}},\bar{m}_{\bar{k}}+\delta'_l\right\}\cup\left\{\bar{\mathbf{\Lambda}}_{\bar{c}},\epsilon'_d\right\}\right)
\end{aligned}
$$

with $P_K, P_L$ are partitions of $\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K,M_K\right\}$, $\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L,M'_L\right\}$:

$$
\begin{aligned}
\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L,M'_L\right\} &= \left\{\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l,m'_{l'}\right\}\cup\left\{\bar{\boldsymbol{\alpha}}'_{\bar{l}},\bar{\mathbf{p}}'_{\bar{l}},\bar{S}'^2_{\bar{l}},\bar{m}'_{\bar{l}'}\right\}\\
\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K,M_K\right\} &= \left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S^2_k,m_k\right\}\cup\left\{\bar{\boldsymbol{\alpha}}_{\bar{k}},\bar{\mathbf{p}}_{\bar{k}},\bar{S}^2_{\bar{k}},\bar{m}_{\bar{k}}\right\}
\end{aligned}
$$

In the commutator (112), we sum over all of these possible partitions.

### 10.3.2 Effective structures

After transformation, the operator version of the action writes:

$$
\begin{aligned}
& \left(S^{(O)}\right)' = \sum_{S\times S}\bar{\mathbf{D}}^{\alpha}_{S^2}\left(\mathbf{A}^+\left(\alpha,p,S^2\right)\mathbf{A}^-\left(\alpha,p,S^2\right)+\frac{1}{2}\right) \qquad (113)\\
& +\sum_{n,n'}\sum_{\left\{S_k,S_l\right\}_{\substack{l=1,\ldots,n'\\k=1\ldots n}}}\sum_{\left\{m'_l,m_k\right\}}\prod_{l=1}^{n'}\left(\hat{\mathbf{A}}^+\left(\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l\right)\right)^{m'_l}\\
& \times V_{n,n'}\left(\left\{\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l,m'_l\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S^2_k,m_k\right\}\right)\prod_{k=1}^{n}\left(\hat{\mathbf{A}}^-\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S^2_k\right)\right)^{m_k}\\
& +\frac{1}{2}\prod_{L=1}^{n'}\prod_{s'=1}^{M'_L}\mathbf{A}^+\left(\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L\right)[I,F]\left(\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L,M'_L\right\},\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K,M_K\right\}\right)\prod_{K=1}^{n}\prod_{s=1}^{M'_k}\mathbf{A}^-\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)
\end{aligned}
$$

In effective action:

$$
[I,F]\left(\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L,M'_L\right\},\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K,M_K\right\}\right)
$$

the structures $\left\{\boldsymbol{\alpha}_c,\mathbf{p}_c,S^2_c\right\}$ $\left\{\bar{\boldsymbol{\alpha}}'_{\bar{d}},\bar{\mathbf{p}}'_{\bar{d}},\bar{S}'^2_{\bar{d}}\right\}$ have been integrated and do not appear anymore in the interaction. They have glued structures $\left\{\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l\right\}$ and $\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S^2_k\right\}$ even if this ones were not interacting initially that is, even if:

$$
V_{n,n'}\left(\left\{\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l,m'_l\right\}_{l\leqslant n'},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S^2_k,m_k\right\}_{l\leqslant n}\right)=0 \qquad (114)
$$

Depending on the form of the resulting interaction (113), some new combined structures may appear.

### 10.3.3 Bound states

Assuming that condition (114) is satisfied, we can describe the combined structures by computing the

the eigenstates of (113) with lowest eigenvalues. It is written as a series involving the $n$ types of sructures:

$$\left|\prod_K \left(\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2, M_K\right)\right)\right\rangle = \sum_{(M_K)} A\left(\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2, M_K\right)\right) \prod_{K=1}^{n} \prod_{s=1}^{M_K} \mathbf{A}^+\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right) \prod_K |Vac\rangle_K$$

**Stt wth lwst gnvls** At the lowest order of the series expansion, the eigenstate writes:

$$\left|\prod_K \left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right\rangle = \sum_K A\left(\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right) \mathbf{A}^+\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right) \prod_K |Vac\rangle_K$$

The coefficients $A\left(\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right)$ are obtained by writing the eigenvalues equation for $\left|\prod_K \left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right\rangle$. The action of $\left(S^{(O)}\right)$ on $\left|\prod_K \left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right\rangle$ yields:

$$\begin{aligned}
&\left(S^{(O)}\right)\left|\prod_K \left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right\rangle \\
= &\sum_K A\left(\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right)\left(\sum_K \bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}} + 2\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}\right) \mathbf{A}^+\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right) \prod_K |Vac\rangle_K \\
&+\sum_{K,L} A\left(\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right) [I,F]\left(\left\{\boldsymbol{\alpha}_L', \mathbf{p}_L', S_L'^2, 1\right\}, \left\{\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2, 1\right\}, \left(\boldsymbol{\alpha}_P, \mathbf{p}_P, S_P^2, 0\right)\right) \mathbf{A}^+\left(\boldsymbol{\alpha}_L', \mathbf{p}_L', S_L'^2\right) \prod_K |Vac\rangle_K
\end{aligned}$$

writing the eigenvalues:

$$\eta\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right) = \sum_{K'} \bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}} + 2\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K} - \varepsilon\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)$$

We show in appendix 4 that for weak interactions, i.e.:

$$[I,F]\left(\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2, 1\right), \left(\boldsymbol{\alpha}_{K'}, \mathbf{p}_{K'}, S_{K'}^2, 1\right)\right) << 1$$

the eigenvalues are:

$$\varepsilon\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right) = \sum_{K'} \frac{[I,F]\left(\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2, 1\right), \left(\boldsymbol{\alpha}_{K'}, \mathbf{p}_{K'}, S_{K'}^2, 1\right)\right) [I,F]\left(\left(\boldsymbol{\alpha}_{K'}, \mathbf{p}_{K'}, S_{K'}^2, 1\right), \left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2, 1\right)\right)}{\left(\bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}} + 2\bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}} - \eta\right)}$$

and the corresponding eigenstates are defined by the following relation on coefficients:

$$A\left(\boldsymbol{\alpha}_L, \mathbf{p}_L, S_L^2\right) \simeq -\frac{[I,F]\left(\left(\boldsymbol{\alpha}_L, \mathbf{p}_L, S_L^2, 1\right), \left(\boldsymbol{\alpha}_{K'}, \mathbf{p}_{K'}, S_{K'}^2, 1\right)\right)}{\bar{\mathbf{D}}_{S_L^2}^{\boldsymbol{\alpha}_L} + 2\bar{\mathbf{D}}_{S_L^2}^{\boldsymbol{\alpha}_L}} A\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)$$

Given one value $A\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)$, all other coefficients of the series are known. In first approximation;

$$\eta_K = \sum_{K'} \bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}} + 2\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K} + \varepsilon_K$$

where:

$$\varepsilon_K A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right)=\sum_L A\left(\left(\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S_L'^2\right)\right)[I,F]\left(\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,1\right\},\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S_L'^2,1\right\},\left(\boldsymbol{\alpha}_P,\mathbf{p}_P,S_P^2,0\right)\right)$$

and the eigenstates writes:

$$\begin{aligned}&\left|\prod_K\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right\rangle \qquad (115)\\ =\ &\mathcal{N}\left\{\sum_K A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right)H_1\left(\frac{\sigma_T}{2\sqrt{2}}\left(\mathbf{\Delta T}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)^t\left(\mathbf{A}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)_{S_K^2}\left(\mathbf{\Delta T}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)_{S_K^2}\right)\right\}\\ &\times\exp\left(-\frac{1}{2}\sum_K\left(\mathbf{\Delta T}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)^t\left(\mathbf{A}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)_{S_K^2}\left(\mathbf{\Delta T}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)_{S_K^2}\right)\end{aligned}$$

where $H_1$ is the Hermite polynomial:

$$H_1\left(\frac{\sigma_T}{2\sqrt{2}}\left(\mathbf{\Delta T}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)^t\left(\mathbf{A}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)_{S_K^2}\left(\mathbf{\Delta T}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)_{S_K^2}\right)=\frac{\sigma_T}{2\sqrt{2}}\left(\mathbf{\Delta T}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)^t\left(\mathbf{A}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)_{S_K^2}\left(\mathbf{\Delta T}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)_{S_K^2}$$

and $\mathcal{N}$ normalization factor. The form of this state is similar to the states obtained while describing independent structures. However, in the present approach, the structures obtained combine initial independent structures. We thus obtain, through the interaction with a third part, an integrated structure encompassing the characteristics of some "primary" collective states.

**Full series expansion** More generally, the state is determined by a series:

$$\begin{aligned}&\left|\prod_K\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\right\rangle\\ =\ &\sum_{(M_K)}A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\right)\prod_{K=1}^n\prod_{s=1}^{M_K}\mathbf{A}^+\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\prod_K|Vac\rangle_K\end{aligned}$$

Given the definition of the creation operators, this writes:

$$\begin{aligned}\left|\prod_K\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\right\rangle\ =\ &\mathcal{N}\sum_{(M_K)}A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_k\right)\right)\prod_{K=1}^n H_n\left(\frac{\sigma_T}{2\sqrt{2}}\left(\mathbf{\Delta T}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)^t\left(\mathbf{A}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)_{S_K^2}\left(\mathbf{\Delta T}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)_{S_K^2}\right)\\ &\times\exp\left(-\frac{1}{2}\sum_K\left(\mathbf{\Delta T}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)^t\left(\mathbf{A}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)_{S_K^2}\left(\mathbf{\Delta T}_{\mathbf{p}_K}^{\boldsymbol{\alpha}_K}\right)_{S_K^2}\right)\end{aligned}$$

where $H_n$ is the n-th Hermite polynomial. The coefficients are computed in appendix 6. These states are similar to (115) but include higher level of activty.

## 11 Exemple with 3 structures

We present both the effective formalism and the operator formalism approach to study the binding of two independent structures through the intermediation of a third one. Binding structures through a third part is described by the diagram:

$$\left|\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\rangle\rightleftarrows\left|\mathbf{T}_l,\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right\rangle\rightleftarrows\left|\mathbf{T}_{k'},\boldsymbol{\alpha}_{k'},\mathbf{p}_{k'},S_{k'}^2\right\rangle\overset{\text{effective}}{\Longrightarrow}\left|\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\rangle\rightleftarrows\left|\mathbf{T}_{k'},\boldsymbol{\alpha}_{k'},\mathbf{p}_{k'},S_{k'}^2\right\rangle$$

Initially, structures $\left|\mathbf{T}_l,\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right\rangle$ interacts both with $\left|\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\rangle$ and $\left|\mathbf{T}_{k'},\boldsymbol{\alpha}_{k'},\mathbf{p}_{k'},S_{k'}^2\right\rangle$ that are a priori not related. If we consider that the time scale of $\left|\mathbf{T}_l,\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right\rangle$ is longer than that of the others, it can be integrated out, which yields the effective interaction between $\left|\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\rangle$ and $\left|\mathbf{T}_{k'},\boldsymbol{\alpha}_{k'},\mathbf{p}_{k'},S_{k'}^2\right\rangle$.

### 11.1 Effective formalism

Within the effective field formalism the situation is described as the follwing. Consider three structures. First, we study independently the three structures:

$$
\begin{aligned}
S_0 =& \sum_{i=1,2} \underline{\Gamma_i}^{\dagger}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right)\left(-\frac{1}{2}\nabla^2_{(\hat{\mathbf{T}})_{S_i^2}}+\frac{1}{2}\left(\left(\mathbf{D}_{S_i^2}+\left(\mathbf{M}_p^{\alpha}\right)_{S_i^2}\right)\Delta\mathbf{T}_p^{\alpha}\right)^2\right)\underline{\Gamma_i}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right)\\
&+V_i\left(\left|\underline{\Gamma_i}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right)\right|^2\right)\\
&+\underline{\Gamma_0}^{\dagger}\left(\mathbf{T}_0, \boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2\right)\left(-\frac{1}{2}\nabla^2_{(\hat{\mathbf{T}})_{S_0^2}}+\frac{1}{2}\left(\left(\mathbf{D}_{S_0^2}+\left(\mathbf{M}_p^{\alpha}\right)_{S_0^2}\right)\Delta\mathbf{T}_p^{\alpha}\right)^2\right)\underline{\Gamma_0}\left(\mathbf{T}_0, \boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2\right)\\
&+V_0\left(\left|\underline{\Gamma_0}\left(\mathbf{T}_0, \boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2\right)\right|^2\right)
\end{aligned}
$$

The structure denoted 0 has a dynamic with lower frequencies. We assume that $\underline{\Gamma_i}$ do not interact with each other. Interactions arise through structure 0. We consider the interaction term:

$$
\begin{aligned}
I =& \sum_{i=0,1,2} \underline{\Gamma_i}^{\dagger}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right)\left(-\frac{1}{2}\nabla^2_{(\hat{\mathbf{T}})_{S_i^2}}+\frac{1}{2}\left(\left(\mathbf{D}_{S_i^2}+\left(\mathbf{M}_{p_i}^{\boldsymbol{\alpha}_i}\right)_{S_i^2}\right)\Delta\mathbf{T}_{p_i}^{\boldsymbol{\alpha}_i}\right)^2\right)\underline{\Gamma_i}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right)\\
&+V_i\left(\left|\underline{\Gamma_i}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right)\right|^2\right)\\
&+\sum_{i=1,2} I\left(\left(\boldsymbol{\alpha}_i, \mathbf{p}_i\right),\left(\boldsymbol{\alpha}_i', \mathbf{p}_i'\right),\left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right), S_i^2, S_0^2\right)\underline{\Gamma_i}^{\dagger}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right)\underline{\Gamma_i}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i', \mathbf{p}_i', S_i^2\right)\underline{\Gamma_0}\left(\mathbf{T}_0, \boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2\right)
\end{aligned}
$$

We compute the vaccum state for $\underline{\Gamma_0}\left(\mathbf{T}_0, \boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2\right)$ as a function of the $\underline{\Gamma_i}$. These two structures act as source terms. The saddle point equations for $S_0+I$ with respect to $\underline{\Gamma_0}$ yield:

$$
\begin{aligned}
\underline{\Gamma_0}\left(\mathbf{T}_0, \boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2\right) =& -\sum_{i=1,2}\int G\left(\left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right),\left(\boldsymbol{\alpha}_0', \mathbf{p}_0'\right)\right) d\left(\boldsymbol{\alpha}_0', \mathbf{p}_0'\right)\\
& I\left(\left(\boldsymbol{\alpha}_i, \mathbf{p}_i\right),\left(\boldsymbol{\alpha}_i', \mathbf{p}_i'\right),\left(\boldsymbol{\alpha}_0', \mathbf{p}_0'\right), S_i^2, S_0^2\right)\underline{\Gamma_i}^{\dagger}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right)\underline{\Gamma_i}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i', \mathbf{p}_i', S_i^2\right)
\end{aligned}
\tag{116}
$$

with the Green function $G\left(\left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right),\left(\boldsymbol{\alpha}_0', \mathbf{p}_0'\right)\right)$ is the inverse of the following operator:

$$
G\left(\left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right),\left(\boldsymbol{\alpha}_0', \mathbf{p}_0'\right)\right)=\left(-\frac{1}{2}\nabla^2_{(\hat{\mathbf{T}})_{S_0^2}}+\frac{1}{2}\left(\left(\mathbf{D}_{S_0^2}+\left(\mathbf{M}_p^{\alpha}\right)_{S_0^2}\right)\Delta\mathbf{T}_p^{\alpha}\right)^2\right)^{-1}
$$

The effective action for $\underline{\Gamma_i}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right)$, $i=1,2$ is obtained by replacing (116) in $S_0+I$:

$$
\begin{aligned}
S_f =& \sum_{i=0,1,2} \underline{\Gamma_i}^{\dagger}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right)\left(-\frac{1}{2}\nabla^2_{(\hat{\mathbf{T}})_{S_i^2}}+\frac{1}{2}\left(\left(\mathbf{D}_{S_i^2}+\left(\mathbf{M}_{p_i}^{\boldsymbol{\alpha}_i}\right)_{S_i^2}\right)\Delta\mathbf{T}_{p_i}^{\boldsymbol{\alpha}_i}\right)^2\right)\underline{\Gamma_i}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right)\\
&+V_i\left(\left|\underline{\Gamma_i}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right)\right|^2\right)\\
&+\sum_{(i,j)\in\{1,2\}^2} I_{i,j}\left(\left(\boldsymbol{\alpha}_i, \mathbf{p}_i\right),\left(\boldsymbol{\alpha}_i', \mathbf{p}_i'\right),\left(\boldsymbol{\beta}_j, \mathbf{t}_j\right),\left(\boldsymbol{\beta}_j', \mathbf{t}_j'\right), S_i^2, S_j^2\right)\\
&\times\underline{\Gamma_i}^{\dagger}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right)\underline{\Gamma_i}\left(\mathbf{T}_i, \boldsymbol{\alpha}_i', \mathbf{p}_i', S_i^2\right)\underline{\Gamma_i}^{\dagger}\left(\mathbf{T}_j, \boldsymbol{\beta}_j, \mathbf{t}_j, S_j^2\right)\underline{\Gamma_i}\left(\mathbf{T}_j', \boldsymbol{\beta}_j', \mathbf{t}_j', S_j^2\right)
\end{aligned}
$$

where:

$$
\begin{aligned}
& I_{i,j}\left(\left(\boldsymbol{\alpha}_i, \mathbf{p}_i\right),\left(\boldsymbol{\alpha}_i', \mathbf{p}_i'\right),\left(\boldsymbol{\beta}_j, \mathbf{t}_j\right),\left(\boldsymbol{\beta}_j', \mathbf{t}_j'\right), S_i^2, S_j^2\right)\\
=& \int I\left(\left(\boldsymbol{\alpha}_i, \mathbf{p}_i\right),\left(\boldsymbol{\alpha}_i', \mathbf{p}_i'\right),\left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right), S_i^2, S_0^2\right)\\
& \times G\left(\left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right),\left(\boldsymbol{\alpha}_0', \mathbf{p}_0'\right)\right) I\left(\left(\boldsymbol{\beta}_j, \mathbf{t}_j\right),\left(\boldsymbol{\beta}_j', \mathbf{t}_j'\right),\left(\boldsymbol{\alpha}_0', \mathbf{p}_0'\right), S_j^2, S_0^2\right) d\left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right) d\left(\boldsymbol{\alpha}_0', \mathbf{p}_0'\right)
\end{aligned}
$$

so that the resulting interaction binds the structures.

### 11.2 Operator formalism

To be more precise, we write the systm in terms of operators:

$$\begin{aligned}
S &= \sum_{i=0,1,2} \sqrt{\left(\bar{\mathbf{D}}_{S_i^2}\right)^2} \left(\mathbf{A}^+\left(\boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right) \mathbf{A}^-\left(\boldsymbol{\alpha}_i, \mathbf{p}_i, S_j^2\right) + \frac{1}{2}\right) \\
&\quad + \sum_{i=1,2} I\left(\left(\boldsymbol{\alpha}_i, \mathbf{p}_i\right), \left(\boldsymbol{\alpha}'_i, \mathbf{p}'_i\right), \left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right), S_i^2, S_0^2\right) \mathbf{A}^+\left(\boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right) \mathbf{A}^-\left(\boldsymbol{\alpha}'_i, \mathbf{p}'_i, S_i^2\right) \\
&\quad \times \left(\mathbf{A}^-\left(\boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2\right) + \mathbf{A}^+\left(\boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2\right)\right)
\end{aligned}$$

We will integrate over degrees of freedom of $\underline{\Gamma_0}\left(\mathbf{T}_0, \boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2\right)$ through a transformation canceling these degrees of freedom.

As explained before, we perform the transformation:

$$\exp\left(-F\right) S \exp\left(F\right)$$

such that:

$$I + \left[F, S_0\right] = 0$$

We show in appendix 4 that using (111),the operator $F$ is given by:

$$\begin{aligned}
F = - & \frac{I\left(\left(\boldsymbol{\alpha}_i, \mathbf{p}_i\right), \left(\boldsymbol{\alpha}'_i, \mathbf{p}'_i\right), \left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right), S_i^2, S_0^2\right)}{\left(\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2}\right)^2} - \sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}'_i, \mathbf{p}'_i, S_i^2}\right)^2}\right)^2 - \left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2}\right)^2} \\
& \times \left(\left(\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2}\right)^2} - \sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}'_i, \mathbf{p}'_i, S_i^2}\right)^2} + \sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2}\right)^2}\right) \mathbf{A}^-\left(\boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2\right)\right. \\
& \left. + \left(\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2}\right)^2} - \sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}'_i, \mathbf{p}'_i, S_i^2}\right)^2} - \sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2}\right)^2}\right) \mathbf{A}^{+-}\left(\boldsymbol{\alpha}_0, \mathbf{p}_0, S_0^2\right)\right)
\end{aligned} \tag{117}$$

Ultimately, we obtain:

$$S' = S_0 + \frac{1}{2}\left[I, F\right]$$

with:

$$\begin{aligned}
\frac{1}{2}\left[I, F\right] &= -\frac{1}{2} \sum_{(i,j)\in\{1,2\}^2} \frac{1}{2} \Delta\left(\left(\boldsymbol{\alpha}_i, \mathbf{p}_i\right), \left(\boldsymbol{\alpha}'_i, \mathbf{p}'_i\right), \left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right), \left(\boldsymbol{\beta}_j, \mathbf{t}_j\right), \left(\boldsymbol{\beta}'_j, \mathbf{t}'_j\right), S_i^2, S_i^2, S_0^2\right) \\
&\quad \times I\left(\left(\boldsymbol{\alpha}_i, \mathbf{p}_i\right), \left(\boldsymbol{\alpha}'_i, \mathbf{p}'_i\right), \left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right), S_i^2, S_0^2\right) I\left(\left(\boldsymbol{\beta}_j, \mathbf{t}_j\right), \left(\boldsymbol{\beta}'_j, \mathbf{t}'_j\right), \left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right), S_j^2, S_0^2\right) \\
&\quad \times \mathbf{A}^+\left(\boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right) \mathbf{A}^-\left(\boldsymbol{\alpha}'_i, \mathbf{p}'_i, S_i^2\right) \mathbf{A}^+\left(\boldsymbol{\beta}_j, \mathbf{t}_j, S_j^2\right) \mathbf{A}^-\left(\boldsymbol{\beta}'_j, \mathbf{t}'_j, S_j^2\right) \\
&= \hat{I}\left(\left(\boldsymbol{\alpha}_i, \mathbf{p}_i\right), \left(\boldsymbol{\alpha}'_i, \mathbf{p}'_i\right), \left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right), S_i^2, S_0^2\right) I\left(\left(\boldsymbol{\beta}_j, \mathbf{t}_j\right), \left(\boldsymbol{\beta}'_j, \mathbf{t}'_j\right), \left(\boldsymbol{\alpha}_0, \mathbf{p}_0\right), S_j^2, S_0^2\right) \\
&\quad \times \mathbf{A}^+\left(\boldsymbol{\alpha}_i, \mathbf{p}_i, S_i^2\right) \mathbf{A}^-\left(\boldsymbol{\alpha}'_i, \mathbf{p}'_i, S_i^2\right) \mathbf{A}^+\left(\boldsymbol{\beta}_j, \mathbf{t}_j, S_j^2\right) \mathbf{A}^-\left(\boldsymbol{\beta}'_j, \mathbf{t}'_j, S_j^2\right)
\end{aligned}$$

and where:

$$\Delta\left(\left(\boldsymbol{\alpha}_i,\mathbf{p}_i\right),\left(\boldsymbol{\alpha}'_i,\mathbf{p}'_i\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),\left(\boldsymbol{\beta}_j,\mathbf{t}_j\right),\left(\boldsymbol{\beta}'_j,\mathbf{t}'_j\right),S_i^2,S_i^2,S_0^2\right)$$
$$= \frac{\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}}{\left(\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}'_i,\mathbf{p}'_i,S_i^2}\right)^2}\right)^2-\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}$$
$$+\frac{\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}}{\left(\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\beta}_j,\mathbf{t}_j,S_j^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\beta}'_j,\mathbf{t}'_j,S_j^2}\right)^2}\right)^2-\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}$$

As a consequence, two states $\left(\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)$ and $\left(\boldsymbol{\beta}_j,\mathbf{t}_j,S_j^2\right)$ that where independent become bound through $\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)$.

Ultimately, the effective operator writes:

$$\begin{aligned}
S_f^O &= \sum_{i=1,2}\sqrt{\left(\bar{\mathbf{D}}_{S_i^2}\right)^2}\left(\mathbf{A}^+\left(\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\mathbf{A}^-\left(\boldsymbol{\alpha}_i,\mathbf{p}_i,S_j^2\right)+\frac{1}{2}\right)\\
&-\frac{1}{2}\sum_{(i,j)\in\{1,2\}^2}\frac{1}{2}\Delta\left(\left(\boldsymbol{\alpha}_i,\mathbf{p}_i\right),\left(\boldsymbol{\alpha}'_i,\mathbf{p}'_i\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),\left(\boldsymbol{\beta}_j,\mathbf{t}_j\right),\left(\boldsymbol{\beta}'_j,\mathbf{t}'_j\right),S_i^2,S_i^2,S_0^2\right)\\
&\times I\left(\left(\boldsymbol{\alpha}_i,\mathbf{p}_i\right),\left(\boldsymbol{\alpha}'_i,\mathbf{p}'_i\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_i^2,S_0^2\right)I\left(\left(\boldsymbol{\beta}_j,\mathbf{t}_j\right),\left(\boldsymbol{\beta}'_j,\mathbf{t}'_j\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_j^2,S_0^2\right)\\
&\times\mathbf{A}^+\left(\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\mathbf{A}^-\left(\boldsymbol{\alpha}'_i,\mathbf{p}'_i,S_i^2\right)\mathbf{A}^+\left(\boldsymbol{\beta}_j,\mathbf{t}_j,S_j^2\right)\mathbf{A}^-\left(\boldsymbol{\beta}'_j,\mathbf{t}'_j,S_j^2\right)
\end{aligned}$$

# 12 Extension: non localized structures

## 12.1 Describing structures with variable spatial extension

We have considered the activities and activities' oscillations of the states as endogeneous, given that they depend solely on the connectivities of the states. Similarly, the spatial extension of the structures is considered as fixed in first approximation. However, we may assume that depending on background, some connections may break and others may be created while maintaining the same properties of the state. This corresponds to describe states experiencing some displacement $S^2\rightarrow S^2+\delta S^2$.

This possibility can be taken into account in the initial formultion (24), where collective states have the form:

$$\prod_{Z,Z'}\left|\Delta T\left(Z,Z'\right),\Delta\hat{T}\left(Z,Z'\right),\alpha\left(Z,Z'\right),p\left(Z,Z'\right)\right\rangle\equiv\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle \tag{118}$$

which implies that switching from $S^2\rightarrow S^2+\delta S^2$, amounts to replace formally:

$$\begin{aligned}
&\prod_{(Z,Z')\in S^2}\left|\Delta T\left(Z,Z'\right),\Delta\hat{T}\left(Z,Z'\right),\alpha\left(Z,Z'\right),p\left(Z,Z'\right)\right\rangle\\
\rightarrow&\prod_{(Z,Z')\in S^2}\left|\Delta T\left(Z,Z'\right),\Delta\hat{T}\left(Z,Z'\right),\alpha\left(Z,Z'\right),p\left(Z,Z'\right)\right\rangle\\
&\times\prod_{(Z,Z')\in S^2+\delta S^2/S^2}\left|\Delta T\left(Z,Z'\right),\Delta\hat{T}\left(Z,Z'\right),\alpha\left(Z,Z'\right),p\left(Z,Z'\right)\right\rangle\\
&\times\prod_{(Z,Z')\in S^2/S^2+\delta S^2}\left|\Delta T\left(Z,Z'\right),\Delta\hat{T}\left(Z,Z'\right),\alpha\left(Z,Z'\right),p\left(Z,Z'\right)\right\rangle^{-1}
\end{aligned}$$

Practically, the power $-1$, amounts to divide $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ by a factor (6) for each point $(Z,Z')\in S^2/S^2+\delta S^2$. In terms of operator formalism, this translates by considering two different structure with spatial extension $S^2$ and $S^2+\delta S^2$, the switching $S^2\rightarrow S^2+\delta S^2$ being described by an interaction term between the different structures whose form is:

$$\int I\left(\boldsymbol{\alpha}',\mathbf{p}',S^2,\boldsymbol{\alpha}',\mathbf{p}',S^2+\delta S^2\right)\mathbf{A}^+\left(\boldsymbol{\alpha}',\mathbf{p}',S^2+\delta S^2\right)\mathbf{A}^-\left(\boldsymbol{\alpha},\mathbf{p},S^2\right)$$

However, to model the possibility of some permanence in the structures characteristics, independently of any localization, we rather consider the dynamic for the same structure described by the states $\left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ with $S^2$ varying. To do so we will consider the fields $\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$ with $S^2$ seen as a dynamic variable.

### 12.2 Fild action with variable spatial extension

We write the following action by considering the frequencies and position as a dynamic variable:

$$\begin{aligned}
S &= \sum_S \underline{\Gamma}^\dagger\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\left(-\frac{1}{2}\nabla^2_{\Upsilon_{S^2}}-\frac{1}{2}\frac{\delta^2}{\delta\left(S^2\right)^2}+V\left(S^2\right)+U\left(\nabla S^2\right)\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\\
&+\sum_S\sum_{\delta S^2}\underline{\Gamma}^\dagger\left(\mathbf{T},\boldsymbol{\alpha}+\delta\boldsymbol{\alpha},\mathbf{p}+\delta\mathbf{p},S^2+\delta S^2\right)U\left(S^2,S^2+\delta S^2\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\\
&+\sum_S \underline{\Gamma}^\dagger\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\left(-\frac{1}{2}\nabla^2_{(\mathbf{T})_{S^2}}+\frac{1}{2}\left(\Delta\mathbf{T}_p^\alpha\right)^t\left(\mathbf{A}_p^\alpha\right)_{S^2}\Delta\mathbf{T}_p^\alpha+\mathbf{C}\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\\
&+\sum_{S_k,S_l}\prod\underline{\Gamma}^\dagger\left(\mathbf{T}'_l,\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right)V\left(\left\{\left|\mathbf{T}',\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S_l'^2\right\rangle\right\},\left\{\left|\mathbf{T},\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\rangle\right\}\right)\prod\underline{\Gamma}\left(\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)
\end{aligned}$$

with:

$$\frac{\delta}{\delta S^2}=\int_{S^2}d\left(Z,Z'\right)\nabla_{(Z,Z')}$$

and:

$$\frac{\delta^2}{\delta\left(S^2\right)^2}=\int_{S^2}d\left(Z,Z'\right)\nabla^2_{(Z,Z')}$$

The potential $V\left(S^2\right)$ should depend on the background on which the structure emerged. It conditions the possible displacement of the structures. The potential:

$$I=\sum_{\delta S^2}\underline{\Gamma}^\dagger\left(\mathbf{T},\boldsymbol{\alpha}+\delta\boldsymbol{\alpha},\mathbf{p}+\delta\mathbf{p},S^2+\delta S^2\right)U\left(S^2,S^2+\delta S^2\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\tag{119}$$

should encompass the structural change arising while displacing the content from one spatial zone to anothr.

Appendix 5 shows that in first approximation we can write the potential:

$$\begin{aligned}
I &= \int_{S^2}\bar{U}\left(S^2,Z,Z'\right)\nabla_{(Z,Z')}\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\\
&\equiv -\bar{U}\left(S^2\right)\frac{\delta}{\delta S^2}\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)
\end{aligned}$$

The expression for $\bar{U}\left(S^2,Z,Z'\right)$ is given in this appendix.

The Green function of:

$$-\frac{1}{2}\nabla^2_{\Upsilon_{S^2}}-\frac{1}{2}\frac{\delta^2}{\delta\left(S^2\right)^2}-\bar{U}\left(S^2\right)\frac{\delta}{\delta S^2}+V\left(S^2\right)+U_\Upsilon\tag{120}$$

should describe the motion of the structure along the whole thread, without breaking the main characteristics of this structure. Since this has to be realized at each points of the trajectory, the green function should itself be written as some path integral.

For the displacement to happen without modifying the structures content, some type of topological content should arise.

Note that such motion along gradient of potential should reproduce continuously what would be obtained by a sequence of deactivations and activations of structures (see discussion after (90)).

We will neglect the activities variations and only consider fluctuations in connectivities and coordinates. The "free" dynamics for single structures is given by the operator (120). Moreover, to study the intrinsic displacement we neglect $V\left(S^2\right)$ and the structures interaction and we should only consider the systm described by:

$$-\frac{1}{2}\frac{\delta^2}{\delta\left(S^2\right)^2}-\bar{U}\left(S^2\right)\frac{\delta}{\delta S^2}-\frac{1}{2}\nabla^2_{(\mathbf{T})_{S^2}}+\frac{1}{2}\left(\Delta\mathbf{T}_p^\alpha\right)^t\left(\mathbf{A}_p^\alpha\right)_{S^2}\Delta\mathbf{T}_p^\alpha+\mathbf{C} \tag{121}$$

The global term:

$$-\frac{1}{2}\frac{\delta^2}{\delta\left(S^2\right)^2}+\bar{U}\left(S^2\right)$$

should favour displacement from $S^2$ to $S^2+\delta S^2$ such that the average connections $\bar{\mathbf{T}}_p^\alpha\left(Z,Z'\right)$ are overall decreasing along the displacement. This corresponds to a loss of content. Moreover this modification of $\bar{\mathbf{T}}_p^\alpha\left(Z,Z'\right)$ may induce a transition of a state $\left|\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$ towards $\left|\mathbf{T},\boldsymbol{\alpha}',\mathbf{p}',S^2\right\rangle$ wth $\bar{\mathbf{T}}_{p'}^{\alpha'}\left(Z,Z'\right)$ closer to the modfied $\bar{\mathbf{T}}_p^\alpha\left(Z,Z'\right)$.

However, the potential:

$$\frac{1}{2}\left(\Delta\mathbf{T}_p^\alpha\right)^t\left(\mathbf{A}_p^\alpha\right)_{S^2}\Delta\mathbf{T}_p^\alpha+\mathbf{C}$$

may avoid this tendency. The first term $\frac{1}{2}\left(\Delta\mathbf{T}_p^\alpha\right)^t\left(\mathbf{A}_p^\alpha\right)_{S^2}\Delta\mathbf{T}_p^\alpha$ is quadratic in $\Delta\mathbf{T}_p^\alpha$. It increases if $\bar{\mathbf{T}}_p^\alpha\left(Z,Z'\right)$ decreases. At least it slows down the decrease in $\bar{\mathbf{T}}_p^\alpha\left(Z,Z'\right)$. The second term is a function of $S^2$:

$$\mathbf{C}=\int_{S^2}C\left(Z,Z'\right)d\left(Z,Z'\right)$$

so that motion should be favoured in a direction of a decreasing $\mathbf{C}$. We write $\mathbf{C}\left(S^2\right)$ for $\mathbf{C}$ to account for this dependency.

The value of $\mathbf{C}\left(S^2\right)$ is given by (52):

$$\mathbf{C}\left(S^2\right)=\int_{S^2}\left(\frac{\tau\omega_0\left(Z\right)}{2}+\frac{\rho\left(C\left(\theta\right)\left|\Psi_0\left(Z\right)\right|^2\omega_0\left(Z\right)+D\left(\theta\right)\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z'\right)\right)}{2\omega_0\left(Z\right)}\right)d\left(Z,Z'\right) \tag{122}$$

so that switching to region with lower average activity is favoured.

### 12.3 Green functions

Action (121) modifies the Green functions (90). The computation is presented in appndx 5. We obtain in first approximation:

$$G\left(S'^2,\mathbf{T}'^{\alpha}_p,S^2,\mathbf{T}^{\alpha}_p\right) \simeq \exp\left(\frac{\delta}{\delta S^2}\bar{U}\left(S^2\right)-\frac{\delta}{\delta S'^2}\bar{U}\left(S'^2\right)-\left(\mathbf{C}\left(S'^2\right)-\mathbf{C}\left(S^2\right)\right)\right) \frac{\exp\left(-\left(\left(\mathbf{Z},\mathbf{Z}'\right)'\right)^t\left\langle\mathbf{A}\right\rangle_{S'^2}\left(\mathbf{Z},\mathbf{Z}'\right)'-\left(\mathbf{Z},\mathbf{Z}'\right)^t\left\langle\mathbf{A}\right\rangle_{S^2}\left(\mathbf{Z},\mathbf{Z}'\right)'\right)}{\left(\left\langle\mathbf{A}\right\rangle_{S'^2}\left\langle\mathbf{A}\right\rangle_{S^2}\right)^{\frac{1}{4}}} \times\frac{\exp\left(-\left(\mathbf{T}'^{\alpha}_p\right)^t\mathbf{A}^{\alpha}_{S'^2}\mathbf{T}'^{\alpha}_p-\left(\mathbf{T}^{\alpha}_p\right)^t\mathbf{A}^{\alpha}_{S^2}\mathbf{T}^{\alpha}_p\right)}{\sqrt{\det\left(\mathbf{A}^{\alpha}_{S^2}\right)}} \tag{123}$$

where $\left(\mathbf{Z},\mathbf{Z}'\right)'=S'^2$ and $\left(\mathbf{Z},\mathbf{Z}'\right)=S^2$ and $\left\langle\mathbf{A}\right\rangle_{S^2}$ is the average of $A\left(Z,Z'\right)$ in $S^2$ and $\left\langle\mathbf{A}\right\rangle_{S'^2}$ is the avrage of $A\left(Z,Z'\right)$ in $S'^2$. The first exponential factor is a trend term that confirms the arguments presented in the previous paragraph: the potential:

$$\frac{\delta}{\delta S^2}\bar{U}\left(S^2\right)-\frac{\delta}{\delta S'^2}\bar{U}\left(S'^2\right)$$

favours displacement towards lower connections, while the second term:

$$-\left(\mathbf{C}\left(S'^2\right)-\mathbf{C}\left(S^2\right)\right)$$

implies a higher probability of displacement towards lower activity region.

### 12.4 Displacement induced transitions

Assume localzid interactions of the form:

$$\underline{\Gamma}^{\dagger}\left(\mathbf{T}'_l,\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l\right)V\left(\left\{\left|\mathbf{T}',\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l\right\rangle\right\},\left\{\left|\mathbf{T},\boldsymbol{\alpha}_k,\mathbf{p}_k,S^2_k\right\rangle\right\}\right)\prod\underline{\Gamma}\left(\mathbf{T}_k,\boldsymbol{\alpha}_k,\mathbf{p}_k,S^2_k\right)$$

where $V$ includes factors $\delta\left(S'^2_l-S^2_k\right)$. Structures experiencing a displacement:

$$\left|\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle\rightarrow\left|\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S'^2_l\right\rangle$$

may be modified to:

$$\left|\mathbf{T}'_l,\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l\right\rangle$$

with amplitude:

$$G\left(S'^2_l,\mathbf{T}'^{\alpha_l}_{p_l},S'^2_l,\mathbf{T}'^{\alpha}_p\right)V\left(\left\{\left|\mathbf{T}'^{\alpha_l}_{p_l},S'^2_l\right\rangle\right\},\left\{\left|\mathbf{T}'^{\alpha}_p,S'^2_l\right\rangle\right\}\right)G\left(S'^2_l,\mathbf{T}'^{\alpha}_p,S^2,\mathbf{T}^{\alpha}_p\right)$$

This amplitude includes a term:

$$\frac{\exp\left(-\left(\mathbf{T}'^{\alpha_l}_{p_l}\right)^t\mathbf{A}^{\alpha}_{S'^2_l}\mathbf{T}'^{\alpha_l}_{p_l}-\left(\mathbf{T}'^{\alpha}_p\right)^t\mathbf{A}^{\alpha}_{S'^2_l}\mathbf{T}'^{\alpha}_p\right)}{\sqrt{\det\left(\mathbf{A}^{\alpha}_{S^2}\right)}}$$

$$\times\exp\left(\frac{\delta}{\delta S^2}\bar{U}\left(S^2\right)-\frac{\delta}{\delta S'^2}\bar{U}\left(S'^2_l\right)-\left(\mathbf{C}\left(S'^2_l\right)-\mathbf{C}\left(S^2\right)\right)\right)\frac{\exp\left(-\left(\mathbf{T}'^{\alpha}_p\right)^t\mathbf{A}^{\alpha}_{S'^2_l}\mathbf{T}'^{\alpha}_p-\left(\mathbf{T}^{\alpha}_p\right)^t\mathbf{A}^{\alpha}_{S^2}\mathbf{T}^{\alpha}_p\right)}{\sqrt{\det\left(\mathbf{A}^{\alpha}_{S^2}\right)}}$$

which, after convolution becomes:

$$\frac{\exp\left(-\left(\mathbf{T}_{p_l}^{\prime\alpha_l}\right)^t \mathbf{A}_{S_l^{\prime 2}}^{\alpha} \mathbf{T}_{p_l}^{\prime\alpha_l}\right)}{\sqrt{\det\left(\mathbf{A}_{S^2}^{\alpha}\right)}} \hat{V}\left(\left\{\left|\mathbf{T}_{p_l}^{\prime\alpha_l}, S_l^{\prime 2}\right\rangle\right\}\right)$$
$$\times \exp\left(\frac{\delta}{\delta S^2}\bar{U}\left(S^2\right) - \frac{\delta}{\delta S^{\prime 2}}\bar{U}\left(S_l^{\prime 2}\right) - \left(\mathbf{C}\left(S_l^{\prime 2}\right) - \mathbf{C}\left(S^2\right)\right)\right) \frac{\exp\left(-\left(\mathbf{T}_p^{\alpha}\right)^t \mathbf{A}_{S^2}^{\alpha} \mathbf{T}_p^{\alpha}\right)}{\sqrt{\det\left(\mathbf{A}_{S^2}^{\alpha}\right)}}$$

with:

$$\hat{V}\left(\left\{\left|\mathbf{T}_{p_l}^{\prime\alpha_l}, S_l^{\prime 2}\right\rangle\right\}\right) = \int V\left(\left\{\left|\mathbf{T}_{p_l}^{\prime\alpha_l}, S_l^{\prime 2}\right\rangle\right\}, \left\{\left|\mathbf{T}_p^{\prime\alpha}, S_l^{\prime 2}\right\rangle\right\}\right) \exp\left(-2\left(\mathbf{T}_p^{\prime\alpha}\right)^t \mathbf{A}_{S_l^{\prime 2}}^{\alpha} \mathbf{T}_p^{\prime\alpha}\right) d\mathbf{T}_{p_l}^{\prime\alpha_l}$$

If $V$ is decreasing function of $\left|\bar{\mathbf{T}}_{p_l}^{\prime\alpha_l} - \bar{\mathbf{T}}_p^{\prime\alpha}\right| = \left|\bar{\mathbf{T}}_{p_l}^{\prime\alpha_l} - \mathbf{T}_p^{\prime\alpha}\right|$, transition is possible under two conditions:

$$\frac{\delta}{\delta S^2}\bar{U}\left(S^2\right) - \frac{\delta}{\delta S^{\prime 2}}\bar{U}\left(S_l^{\prime 2}\right) - \left(\mathbf{C}\left(S_l^{\prime 2}\right) - \mathbf{C}\left(S^2\right)\right) > 0$$

and:

$$\left|\bar{\mathbf{T}}_{p_l}^{\prime\alpha_l} - \mathbf{T}_p^{\prime\alpha}\right| \simeq 0$$

That is, transition occurs for motion towards relatively low actvity region with similar level of connectivity.

# 13 Conclusion

The final section of the present work has extended our formalism to encompass a field-based representation for non-localized interacting structures. Activated states can undergo shifts from one region to another, the driving forces behind such displacements being the interactions. The system's background state, conditions these displacements and functions as a landscape. In our formalism, we have primarily focused on activity and frequencies as the key elements. Consequently, the spatial extension of a state should be determined by these variables, and the spatial extent of a state remains either undefined or, at the very least, variable. Exploring these possibilities is a topic for future research.

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# Appendix 1

## 1.1 Full system action

In addition to the neurons activity field action, we add the functionals that describe the dynamics for connectivity dynamics (see ([8])).

$$S_{\Gamma}^{(1)}=\int\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)\nabla_{T}\left(\frac{\sigma_{T}^{2}}{2}\nabla_{T}+O_{T}\right)\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right) \tag{124}$$

$$S_{\Gamma}^{(2)}=\int\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)\nabla_{\hat{T}}\left(\frac{\sigma_{\hat{T}}^{2}}{2}\nabla_{\hat{T}}+O_{\hat{T}}\right)\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right) \tag{125}$$

$$S_{\Gamma}^{(3)}=\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)\nabla_{C}\left(\frac{\sigma_{C}^{2}}{2}\nabla_{C}+O_{C}\right)\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)$$

$$S_{\Gamma}^{(4)}=\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)\nabla_{D}\left(\frac{\sigma_{D}^{2}}{2}\nabla_{D}+O_{D}\right)\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right) \tag{126}$$

where:

$$O_{C}=\frac{C}{\tau_{C}\omega\left(J,\theta,Z,|\Psi|^{2}\right)}-\frac{\alpha_{C}\left(1-C\right)\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^{2}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}}{\omega\left(J,\theta,Z,|\Psi|^{2}\right)} \tag{127}$$

$$O_{D}=\frac{D}{\tau_{D}\omega\left(J,\theta,Z,|\Psi|^{2}\right)}-\alpha_{D}\left(1-D\right)\left|\Psi\left(\theta,Z\right)\right|^{2}$$

$$\begin{aligned}O_{\hat{T}}=&-\frac{\rho}{\omega\left(J,\theta,Z,|\Psi|^{2}\right)}\left(\left(h\left(Z,Z'\right)-\hat{T}\right)C\left|\Psi\left(\theta,Z\right)\right|^{2}h_{C}\left(\omega\left(J,\theta,Z,|\Psi|^{2}\right)\right)\right.\\&\left.-D\hat{T}\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}h_{D}\left(\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^{2}\right)\right)\right)\end{aligned}$$

$$O_{T}=-\left(-\frac{1}{\tau\omega\left(J,\theta,Z,|\Psi|^{2}\right)}T+\frac{\lambda}{\omega\left(J,\theta,Z,|\Psi|^{2}\right)}\hat{T}\right)$$

The functional $S_{\Gamma}^{(1)}$ describes the connectivities, while $S_{\Gamma}^{(i)}$ describe the accumulation of incoming and outcoming current that influence the connectivity dynamics.

## 1.2 Replacing activity field

To find the background state for connectivity, we first derived in ([6]) that activty can be in average replaced as functional of the connectivities. The inverse activities satisfy:

$$\begin{aligned}&\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right) \quad (128)\\=&\;G\left(J\left(\theta,Z\right)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_{1}|}{c},Z_{1},\Psi\right)T\left|\Gamma\left(T,\hat{T},\theta,Z,Z_{1}\right)\right|^{2}}{\omega\left(J,\theta,Z,|\Psi|^{2}\right)}\left|\Psi\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1}\right)\right|^{2}dZ_{1}\right)\end{aligned}$$

where we have assumed that the field $\left|\Psi\left(\theta,Z\right)\right|^{2}$ is constrained by a potential limiting the activit around some average $\left|\Psi_{0}\left(Z\right)\right|^{2}$. We choose:

$$V=\frac{1}{2}\left(\left|\Psi\left(Z\right)\right|^{2}-\int T\left(Z',Z_{1}\right)\left|\Psi_{0}\left(Z\right)\right|^{2}dZ_{1}\right)^{2}$$

We showed in ([6]) that in a static first approximation, corresponding to an averaging over individual neurons sgnls time scale:

$$|\Psi(Z)|^2 = \frac{2T(Z)\left\langle |\Psi_0(Z')|^2 \right\rangle_Z}{\left(1+\sqrt{1+4\left(\frac{\lambda\tau\nu c - T(Z)}{\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega\right)T(Z)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}\right)^2 T(Z)\left\langle |\Psi_0(Z')|^2\right\rangle_Z}\right)} \tag{129}$$

which allows to derive the connectivities background states.

### 1.3 Full system background state

Replacing $\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)$ and $|\Psi(Z)|^2$ allowed to minimize $\sum_i S_\Gamma^{(1)}$. The background state for connectivity functions may have two forms, for activated connections or unactivated ones (those with $\langle T\rangle = 0$). In quasi-static approximation, we find:

$$\begin{aligned}
&|\Gamma|_a^2\left(T,\hat{T},\theta,C,D\right) \\
\simeq\ & \left\{\mathcal{N}\exp\left(-\frac{a_C(Z,Z')}{2}(C-C(\theta))^2\right)\exp\left(-\frac{a_D(Z,Z')}{2}(D-D(\theta))^2\right)\right. \\
&\times\exp\left(-\frac{\rho\left|\bar{\Psi}(\theta,Z,Z')\right|^2}{2}\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)^2\right) \\
&\left.\times\|\Gamma_0(\theta,Z,Z')\|\exp\left(-\frac{|\Psi(\theta,Z)|^2}{2\tau\omega}(T-\langle T\rangle)^2\right)\right\}_{(Z,Z'),\langle T(Z,Z')\rangle\neq 0}
\end{aligned} \tag{130}$$

$$\begin{aligned}
&|\Gamma|_u^2\left(T,\hat{T},\theta,C,D\right) \\
\simeq\ & \left\{\mathcal{N}\exp\left(-\frac{a_C(Z,Z')}{2}(C-C(\theta))^2\right)\exp\left(-\frac{a_D(Z,Z')}{2}(D-\langle D\rangle)^2\right)\right. \\
&\left.\times\exp\left(-\frac{\rho\left|\bar{\Psi}(\theta,Z,Z')\right|^2}{2}\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)^2\right)\times\delta(T)\right\}_{(Z,Z'),\langle T(Z,Z')\rangle\neq 0}
\end{aligned}$$

where $\mathcal{N}$ is a normalization factor ensuring that the constraint over the number of connections is satisfied and where:

$$\begin{aligned}
\left|\bar{\Psi}(\theta,Z,Z')\right|^2 &= C(\theta)|\Psi(\theta,Z)|^2 h_C + D(\theta)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2 h_D \\
a_C(Z,Z') &= \frac{1}{\tau_C\omega}+\alpha_C\frac{\omega'\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}{\omega} \\
a_D(Z,Z') &= \frac{1}{\tau_D\omega}+\alpha_D|\Psi(\theta,Z)|^2
\end{aligned}$$

The averages for $C$ and $D$ are:

$$C \rightarrow \langle C(\theta)\rangle = \frac{\alpha_C \frac{\omega' \left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right)\right|^2}{\omega}}{\frac{1}{\tau_C \omega} + \alpha_C \frac{\omega' \left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right)\right|^2}{\omega}} = \frac{\alpha_C \omega' \left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right)\right|^2}{\frac{1}{\tau_C} + \alpha_C \omega' \left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right)\right|^2} \equiv C(\theta) \quad (131)$$

$$D \rightarrow \langle D(\theta)\rangle = \frac{\alpha_D \omega |\Psi(\theta, Z)|^2}{\frac{1}{\tau_D} + \alpha_D \omega |\Psi(\theta, Z)|^2} \equiv D(\theta) \quad (132)$$

We showed that under some approximations, the average values in this background states present severl possbl patterns:

$$T(Z_-, Z'_+) = \frac{\lambda\tau \exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left(\frac{1}{\tau_D \alpha_D} + \frac{1}{b\bar{T}^2 \left\langle |\Psi_0(Z')|^2\right\rangle_Z}\right)}{\frac{1}{\tau_D \alpha_D} + \frac{1}{\alpha_C \tau_C} + \frac{1}{b\bar{T}^2 \left\langle |\Psi_0(Z')|^2\right\rangle_Z} + b\bar{T}\left(\bar{T}\left\langle |\Psi_0(Z')|^2\right\rangle_{Z'}^2\right)^2} \simeq 0 \quad (133)$$

$$T(Z_+, Z'_+) = \frac{\lambda\tau \exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left(\frac{1}{\tau_D \alpha_D} + b\bar{T}\left(\bar{T}\left\langle |\Psi_0(Z')|^2\right\rangle_Z^2\right)^2\right)}{\frac{1}{\tau_D \alpha_D} + \frac{1}{\alpha_C \tau_C} + b\bar{T}\left(\bar{T}\left\langle |\Psi_0(Z')|^2\right\rangle_Z^2\right)^2 + b\bar{T}\left(\bar{T}\left\langle |\Psi_0(Z')|^2\right\rangle_{Z'}^2\right)^2} \simeq \frac{\lambda\tau \exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{2}$$

$$T(Z_+, Z'_-) = \frac{\lambda\tau \exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left(\frac{1}{\tau_D \alpha_D} + b\bar{T}\left(\bar{T}\left\langle |\Psi_0(Z')|^2\right\rangle_Z^2\right)^2\right)}{\frac{1}{\tau_D \alpha_D} + \frac{1}{\alpha_C \tau_C} + b\bar{T}\left(\bar{T}\left\langle |\Psi_0(Z')|^2\right\rangle_Z^2\right)^2 + \frac{1}{b\bar{T}^2 \left\langle |\Psi_0(Z')|^2\right\rangle_{Z'}}} \simeq \lambda\tau \exp\left(-\frac{|Z-Z'|}{\nu c}\right)$$

$$T(Z_-, Z'_-) \simeq \frac{\lambda\tau \exp\left(-\frac{|Z-Z'|}{\nu c}\right) + \frac{1}{b\bar{T}^2 \left\langle |\Psi_0(Z')|^2\right\rangle_Z}}{1 + \frac{\tau_D \alpha_D}{\alpha_C \tau_C} + \frac{1}{b\bar{T}^2 \left\langle |\Psi_0(Z')|^2\right\rangle_Z} + \frac{1}{b\bar{T}^2 \left\langle |\Psi_0(Z')|^2\right\rangle_{Z'}}} \simeq \frac{\lambda\tau \exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{2}$$

with $\bar{T} = \frac{\lambda\tau\nu cb}{2}$, $b$ a coefficient characterizing the function $G$ in the linear approximation[6] and $\left\langle |\Psi_0(Z')|^2\right\rangle_{Z'}^2$, $\left\langle |\Psi_0(Z')|^2\right\rangle_Z^2$ are some averaged background fields in regions surrounding $Z$ and $Z'$ respctvl, given by a potential describing some average activity depending on the points. These results are under the hypothesis the field $\Psi_0(Z)$ is static.

## 1.4 Eigenstates of the effective action

### 1.4.1 Rewriting saddle point equation

After change of variable:

$$\Delta\Gamma\left(T, \hat{T}, \theta, Z, Z'\right) \rightarrow \exp\left(-\frac{\rho \left|\bar{\Psi}(Z, Z')\right|^2 \left(\hat{T} - \left\langle \hat{T}\right\rangle\right)^2}{4\sigma_{\hat{T}}^2} - \frac{V_0\left(\hat{T} - \left\langle \hat{T}\right\rangle\right)}{2\sigma_{\hat{T}}^2 \omega\left(\theta, Z, |\Psi|^2\right)}\right)$$
$$\times \exp\left(-\frac{\left((T - \langle T\rangle)^2 - 2\lambda\tau\left(\hat{T} - \left\langle \hat{T}\right\rangle\right)(T - \langle T\rangle)\right)}{4\sigma_T^2 \tau\omega}\right) \Delta\Gamma\left(T, \hat{T}, \theta, Z, Z'\right) \quad (134)$$

[6] $b \simeq G'(0)$

and:

$$\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right) \rightarrow \exp\left(\frac{\rho\left|\bar{\Psi}\left(Z,Z'\right)\right|^{2}\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)^{2}}{4\sigma_{\hat{T}}^{2}}+\frac{V_{0}\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)}{2\sigma_{\hat{T}}^{2}\omega\left(\theta,Z,\left|\Psi\right|^{2}\right)}\right) \times\exp\left(\frac{\left(\frac{\left(T-\langle T\rangle\right)^{2}}{\tau}-2\lambda\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)\left(T-\langle T\rangle\right)\right)}{4\sigma_{T}^{2}\tau\omega}\right)\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right) \tag{135}$$

with:

$$V_{0}=\left(\frac{\rho D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_{0}\left(Z'\right)\right|^{2}}{\omega_{0}\left(Z\right)}\hat{T}\left(1-\left(1+\left\langle\left|\Psi_{\Gamma}\right|^{2}\right\rangle\right)\hat{T}\right)^{-1}\left[O\frac{\Delta T\left|\Delta\Gamma\left(\theta_{1},Z_{1},Z_{1}'\right)\right|^{2}}{T}\right]\right)$$

the saddle point equation of the action (3) for the connectivity fields $\Delta\Gamma$ and $\Delta\Gamma^{\dagger}$ becomes:

$$\begin{aligned}0=&\left(-\sigma_{\hat{T}}^{2}\nabla_{\hat{T}}^{2}+\frac{1}{4\sigma_{\hat{T}}^{2}}\left(\left|\bar{\Psi}\left(Z,Z'\right)\right|^{2}\Delta\hat{T}+\frac{\rho\left(V_{0}-\frac{\sigma_{\hat{T}}^{2}}{\sigma_{T}^{2}}\lambda\Delta T\left|\Psi\left(Z\right)\right|^{2}\right)}{\omega_{0}\left(Z\right)}\right)^{2}\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\&+\left(-\sigma_{T}^{2}\nabla_{T}^{2}+\frac{1}{4\sigma_{T}^{2}}\left(\frac{\Delta T-\lambda\tau\Delta\hat{T}}{\tau\omega_{0}\left(Z\right)}\right)^{2}\right.\\&\left.-\left(\frac{\left|\bar{\Psi}\left(Z,Z'\right)\right|^{2}}{2}+\frac{\left|\Psi\left(Z\right)\right|^{2}}{2\tau\omega_{0}\left(Z\right)}+V\left(\theta,Z,Z',\Delta\Gamma\right)\Delta T-\alpha\right)\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\end{aligned} \tag{136}$$

with:

$$V_{0}\left(Z,Z'\right)=\left(\frac{\rho D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_{0}\left(Z'\right)\right|^{2}}{\omega_{0}\left(Z\right)}\hat{T}\left(1-\left(1+\left\langle\left|\Psi_{\Gamma}\right|^{2}\right\rangle\right)\hat{T}\right)^{-1}\left[O\frac{\Delta T\left|\Delta\Gamma\left(\theta_{1},Z_{1},Z_{1}'\right)\right|^{2}}{T}\right]\right) \tag{137}$$

$$V\left(\theta,Z,Z',\Delta\Gamma\right)=V_{1}\left(\theta,Z,Z',\Delta\Gamma\right)\left(1+V_{2}\left(\theta,Z,Z',\Delta\Gamma\right)\right)$$

with $V_1$ and $V_2$ given by:

$$\begin{aligned}&V_{1}\left(\theta,Z,Z',\Delta\Gamma\right)\\=&\int\Delta\Gamma^{\dagger}\left(T_{2},\hat{T}_{2},\theta_{2},Z_{2},Z_{2}'\right)\nabla_{\hat{T}_{2}}\left(\frac{\rho D\left(\theta\right)\left\langle\hat{T}_{2}\right\rangle\left|\Psi_{0}\left(Z_{2}'\right)\right|^{2}}{\omega_{0}\left(Z_{2}\right)}\left[\check{T}\left(1-\left(1+\left\langle\left|\Psi_{\Gamma}\right|^{2}\right\rangle\right)\check{T}\right)^{-1}O\right]_{\left(T,\hat{T},\theta,Z,Z'\right)}^{\left(T_{2},\hat{T}_{2},\theta_{2},Z_{2},Z_{2}'\right)}\right)\\&\times\Delta\Gamma\left(T_{2},\hat{T}_{2},\theta_{2},Z_{2},Z_{2}'\right)d\left(T_{2},\hat{T}_{2},\theta_{2},Z_{2},Z_{2}'\right)\end{aligned} \tag{138}$$

and:

$$V_{2}\left(\theta,Z,Z',\Delta\Gamma\right)=\int\left[\check{T}\left(1-\left(1+\left\langle\left|\Psi_{\Gamma}\right|^{2}\right\rangle\right)\check{T}\right)^{-1}\right]_{\left(T_{1},\hat{T}_{1},\theta_{1},Z_{1},Z_{1}'\right)}^{\left(T,\hat{T},\theta,Z,Z'\right)}\left[\frac{\Delta T\left|\Delta\Gamma\left(\theta_{1},Z_{1},Z_{1}'\right)\right|^{2}}{T}\right]d\left(T_{1},\hat{T}_{1},\theta_{1},Z_{1},Z_{1}'\right)$$

respectively.

Equation (136) encapsulates the main characteristics of connectivity states:

$$\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$$

The two first terms describes the individual connectivities. The term $V(\theta, Z, Z', \Delta\Gamma)$ measures the interactions between connectivities

Operator $O$ is defined by (5). Recall that $\alpha$ implements the constraint $\|\Delta\Gamma\| = \overline{\|\Delta\Gamma\|}$. As in the previous paragraph $\alpha$ stands for:

$$\alpha_0 + U'\left(|\Delta\Gamma(Z, Z')|^2\right) \tag{139}$$

where $\alpha_0$ is the Lagrange multiplier for the overall constraint, and $U(\Delta\Gamma(Z, Z'))$ the potential.

After diagonalization of the potential by a matrix $P = \begin{pmatrix} w_1 & w_2 \\ w_1' & w_2' \end{pmatrix}$, whose components are:

$$\begin{aligned} w_1 &= \sqrt{\frac{1}{2}\left(1+\sqrt{\frac{\left(\frac{(v-u)}{2s}\right)^2}{1+\left(\frac{(v-u)}{2s}\right)^2}}\right)}, \; w_2 = \sqrt{\frac{1}{2}\left(1-\sqrt{\frac{\left(\frac{(v-u)}{2s}\right)^2}{1+\left(\frac{(v-u)}{2s}\right)^2}}\right)} \\ w_1' &= -\sqrt{\frac{1}{2}\left(1-\sqrt{\frac{\left(\frac{(v-u)}{2s}\right)^2}{1+\left(\frac{(v-u)}{2s}\right)^2}}\right)}, \; w_2' = \sqrt{\frac{1}{2}\left(1+\sqrt{\frac{\left(\frac{(v-u)}{2s}\right)^2}{1+\left(\frac{(v-u)}{2s}\right)^2}}\right)} \end{aligned}$$

we show that this background state equation becomes:

$$\begin{aligned} 0 &= \left(-\sigma_{\hat{T}}^2 \nabla_{\hat{T}'}^2 + \frac{\lambda_+^2}{4\sigma_{\hat{T}}^2}\left(\Delta\hat{T}' - \Delta\hat{T}_0' - \frac{w_2}{\lambda_+}V\right)^2\right)\Delta\Gamma\left(T, \hat{T}, \theta, Z, Z'\right) \\ &+ \left(-\sigma_T^2 \nabla_{T'}^2 + \frac{\lambda_-^2}{\sigma_T^2}\left(\Delta T' - \Delta T_0' - \frac{w_1}{\lambda_-}V\right)^2\right)\Delta\Gamma\left(T, \hat{T}, \theta, Z, Z'\right) \\ &- \left(u + v + \left(\frac{w_1^2}{\lambda_+}V^2 + \frac{w_2^2}{\lambda_-}V^2\right) - \alpha\right)\Delta\Gamma\left(T, \hat{T}, \theta, Z, Z'\right) \end{aligned} \tag{140}$$

where:

$$\lambda_{\pm} = \sqrt{\frac{1}{2}\left(u^2 + v^2\right) + s^2 \pm \frac{(u+v)}{2}\sqrt{(u-v)^2 + 4s^2}}$$

$$\begin{aligned} u &= \frac{|\Psi_0(Z)|^2}{\tau\omega_0(Z)} \\ v &= \rho\left|\bar{\Psi}_0(Z, Z')\right|^2 \\ s &= -\frac{\lambda|\Psi_0(Z)|^2}{\omega_0(Z)}\frac{\sigma_{\hat{T}}}{\sigma_T} \end{aligned}$$

$$\left(\Delta T_0, \Delta\hat{T}_0\right) \simeq \left(-\frac{\lambda\tau V_0}{\sigma_T\omega_0(Z)\left|\bar{\Psi}_0(Z, Z')\right|^2}, \frac{\Delta T_0}{\lambda\tau}\frac{\sigma_T}{\sigma_{\hat{T}}}\right) \tag{141}$$

and $\left(X', \hat{X}'\right)$ are the coordinates of any vector in the diagonal basis of the potential:

$$\left(X', \hat{X}'\right)^t = P^{-1}\left(X, \hat{X}\right)$$

Note that:

$$\lambda_+ + \lambda_- = u + v$$

The eigenstates and averages are obtained in ([8]). Under several approximations we obtain that:

### 1.4.2 Eigenstates

The solutions of (136) become:

$$\begin{aligned}
& \Delta\Gamma_{\delta}\left(T,\hat{T},\theta,Z,Z'\right) && (142)\\
= & \exp\left(-\frac{1}{2}\left(\mathbf{\Delta T}-\mathbf{\Delta\bar{T}}\right)^{t}\hat{U}\left(\mathbf{\Delta T}-\mathbf{\Delta\bar{T}}\right)\right)\\
& \times H_{p}\left(\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\frac{\sigma_{T}\lambda_{+}}{2\sqrt{2}}\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\right)H_{p-\delta}\left(\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\frac{\sigma_{\hat{T}}\lambda_{-}}{2\sqrt{2}}\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\right)
\end{aligned}$$

and their conjugate:

$$\begin{aligned}
& \Delta\Gamma_{\delta}^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right) && (143)\\
= & H_{p}\left(\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\frac{\sigma_{T}\lambda_{+}}{2\sqrt{2}}\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\right)H_{p-\delta}\left(\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\frac{\sigma_{\hat{T}}\lambda_{-}}{2\sqrt{2}}\left(\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}'\right)_{2}\right)
\end{aligned}$$

where $H_p$ and $H_{p-\delta}$ are Hermite polynomials and the variables are:

$$\begin{aligned}
\mathbf{\Delta T}-\mathbf{\Delta\bar{T}} &= \begin{pmatrix}\Delta T-\langle\Delta T\rangle\\ \Delta\hat{T}-\left\langle\Delta\hat{T}\right\rangle\end{pmatrix} && (144)\\
\mathbf{\Delta T}'-\mathbf{\Delta\bar{T}}' &= P^{-1}\left(\mathbf{\Delta T}-\mathbf{\Delta\bar{T}}\right)
\end{aligned}$$

with parameters:

$$\begin{aligned}
\Delta T_{0} &\simeq -\frac{\lambda\tau V_{0}}{\omega_{0}(Z)\left|\Psi_{0}(Z)\right|^{2}}\\
\Delta\hat{T}_{0} &\simeq \frac{\Delta T_{0}}{\lambda\tau}
\end{aligned}$$

$$\begin{pmatrix}\Delta T_{1}\\ \Delta\hat{T}_{1}\end{pmatrix}=PD^{-1}P^{-1}\begin{pmatrix}V\\ 0\end{pmatrix}=U^{-1}\begin{pmatrix}V\\ 0\end{pmatrix}$$

and the matrix $\hat{U}$ given by:

$$\hat{U}=\begin{pmatrix}\frac{1}{\sigma_{T}} & 0\\ 0 & \frac{1}{\sigma_{\hat{T}}}\end{pmatrix}U\begin{pmatrix}\frac{1}{\sigma_{T}} & 0\\ 0 & \frac{1}{\sigma_{\hat{T}}}\end{pmatrix}=\begin{pmatrix}\frac{s^{2}+u^{2}}{\sigma_{T}^{2}} & -\frac{s(u+v)}{\sigma_{T}\sigma_{\hat{T}}}\\ -\frac{s(u+v)}{\sigma_{T}\sigma_{\hat{T}}} & \frac{s^{2}+v^{2}}{\sigma_{\hat{T}}^{2}}\end{pmatrix}$$

The potential background field of the sytem are thus defined by considering the set:

$$W=\left\{(Z,Z'),p_{1}(Z,Z')\lambda_{+}+p_{2}(Z,Z')\lambda_{-}=\frac{u+v}{2}+\left(\frac{w_{1}^{2}}{\lambda_{+}}+\frac{w_{2}^{2}}{\lambda_{-}}\right)V^{2}-\alpha>0\right\} \tag{145}$$

and associating to each function $\delta(Z,Z'):W\rightarrow[0,p(Z,Z')]$, the potential background state:

$$\prod_{W}\Delta\Gamma_{\delta(Z,Z')}\left(T,\hat{T},\theta,Z,Z'\right)$$

and:

$$\prod_{W}\Delta\Gamma_{\delta(Z,Z')}^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)$$

We conclude by considering an example of solutions of (140), and consider $(Z,Z')$ such that:

$$\frac{u+v}{2}+\left(\frac{w_{1}^{2}}{\lambda_{+}}+\frac{w_{2}^{2}}{\lambda_{-}}\right)V^{2}-\alpha=0$$

Taking into account (134) and (135), the background state for (140) is:

$$\Delta\Gamma\left(T,\hat{T},Z,Z'\right)=\exp\left(-\frac{1}{2}\begin{pmatrix}\Delta T'-\Delta T_0'-\frac{w_2}{\lambda_+}V\\ \Delta\hat{T}'-\Delta\hat{T}_0'-\frac{w_1}{\lambda_-}V\end{pmatrix}^t D\begin{pmatrix}\Delta T'-\Delta T_0'-\frac{w_2}{\lambda_+}V\\ \Delta\hat{T}'-\Delta\hat{T}_0'-\frac{w_1}{\lambda_-}V\end{pmatrix}\right)$$

Coming back to the initial variables and reintroducing $\sigma_T$ and $\sigma_{\hat{T}}$, it yields:

$$\Delta\Gamma\left(T,\hat{T},Z,Z'\right)=\exp\left(-\frac{1}{2}\begin{pmatrix}\Delta T-\Delta T_0-\Delta T_1\\ \Delta\hat{T}-\Delta\hat{T}_0-\Delta\hat{T}_1\end{pmatrix}^t \hat{U}\begin{pmatrix}\Delta T-\Delta T_0-\Delta T_1\\ \Delta\hat{T}-\Delta\hat{T}_0-\Delta\hat{T}_1\end{pmatrix}\right)\tag{146}$$

and for $\Delta\Gamma^{\dagger}\left(T,\hat{T},Z,Z'\right)$:

$$\Delta\Gamma^{\dagger}\left(T,\hat{T},Z,Z'\right)=1\tag{147a}$$

### 1.4.3 Estimation of $\Delta T\left(Z,Z'\right)$ and $\Delta\hat{T}\left(Z,Z'\right)$

Average connectivities are given by:

$$\langle\Delta T\rangle\simeq\frac{\omega_0\left(Z\right)\langle T\rangle}{\rho D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2 k\underline{A_1}\left\|\Delta\Gamma\right\|^6}\left\langle\rho\frac{\left|\Psi_0\left(Z\right)\right|^2}{A}\right\rangle^2\langle T\rangle\tag{148}$$

with:

$$A_1\left(Z,Z'\right)=\frac{\rho D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2}{\omega_0\left(Z\right)}\underline{A_1}\left(Z,Z'\right)$$

and:

$$\underline{A_1}\left(Z,Z'\right)=\left\langle\left[F\left(Z_2,Z_2'\right)\left[\check{T}\left(1-\left\langle\left|\Psi_0\right|^2\right\rangle\frac{\langle\Delta T\rangle}{T}\left\|\Delta\Gamma\right\|^2\right)^{-1}O\right]\right]_{\left(T,\hat{T},\theta,Z,Z'\right)}\right\rangle$$

$$\left\langle\Delta\hat{T}\right\rangle=\hat{A}\langle\Delta T\rangle\tag{149}$$

$$\hat{A}\simeq-\frac{1}{v}A\frac{\left\|\Delta\Gamma\right\|^2}{\langle T\rangle}\tag{150}$$

With our approximations and using (150) this becomes:

$$\begin{aligned}\Delta T\left(Z,Z'\right)&=\frac{A_1\left(Z,Z'\right)A\left(Z,Z'\right)\left\langle v^2\right\rangle}{\left\langle A_1\left(Z,Z'\right)\right\rangle\left\langle A_0\left(Z,Z'\right)\right\rangle v^2}\langle\Delta T\rangle\\ \Delta\hat{T}\left(Z,Z'\right)&=\frac{A_0\left(Z,Z'\right)}{\left\langle A_0\left(Z,Z'\right)\right\rangle}\left\langle\Delta\hat{T}\right\rangle\end{aligned}$$

## Appendix 2. Dynamics in activities and emerging collective state

This appendix incorporates several elements of ([5]), ([6]) and ([7]). We revisit how activity may manifest stable oscillation patterns, and that these oscillations, induced by external signals, may bind individual elements to produce an emerging activated state. The link between the connectivities of the state and its average activities is provided. Ultimately, synthetizing the results, we justify the reverse point of view adopted in the text, namely, that a collective state can be characterized by some possible stable oscillating activities.

### 2.1 Dynamic wave equation for activities

This section seeks dynamic solutions for (2). We use the relation (151) provided below to substitute the non-static part of the field $\Psi$ as a function of the activities, and subsequently derive a wave equation for these activities. The outcomes of this section will be instrumental in deducing the internal activity of a collective state and justifying the possibility of sustained oscillating internal activity in such states.

#### 2.1.1 Differential equation for activities in the local approximation

A local approximation of (2) around a position-independent static equilibrium can be derived for non-static activities. Assuming a static background field $\Psi_0$, we showed in ([5]) (see ([6]) for an account) the following relation between the fluctuations $\delta\Psi(\theta, Z)$ around this background and the time-dependent part of the activities $\omega\left(J, Z, |\Psi|^2\right)$ is given by:

$$\delta\Psi(\theta, Z) \simeq \frac{\nabla_\theta \omega\left(J, Z, |\Psi|^2\right)}{V''(\Psi_0(Z))\,\omega_0^2\left(J, Z, |\Psi|^2\right)}\Psi_0 \tag{151}$$

where $\omega\left(J, Z, |\Psi|^2\right)$ is the time-dependent activity.

We can find a local approximation of (2) if we expand $\omega\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)$ to the second-order in $Z - Z_1$, and consider the other terms in the right-hand side of (2) as corrections. Neglecting the perturbative corrections in the effective action for $\Psi(\theta, Z)$, the cells'system is described by the "classical" action:

$$\hat{S} = -\frac{1}{2}\Psi^\dagger(\theta, Z, \omega)\nabla\left(\frac{\sigma_\theta^2}{2}\nabla - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right)\Psi(\theta, Z) + V(\Psi) \tag{152}$$

and the equation for $\omega\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)$ is:

$$\begin{aligned} F^{-1}(\omega(J(\theta), \theta)) &= J(\theta, Z) + \int \frac{\kappa}{N}\frac{\omega\left(J, \theta - \frac{|Z-Z_1|}{c}, Z_1, \Psi\right)T\left(Z, \theta, Z_1, \theta - \frac{|Z-Z_1|}{c}\right)}{\omega\left(J, \theta, Z, |\Psi|^2\right)} \\ &\quad \times \left(\left|\Psi_0 + \delta\Psi\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)\right|^2\right)dZ_1 \end{aligned} \tag{153}$$

We then expand $\omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)$ around $\omega(\theta, Z)$ to the second-order in $Z - Z_1$ and compute the integrals, which yields for the right-hand side of (153):

$$\begin{aligned} & J(\theta) + \int \frac{\kappa}{N}\frac{\omega\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)}{\omega(\theta, Z)}T\left(Z, \theta, Z_1, \theta - \frac{|Z-Z_1|}{c}\right) \\ & \times \left|\Psi_0(Z_1) + \delta\Psi\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)\right|^2 dZ_1 \\ \simeq\ & J(\theta) + \frac{TW(1)}{\bar{\Lambda}} + \frac{\hat{f}_1\nabla_\theta\omega(\theta, Z)}{\omega(\theta, Z)} + \frac{\hat{f}_3\nabla_\theta^2\omega(\theta, Z)}{\omega(\theta, Z)} + c^2\frac{\hat{f}_3\nabla_Z^2\omega(\theta, Z)}{\omega(\theta, Z)} + T\Psi_0\delta\Psi(\theta, Z) \end{aligned}$$

where we defined:

$$\begin{aligned}
\hat{f}_1 &= -\frac{\Gamma_1}{c},\ \hat{f}_3 = \frac{\Gamma_2}{c^2} \\
\Gamma_1 &= \frac{\kappa}{NX_r}\int |Z - Z_1| T(Z, Z_1) |\Psi_0(Z_1)|^2 dZ_1 \\
\Gamma_2 &= \frac{\kappa}{2NX_r}\int (Z - Z_1)^2 T(Z, Z_1) |\Psi_0(Z_1)|^2 dZ_1
\end{aligned} \tag{154}$$

and:

$$T\Psi_0\delta\Psi(\theta, Z) = \int \frac{\kappa T(Z, Z_1)}{N}\Psi_0(Z_1)\delta\Psi\left(\theta - \frac{|Z - Z_1|}{c}, Z_1\right)dZ_1$$

$$\delta T\Psi_0 = \int \frac{\kappa\delta T(Z, Z_1, \theta)}{N}\Psi_0(Z_1) dZ_1$$

Substracting $F^{-1}(\omega_0)$, where $\omega_0$ is the static solution for activity, equation (153) becomes:

$$F^{-1}(\omega(J(\theta), \theta)) - F^{-1}(\omega_0) = J(\theta, Z) + \frac{\hat{f}_1\nabla_\theta\omega(\theta, Z)}{\omega(\theta, Z)} + \frac{\hat{f}_3\nabla_\theta^2\omega(\theta, Z)}{\omega(\theta, Z)} + c^2\hat{f}_3\frac{\nabla_Z^2\omega(\theta, Z)}{\omega(\theta, Z)} + T\Psi_0\delta\Psi(\theta, Z) \tag{155}$$

Using also (151):

$$\delta\Psi(\theta, Z) \simeq \frac{\nabla_\theta\omega\left(J, Z, |\Psi|^2\right)}{V''(\Psi_0(Z))\,\omega_0^2\left(J, Z, |\Psi_0|^2\right)}\Psi_0 \tag{156}$$

leads to rewrite the last term in (155):

$$\begin{aligned}
T\delta\Psi(\theta, Z) &\simeq \delta\Psi(\theta, Z) - \Gamma_1\nabla_\theta\delta\Psi(\theta, Z) \\
&\simeq N_1\nabla_\theta\omega_0\left(J, Z, |\Psi_0|^2\right) - N_2\nabla_\theta\omega_0\left(J, Z, |\Psi_0|^2\right)
\end{aligned}$$

with:

$$\begin{aligned}
N_1 &= \frac{\Psi_0(Z)}{U''(X_0)\,\omega^2\left(J, Z, |\Psi_0|^2\right)} \\
N_2 &= \frac{\Gamma_1\Psi_0(Z)}{U''(X_0)\,\omega^2\left(J, Z, |\Psi_0|^2\right)}
\end{aligned}$$

We assume that $F^{-1}$ is slowly varying, so that:

$$F^{-1}(\omega(J(\theta), \theta)) - F^{-1}(\omega_0) \simeq \Gamma_0(\omega(J(\theta), \theta) - \omega_0)$$

with[7]:

$$f = \left(F^{-1}\right)'\left(\frac{\kappa}{N}\int T(Z, Z_1) W(1) dZ_1\bar{\mathcal{G}}_0(0, Z_1)\right)$$

and define:

$$\Omega(\theta, Z) = \omega(\theta, Z) - \omega_0$$

As a result, the expansion of (155) for a non-static current is then:

$$f\Omega(\theta, Z) = J(\theta, Z) + \left(\frac{\hat{f}_1}{\omega(\theta, Z)} + N_1\right)\nabla_\theta\Omega(\theta, Z) + \left(\frac{\hat{f}_3}{\omega(\theta, Z)} - N_2\right)\nabla_\theta^2\Omega(\theta, Z) + \frac{c^2\hat{f}_3}{\omega(\theta, Z)}\nabla_Z^2\Omega(\theta, Z) \tag{157}$$

[7]Given our assumption that $F$ is an increasing function, $f > 0$.

A careful study of this equation is performed in ([5]). We show that this equation has non sinusoidal stable traveling wave solutions and that in first approximation it can be replaced by an usual wave equation:

$$f\Omega(\theta,Z)-\left(\frac{\hat{f}_3}{\omega_0}-N_2\right)\nabla_\theta^2\Omega(\theta,Z)-\frac{c^2\hat{f}_3}{\omega_0}\nabla_Z^2\Omega(\theta,Z)=J(\theta,Z) \tag{158}$$

where $\omega_0$ is the average of the static activity.

### 2.1.2 Perturbative corrections to the local frequency equation

The perturbative expansion of the path integral for the field action (1) modifies the activities equation. We computed this effective action, written $\Gamma(\Psi^\dagger,\Psi)$, in ([5]). It is not equal to $\hat{S}(\Psi^\dagger,\Psi)$ defined in (1) since the dependency of $\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)$ in $|\Psi|^2$ introduces self interaction trms.

In the local approximation, this effective action corrects (1) by a series expansion in field:

$$\Gamma\left(\Psi^\dagger,\Psi\right)\simeq\int\Psi^\dagger(\theta,Z)\left(-\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)\right)\delta(\theta_f-\theta_i)+\Omega(\theta,Z)\right)\Psi(\theta,Z) \tag{159}$$

where $\Omega(\theta,Z)$ is a corrective term depending on the successive derivatives of the field (the constants $a_j$ are derived in ([5])):

$$\begin{aligned}\Omega(\theta,Z) &= \int\sum_{\substack{j\geqslant1\\m\geqslant1}}\sum_{\substack{(p_l^i)_{m\times j}\\p_l+\sum_i p_l^i\geqslant2}}\frac{a_j}{j!}\frac{\delta^{\sum_l p_l}\left[-\frac{1}{2}\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}\left(|\Psi(\theta,Z)|^2\right)\right)\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l}\delta\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\Psi(\theta,Z) \\ &\quad\times\left(\prod_{l=1}^{j}\Psi^\dagger\left(\theta_f^{(l)},Z_l\right)\right)\prod_{i=1}^{m}\left[\frac{\delta^{\sum_l p_l^i}\left[\hat{S}_{cl,\theta}\left(\Psi^\dagger,\Psi\right)\right]}{\prod_{l=1}^{j}\prod_{k_l^i=1}^{p_l^i}\delta\left|\Psi\left(\theta^{(l)},Z_l\right)\right|^2}\right]\left(\prod_{l=1}^{j}\Psi\left(\theta_i^{(l)},Z_l\right)\right)\end{aligned} \tag{160}$$

The term $\mathcal{G}_0$ is a function of $Z$ and represents a two points free Green function (see ([5])).

The previous equation (159) defines an effective activity that can be identified as:

$$\omega_e^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)=\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)+\int^\theta\Omega(\theta,Z) \tag{161}$$

where $\omega\left(J(\theta),\theta,Z,\bar{\mathcal{G}}_0+|\Psi|^2\right)$ is the solution of:

$$\begin{aligned}\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right) &= G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\right. \\ &\quad\left.\times\left(\bar{\mathcal{G}}_0(0,Z_1)+\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\right)\end{aligned}$$

Which is the classical activity equation, up to the inclusion of the Green function $\bar{\mathcal{G}}_0(0,Z_1)$.

The second term $\int^{\theta} \Omega(\theta, Z)$ in (161) represents corrections due to the interactions. Using (160), we can find its expression as a series expansion in terms of activities and field. The computations of these corrections to the classical equation are presented in ([5]) and confirm the possibility of traveling wave solutions. At the lowest order, we find:

$$\int^{\theta} \Omega(\theta, Z) = \frac{1}{4} \int \int^{\theta} \frac{\delta\left(\nabla_{\theta}\omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \times \frac{\delta\left(\nabla_{\theta}\omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{\delta\left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2} \left|\Psi\left(\theta^{(l)}, Z_l\right)\right|^2$$

and we show that this term counters the variations of $\omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)$ and thus stabilizes the oscillations.

To sum up, the perturbative corrections account for interactions between the classical solutions and the entire thread. Moreover these interactions stabilize the traveling waves.

## 2.3 External signals and connectivity switching

We have given in (133) a first approximation of equilibrium values for connectivity functions. Based on wave dynamics in the previous paragraph, this section presents the effect of external signals on these values. The complete derivation is given in part I and II. The result obtained will also apply to the modified connectivities (148) as these ones are considered in this work as independent activated states.

### 2.3.1 Sources induced activities

In the perspective of this work, we are looking at the solutions of (158) induced by some ponctual sources. Assume several signals arising at some points $(Z_1, \theta_1), ..., (Z_N, \theta_N)$.

The solution to (158) are then:

$$\Omega(Z, \theta) = \sum_{i=1}^{N} G((Z, \theta), (Z_i, \theta_i)) J(Z_i, \theta_i) \tag{162}$$

Where $G((Z, \theta), (Z_i, \theta_i))$ is the Green function of:

$$f - \left(\frac{\hat{f}_3}{\omega_0} - N_2\right) \nabla_{\theta}^2 - \frac{c^2 \hat{f}_3}{\omega_0} \nabla_Z^2$$

The non local equation (153) in presence of sources has solutions:

$$\Omega(\theta, Z) \simeq \sum_{i=1}^{N} G_T((Z, \theta), (Z_i, \theta_i)) \frac{J(Z_i, \theta_i)}{1 + \langle T \rangle_{\Psi_0}} \tag{163}$$

where $G_T((Z, \theta), (Z_i, \theta_i))$ is defined by:

$$G_T((Z, \theta), (Z_i, \theta_i)) = \left(\frac{1}{1 - G_T}\right)((Z, \theta), (Z_i, \theta_i))$$

and $G_T$ is the operator with kernel $G_T\left(Z, \theta, Z_1, \theta - \frac{|Z - Z_1|}{c}\right)$. Equation (163) is the local version of (162). Both solutions present interference phenomena. When the number of sources is large, we may expect that solutions of (163) and (162) locate mainly at some maxima depending both on the connectivity field $\left|\Gamma\left(T, \hat{T}, \theta, Z, Z'\right)\right|^2$ and neuron field. In the sequel, we will write:

$$Z_M^{(\varepsilon)}\left(|\Gamma|^2, |\Psi_0|^2\right)$$

the location of these maxima, with $\varepsilon = 1, ...$ indexing these maxima. We will also assume that at these maxima, the activities are all equal to some value:

$$\omega \simeq \omega' \simeq \omega_M$$

so that:

$$\begin{aligned}
h_C(\omega) &\simeq h_C(\omega_M) \\
h_D(\omega) &\simeq h_D(\omega_M)
\end{aligned}$$

The precise derivation of the interference phenomenom has been presented in a field theoretic context in part II. It is sufficient for the rest of the section to build on the qualititative argument we presented here.

### 2.3.2 Effective action and background state for given sources states

We have seen that for a given external state, interference arise, and activities localize at some points

$$Z_M^{(\varepsilon)}\left(|\Gamma|^2, |\Psi_0|^2\right)$$

with $\varepsilon = 1, ...$ indexing these points. We will also assumed for the sake of simplicity, that that at these maxima, the activities are all equal to some value:

$$\omega \simeq \omega' \simeq \omega_M$$

so that:

$$\begin{aligned}
h_C(\omega) &\simeq h_C(\omega_M) \\
h_D(\omega) &\simeq h_D(\omega_M)
\end{aligned}$$

Assuming that functions $h_C(\omega)$ and $h_D(\omega)$ are proportional to some positive power of $\omega$ implies that outside the set of points $U_M = \left\{Z_M^{(\varepsilon)}\left(|\Gamma|^2, |\Psi_0|^2\right)\right\}$, $h_C(\omega)$ and $h_D(\omega)$ can be considered as nul. We will write $Z$ the generic points of the complementary set of $U_M$, written $CU_M$. In the context of this work, the points $U_M$ correspond to an activated collective state.

We compute average connectivity between points of $U_M$, between points of $CU_M$, and between points of $U_M$ and $CU_M$.

### 2.3.3 Connectivity between points of $U_M$

The background state at points $\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right) \subset U_M$ is similar to (130):

$$\begin{aligned}
&\Gamma\left(T, \hat{T}, \theta, Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right) \\
= \;& \exp\left(-\left(\left(-\frac{1}{\tau\omega_M}T + \frac{\lambda}{\omega_M}\left\langle\hat{T}\right\rangle\right)\left|\Psi\left(\theta, Z_M^{(\varepsilon_1)}\right)\right|^2\right)^2\right) \\
&\times \exp\left(-\left(\frac{\rho}{\omega_M}\left(\left(h\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right) - \hat{T}\right)C(\theta)\, h_C - D(\theta)\,\hat{T} h_D\right)\left|[\Psi.\Psi]\left(\theta, Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right)\right|^2\right)^2\right)
\end{aligned}$$

wth:

$$\left|[\Psi.\Psi]\left(\theta,Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)\right|^2=\left|\Psi\left(\theta,Z_M^{(\varepsilon_1)}\right)\right|^2\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2$$

and the average values in this background states satisfy:

$$\begin{aligned}
C_{Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}} &= \frac{\alpha_C\omega_M\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2}{\frac{1}{\tau_C}+\alpha_C\omega_M\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2}\\
D_{Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}} &= \frac{\alpha_D\omega_M}{\frac{1}{\tau_D}+\alpha_D\omega_M\left|\Psi\left(\theta,Z_M^{(\varepsilon_1)}\right)\right|^2}
\end{aligned}$$

$$T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)=\lambda\tau\hat{T}\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)=\lambda\tau\frac{h\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)C_{Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}}(\theta)h_C}{C_{Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}}(\theta)h_C+D_{Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}}(\theta)h_D}$$

For an exponential dependency of connectivities in the distance between the connected points:

$$h\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)\simeq\exp\left(-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{\nu c}\right)$$

we obtain the average connectivity at points impacted by signals:

$$T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)=\frac{\lambda\tau\exp\left(-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{\nu c}\right)\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2h_C}{\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2h_C+\frac{\left(\frac{1}{\alpha_C\tau_C}+\omega_M\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2\right)\alpha_Dh_D}{\frac{1}{\tau_D}+\alpha_D\omega_M\left|\Psi\left(\theta,Z_M^{(\varepsilon_1)}\right)\right|^2}}\tag{164}$$

In a lon- run static perspective, it becomes:

$$T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)=\frac{\lambda\tau\exp\left(-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{\nu c}\right)\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2h_C}{\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2h_C+\left(\frac{1}{\alpha_C\tau_C}+\omega_M\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2\right)\frac{\alpha_Dh_D}{\frac{1}{\tau_D}+\alpha_D\omega_M\left|\Psi_0\left(Z_M^{(\varepsilon_1)}\right)\right|^2}}$$

The system has to be supplemented with long-term determination of activities $\omega_M$:

$$\begin{aligned}
\omega_M^{-1}\left(Z_M^{(\varepsilon_1)},|\Psi|^2\right) &\simeq G\left(\frac{\kappa}{N}\sum_{Z_M^{(\varepsilon_2)}}T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2\right)\\
&\simeq G\left(C\frac{\left|\Psi_0\left(Z_M\right)\right|^4h_C}{\left|\Psi_0\left(Z_M\right)\right|^2h_C+\left(\frac{1}{\alpha_C\tau_C}+\omega_M\left|\Psi_0\left(Z_M\right)\right|^2\right)\frac{\alpha_Dh_D}{\frac{1}{\tau_D}+\alpha_D\omega_M\left|\Psi_0(Z_M)\right|^2}}\right)
\end{aligned}$$

where:

$$C=\frac{\kappa\lambda\tau}{N\left(\sharp\left\{Z_{M}^{(\varepsilon_{1})}\right\}\right)}\sum_{Z_{M}^{(\varepsilon_{1})},Z_{M}^{(\varepsilon_{2})}}\exp\left(-\frac{\left|Z_{M}^{(\varepsilon_{1})}-Z_{M}^{(\varepsilon_{2})}\right|}{\nu c}\right)$$

and $|\Psi_{0}(Z_{M})|^{2}$ is the average of $\left|\Psi_{0}\left(Z_{M}^{(\varepsilon_{2})}\right)\right|^{2}$ over $\left\{Z_{M}^{(\varepsilon_{2})}\right\}$. The value of $|\Psi_{0}(Z_{M})|^{2}$ can be approximated in the following way. We have seen in part II in the field theoretic approach to the interferences that the signals modify the potential for $|\Psi_{0}(Z)|^{2}$ but that in first approximation, this modification can be neglected. Thus, the value of $|\Psi_{0}(Z)|^{2}$ after interferences may be replaced in first approximation by the background field before interferences. This is formula (129):

$$|\Psi_{0}(Z)|^{2}=\frac{2T(Z)\left\langle|\Psi_{0}(Z')|^{2}\right\rangle_{Z}}{\left(1+\sqrt{1+4\left(\frac{\lambda\tau\nu c-T(Z)}{\left(\frac{1}{\tau_{D}\alpha_{D}}+\frac{1}{\tau_{C}\alpha_{C}}+\Omega\right)T(Z)-\frac{1}{\tau_{D}\alpha_{D}}\lambda\tau\nu c}\right)^{2}T(Z)\left\langle|\Psi_{0}(Z')|^{2}\right\rangle_{Z}}\right)}$$

where all quantities are computed in the initial background state. The system emrging from the interferences thus depends on the whole initial structure.

### 2.3.4 Connectivity between points of $CU_{M}$

The connectivity function for two points in $CU_{M}$ is obtained by setting $\omega<<1$ and $\omega'<<1$:

$$T(Z,Z')\simeq\frac{\alpha_{C}\omega\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}h_{C}}{\alpha_{C}\omega\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}h_{C}+\left(\frac{\omega'}{\tau_{C}}\right)\frac{\alpha_{D}h_{D}}{\frac{|\Psi(\theta,Z)|^{2}}{\tau_{D}}}}$$

and these values are identitical to those computed for the static background state in the previous section (see (197)), up to some global modifications of the system by the interfering signals. These modifications are encompassed in the values of the constants $\Omega$, $\bar{\Omega}$... in (197). These modifications are negligible in general.

### 2.3.5 Connectivity between points of $U_{M}$ and points of $CU_{M}$

Two cases arise. The connectivity function for two points in $CU_{M}$ are obtained by setting $\omega=\omega_{M}$ and $\omega'<<1$ or $\omega<<1$ and $\omega'=\omega_{M}$.

$$T\left(Z_{M}^{(\varepsilon)},Z'\right)\simeq\frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}h_{C}}{\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}h_{C}+\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}\frac{\alpha_{D}\omega_{M}h_{D}}{\frac{|\Psi(\theta,Z)|^{2}}{\tau_{D}}+\alpha_{D}\omega_{M}}}\tag{165}$$

$$T\left(Z,Z_{M}^{(\varepsilon)}\right)\simeq\frac{\alpha_{C}\omega\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}h_{C}}{\alpha_{C}\omega\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}h_{C}+\frac{\omega_{M}}{\tau_{C}}\frac{\alpha_{D}h_{D}\tau_{D}}{|\Psi(\theta,Z)|^{2}}}<<1\tag{166}$$

As a consequence, points of the set $U$ do not connect with elements of $CU$. On the contrary, elements of $CU$ send signals and connect to elements of $U$ but their firing rate being slow, they do not influence the whole set that remains unaffected.

In the perspective of this article, this means that we can consider external sources affecting a single collective state.

## 2.4 Remark: Field theoretic transcription of source-induced modifications

Note that the switches:

$$T\left(Z_{M},Z_{M}\right)\rightarrow T\left(Z_{M}^{(\varepsilon_{1})},Z_{M}^{(\varepsilon_{2})}\right)$$

due to the sources can be implemented in the following manner. At points $\left(Z_{M}^{(\varepsilon_{1})},Z_{M}^{(\varepsilon_{2})}\right)$, the saddle point equation (136):

$$0=\left(-\sigma_{\hat{T}}^{2}\nabla_{\hat{T}}^{2}+\frac{1}{4\sigma_{\hat{T}}^{2}}\left(\left|\bar{\Psi}\left(Z,Z'\right)\right|^{2}\Delta\hat{T}+\frac{\rho\left(V_{0}-\frac{\sigma_{\hat{T}}^{2}}{\sigma_{T}^{2}}\lambda\Delta T\left|\Psi\left(Z\right)\right|^{2}\right)}{\omega_{0}\left(Z\right)}\right)^{2}\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right) \tag{167}$$
$$+\left(-\sigma_{T}^{2}\nabla_{T}^{2}+\frac{1}{4\sigma_{T}^{2}}\left(\frac{\Delta T-\lambda\tau\Delta\hat{T}}{\tau\omega_{0}\left(Z\right)}\right)^{2}\right.$$
$$\left.-\left(\frac{\left|\bar{\Psi}\left(Z,Z'\right)\right|^{2}}{2}+\frac{\left|\Psi\left(Z\right)\right|^{2}}{2\tau\omega_{0}\left(Z\right)}+V\left(\theta,Z,Z',\Delta\Gamma\right)\Delta T-\alpha\right)\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$$

is quadratic in connectivities. As a consequence adding to this equation a term:

$$\frac{\left(\Delta T\left(Z_{M},Z_{M}\right)-T\left(Z_{M}^{(\varepsilon_{1})},Z_{M}^{(\varepsilon_{2})}\right)\right)\Delta T}{2\sigma_{T}^{2}\left(\tau\omega_{0}\left(Z\right)\right)^{2}}\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$$

with:

$$\Delta T\left(Z_{M},Z_{M}\right)=T\left(Z_{M}^{(\varepsilon_{1})},Z_{M}^{(\varepsilon_{2})}\right)-T\left(Z_{M},Z_{M}\right)$$

will shift the average, in first approximation, by an amount:

$$\Delta T\rightarrow\Delta T+\Delta T\left(Z_{M},Z_{M}\right)$$

In terms of action functional, this corresponds to add a term in (84):

$$\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)J\left(\Delta T,Z,Z'\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$$

where:

$$J\left(\Delta T,Z,Z'\right)=\frac{\left(\Delta T\left(Z_{M},Z_{M}\right)-T\left(Z_{M}^{(\varepsilon_{1})},Z_{M}^{(\varepsilon_{2})}\right)\right)\Delta T}{2\sigma_{T}^{2}\left(\tau\omega_{0}\left(Z\right)\right)^{2}}$$

This modificaion of action generalizes easily to the collective field describing the collective states. We introduce in action (84) a term:

$$\underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)J\left(\Delta\mathbf{T},S^{2},\theta\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)$$

where $J\left(\Delta\mathbf{T},S^{2},\theta\right)$ encompass the contributions at every point of:

$$J\left(\Delta\mathbf{T},S^{2},\theta\right)=\sum_{Z\in S}J\left(\Delta\mathbf{T},Z,\theta\right)J\left(\Delta\mathbf{T},S^{2},\theta\right)$$

### 2.5 Synthesis: activities of activated states

We can gather the various results from this section. External signals induce oscillations in activity that propagate along the thread. Some cells may bind together at points of positive interference. The activation of a collective state corresponds to an average activity, as given by the formula above. Reversing the point of view, we can consider that collective states of bound cells are characterized by an activity characteristic of the collective state. The average of this activity can be computed, but we may also consider that these activities themselves may present oscillating characteristics. In the next section, we will compute both average levels and frequencies in activities. An important characteristic will emerge in this derivation. The solutions for average levels of activity amplitudes and frequencies may be multiple. A given collective state will thus possibly exist in multiple (possible infinite) states, labelled by integers. Such states may undergo transitions, and the collective state may switch from state to another.

# Appendix 3. Activities for collective field and averages

As explained in the text, we begin with a certain collective state, i.e. we consider a large number of elements connected with some average values of connectivity functions that will be determined later, based on consistency conditions. Writing the equations for activities in this state leads us to discover multiple oscillating solutions characterizing the collective state.

To find the activities $\omega\left(\theta, Z, |\Psi|^2\right)$, we start with the defining equation:

$$\begin{aligned}&\omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\\ = \;& G\left(J(\theta, Z)+\int \frac{\kappa}{N} \frac{\omega\left(J, \theta-\frac{|Z-Z_1|}{c}, Z_1, \Psi\right) \Delta T\left(Z, \theta, Z_1, \theta-\frac{|Z-Z_1|}{c}\right)}{\omega\left(J, \theta, Z, |\Psi|^2\right)}\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c}, Z_1\right)\right|^2 dZ_1\right)\end{aligned}$$

As in ([5]), we consider a static part that satisfies:

$$\omega^{-1}\left(Z, |\Psi|^2\right)=G\left(\int \frac{\kappa}{N} \frac{\Delta \omega\left(Z_1, \Psi\right) \Delta T\left(Z, Z_1\right)}{\Delta \omega\left(Z, |\Psi|^2\right)}\left|\Psi\left(Z_1\right)\right|^2 dZ_1\right)$$

Using that the modifications states is a group over a bounded domain and involve some finite number of points we find the equation for the modification modification at each point $Z_i$ of this group:

$$\omega_i^{-1}\left(Z, |\Psi|^2\right)=G\left(\sum_j \frac{\kappa}{N} \frac{\Delta \omega\left(Z_j, \Psi\right) \Delta T\left(Z_i, Z_j\right)}{\Delta \omega\left(Z_i, |\Psi|^2\right)}\left|\Psi\left(Z_j\right)\right|^2\right)$$

with solution:

$$\omega\left(\mathbf{Z}, \mathbf{T}, |\Psi|^2\right)$$

where $\omega\left(\mathbf{Z}, \mathbf{T}, |\Psi|^2\right)$ is the vector with coordinates $\omega\left(Z_i, T_i, |\Psi|^2\right)$.

The first order variation around the static background solution becomes:

$$\begin{aligned}
& \Delta\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right) \qquad (168)\\
= \quad & G'\left(G^{-1}\left(\omega_0^{-1}\left(J,Z,|\Psi|^2\right)\right)\right)\\
& \times\Delta\left(\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2 dZ_1\right)
\end{aligned}$$

where the variation $\Delta$ is given by:

$$\begin{aligned}
& \Delta\left(\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2 dZ_1\right)\\
\simeq \quad & -\int\frac{\kappa}{N}\frac{T(Z,Z_1)\,\omega_0(J,Z_1,\Psi)\,|\Psi_0(Z_1)|^2\,dZ_1}{\omega_0\left(J,Z,|\Psi|^2\right)}\frac{\Delta\omega^{-1}\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)}{\omega_0^{-1}(J,Z_1,\Psi)}\\
& +\int\frac{\kappa}{N}\frac{T(Z,Z_1)\,\omega_0(J,Z_1,\Psi)\,|\Psi_0(Z_1)|^2\,dZ_1}{\omega_0\left(J,Z,|\Psi|^2\right)}\frac{\Delta\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)}{\omega_0^{-1}\left(J,Z,|\Psi|^2\right)}\\
& +\int\frac{\kappa}{N}\frac{T(Z,Z_1)\,\omega_0(J,Z_1,\Psi)\,dZ_1}{\omega_0\left(J,Z,|\Psi|^2\right)}\Delta\,|\Psi_0(Z_1,\theta)|^2
\end{aligned}$$

The important point at this stage is the following. We showed in ([5]) (see an account in appendix0B) that wave dynamics for connectivities implies that the contribution $\Delta|\Psi_0(Z_1,\theta)|^2$ stabilizes the dynamics for $\Delta\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)$[8]. As a consequence, we can consider in first approximation that $\Delta|\Psi_0(Z_1,\theta)|^2=0$ and assume that activities will be stably oscillating. Moreover, at the level of activities oscillations, connectivities $T(Z,Z_1)$ will also be considered as static for average oscillations:

The oscillation part is obtained by:

$$\begin{aligned}
& \Delta\left(\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2 dZ_1\right)\\
\simeq \quad & -\int\frac{\kappa}{N}\frac{T(Z,Z_1)\,\omega(J,Z_1,\Psi)\,|\Psi_0(Z_1)|^2\,dZ_1}{\omega\left(J,Z,|\Psi|^2\right)}\frac{\Delta\omega^{-1}\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)}{\omega^{-1}(J,Z_1,\Psi)}\\
& +\int\frac{\kappa}{N}\frac{T(Z,Z_1)\,\omega(J,Z_1,\Psi)\,|\Psi_0(Z_1)|^2\,dZ_1}{\omega\left(J,Z,|\Psi|^2\right)}\frac{\Delta\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)}{\omega^{-1}\left(J,Z,|\Psi|^2\right)}
\end{aligned}$$

and equation (168) rewrites:

[8] See also the discussion after(170).

$$\left(1-\frac{G'\left(G^{-1}\left(\omega^{-1}\left(J,Z,|\Psi|^{2}\right)\right)\right)}{\omega^{-1}\left(J,Z,|\Psi|^{2}\right)}\right)\Delta\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)$$

$$= -G'\left(G^{-1}\left(\omega^{-1}\left(J,Z,|\Psi|^{2}\right)\right)\right)\int\frac{\kappa}{N}\frac{T\left(Z,Z_{1}\right)\omega\left(J,Z_{1},\Psi\right)\left|\Psi_{0}\left(Z_{1}\right)\right|^{2}dZ_{1}}{\omega\left(J,Z,|\Psi|^{2}\right)}\frac{\Delta\omega^{-1}\left(J,\theta-\frac{|Z-Z_{1}|}{c},Z_{1},\Psi\right)}{\omega^{-1}\left(J,Z_{1},\Psi\right)}$$

with solution:

$$\begin{aligned}\Delta\omega\left(J,\theta,Z,|\Psi|^{2}\right) &= G'\left(G^{-1}\left(\omega^{-1}\left(J,Z,|\Psi|^{2}\right)\right)\right)\\ &\times\int\frac{\kappa}{N}\frac{T\left(Z,Z_{1}\right)\omega\left(J,Z_{1},\Psi\right)\left|\Psi_{0}\left(Z_{1}\right)\right|^{2}dZ_{1}}{G'\left(G^{-1}\left(\omega^{-1}\left(J,Z,|\Psi|^{2}\right)\right)\right)-\omega^{-1}\left(J,Z,|\Psi|^{2}\right)}\Delta\omega\left(J,\theta-\frac{|Z-Z_{1}|}{c},Z_{1},\Psi\right)\end{aligned}$$

We use that the modified state form a group over a bounded domain and involve some finite number of points. For such group, we replace the integral by a sum and we have:

$$\Delta\omega\left(\theta,Z_{i},|\Psi|^{2}\right)=\sum_{j}\hat{T}\left(Z_{i},Z_{j}\right)\Delta\omega\left(\theta-\frac{|Z_{i}-Z_{j}|}{c},Z_{j},\Psi\right) \tag{169}$$

where:

$$\hat{T}\left(Z_{i},Z_{j}\right)=\frac{\kappa}{N}\frac{T\left(Z_{i},Z_{j}\right)\omega\left(J,Z_{j},\Psi\right)\left|\Psi_{0}\left(Z_{j}\right)\right|^{2}}{G^{-1}\left(\omega^{-1}\left(J,Z_{i},|\Psi|^{2}\right)\right)-\omega^{-1}\left(J,Z_{i},|\Psi|^{2}\right)}$$

We look for oscillatory solutions:

$$\Delta\omega\left(J,\theta,Z,|\Psi|^{2}\right)=A\left(Z_{i}\right)\exp\left(i\Upsilon\theta\right)$$

and for such functions, equation (169) writes:

$$A\left(Z_{i}\right)=\sum_{j}\hat{T}\left(Z_{i},Z_{j}\right)\exp\left(-i\Upsilon\frac{|Z_{i}-Z_{j}|}{c}\right)A\left(Z_{j}\right) \tag{170}$$

Equation (170) has non-nul solutions for frequencies $\Upsilon_{p}$ satisfying:

$$\det\left(1-T\left(Z_{i},Z_{j}\right)\exp\left(-i\Upsilon_{p}\frac{|Z_{i}-Z_{j}|}{c}\right)\right)=0$$

Generally, the $\Upsilon_{p}$ are complex, and oscillations are dampened, However we showed in ([5]), that the perturbative corrections to the effective action for $\left|\Psi_{0}\left(Z_{l}\right)\right|^{2}$ shifts the background:

$$\left|\Psi_{0}\left(Z_{l}\right)\right|^{2}\rightarrow\left|\Psi_{0}\left(Z_{l}\right)\right|^{2}+\Delta\left|\Psi_{0}\left(Z_{l}\right)\right|^{2}$$

and this shifts allows for stable oscillations. In first proximation, this is equivalent to consider that for some parameters, the $\Upsilon_{p}$ can be considered as real, and $\left|\Psi_{0}\left(Z_{1}\right)\right|^{2}$ as time independent.

The possible osillatory activities associated to the assembly is thus given by the sets:

$$\left\{\left\{A\left(Z_{i}\right)\right\}_{i=1,\ldots n},\Upsilon_{p}\left(\left\{\hat{T}\left(Z_{i},Z_{j}\right)\right\}\right)\right\}_{p}$$

where $p$ refers to the frequencies $\Upsilon$, and the $A(Z_i)$ satisfy:

$$A(Z_i) = \sum_{j \neq i} A(Z_j) \hat{T}(Z_i, Z_j) \exp\left(-i\Upsilon_p\left(\left\{\hat{T}(Z_i, Z_j)\right\}\right)\frac{|Z_i - Z_j|}{c}\right)$$

In these equation, one amplitude is a free parameter. We can choose $A(Z_1)$ and set:

$$A_1(Z_i) = \frac{A(Z_i)}{A(Z_1)}$$

Thus, the $A_1(Z_i)$ are solutions of $n-1$ systems of equation with $i \geqslant 2$:

$$A_1(Z_i) = \sum_{j \neq i} A_1(Z_j) \hat{T}(Z_i, Z_j) \exp\left(-i\Upsilon_p \frac{|Z_i - Z_j|}{c}\right) \tag{171}$$

For $i \neq 1$, $j \neq 1$, the solutions of(171) are:

$$\left(\delta_{ij} - \hat{T}(Z_i, Z_j) \exp\left(-i\Upsilon_p \frac{|Z_i - Z_j|}{c}\right)\right) A_1(Z_j) = \hat{T}(Z_i, Z_1) \exp\left(-i\Upsilon_p \frac{|Z_i - Z_1|}{c}\right) \tag{172}$$

We rewrite these equations (172) matricially by setting:

$$\left(1 - \hat{\mathbf{T}} \exp\left(-i\Upsilon_p \frac{|\mathbf{\Delta Z}|}{c}\right)\right)_{ij} = \left(\delta_{ij} - \hat{T}(Z_i, Z_j) \exp\left(-i\Upsilon_p \frac{|Z_i - Z_j|}{c}\right)\right)$$

and

$$\left(\hat{T}_1(\mathbf{Z}) \exp\left(-i\Upsilon_p \frac{|\mathbf{\Delta Z}_1|}{c}\right)\right)_i = \hat{T}(Z_i, Z_1) \exp\left(-i\Upsilon_p \frac{|Z_i - Z_1|}{c}\right)$$

As a consequence, equation (172) has the form:

$$A_1(\mathbf{Z}) = \left(1 - \hat{\mathbf{T}} \exp\left(-i\Upsilon_p \frac{|\mathbf{\Delta Z}|}{c}\right)\right)^{-1} \hat{T}_1(\mathbf{Z}) \exp\left(-i\Upsilon_p \frac{|\mathbf{\Delta Z}_1|}{c}\right)$$

and the modified dynamic of activities writes;

$$\Delta\omega\left(\theta, \mathbf{Z}, |\Psi|^2\right) = A(Z_1)\left(1, \left(1 - \hat{\mathbf{T}} \exp\left(-i\Upsilon_p \frac{|\mathbf{\Delta Z}|}{c}\right)\right)^{-1} \hat{T}_1(\mathbf{Z}) \exp\left(-i\Upsilon_p \frac{|\mathbf{\Delta Z}_1|}{c}\right)\right)^t \exp\left(i\Upsilon_p\left(\hat{\mathbf{T}}\right)\theta\right)$$

where:

$$\Upsilon_p\left(\hat{\mathbf{T}}\right) = \Upsilon_p\left(\left\{\hat{T}(Z_i, Z_j)\right\}\right)$$

Ultimately, adding also the average static solution with respect to $\hat{\mathbf{T}}$ yields the result:

$$\begin{aligned}&\Delta\omega(Z, \mathbf{T}) \\ &+A(Z_1)\left(1, \left(1 - \hat{\mathbf{T}} \exp\left(-i\Upsilon_p\left(\hat{\mathbf{T}}\right) \frac{|\mathbf{\Delta Z}|}{c}\right)\right)^{-1} \hat{T}_1(\mathbf{Z}) \exp\left(-i\Upsilon_p\left(\hat{\mathbf{T}}\right) \frac{|\mathbf{\Delta Z}_1|}{c}\right)\right)^t \exp\left(i\Upsilon_p\left(\hat{\mathbf{T}}\right)\theta\right)\end{aligned}$$

## 3.1 Averages computations and activated states

### 3.1.1 Averages

Now, considering (14) for the group of shifted states, we rewrite the action functional for this group by taking into account their particular interactions. Given the activities, we can compute the average connectivities. To do so, we replace in (15) (see ([8])):

$$\left(\left(\left(Z-Z'\right)\left(\nabla_Z+\nabla_Z\omega_0\left(Z\right)\right)+\frac{\left|Z-Z'\right|}{c}\right)\Delta\omega\left(\theta,Z,\left|\Psi\right|^2\right)\right) \rightarrow \frac{\left|Z-Z'\right|}{c}\Delta\omega\left(\theta,Z,\left|\Psi\right|^2\right) \rightarrow g\left|Z-Z'\right|\Delta\omega\left(\theta,Z,\left|\Psi\right|^2\right)$$

assume $A\left(Z_1\right)$ proportional to activity on the group:

$$\begin{aligned}&\left(\left(Z-Z'\right)\left(\nabla_Z+\nabla_Z\omega_0\left(Z\right)\right)+\frac{\left|Z-Z'\right|}{c}\right)\Delta\omega\left(\theta,\mathbf{Z},\left|\Psi\right|^2\right)\\ =\;& \left|Z-Z'\right|k\left\langle\left|\Psi\right|^2\right\rangle\left(1,\left(1-\mathbf{T}\exp\left(-i\Upsilon_p\frac{\left|\mathbf{\Delta Z}\right|}{c}\right)\right)^{-1}T\left(\mathbf{Z}\right)\exp\left(-i\Upsilon_p\frac{\left|\mathbf{\Delta Z}_1\right|}{c}\right)\right)^t\exp\left(i\Upsilon_p\left(\mathbf{T}\right)\theta\right)\end{aligned}$$

In these equations, we decompose $\mathbf{T}\rightarrow\langle\mathbf{T}\rangle+\mathbf{\Delta T}$ where $\langle\mathbf{T}\rangle$ is defined by the background.

We thus consider the effective action resulting from the shift. It leads to rewrite (14):

$$\begin{aligned}&-\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\nabla_T\left(\sigma_T^2\nabla_T+\frac{\left(\Delta T-\underline{\Delta\left\langle T\right\rangle}\right)-\lambda\left(\Delta\hat{T}-\underline{\Delta\left\langle\hat{T}\right\rangle}\right)}{\tau\omega_0\left(Z\right)}\left|\Psi\left(\theta,Z\right)\right|^2\right)\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\&-\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\nabla_{\hat{T}}\left(\sigma_{\hat{T}}^2\nabla_{\hat{T}}-\rho\left|\bar{\Psi}_0\left(Z,Z'\right)\right|^2\left(\Delta\hat{T}-\underline{\Delta\left\langle\hat{T}\right\rangle}\right)\right.\\&+\frac{\rho D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2}{\omega_0\left(Z\right)}\left.\left(\left(Z-Z'\right)\left(\nabla_Z+\nabla_Z\omega_0\left(Z\right)\right)+\frac{\left|Z-Z'\right|}{c}\right)\Delta\omega\left(\theta,Z,\left|\Psi\right|^2\right)\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\end{aligned} \tag{173}$$

At the scale considered, system's frequencies can be replaced by their averages:

$$\Delta\omega\left(\theta,Z,\left|\Psi\right|^2\right)\simeq\Delta\omega\left(Z,\mathbf{T}\right)$$

and the second term of (173) rewrites:

$$\begin{aligned}&\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\nabla_{\hat{T}}\left(\rho\left|\Psi_0\left(Z\right)\right|^2\left(\Delta\hat{T}-\underline{\Delta\left\langle\hat{T}\right\rangle}\right)\right.\\&\left.+\frac{\rho}{\omega_0\left(Z\right)}\left(D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2A\left|Z-Z'\right|\Delta\omega\left(\theta,Z,\left|\Psi\right|^2\right)\right)\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\ =\;&\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\\&\times\nabla_{\hat{T}}\left(\frac{\rho\left(C\left(\theta\right)\left|\Psi_0\left(Z\right)\right|^2\omega_0\left(Z\right)+D\left(\theta\right)\hat{T}\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z'\right)\right)\left(\Delta\hat{T}-\Delta'\hat{T}\right)}{\omega_0\left(Z\right)}\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\end{aligned}$$

wth:

$$\begin{aligned}\Delta'\hat{T}&=\underline{\Delta\left\langle\hat{T}\right\rangle}+\frac{\left(D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2\left|Z-Z'\right|g\Delta\omega\left(\theta,Z,\left|\Psi\right|^2\right)\right)}{\left(C\left(\theta\right)\left|\Psi_0\left(Z\right)\right|^2\omega_0\left(Z\right)+D\left(\theta\right)\hat{T}\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z'\right)\right)}\\&\simeq\underline{\Delta\left\langle\hat{T}\right\rangle}+\Delta^{\omega}\hat{T}\end{aligned}$$

whr:

$$\Delta^{\omega}\hat{T}=\frac{\left(D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2\left|Z-Z'\right|g\Delta\omega\left(Z,\mathbf{T}\right)\right)}{\left(C\left(\theta\right)\left|\Psi_0\left(Z\right)\right|^2\omega_0\left(Z\right)+D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z'\right)\right)}$$

Dfn ls:

$$\Delta'T=\underline{\Delta\left\langle\hat{T}\right\rangle}+\lambda\Delta^{\omega}\hat{T}$$

### 3.1.2 Change of variables and activated states

Performing the change of variable in (173):

$$\begin{aligned}\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\quad&\rightarrow\quad\exp\left(-\frac{\rho\left(\left(C\left(\theta\right)\left|\Psi\left(Z\right)\right|^2h_C+D\left(\theta\right)\left|\Psi\left(Z'\right)\right|^2h_D\right)\left(\Delta\hat{T}-\Delta'\hat{T}\right)^2\right)}{4\sigma_{\hat{T}}^2\omega\left(\theta,Z,\left|\Psi\right|^2\right)}\right)\\&\times\exp\left(-\frac{\left(\left(\Delta T-\Delta'T\right)^2-2\lambda\tau\left(\Delta\hat{T}-\Delta'\hat{T}\right)\left(\Delta T-\Delta'T\right)\right)}{4\sigma_T^2\tau\omega}\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\end{aligned}\tag{174}$$

and:

$$\begin{aligned}\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\quad&\rightarrow\quad\exp\left(\frac{\rho\left(\left(C\left(\theta\right)\left|\Psi\left(Z\right)\right|^2h_C+D\left(\theta\right)\left|\Psi\left(Z'\right)\right|^2h_D\right)\left(\Delta\hat{T}-\Delta'\hat{T}\right)^2\right)}{4\sigma_{\hat{T}}^2\omega\left(\theta,Z,\left|\Psi\right|^2\right)}\right)\\&\times\exp\left(\frac{\left(\frac{(T-\langle T\rangle)^2}{\tau}-2\lambda\left(\Delta\hat{T}-\Delta'\hat{T}\right)\left(\Delta T-\Delta'T\right)\right)}{4\sigma_T^2\tau\omega}\right)\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\end{aligned}\tag{175}$$

leads to the action in first approximation in $\frac{\sigma_{\hat{T}}^2}{\sigma_T^2}$:

$$\begin{aligned}&S\left(\left\{\Delta\Gamma_{S_\alpha^2}\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_\alpha^2},\theta\right)\right\}\right)\\=\;&-\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\sigma_T^2\nabla_T^2-\frac{1}{2\sigma_T^2}\left(\frac{\left(\Delta T-\Delta T'\right)-\lambda\left(\Delta\hat{T}-\Delta'\hat{T}\right)}{\tau\omega_0\left(Z\right)}\left|\Psi_0\left(\theta,Z\right)\right|^2\right)^2\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\&-\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\\&\times\left(\sigma_{\hat{T}}^2\nabla_{\hat{T}}^2-\frac{1}{2\sigma_{\hat{T}}^2}\left(\frac{\rho\left(C\left(\theta\right)\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z\right)+D\left(\theta\right)\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z'\right)\right)\left(\Delta\hat{T}-\Delta'\hat{T}\right)}{\omega_0\left(Z\right)}\right)^2\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\&+\left(\tau\omega_0\left(Z\right)+\frac{\rho\left(C\left(\theta\right)\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z\right)+D\left(\theta\right)\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z'\right)\right)}{\omega_0\left(Z\right)}\right)\left\|\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right\|^2\end{aligned}\tag{176}$$

The average values satisfy:

$$\begin{aligned}\left\langle\Delta T\right\rangle\quad&=\quad\left\langle\Delta'T\right\rangle=\underline{\Delta\left\langle T\right\rangle}+\lambda\frac{\left(D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2\left|Z-Z'\right|g\Delta\omega\left(Z,\left\langle\mathbf{\Delta T}\right\rangle\right)\right)}{\left(C\left(\theta\right)\left|\Psi_0\left(Z\right)\right|^2\omega_0\left(Z\right)+D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z'\right)\right)}\\\left\langle\Delta\hat{T}\right\rangle\quad&=\quad\left\langle\Delta'\hat{T}\right\rangle=\underline{\Delta\left\langle\hat{T}\right\rangle}+\frac{\left(D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2\left|Z-Z'\right|g\Delta\omega\left(Z,\left\langle\mathbf{\Delta T}\right\rangle\right)\right)}{\left(C\left(\theta\right)\left|\Psi_0\left(Z\right)\right|^2\omega_0\left(Z\right)+D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z'\right)\right)}\end{aligned}\tag{177}$$

and the effective action wrts:

$$\begin{aligned}
&\hat{S}\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right) \qquad (178)\\
=\; &-\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\\
&\left(\sigma_T^2\nabla_T^2-\frac{1}{2\sigma_T^2}\left(\frac{\left(\Delta T-\langle\Delta T\left(Z,Z'\right)\rangle_p^{\alpha}\right)-\lambda\left(\Delta\hat{T}-\left\langle\Delta\hat{T}\left(Z,Z'\right)\right\rangle_p^{\alpha}\right)}{\tau\omega_0\left(Z\right)}\left|\Psi_0\left(\theta,Z\right)\right|^2\right)^2\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\
&-\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\\
&\times\left(\sigma_{\hat{T}}^2\nabla_{\hat{T}}^2-\frac{\left(D\left(Z,Z'\right)\left(\Delta\hat{T}-\left\langle\Delta\hat{T}\left(Z,Z'\right)\right\rangle_p^{\alpha}\right)+\mathbf{M}^{\alpha}\left(Z,Z'\right)\left(\Delta T-\langle\Delta T\left(Z,Z'\right)\rangle_p^{\alpha}\right)\right)^2}{2\sigma_{\hat{T}}^2}\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\
&+C\left(Z,Z'\right)\left\|\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right\|^2
\end{aligned}$$

with:

$$C\left(Z,Z'\right)=\tau\omega_0\left(Z\right)+\frac{\rho\left(C\left(\theta\right)\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z\right)+D\left(\theta\right)\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z'\right)\right)}{\omega_0\left(Z\right)}$$

and:

$$\mathbf{M}^{\alpha}\left(Z,Z'\right)=\left(\frac{\rho D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2A\left|Z-Z'\right|}{\omega_0\left(Z\right)}\left(\nabla_{\mathbf{\Delta T}_{\left(Z_1,Z_1'\right)}}\left(\Delta\omega\left(Z,\langle\mathbf{\Delta T}\rangle\right)\right)_{\left(\left\langle\mathbf{\Delta T}_{\left(Z_1,Z_1'\right)}\right\rangle_p^{\alpha}\right)}\right)\right)$$

and action (178) is rewritten as:

$$\begin{aligned}
&\hat{S}\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right) \qquad (179)\\
=\; &-\Delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\nabla_T^2+\nabla_{\hat{T}}^2-\frac{1}{2}\left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^{\alpha}\right)^t\mathbf{A}_p^{\alpha}\left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^{\alpha}\right)\right)\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\
&+C\left(Z,Z'\right)\left\|\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right\|^2
\end{aligned}$$

where the variables are:

$$\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^{\alpha}=\begin{pmatrix}\Delta T-\langle\Delta T\rangle\\ \Delta\hat{T}-\left\langle\Delta\hat{T}\right\rangle\end{pmatrix} \qquad (180)$$

and:

$$\mathbf{A}_p^{\alpha}=\begin{pmatrix}\left(\frac{1}{\tau\omega_0(Z)}\right)^2+\left(\mathbf{M}^{\alpha}\left(Z,Z'\right)\right)^2 & -\lambda\left(\frac{1}{\tau\omega_0(Z)}\right)^2+D\left(Z,Z'\right)\mathbf{M}^{\alpha}\left(Z,Z'\right)\\ -\lambda\left(\frac{1}{\tau\omega_0(Z)}\right)^2+D\left(Z,Z'\right)\mathbf{M}^{\alpha}\left(Z,Z'\right) & \left(\frac{\lambda}{\tau\omega_0(Z)}\right)^2+D^2\left(Z,Z'\right)\end{pmatrix}$$

The matrix $\hat{\mathbf{A}}_p^\alpha$ is defined by:

$$\begin{aligned}
\hat{\mathbf{A}}_p^\alpha &= \begin{pmatrix} \frac{1}{\sigma_T} & 0 \\ 0 & \frac{1}{\sigma_{\hat{T}}} \end{pmatrix} \mathbf{A}_p^\alpha \begin{pmatrix} \frac{1}{\sigma_T} & 0 \\ 0 & \frac{1}{\sigma_{\hat{T}}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\left(\frac{1}{\tau\omega_0(Z)}\right)^2 + \left(\mathbf{M}^\alpha(Z,Z')\right)^2}{\sigma_T^2} & \frac{-\lambda\left(\frac{1}{\tau\omega_0(Z)}\right)^2 + D(Z,Z')\mathbf{M}^\alpha(Z,Z')}{\sigma_T\sigma_{\hat{T}}} \\ \frac{-\lambda\left(\frac{1}{\tau\omega_0(Z)}\right)^2 + D(Z,Z')\mathbf{M}^\alpha(Z,Z')}{\sigma_T\sigma_{\hat{T}}} & \frac{\left(\frac{\lambda}{\tau\omega_0(Z)}\right)^2 + D^2(Z,Z')}{\sigma_{\hat{T}}^2} \end{pmatrix}
\end{aligned}$$

The minimization of $\hat{S}\left(\Delta\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right)$ is similar to (140) and leads to the solutions:

$$\begin{aligned}
\Delta\Gamma_{n,n'}^{\alpha,p}\left(T,\hat{T},\theta,Z,Z'\right) &= \mathcal{N}\exp\left(-\frac{1}{2}\left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)^t \hat{\mathbf{A}}_p^\alpha \left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)\right) \\
&\quad\times H_n\left(\frac{\sigma_T}{2\sqrt{2}}\left(\left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)^t \hat{\mathbf{A}}_p^\alpha \left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)\right)_2\right) \\
&\quad\times H_{n'}\left(\frac{\sigma_{\hat{T}}}{2\sqrt{2}}\left(\left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)^t \hat{\mathbf{A}}_p^\alpha \left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)\right)_1\right)
\end{aligned} \tag{181}$$

and:

$$\begin{aligned}
\Delta\Gamma_{n,n'}^{\dagger\alpha,p}\left(T,\hat{T},\theta,Z,Z'\right) &= H_n\left(\frac{\sigma_T}{2\sqrt{2}}\left(\left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)^t \hat{\mathbf{A}}_p^\alpha \left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)\right)_+\right) \\
&\quad\times H_{n'}\left(\frac{\sigma_{\hat{T}}}{2\sqrt{2}}\left(\left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)^t \hat{\mathbf{A}}_p^\alpha \left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)\right)_-\right)
\end{aligned} \tag{182}$$

where $H_p$ and $H_{p-\delta}$ are Hermite polynomials. Given the diagonalization of $\hat{\mathbf{A}}_p^\alpha$

$$\hat{\mathbf{A}}_p^\alpha = PDP^{-1}$$

$$\begin{aligned}
\left(\left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)^t \hat{\mathbf{A}}_p^\alpha \left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)\right)_+ &= P\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} D \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P^{-1} \\
\left(\left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)^t \hat{\mathbf{A}}_p^\alpha \left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)\right)_- &= P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} D \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P^{-1}
\end{aligned}$$

The constant $\mathcal{N}$ is the normalization factor.

The lowest eigenvalue state is:

$$\Delta\Gamma_0^{\alpha,p}\left(T,\hat{T},\theta,Z,Z'\right) = \mathcal{N}\exp\left(-\frac{1}{2}\left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)^t \hat{\mathbf{A}}_p^\alpha \left(\mathbf{\Delta T}-\langle\mathbf{\Delta T}\rangle_p^\alpha\right)\right) \tag{183}$$

and the state associated to the system is:

$$\prod_{Z,Z'}\Delta\Gamma_\delta^{\alpha,p}\left(T,\hat{T},\theta,Z,Z'\right) \equiv \prod_{Z,Z'}\left|\Delta T\left(Z,Z'\right),\Delta\hat{T}\left(Z,Z'\right),\alpha\left(Z,Z'\right),p\left(Z,Z'\right),S^2\right\rangle \equiv \left|\boldsymbol{\alpha},\mathbf{p},S^2\right\rangle$$

To simplify the arguments, we assume as before that $\left\langle\Delta\hat{T}\right\rangle \simeq \frac{\langle\Delta T\rangle}{\lambda}$. This yields the equation for the average $\left\langle\Delta\hat{T}\right\rangle$ by writing:

$$\left\langle\Delta\hat{T}\right\rangle = \left\langle\Delta'\hat{T}\right\rangle$$

which leads to the relation:There are several sets of solutions:

$$\left(\langle\Delta\mathbf{T}\rangle^{\alpha},\left\langle\Delta\hat{\mathbf{T}}\right\rangle^{\alpha}=\frac{\langle\Delta\mathbf{T}\rangle^{\alpha}}{\lambda}\right)$$

For each of these solutions, a sequence of frequencies $\left(\Upsilon_{p}^{\alpha}\right)$ are compatible, and the variable part of activities is:

$$\Delta\omega_{p}^{\alpha}\left(\theta,\mathbf{Z}\right)=A\left(Z_{1}\right)\left(1,\left(1-\left(\langle\Delta\mathbf{T}\rangle_{p}^{\alpha}\right)\exp\left(-i\Upsilon_{p}\frac{|\mathbf{\Delta Z}|}{c}\right)\right)^{-1}\langle\Delta\mathbf{T}\rangle_{p}^{\alpha}\exp\left(-i\Upsilon_{p}\frac{|\mathbf{\Delta Z}|}{c}\right)\right)^{t}\exp\left(i\Upsilon_{p}^{\alpha}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)\theta\right)$$

Ultimately, the effective action $S\left(\left\{\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\}\right)$ for the state writes:

$$\begin{aligned}&S\left(\left\{\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\}\right) \qquad \text{(FCT)}\\=\;&\underline{\Gamma}^{\dagger}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\left(-\nabla_{\Delta\mathbf{T}}^{2}+\frac{1}{2}\left(\mathbf{A}\left(\Delta\mathbf{T}-\langle\Delta\mathbf{T}\rangle^{\alpha}\right)\right)^{2}+C\right)\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)+U\left(\left\|\underline{\Gamma}\left(\Delta\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2},\theta\right)\right\|^{2}\right)\end{aligned}$$

where $\mathbf{D}$ is diagonal with elements:

$$\mathbf{D}\left(Z,Z'\right)=D\left[\frac{\rho\left(C\left(\theta\right)\left|\Psi_{0}\left(Z\right)\right|^{2}\omega_{0}\left(Z\right)+D\left(\theta\right)\hat{T}\left|\Psi_{0}\left(Z'\right)\right|^{2}\omega_{0}\left(Z'\right)\right)}{\omega_{0}\left(Z\right)}\right]$$

The matrix $\mathbf{A}^{\alpha}$ and the constant $C^{\alpha}\left(Z,Z'\right)$ are defined by:

$$\mathbf{A}^{\alpha}=\sqrt{\mathbf{D}^{2}+\left(\mathbf{M}^{\alpha}\right)^{t}\mathbf{M}^{\alpha}}$$

$$C=\sum_{\left(Z,Z'\right)}\mathbf{C}\left(Z,Z'\right)$$

with:

$$\mathbf{C}\left(Z,Z'\right)=\frac{\tau\omega_{0}\left(\mathbf{Z}_{1}\right)}{2}+\frac{\rho\left(C\left(\theta\right)\left|\Psi_{0}\left(\mathbf{Z}_{1}\right)\right|^{2}\omega_{0}\left(\mathbf{Z}_{1}\right)+D\left(\theta\right)\left|\Psi_{0}\left(\mathbf{Z}_{2}\right)\right|^{2}\omega_{0}\left(\mathbf{Z}_{2}\right)\right)}{2\omega_{0}\left(\mathbf{Z}_{1}\right)}$$

### 3.2 Rewriting effective action for collective state

Using a change of variable similar to ([8]), we obtain:

$$\begin{aligned}&S\left(\left\{\Delta\Gamma_{S_{\alpha}^{2}}\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_{\alpha}^{2}},\theta\right)\right\}\right) \qquad (184)\\=\;&-\Delta\Gamma_{S_{\alpha}^{2}}^{\dagger}\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_{\alpha}^{2}},\theta\right)\\&\left(\nabla_{\mathbf{T}}^{2}-\frac{1}{2}\left(\frac{\left(\Delta\mathbf{T}-\underline{\langle\Delta\mathbf{T}\left(\mathbf{Z}\right)\rangle}\right)-\lambda\left(\Delta\hat{\mathbf{T}}-\underline{\left\langle\Delta\hat{\mathbf{T}}\left(\mathbf{Z}\right)\right\rangle}\right)}{\tau\omega_{0}\left(\mathbf{Z}_{1}\right)}\left|\Psi_{0}\left(\theta,\mathbf{Z}_{1}\right)\right|^{2}\right)^{2}\right)\Delta\Gamma_{S_{\alpha}^{2}}\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_{\alpha}^{2}},\theta\right)\\&-\Delta\Gamma^{\dagger}\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_{\alpha}^{2}},\theta\right)\\&\times\left(\nabla_{\hat{\mathbf{T}}}^{2}-\frac{1}{2}\left(\frac{\rho\left(C\left(\theta\right)\left|\Psi_{0}\left(\mathbf{Z}_{1}\right)\right|^{2}\omega_{0}\left(\mathbf{Z}_{1}\right)+D\left(\theta\right)\left|\Psi_{0}\left(\mathbf{Z}_{2}\right)\right|^{2}\omega_{0}\left(\mathbf{Z}_{2}\right)\right)\left(\Delta\hat{\mathbf{T}}-\Delta\hat{\mathbf{T}}'\left(\mathbf{Z}\right)\right)}{\omega_{0}\left(\mathbf{Z}_{1}\right)}\right)^{2}\right)\Delta\Gamma\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_{\alpha}^{2}},\theta\right)\\&+\left(\tau\omega_{0}\left(\mathbf{Z}_{1}\right)+\frac{\rho\left(C\left(\theta\right)\left|\Psi_{0}\left(\mathbf{Z}_{1}\right)\right|^{2}\omega_{0}\left(\mathbf{Z}_{1}\right)+D\left(\theta\right)\left|\Psi_{0}\left(\mathbf{Z}_{2}\right)\right|^{2}\omega_{0}\left(\mathbf{Z}_{2}\right)\right)}{\omega_{0}\left(\mathbf{Z}_{1}\right)}\right)\left\|\Delta\Gamma_{S_{\alpha}^{2}}\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_{\alpha}^{2}},\theta\right)\right\|^{2} \qquad (185)\end{aligned}$$

where:

$$\Delta\hat{\mathbf{T}}-\Delta\hat{\mathbf{T}}'=\Delta\hat{\mathbf{T}}-\left\langle\Delta\hat{\mathbf{T}}\right\rangle+\frac{\left(D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2\left|Z-Z'\right|g\Delta\omega\left(Z,\Delta\mathbf{T}\right)-\omega\left(Z,\left\langle\Delta\mathbf{T}\right\rangle\right)\right)}{\left(C\left(\theta\right)\left|\Psi_0\left(Z\right)\right|^2\omega_0\left(Z\right)+D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z'\right)\right)}$$

We can rewrite the differences in (184). We define:

$$\left[\mathbf{N}_p^{\alpha}\right]_{\left(Z_i,Z_j\right)}=\left(\delta_{ij}-\left[\mathbf{\Delta T}\right]_{\left(Z_i,Z_j\right)}\exp\left(-i\Upsilon_p\frac{\left|Z_i-Z_j\right|}{c}\right)\right)^{-1}$$

so that in first approximation we have:

$$\begin{aligned}\Delta\hat{\mathbf{T}}-\Delta\hat{\mathbf{T}}' &= \Delta\hat{\mathbf{T}}-\left\langle\Delta\hat{\mathbf{T}}\right\rangle+\frac{\left(D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2\left|Z-Z'\right|g\Delta\omega\left(Z,\Delta\mathbf{T}\right)-\omega\left(Z,\left\langle\Delta\mathbf{T}\right\rangle\right)\right)}{\left(C\left(\theta\right)\left|\Psi_0\left(Z\right)\right|^2\omega_0\left(Z\right)+D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z'\right)\right)}\\ &= \Delta\hat{\mathbf{T}}-\left\langle\Delta\hat{\mathbf{T}}\right\rangle+\left(\nabla_{\mathbf{\Delta T}_{\left(Z_1,Z_1'\right)}}\left(\Delta\omega\left(Z,\left\langle\mathbf{\Delta T}\right\rangle\right)\right)_{\left(\left\langle\mathbf{\Delta T}_{\left(Z_1,Z_1'\right)}\right\rangle_p^{\alpha}\right)}\right)\left(\Delta\mathbf{T}-\left\langle\Delta\mathbf{T}\right\rangle^{\alpha}\right)\end{aligned}$$

and:

$$\frac{\left(\Delta\mathbf{T}-\underline{\left\langle\Delta\mathbf{T}\left(\mathbf{Z}\right)\right\rangle}\right)-\lambda\left(\Delta\hat{\mathbf{T}}-\underline{\left\langle\Delta\hat{\mathbf{T}}\left(\mathbf{Z}\right)\right\rangle}\right)}{\tau\omega_0\left(\mathbf{Z}_1\right)}\left|\Psi_0\left(\theta,\mathbf{Z}_1\right)\right|^2=\frac{\left(\Delta\mathbf{T}-\left\langle\Delta\mathbf{T}\left(\mathbf{Z}\right)\right\rangle\right)-\lambda\left(\Delta\hat{\mathbf{T}}-\left\langle\Delta\hat{\mathbf{T}}\left(\mathbf{Z}\right)\right\rangle\right)}{\tau\omega_0\left(\mathbf{Z}_1\right)}\left|\Psi_0\left(\theta,\mathbf{Z}_1\right)\right|^2$$

Then, we rewrite:

$$\begin{aligned}&\frac{\rho\left(C\left(\theta\right)\left|\Psi_0\left(\mathbf{Z}_1\right)\right|^2\omega_0\left(\mathbf{Z}_1\right)+D\left(\theta\right)\left|\Psi_0\left(\mathbf{Z}_2\right)\right|^2\omega_0\left(\mathbf{Z}_2\right)\right)\left(\Delta\hat{\mathbf{T}}-\Delta\hat{\mathbf{T}}'\left(\mathbf{Z}\right)\right)}{\omega_0\left(\mathbf{Z}_1\right)}\\ =\; &\mathbf{D}\left(\Delta\hat{\mathbf{T}}-\left\langle\Delta\hat{\mathbf{T}}\right\rangle\right)+\mathbf{M}^{\alpha}\left(\Delta\mathbf{T}-\left\langle\Delta\mathbf{T}\right\rangle^{\alpha}\right)\end{aligned}$$

with:

$$\mathbf{D}\left(\Delta\hat{\mathbf{T}}-\left\langle\Delta\hat{\mathbf{T}}\right\rangle\right)=D\left[\frac{\rho\left(C\left(\theta\right)\left|\Psi_0\left(Z\right)\right|^2\omega_0\left(Z\right)+D\left(\theta\right)\hat{T}\left|\Psi_0\left(Z'\right)\right|^2\omega_0\left(Z'\right)\right)}{\omega_0\left(Z\right)}\right]\left(\Delta\hat{\mathbf{T}}-\left\langle\Delta\hat{\mathbf{T}}\right\rangle\right)$$

and:

$$\begin{aligned}&\mathbf{M}^{\alpha}\left(\Delta\mathbf{T}-\left\langle\Delta\mathbf{T}\right\rangle^{\alpha}\right)\\ =\; &\frac{\rho}{\omega_0\left(Z\right)}\left(D\left(\theta\right)\left\langle\hat{T}\right\rangle\left|\Psi_0\left(Z'\right)\right|^2A\left|Z-Z'\right|\left(\nabla_{\mathbf{\Delta T}_{\left(Z_1,Z_1'\right)}}\left(\Delta\omega\left(Z,\left\langle\mathbf{\Delta T}\right\rangle\right)\right)_{\left(\left\langle\mathbf{\Delta T}_{\left(Z_1,Z_1'\right)}\right\rangle_p^{\alpha}\right)}\right)\right)\left(\Delta\mathbf{T}-\left\langle\Delta\mathbf{T}\right\rangle^{\alpha}\right)\end{aligned}$$

For a given $\langle\Delta\mathbf{T}\rangle_p^\alpha$, the action $S\left(\left\{\Delta\Gamma_{S_\alpha^2}\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_\alpha^2},\theta\right)\right\}\right)$ becomes in frst approximation:

$$\begin{aligned}
& S\left(\left\{\Delta\Gamma_{S_\alpha^2}\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_\alpha^2},\theta\right)\right\}\right) \qquad (186)\\
= & -\Delta\Gamma_{S_\alpha^2}^{\dagger}\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_\alpha^2},\theta\right)\\
& \times\left(\nabla_{\mathbf{T}}^2-\frac{1}{2}\left(\frac{\left(\Delta\mathbf{T}-\langle\Delta\mathbf{T}\rangle^\alpha\right)-\lambda\left(\Delta\hat{\mathbf{T}}-\left\langle\Delta\hat{\mathbf{T}}\right\rangle\right)}{\tau\omega_0\left(\mathbf{Z}_1\right)}\left|\Psi_0\left(\theta,\mathbf{Z}_1\right)\right|^2\right)^2\right)\Delta\Gamma_{S_\alpha^2}\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_\alpha^2},\theta\right)\\
& -\Delta\Gamma^{\dagger}\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_\alpha^2},\theta\right)\left(\nabla_{\hat{\mathbf{T}}}^2-\frac{1}{2}\left(\mathbf{D}\left(\Delta\hat{\mathbf{T}}-\left\langle\Delta\hat{\mathbf{T}}\right\rangle\right)+\mathbf{M}^\alpha\left(\Delta\mathbf{T}-\langle\Delta\mathbf{T}\rangle^\alpha\right)\right)^2\right)\Delta\Gamma\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_\alpha^2},\theta\right)
\end{aligned}$$

Given our hypotheses, we have:

$$\left\|\Delta\hat{\mathbf{T}}-\left\langle\Delta\hat{\mathbf{T}}\right\rangle\right\|<<\left\|\Delta\mathbf{T}-\langle\Delta\mathbf{T}\rangle^\alpha\right\|$$

Ultimately, considering the potential, the field action simplifies in first apprximation:

$$\begin{aligned}
& S\left(\left\{\Delta\Gamma_{S_\alpha^2}\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_\alpha^2},\theta\right)\right\}\right) \qquad (187)\\
\simeq & S\left(\left\{\Delta\Gamma_{S_\alpha^2}\left(\left(\mathbf{T},\mathbf{Z}\right)_{S_\alpha^2},\theta\right)\right\}\right)\\
= & -\Delta\Gamma_{S_\alpha^2}^{\dagger}\left(\left(\mathbf{T},\mathbf{Z}\right)_{S_\alpha^2},\theta\right)\left(\nabla_{\mathbf{T}}^2-\frac{1}{2}\left(\mathbf{A}\left(\Delta\mathbf{T}-\langle\Delta\mathbf{T}\rangle^\alpha\right)\right)^2-\mathbf{C}\right)\Delta\Gamma_{S_\alpha^2}\left(\left(\mathbf{T},\mathbf{Z}\right)_{S_\alpha^2},\theta\right)\\
& +C\left\|\Delta\Gamma_{S_\alpha^2}\left(\left(\mathbf{T},\mathbf{Z}\right)_{S_\alpha^2},\theta\right)\right\|^2+U\left(\left\|\Gamma\left(\left(\mathbf{T},\hat{\mathbf{T}},\mathbf{Z}\right)_{S_\alpha^2},\theta\right)\right\|^2\right) \qquad (188)
\end{aligned}$$

where:

$$\mathbf{A}^\alpha=\sqrt{\mathbf{D}^2+\left(\mathbf{M}^\alpha\right)^t\mathbf{M}^\alpha}$$

and:

$$\mathbf{C}\left(Z,Z'\right)=\frac{\tau\omega_0\left(\mathbf{Z}_1\right)}{2}+\frac{\rho\left(C\left(\theta\right)\left|\Psi_0\left(\mathbf{Z}_1\right)\right|^2\omega_0\left(\mathbf{Z}_1\right)+D\left(\theta\right)\left|\Psi_0\left(\mathbf{Z}_2\right)\right|^2\omega_0\left(\mathbf{Z}_2\right)\right)}{2\omega_0\left(\mathbf{Z}_1\right)}$$

# Appendix 4. n interacting fields

The Equations for activities are similar to the n field case and are defined by:

$$\begin{aligned}
& \omega_i^{-1}\left(J,\theta,Z,|\Psi|^2\right)\\
= & G\left(\int\sum\frac{\kappa}{N}g^{ij}\frac{\omega_j\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T_{ij}\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega_i\left(J,\theta,Z,|\Psi|^2\right)}\left|\Psi_j\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2dZ_1\right)
\end{aligned}$$

The $n\times n$ matrix $G$ has coefficients in the interval $[-1,1]$. In the sequel, the sum over index $j$ is implicit. For instance, if $n=2$, the matrix $g$:

$$G=\begin{pmatrix}1 & -g\\ -g & 1\end{pmatrix}$$

represents inhibitory interactions between two populations, with, similarly to the one field case:

$$T_{ij}\left(Z,Z_1\right)=\int T_{ij}\left|\Gamma_{ij}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z',C_{ij},D_{ij}\right)\right|^2$$

The field action for connectivity is recalled in appendix 2 and decomposed in four terms plus potential:

$$S_{\Gamma}^{(1)}+S_{\Gamma}^{(2)}+S_{\Gamma}^{(3)}+S_{\Gamma}^{(4)}+U\left(\left\{\left|\Gamma\left(\theta,Z,Z',C,D\right)\right|^2\right\}\right)$$

We also derive an expression for the average connectivity functions $T_{ij}$ and the equilibrium static activities as a function of the non interacting frquencies, by assuming:

$$G^{ij}T_{ij}<<T_{ii}$$

This is valid if structures $i$ and $j$ are distant so that we have:

$$G^{j}=\exp\left(-\frac{d}{\nu}\right)\bar{G}^{j}$$

for $d$ the average distant between structures. In such case, we also have $T_{ij}<<T_{ii}$ for $i\neq j$.

The field action for connectivity flds is decomposed in four terms:

$$S_{\Gamma}^{(1)}+S_{\Gamma}^{(2)}+S_{\Gamma}^{(3)}+S_{\Gamma}^{(4)}+U\left(\left\{\left|\Gamma\left(\theta,Z,Z',C,D\right)\right|^2\right\}\right)$$

where:

$$\begin{aligned}S_{\Gamma}^{(1)}&=\int\Gamma_{ij}^{\dagger}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z',C_{ij},D_{ij}\right)\\&\times\nabla_{T_{ij}}\left(\frac{\sigma_{T}^{2}}{2}\nabla_{T_{ij}}-\left(-\frac{1}{\tau\omega}T_{ij}+\frac{\lambda}{\omega}\hat{T}_{ij}\right)\left|\Psi_{i}\left(\theta,Z\right)\right|^{2}\right)\Gamma_{ij}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z',C_{ij},D_{ij}\right)\end{aligned}\tag{189}$$

$$\begin{aligned}S_{\Gamma}^{(2)}&=\int\Gamma_{ij}^{\dagger}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z',C_{ij},D_{ij}\right)\\&\times\nabla_{\hat{T}_{ij}}\left(\frac{\sigma_{\hat{T}_{ij}}^{2}}{2}\nabla_{\hat{T}_{ij}}-\frac{\rho}{\omega_{i}\left(J,\theta,Z,\left|\Psi\right|^{2}\right)}\left(\left(h_{ij}\left(Z,Z'\right)-\hat{T}_{ij}\right)C\left|\Psi\left(\theta,Z\right)\right|^{2}h_{C}\left(\omega_{i}\left(J,\theta,Z,\left|\Psi\right|^{2}\right)\right)\right.\right.\\&\left.\left.-D\hat{T}_{ij}\left|\Psi_{j}\left(\theta-\frac{\left|Z-Z'\right|}{c},Z'\right)\right|^{2}h_{D}\left(\omega_{j}\left(J,\theta-\frac{\left|Z-Z'\right|}{c},Z',\left|\Psi\right|^{2}\right)\right)\right)\right)\Gamma_{ij}\left(T_{ij},\hat{T}_{ij},\theta,Z,Z',C_{ij},D_{ij}\right)\end{aligned}\tag{190}$$

$$\begin{aligned}S_{\Gamma}^{(3)}&=\Gamma_{ij}^{\dagger}\left(T,\hat{T},\theta,Z,Z',C_{ij},D_{ij}\right)\\&\times\nabla_{C_{ij}}\left(\frac{\sigma_{C_{ij}}^{2}}{2}\nabla_{C}\right.\\&\left.+\left(\frac{C}{\tau_{C}\omega_{i}\left(J,\theta,Z,\left|\Psi\right|^{2}\right)}-\alpha_{C}\left(1-C\right)\frac{\omega_{j}\left(J,\theta-\frac{\left|Z-Z'\right|}{c},Z',\left|\Psi\right|^{2}\right)\left|\Psi_{j}\left(\theta-\frac{\left|Z-Z'\right|}{c},Z'\right)\right|^{2}}{\omega_{i}\left(J,\theta,Z,\left|\Psi\right|^{2}\right)}\right)\left|\Psi_{i}\left(\theta,Z\right)\right|^{2}\right)\\&\times\Gamma_{ij}\left(T,\hat{T},\theta,Z,Z',C,D\right)\end{aligned}\tag{191}$$

and:

$$S_{\Gamma}^{(4)} = \Gamma_{ij}^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right) \tag{192}$$
$$\times\nabla_D\left(\frac{\sigma_D^2}{2}\nabla_D+\left(\frac{D}{\tau_D\omega_i\left(J,\theta,Z,|\Psi|^2\right)}-\alpha_D\left(1-D\right)\left|\Psi\left(\theta,Z\right)\right|^2\right)\right)\Gamma_{ij}\left(T,\hat{T},\theta,Z,Z',C,D\right)$$

In (1), we added a potential:

$$U\left(\left\{\left|\Gamma\left(\theta,Z,Z',C,D\right)\right|^2\right\}\right)=U\left(\int T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2 dTd\hat{T}\right)$$

that models the constraint about the number of active connections in the system.

## 4.1 Equilibrium activities

We saw in part III how to compute background activities. Writing activities equations:

$$\omega_i\left(Z\right)=G\left(\sum_j\int\frac{\kappa}{N}\frac{G^{ij}T_{ij}\left|\Gamma_{ij}\left(T,\hat{T},Z,Z_1\right)\right|^2\omega_j\left(J,Z_1\right)}{\omega_i\left(Z\right)}\left(\mathcal{G}_{j0}+\left|\Psi_j\left(Z_1\right)\right|^2\right)dZ_1\right) \tag{193}$$

For a given structures with specific points, we replace the integrals in the previous formula (193) by sums:

$$\omega_i^{-1}\left(Z_{a_i},|\Psi|^2\right)=G\left(\sum_j\sum_{b_j}\frac{\kappa}{N}G^{ij}\frac{\omega_j\left(Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\omega_i\left(Z_{a_i},|\Psi|^2\right)}\left(\mathcal{G}_{j0}+\left|\Psi_j\left(Z_{b_j}\right)\right|^2\right)\right) \tag{194}$$

Assuming:

$$G^{ij}T_{ij}<<T_{ii}$$

we expand (194) to the first order:

$$\omega_i^{-1}\left(Z_{a_i}\right) \simeq G\left(\sum_{a_i}\frac{\kappa}{N}\frac{\omega_i\left(Z_{\alpha_j}\right)T_{ii}\left(Z_{a_i},Z_{\alpha_j}\right)}{\omega_i\left(Z_{a_i},|\Psi|^2\right)}\left(\mathcal{G}_{i0}+\left|\Psi_i\left(Z_{a_j}\right)\right|^2\right)\right)$$
$$+\sum_{j\neq i,\{\{b_j\}\}}G'\left(\sum_{a_i}\frac{\kappa}{N}\frac{\omega_i\left(Z_{\alpha_j}\right)T_{ii}\left(Z_{a_i},Z_{b_j}\right)}{\omega_i\left(Z_{a_i},|\Psi|^2\right)}\left(\mathcal{G}_{i0}+\left|\Psi_i\left(Z_{b_j}\right)\right|^2\right)\right)$$
$$\times\frac{\kappa}{N}G^{ij}\frac{\omega_j\left(Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\omega_i\left(Z_{a_i},|\Psi|^2\right)}\left(\mathcal{G}_{j0}+\left|\Psi_j\left(Z_{b_j}\right)\right|^2\right)$$

where $G^{ii}=1$, and this becomes:

$$\omega_i^{-1}\left(Z_{a_i}\right) \simeq \omega_{i0}^{-1}\left(Z_{a_i}\right) \tag{195}$$
$$+G'\left(G^{-1}\left(\omega_{i0}\left(Z_{a_i}\right)\right)\right)$$
$$\times\sum_{j\neq i,\{\{b_j\}\}}\frac{\kappa}{N}G^{ij}\frac{\omega_j\left(Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\omega_i\left(Z_{a_i},|\Psi|^2\right)}\left(\mathcal{G}_{j0}+\left|\Psi_j\left(Z_{b_j}\right)\right|^2\right)$$

where $\omega_{i0}(Z)$ are activities without interactions (where $G^{ii}=1$):

$$\omega_{i0}\left(Z_{a_i}\right)=G\left(\sum_{a_i'}\frac{\kappa}{N}\frac{\omega_j\left(Z_{a_i'}\right)T_{ii}\left(Z_{a_i},Z_{a_i'}\right)}{\omega_i\left(Z_{a_i},|\Psi|^2\right)}\left(\mathcal{G}_{i0}+\left|\Psi_i\left(Z_{a_i'}\right)\right|^2\right)\right)$$

Equation (195) has solutions:

$$\begin{aligned}&\omega_i\left(Z_{a_i}\right)\\ =&\sum_{\{\{b_j\}\}}\left(\delta_{(i,a_i)(j,b_j)}\right.\\ &\left.-G'\left(G^{-1}\left(\omega_{0i}\left(Z_{a_i}\right)\right)\right)\left(\frac{\kappa}{N}\frac{G^{ij}\omega_{0j}\left(Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\omega_{0i}\left(Z_{a_i}\right)}\left(\mathcal{G}_{j0}+\left|\Psi_j\left(Z_{b_j}\right)\right|^2\right)\right)_{j\neq i}\right)_{ij}^{-1}\omega_{0j}\left(Z_{b_j}\right)\end{aligned}\tag{196}$$

Remark that for inhibitory interactions $\omega_i(Z)<\omega_{0i}(Z)$.

## 4.2 Connectivity functions

In part 1 we obtained average connectivity functions:

$$\begin{aligned}T_{ij}\left(Z_-,Z_+'\right)&=\frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{bG^i\bar{T}\left\langle\bar{T}\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z}\right)}{\frac{1}{\tau_D\alpha_D}+\frac{1}{\alpha_C\tau_C}+\frac{1}{bG^i\bar{T}\left\langle\bar{T}\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z}+\frac{\left(bG^j\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{j0}(Z')\right|^2\right\rangle_{Z'}\right)^2}{2}}\simeq 0\\
T_{ij}\left(Z_+,Z_+'\right)&=\frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left(\frac{1}{\tau_D\alpha_D}+\frac{\left(bG^i\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z\right)^2}{2}\right)}{\frac{1}{\tau_D\alpha_D}+\frac{1}{\alpha_C\tau_C}+\frac{\left(bG^i\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z\right)^2}{2}+\frac{\left(bG^j\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{j0}(Z')\right|^2\right\rangle_{Z'}\right)^2}{2}}\simeq\frac{G^i\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{\left(G^i+G^j\right)}\\
T_{ij}\left(Z_+,Z_-'\right)&=\frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left(\frac{1}{\tau_D\alpha_D}+\frac{\left(bG^i\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z\right)^2}{2}\right)}{\frac{1}{\tau_D\alpha_D}+\frac{1}{\alpha_C\tau_C}+\frac{\left(bG^i\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z\right)^2}{2}+\frac{1}{bG^j\bar{T}\left\langle\bar{T}\left|\bar{\Psi}_{j0}(Z')\right|^2\right\rangle_Z}}\simeq\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\\
T\left(Z_-,Z_-'\right)&\simeq\frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)+\frac{1}{bG^i\bar{T}\left\langle\bar{T}\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z}}{1+\frac{\tau_D\alpha_D}{\alpha_C\tau_C}+\frac{1}{bG^i\bar{T}\left\langle\bar{T}\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z}+\frac{1}{bG^j\bar{T}\left\langle\bar{T}\left|\bar{\Psi}_{j0}(Z')\right|^2\right\rangle_Z}}\simeq\frac{G^j\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{\left(G^i+G^j\right)}\end{aligned}\tag{197}$$

with $\bar{T}=\frac{\lambda\tau\nu cb}{2}$, $b$ a coefficient characterizing the function $G$ in the linear approximation[9], and the coefficient $G^i$ measuring the connectivity of the $i$-th field with the other types. The expressions $\left\langle\left|\Psi_{i0}(Z')\right|^2\right\rangle_{Z'}^2$, $\left\langle\left|\Psi_{i0}(Z')\right|^2\right\rangle_Z^2$ are some averaged background fields for the $t$-th types of cells in regions surrounding $Z$ and $Z'$ respctvl. They are determined by a potential describing some average activity depending on the points. These results are derived under the assumption of static fields $\Psi_{i0}(Z)$.

[9] $b\simeq G'(0)$

Here we also assumed $G^j << G^i$. This is valid if structures $i$ and $j$ are distant so that we have:

$$G^j = \exp\left(-\frac{d}{\nu}\right)\bar{G}^j$$

for $d$ the average distant between structures.

# Appendix 5. Activities for cllective states

We look for activities equations for collective states. We write the equations for activities:

$$\begin{aligned}
&\Delta\omega_i^{-1}\left(Z_{a_i},\theta\right) \\
=\ & G\left(\sum_j\sum_{b_j}\frac{\kappa}{N}g^{ij}\frac{\Delta\omega_j\left(\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c},Z_{b_j}\right)\Delta T_{ij}\left(Z_{a_i},Z_{b_j},\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}\right)}{\Delta\omega_i\left(Z_{a_i},\theta\right)}\left|\Psi_j\left(\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c},Z_{b_j}\right)\right|^2\right)
\end{aligned} \tag{198}$$

Assuming:

$$G^{ij}\Delta T_{ij} << \Delta T_{ii}$$

leads to a first order expansion with respect to the activities of the groups considered individually. The solutions will thus be described by corrections to the individual groups.

### 5.1 Static part

The static part is derived as for (196). The static form of (198) is:

$$\Delta\omega_i^{-1}\left(Z_{a_i}\right) = G\left(\sum_j\sum_{b_j}\frac{\kappa}{N}g^{ij}\frac{\Delta\omega_j\left(Z_{b_j}\right)\Delta T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\Delta\omega_i\left(Z_{a_i},\theta\right)}\left|\Psi_j\left(Z_{b_j}\right)\right|^2\right) \tag{199}$$

The expansion (198) around non interacting solution yields an equation similar to (195):

$$\begin{aligned}
\overline{\Delta\omega}_i^{-1}\left(Z_{a_i}\right) \simeq\ & \Delta\omega_{i0}^{-1}\left(Z_{a_i}\right) \\
&+G'\left(G^{-1}\left(\overline{\Delta\omega}_{i0}\left(Z_{a_i}\right)\right)\right) \\
&\times\sum_{j\neq i}\sum_{b_j}\frac{\kappa}{N}G^{ij}\frac{\overline{\Delta\omega}_j\left(Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\overline{\Delta\omega}_i\left(Z_{a_i},\left|\Psi\right|^2\right)}\left(\mathcal{G}_{j0}+\left|\Psi_j\left(Z_{b_j}\right)\right|^2\right)
\end{aligned} \tag{200}$$

with solution:

$$\begin{aligned}
&\overline{\Delta\omega}_i\left(Z_{a_i}\right) \\
=\ & \sum_{j,b_j}\left(\delta_{(i,a_i)(j,b_j)}-G'\left(G^{-1}\left(\overline{\Delta\omega}_{0i}\left(Z\right)\right)\right)\left(\frac{\kappa}{N}\frac{G^{ij}\overline{\Delta\omega}_{0j}\left(Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\overline{\Delta\omega}_{0i}\left(Z_{a_i}\right)}\left(\mathcal{G}_{j0}+\left|\Psi_j\left(Z_{b_j}\right)\right|^2\right)\right)_{j\neq i}\right)_{ij}^{-1} \\
&\times\overline{\Delta\omega}_{0j}\left(Z_{b_j}\right)
\end{aligned} \tag{201}$$

where $\omega_{0j}\left(Z_{a_i}\right)$ activities without interactions satisfy:

$$\overline{\Delta\omega}_{i0}^{-1}\left(Z_{a_i}\right) = G\left(\sum_{\beta_i}\frac{\kappa}{N}\frac{\overline{\Delta\omega}_{i0}\left(Z_{\beta_i}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\overline{\Delta\omega}_{i0}\left(Z_{a_i},\theta\right)}\left|\Psi_j\left(Z_{b_j}\right)\right|^2\right)$$

with $g^{ii}=1$. The solutions (182) mix the static activities of the several groups, showing that composed structures and individual ones are binded by some consistencies conditions.

## 5.2 Non static part

### 5.2.1 Equation for non static frequencies

Expand (36) around (37):

$$\begin{aligned} &\Delta\omega_i^{-1}\left(Z_{a_i}, |\Psi|^2\right) \\ = \quad &\Delta G\left(\sum_j \sum_{b_j} \frac{\kappa}{N} g^{ij} \frac{\omega_j\left(\theta - \frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}, Z_{b_j}\right) T_{ij}\left(Z_{a_i}, Z_{b_j}, \theta - \frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}\right)}{\omega_i\left(Z_{a_i}, \theta\right)} \left|\Psi_j\left(\theta - \frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}, Z_{b_j}\right)\right|^2\right) \end{aligned}$$

is given at the first order:

$$\begin{aligned} &\Delta\omega_i^{-1}\left(Z_{a_i}, \theta\right) \\ = \quad &G'\left(G^{-1}\left(\omega^{-1}\left(Z_{a_i}, |\Psi|^2\right)\right)\right) \\ &\times\Delta\left(\sum_j \sum_{b_j} \frac{\kappa}{N} g^{ij} \frac{\omega_j\left(\theta - \frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}, Z_{b_j}\right) T_{ij}\left(Z_{a_i}, Z_{b_j}, \theta - \frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}\right)}{\omega_i\left(Z_{a_i}, \theta\right)} \left|\Psi_j\left(\theta - \frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}, Z_{b_j}\right)\right|^2\right) \end{aligned}$$

As for one field case, we can neglect:

$$\Delta\left|\Psi_j\left(\theta - \frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}, Z_{b_j}\right)\right|^2$$

and we obtain:

$$\begin{aligned} &\Delta\omega_i^{-1}\left(Z_{a_i}, \theta\right) \\ = \quad &G'\left(G^{-1}\left(\omega^{-1}\left(Z_{a_i}, |\Psi|^2\right)\right)\right) \\ &\times\Delta\left(\sum_j \sum_{b_j} \frac{\kappa}{N} g^{ij} \frac{\omega_j\left(\theta - \frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}, Z_{b_j}, \Psi\right) T_{ij}\left(Z_{a_i}, Z_{b_j}, \theta - \frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}\right)}{\omega_i\left(Z_{a_i}, \theta\right)} \left|\Psi_j\left(\theta - \frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}, Z_{b_j}\right)\right|^2\right) \end{aligned}$$

The variation in the last equation is given by:

$$\Delta\left(\sum_j\sum_{b_j}\frac{\kappa}{N}g^{ij}\frac{\omega_j\left(\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c},Z_{b_j},\Psi\right)T_{ij}\left(Z_{a_i},Z_{b_j},\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}\right)}{\omega_i\left(Z_{a_i},\theta\right)}\left|\Psi_j\left(\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c},Z_{b_j}\right)\right|^2\right)$$

$$\begin{aligned}\simeq\quad & -\sum_{\{\{b_j\}\}}\frac{\kappa}{N}g^{ij}\frac{\omega_j\left(Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\omega_i\left(Z_{a_i}\right)}\left|\Psi_j\left(Z_{b_j}\right)\right|^2\frac{\Delta\omega_j^{-1}\left(\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c},Z_{b_j},\Psi\right)}{\omega_j^{-1}\left(Z_{b_j}\right)}\\ & +\sum_{\{\{b_j\}\}}\frac{\kappa}{N}g^{ij}\frac{\omega_j\left(Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\omega_i\left(Z_{a_i}\right)}\left|\Psi_j\left(Z_{b_j}\right)\right|^2\frac{\Delta\omega_i^{-1}\left(Z_{a_i},\theta\right)}{\omega_i\left(Z_{a_i}\right)}\\ & +\sum_{\{\{b_j\}\}}\frac{\kappa}{N}g^{ij}\frac{\omega_j\left(Z_{b_j},\Psi\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\omega_i\left(Z_{a_i},\theta\right)}\Delta\left|\Psi_j\left(\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c},Z_{b_j}\right)\right|^2\end{aligned}$$

We neglect $\Delta\left|\Psi_j\left(\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c},Z_{b_j}\right)\right|^2$ in first approximation. Consequently, the activities equation writes:

$$\begin{aligned}&\left(1-G'\left(G^{-1}\left(\omega^{-1}\left(Z_{a_i},|\Psi|^2\right)\right)\right)\right)\Delta\omega_i^{-1}\left(Z_{a_i},\theta\right)\\ =\quad & -G'\left(G^{-1}\left(\omega^{-1}\left(Z_{a_i},|\Psi|^2\right)\right)\right)\sum_j\sum_{b_j}\frac{\kappa}{N}g^{ij}\frac{\omega_j\left(Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{\omega_i\left(Z_{a_i}\right)}\left|\Psi_j\left(Z_{b_j}\right)\right|^2\frac{\Delta\omega_j^{-1}\left(\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c},Z_{b_j},\Psi\right)}{\omega_j^{-1}\left(Z_{b_j}\right)}\end{aligned}$$

with solution:

$$\Delta\omega_i^{-1}\left(Z_{a_i},\theta\right)=\sum_j\sum_{b_j}\hat{T}_{ij}\left(Z_{a_i},Z_{b_j}\right)\Delta\omega_j\left(\theta-\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c},Z_{b_j}\right)\tag{202}$$

where:

$$\hat{T}_{ij}\left(Z_{a_i},Z_{b_j}\right)=G'\left(G^{-1}\left(\omega^{-1}\left(Z_{a_i},|\Psi|^2\right)\right)\right)\frac{\kappa}{N}g^{ij}\frac{\omega_j\left(Z_{b_j}\right)T_{ij}\left(Z_{a_i},Z_{b_j}\right)}{G'\left(G^{-1}\left(\omega^{-1}\left(Z_{a_i},|\Psi|^2\right)\right)\right)-\omega_i^{-1}\left(Z_{a_i}\right)}\left|\Psi_j\left(Z_{b_j}\right)\right|^2$$

### 5.2.2 Equation for oscillating solutions

Rewriting the solutions (202) as a vector:

$$\left(\Delta\omega_i^{-1}\left(Z_{a_i},\theta\right)\right)_i\equiv\Delta\boldsymbol{\omega}^{-1}\left(Z_{\boldsymbol{\alpha}},\theta\right)$$

we look for oscillatory solutions:

$$\Delta\boldsymbol{\omega}^{-1}\left(Z_{\boldsymbol{\alpha}},\theta\right)=\Delta\boldsymbol{\omega}^{-1}\left(Z_{\boldsymbol{\alpha}}\right)\exp\left(i\Upsilon\theta\right)$$

which implies that for these solutions, (202) rewrites as :

$$\Delta\boldsymbol{\omega}^{-1}\left(Z_{\boldsymbol{\alpha}}\right)=M\Delta\boldsymbol{\omega}^{-1}\left(Z_{\boldsymbol{\alpha}}\right)$$

with:

$$M_{(ia_i),(jb_j)} = \hat{T}_{ij}\left(Z_{a_i}, Z_{b_j}\right)\exp\left(-i\Upsilon\frac{\left|Z_{a_i}-Z_{b_j}\right|}{c}\right)$$

We decompose the matrix $M_{(ia_i),(jb_j)}$ into a diagonal and a non-diagonal part:

$$M = \left(M_{(ia_i),(jb_j)}\right) + \left(M_{(ia_i),(jb_j)}\right)_{i\neq j}$$

so that $\left(M_{(ia_i),(jb_j)}\right)_{i\neq j}$ can be studied perturbatively. The condition for oscillatory solutions writes:

$$\det\left(1-M\right) = 0 \tag{203}$$

Given our order of approximation, the elements of:

$$\left(M_{(ia_i),(jb_j)}\right)_{i\neq j}$$

are of lower magnitude than that of:

$$1-\left(M_{(ia_i),(jb_j)}\right)$$

and we write (203) as:

$$\begin{aligned}
\det\left(1-M\right) &= \prod\det\left(1-\left(M_{(ia_i),(ia_i)}\right)\right)\exp\left(-\frac{1}{2}Tr\left(\left(1-\left(M_{(ia_i),(ia_i)}\right)\right)^{-1}\left(M_{(ia_i),(jb_j)}\right)_{i\neq j}\right)^2\right) \\
&= \prod\det\left(1-\left(M_{i,i}\right)\right)\left(1-\frac{1}{2}Tr\left(\left(1-M_{i,i}\right)^{-1}\left(M_{i,j}\right)_{i\neq j}\right)^2\right) \\
&= \prod\det\left(1-\left(M_{i,i}\right)\right)\left(1-\frac{1}{2}\sum_i\sum_{j\neq i}Tr\left(\left(1-M_{j,j}\right)^{-1}\left(M_{j,i}\right)\left(1-M_{i,i}\right)^{-1}\left(M_{i,j}\right)\right)\right)
\end{aligned} \tag{204}$$

To solve further (204), we assume that $1-M$ can be diagonalized:

$$1-M_{i,i} = U_i\left(\left(1-f_{l_i}\left(\gamma\right)\right)\right)_{l_i}U_i^{-1}$$

where:

$$\left(\left(1-f_{l_i}\left(\gamma\right)\right)\right)_{l_i}$$

is diagonal with elements:

$$1-f_{l_i}\left(\gamma\right)$$

Now, define the frequencies $\gamma_{l_i}$ by the following equation:

$$f_{l_i}\left(\gamma_{l_i}\right) = 1$$

Then, $\det\left(1-M\right)$ becomes:

$$\begin{aligned}
&\det\left(1-M\right) \\
&= \prod_i\left(\prod_{l_i}\left(1-f_{l_i}\left(\gamma\right)\right)\right)_i\left(1-\frac{1}{2}Tr\left(\left(1-f_{l_j}\left(\gamma\right)\right)_j\right)^{-1}U_j^{-1}M_{j,i}U_i\left(\left(1-f_{l_i}\left(\gamma\right)\right)_i\right)^{-1}U_i^{-1}M_{i,j}U_j\right) \\
&= \prod_i\left(\prod_{l_i}\left(1-f_{l_i}\left(\gamma\right)\right)\right)\left(1-\frac{1}{2}\sum_i\sum_{j\neq i}\sum_{k,l}\left(\left(1-f_{j,l}\left(\gamma\right)\right)\right)^{-1}\left[\hat{M}_{j,i}\right]_{l,k}\left(1-f_{i,k}\left(\gamma\right)\right)^{-1}\left[\hat{M}_{i,j}\right]_{k,l}\right)
\end{aligned}$$

where:

$$\left[\hat{M}_{j,i}\right] = U_j^{-1} M_{j,i} U_i$$
$$\left[\hat{M}_{i,j}\right] = U_i^{-1} M_{i,j} U_j$$

Define similarly the frequencies $\gamma_{i,k}$ by:

$$f_{i,k}\left(\gamma_{i,k}\right) = 1$$

and the equation for the frequencies:

$$\det\left(1 - M\right) = 0$$

writes ultimately:

$$1 - \frac{1}{2}\sum_i \sum_{j\neq i} \sum_{k,l} \left(\left(f_{j,l}\left(\gamma_{j,l}\right) - f_{j,l}\left(\gamma\right)\right)\right)^{-1} \left[\hat{M}_{j,i}\right]_{l,k} \left(f_{i,k}\left(\gamma_{i,k}\right) - f_{i,k}\left(\gamma\right)\right)^{-1} \left[\hat{M}_{i,j}\right]_{k,l} = 0 \tag{205}$$

The $\gamma_{i,k}$ are the possible frequencies for structure $i$ without interaction.

### 5.2.3 Solutions for interacting structures with close frequencies

If the structures before interaction are in states $\gamma_{j,l_j}$ and these frequencies are relatively close from each other, that is, if we assume:

$$\left|\gamma_{j,l_j} - \gamma_{j',l_{j'}}\right| << 1$$

we start by defining:

$$\bar{\gamma} = \frac{1}{m}\sum \gamma_{j,l_j}$$

where $m$ is the number of structures involved in the interactions, we can compute the expression arising in (205):

$$\begin{aligned}
&\sum_i \sum_{j\neq i} \sum_{k,l} \left(\left(f_{j,l}\left(\gamma_{j,l}\right) - f_{j,l}\left(\gamma\right)\right)\right)^{-1} \left[\hat{M}_{j,i}\right]_{l,k} \left(f_{i,k}\left(\gamma_{i,k}\right) - f_{i,k}\left(\gamma\right)\right)^{-1} \left[\hat{M}_{i,j}\right] \qquad (206)\\
\simeq\; &\sum_i \sum_{j\neq i} \sum_{\substack{k,l\\ l\neq l_j, k\neq k_i}} \left(\left(f_{j,l}\left(\gamma_{j,l}\right) - f_{j,l}\left(\bar{\gamma}\right)\right)\right)^{-1} \left[\hat{M}_{j,i}\right]_{l,k} \left(f_{i,k}\left(\gamma_{i,k}\right) - f_{i,k}\left(\bar{\gamma}\right)\right)^{-1} \left[\hat{M}_{i,j}\right]_{k,l}\\
&+\sum_i \sum_{j\neq i} \sum_{\substack{k\\ k\neq k_i}} \left(\left(f_{j,l_j}\left(\gamma_{j,l_j}\right) - f_{j,l_j}\left(\gamma\right)\right)\right)^{-1} \left[\hat{M}_{j,i}\right]_{l,k} \left(f_{i,k}\left(\gamma_{i,k}\right) - f_{i,k}\left(\bar{\gamma}\right)\right)^{-1} \left[\hat{M}_{i,j}\right]_{k,l}\\
&+\sum_i \sum_{j\neq i} \sum_{\substack{l\\ l\neq l_j}} \left(\left(f_{j,l}\left(\gamma_{j,l}\right) - f_{j,l}\left(\bar{\gamma}\right)\right)\right)^{-1} \left[\hat{M}_{j,i}\right]_{l,k} \left(f_{i,k_i}\left(\gamma_{i,k_i}\right) - f_{i,k_i}\left(\gamma\right)\right)^{-1} \left[\hat{M}_{i,j}\right]_{k,l}\\
&+\sum_i \sum_{j\neq i} \left(f_{j,l_j}\left(\gamma_{j,l_j}\right) - f_{j,l_j}\left(\gamma\right)\right)^{-1} \left[\hat{M}_{j,i}\right]_{l_j,k_i} \left(f_{i,k_i}\left(\gamma_{i,k_i}\right) - f_{i,k_i}\left(\gamma\right)\right)^{-1} \left[\hat{M}_{i,j}\right]_{k_i,l_j}
\end{aligned}$$

the last term in (206) dominates and (205) writes:

$$1 - \frac{1}{2}\sum_i \sum_{j\neq i} \left(f_{j,l_j}\left(\gamma_{j,l_j}\right) - f_{j,l_j}\left(\gamma\right)\right)^{-1} \left[\hat{M}_{j,i}\right]_{l_j,k_i} \left(f_{i,k_i}\left(\gamma_{i,k_i}\right) - f_{i,k_i}\left(\gamma\right)\right)^{-1} \left[\hat{M}_{i,j}\right]_{k_i,l_j} \tag{207}$$

To solve (207), we consider each term of the sum individually, and decompose (207) in individual equations by wrtng:

$$\left(f_{j,l_j}\left(\gamma_{j,l_j}\right)-f_{j,l_j}\left(\gamma\right)\right)^{-1}\left[\hat{M}_{j,i}\right]_{l_j,k_i}\left(f_{i,k_i}\left(\gamma_{i,k_i}\right)-f_{i,k_i}\left(\gamma\right)\right)^{-1}\left[\hat{M}_{i,j}\right]_{k_i,l_j}=d_{i,j} \tag{208}$$

with:

$$\sum d_{i,j}=1$$

Solving for the frequencies imply to find the $d_{i,j}$. This is performed by writing in first approximation:

$$f_{j,l_j}\left(\gamma_{j,l_j}\right)-f_{j,l_j}\left(\gamma\right)\simeq\left(\frac{\partial}{\partial\gamma}f_{j,l_j}\left(\gamma\right)\right)_{\gamma_{j,l_j}}\left(\frac{\partial}{\partial\gamma}f_{i,k_i}\left(\gamma\right)\right)_{\gamma_{i,k_i}}\left(\gamma_{j,l_j}-\gamma\right)\left(\gamma_{i,k_i}-\gamma\right)$$

and (208) becomes:

$$\left(\gamma_{j,l_j}-\gamma\right)\left(\gamma_{i,k_i}-\gamma\right)=\frac{\left[\hat{M}_{j,i}\right]_{l_j,k_i}\left[\hat{M}_{i,j}\right]_{k_i,l_j}}{d_{i,j}\left(\frac{\partial}{\partial\gamma}f_{j,l_j}\left(\gamma\right)\right)_{\gamma_{j,l_j}}\left(\frac{\partial}{\partial\gamma}f_{i,k_i}\left(\gamma\right)\right)_{\gamma_{i,k_i}}} \tag{209}$$

There are thus $\frac{m(m-1)}{2}$ equations, $\frac{m(m-1)}{2}-1$ coefficients, $\frac{m(m-1)}{2}$ variables including the variable $\gamma$. We first start by solving for $\gamma$, then the $d_{i,j}$, and ultimately for the variables $\gamma_{i,k_i}$.

Multiplying (209) by $d_{i,j}$ and summing over $i$ nd $j\neq i$, leads to:

$$\gamma^2-\gamma\sum_{i,j\neq i}d_{i,j}\left(\gamma_{i,k_i}+\gamma_{j,l_j}\right)+\sum_{i,j\neq i}d_{i,j}\gamma_{i,k_i}\gamma_{j,l_j}-\sum_{i,j\neq i}\frac{\left[\hat{M}_{j,i}\right]_{l_j,k_i}\left[\hat{M}_{i,j}\right]_{k_i,l_j}}{\left(\frac{\partial}{\partial\gamma}f_{j,l_j}\left(\gamma\right)\right)_{\gamma_{j,l_j}}\left(\frac{\partial}{\partial\gamma}f_{i,k_i}\left(\gamma\right)\right)_{\gamma_{i,k_i}}}$$

with solution:

$$\gamma=\sum_{i,j\neq i}d_{i,j}\frac{\gamma_{i,k_i}+\gamma_{j,l_j}}{2}\pm\sqrt{\left(\sum_{i,j\neq i}d_{i,j}\frac{\gamma_{i,k_i}-\gamma_{j,l_j}}{2}\right)^2+\sum_{i,j\neq i}\frac{\left[\hat{M}_{j,i}\right]_{l_j,k_i}\left[\hat{M}_{i,j}\right]_{k_i,l_j}}{\left(\frac{\partial}{\partial\gamma}f_{j,l_j}\left(\gamma\right)\right)_{\gamma_{j,l_j}}\left(\frac{\partial}{\partial\gamma}f_{i,k_i}\left(\gamma\right)\right)_{\gamma_{i,k_i}}}} \tag{210}$$

To complete the derivation, we have to find $d_{i,j}$, which will yield the $\gamma_{i,k_i}$. Using again (209) leads to:

$$C_{l_j,k_i}d_{i,j}=\frac{\left[\hat{M}_{j,i}\right]_{l_j,k_i}\left[\hat{M}_{i,j}\right]_{k_i,l_j}}{\left(\frac{\partial}{\partial\gamma}f_{j,l_j}\left(\gamma\right)\right)_{\gamma_{j,l_j}}\left(\frac{\partial}{\partial\gamma}f_{i,k_i}\left(\gamma\right)\right)_{\gamma_{i,k_i}}} \tag{211}$$

with:

$$C_{l_j,k_i}=\left(\gamma_{j,l_j}-\gamma\right)\left(\gamma_{i,k_i}-\gamma\right)$$

Using our assumption:

$$\left|\gamma_{j,l_j}-\gamma_{j',l_{j'}}\right|<<1$$

we have:

$$C_{l_j,k_i}\simeq C=\left\langle\left(\gamma_{j,l_j}-\gamma\right)\left(\gamma_{i,k_i}-\gamma\right)\right\rangle$$

and summing (211) over $i$ nd $j\neq i$, leads to:

$$C=\sum_{i,j}\left[\left[M\right]\right]_{i,j}$$

where:

$$[[M]]_{i,j}=\frac{\left[\hat{M}_{j,i}\right]_{l_j,k_i}\left[\hat{M}_{i,j}\right]_{k_i,l_j}}{\left(\frac{\partial}{\partial\gamma}f_{j,l_j}\left(\gamma\right)\right)_{\gamma_{j,l_j}}\left(\frac{\partial}{\partial\gamma}f_{i,k_i}\left(\gamma\right)\right)_{\gamma_{i,k_i}}}$$

and as a consequence:

$$d_{i,j}\simeq\frac{[[M]]_{i,j}}{\sum_{i,j}[[M]]_{i,j}}$$

so that, writing $\gamma_{(i,l_i)}$ to label the resulting frequency for the new structure, we obtain:

$$\gamma_{(i,l_i)}=\sum[[M]]_{i,j}\frac{\gamma_{i,k_i}+\gamma_{j,l_j}}{2}\pm\sqrt{\left(\sum\frac{[[M]]_{i,j}}{\sum_{i,j}[[M]]_{i,j}}\frac{\gamma_{i,k_i}-\gamma_{j,l_j}}{2}\right)^2+\sum[[M]]_{i,j}}$$

Ultimately, the possible activities are:

$$\overline{\Delta\omega}_{i0}^{-1}\left(Z_{a_i}\right)+\Delta\boldsymbol{\omega}^{-1}\left(Z_{\boldsymbol{\alpha}}\right)\exp\left(i\Upsilon_{\{i,l_i\}}\theta\right)$$

# Appendix 6

We present the change of basis in the operator formalism. In a first step, we derive the transformation that cancels the interactions with the intermediate structure at the first order. Then we rewrite the transformed operator at the second order. Interactions reappear as effective terms between structure at the same time-scale in which the intermediate structures action is hidden.

## 6.1 Finding $F$

As explained in the text, we find operator $F$ by solving (109):

$$[F,S_0]+I=0 \tag{212}$$

where:

$$S_0=\sum_{S\times S}\bar{\mathbf{D}}_{S^2}^{\alpha}\left(\mathbf{A}^+\left(\alpha,p,S^2\right)\mathbf{A}^-\left(\alpha,p,S^2\right)+\frac{1}{2}\right)$$

and by postulating form of $F$ which is similar to that of $I$:

$$\begin{aligned}F\quad=\quad&\sum_{n,n'}\sum_{\substack{k=1\ldots n\\ l=1,\ldots,n'}}\sum_{\{S_k,S_l\}_{\substack{l=1,\ldots,n'\\ k=1\ldots n}}}\prod_{l=1}^{n'}\prod_{s=1}^{m_l'}\mathbf{A}^+\left(\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2\right)\\&\times F_{n,n'}\left(\left\{\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2,m_l'\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2,m_k\right\}\right)\prod_{k=1}^{n}\prod_{s=1}^{m_k}\mathbf{A}^-\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)\end{aligned} \tag{213}$$

To solve (212), we will use the commutation relations:

$$\left[\mathbf{A}^+\left(\alpha,p,S^2\right)\mathbf{A}^-\left(\alpha,p,S^2\right),\prod_{s=1}^{m_l'}\mathbf{A}^+\left(\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2\right)\right]=m_l'\bar{\mathbf{D}}_{S_l'^2}^{\boldsymbol{\alpha}_l'}\delta\left(\left(\alpha,p,S^2\right)-\left(\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2\right)\right)$$

and:

$$\left[\mathbf{A}^+\left(\alpha,p,S^2\right)\mathbf{A}^-\left(\alpha,p,S^2\right),\prod_{s=1}^{m_k}\mathbf{A}^-\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)\right]=-m_k\bar{\mathbf{D}}_{S_k^2}^{\boldsymbol{\alpha}_k}\delta\left(\left(\alpha,p,S^2\right)-\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)\right)$$

Inserting these relations in (212) leads to:

$$\begin{aligned}0 &= \sum_{l=1}^{n'}\sum_{k=1}^{n}\left(m_l'\bar{\mathbf{D}}_{S_l'^2}^{\boldsymbol{\alpha}_l'}-m_k\bar{\mathbf{D}}_{S_k^2}^{\boldsymbol{\alpha}_k}\right)F_{m_1,...,m_n,m_1',...,m_{n'}'}\left(\left\{\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right)\\&+V_{m_1,...,m_n,m_1',...,m_{n'}'}\left(\left\{\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right)\end{aligned}$$

and this yields the matrices elements of the trnsfrmtn:

$$F_{m_1,...,m_n,m_1',...,m_{n'}'}\left(\left\{\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right)=-\frac{V_{m_1,...,m_n,m_1',...,m_{n'}'}\left(\left\{\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\right)}{\sum_{l=1}^{n'}m_l'\bar{\mathbf{D}}_{S_l'^2}^{\boldsymbol{\alpha}_l'}-\sum_{k=1}^{n}m_k\bar{\mathbf{D}}_{S_k^2}^{\boldsymbol{\alpha}_k}} \tag{214}$$

## 6.2 Computing transformed action

Once the transformation matrix is found, we compute the transformed prtr $\left(S^{(O)}\right)'$:

$$\left(S^{(O)}\right)'=S_0+\frac{1}{2}\left[I,F\right]$$

by writing $[I,F]$ as a series expansion:

$$\begin{aligned}\left[I,F\right] &= \sum\prod_{L=1}^{n'}\prod_{s'=1}^{M_L'}\mathbf{A}^+\left(\boldsymbol{\alpha}_L',\mathbf{p}_L',S_L'^2\right) &(215)\\&\times\left[I,F\right]_{M_1,...,M_n,M_1',...,M_{n'}'}\left(\left\{\boldsymbol{\alpha}_L',\mathbf{p}_L',S_L'^2\right\},\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right\}\right)\prod_{K=1}^{n}\prod_{s=1}^{M_k'}\mathbf{A}^-\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\end{aligned}$$

Using the series (213) for $F$ and (107) for $I$. The commutator involves contributions of the form:

$$A=\left[\prod_{l=1}^{n'}\prod_{s=1}^{m_l'}\mathbf{A}^+\left(\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2\right)\prod_{k=1}^{n}\prod_{s=1}^{m_k}\mathbf{A}^-\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right),\prod_{q=1}^{n_1'}\prod_{s=1}^{m_q'}\mathbf{A}^+\left(\boldsymbol{\alpha}_q',\mathbf{p}_q',S_q'^2\right)\prod_{p=1}^{n_1}\prod_{s=1}^{m_p}\mathbf{A}^-\left(\boldsymbol{\alpha}_p,\mathbf{p}_p,S_p^2\right)\right] \tag{216}$$

and $A$ is obtained by successive derivation of a given commutator. Actually:

$$A=\prod_{l=1}^{n'}\left[\frac{\partial^{m_l'}}{\partial t_l^{m_l'}}\right]_{t_l=0}\prod_{k=1}^{n}\left[\frac{\partial^{m_k}}{\partial t_k^{m_k}}\right]_{t_k=0}C \tag{217}$$

where $C$ is defined by:

$$\begin{aligned}C &= \left[\exp\left(\sum t_k\mathbf{A}^-\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)\right)\exp\left(\sum t_k'\mathbf{A}^+\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)\right),\right. &(218)\\&\left.\exp\left(\sum t_l\mathbf{A}^-\left(\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right)\right)\exp\left(\sum t_l'\mathbf{A}^+\left(\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right)\right)\right] &(219)\end{aligned}$$

The commutator (218) is calculated by using the Campbell Hausdorf formula:

$$\begin{aligned}&\exp\left(\sum t_k\mathbf{A}^-\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)\right)\exp\left(\sum t_k'\mathbf{A}^+\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)\right)\\&\times\exp\left(\sum t_l\mathbf{A}^-\left(\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right)\right)\exp\left(\sum t_l'\mathbf{A}^+\left(\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right)\right)\\=&\exp\left(\sum t_k\mathbf{A}^-\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)+\sum t_l\mathbf{A}^-\left(\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right)\right)\\&\times\exp\left(\sum t_k'\mathbf{A}^+\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)+\sum t_l'\mathbf{A}^+\left(\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right)\right)\exp\left(\sum t_k't_l\delta\left(\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)-\left(\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right)\right)\right)\end{aligned}$$

so that, we find for (218):

$$\begin{aligned}
C &= \exp\left(\sum t_k \mathbf{A}^-\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)+\sum t_l \mathbf{A}^-\left(\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right)\right) \qquad (220)\\
&\times\exp\left(\sum t'_k \mathbf{A}^+\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)+\sum t'_l \mathbf{A}^+\left(\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right)\right)\\
&\times\left(\exp\left(\sum t'_k t_l \delta\left(\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)-\left(\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right)\right)\right)-\exp\left(\sum t_k t'_l \delta\left(\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)-\left(\boldsymbol{\alpha}_l,\mathbf{p}_l,S_l^2\right)\right)\right)\right)
\end{aligned}$$

Applying (217) to (220) leads to the expression of the coefficient (216) involved in (215). As a consequence, we find:

$$\begin{aligned}
&[I,F]_{M_1,\ldots,M_n,M'_1,\ldots,M'_{n'}}\left(\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L\right\},\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right\}\right) \qquad (221)\\
=&\sum_{P_K,P_L}\sum_{\{\epsilon'_d\},\{\epsilon_c\}}\sum_{\{\delta_k\},\{\delta'_l\}}\prod\left(\epsilon'_d!\right)^2\left(\epsilon_c!\right)^2\prod_{\bar{k}\bar{l}}\prod_{kl}(-1)^{\delta'_l}C_{m'_l+\delta'_l}^{\delta'_l}C_{m_k+\delta_k}^{\delta_k}\bar{\mathbf{D}}_{S_k^2}^{\boldsymbol{\alpha}_k}\bar{\mathbf{D}}_{S_k^2}^{\boldsymbol{\alpha}'_l}\bar{\mathbf{D}}_{S_c^2}^{\boldsymbol{\alpha}_c}\bar{\mathbf{D}}_{S_d^2}^{\boldsymbol{\alpha}'_d}\\
\times\;&\delta\left(\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)-\left(\bar{\boldsymbol{\alpha}}'_{\bar{l}},\bar{\mathbf{p}}'_{\bar{l}},\bar{S}'^2_{\bar{l}}\right)\right)\delta\left(\left(\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l\right)-\left(\bar{\boldsymbol{\alpha}}_{\bar{k}},\bar{\mathbf{p}}_{\bar{k}},\bar{S}^2_{\bar{k}}\right)\right)\\
&\delta\left(\left(\boldsymbol{\alpha}_c,\mathbf{p}_c,S_c^2\right)-\left(\bar{\boldsymbol{\alpha}}'_{\bar{d}},\bar{\mathbf{p}}'_{\bar{d}},\bar{S}'^2_{\bar{d}}\right)\right)\delta\left(\left(\boldsymbol{\alpha}'_d,\mathbf{p}'_d,S'^2_d\right)-\bar{\boldsymbol{\alpha}}_{\bar{c}},\bar{\mathbf{p}}_{\bar{c}},\bar{S}^2_{\bar{c}}\right)\\
&\times F\left(\left\{\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l,m'_l+\delta'_l\right\}\cup\left\{\boldsymbol{\alpha}'_d,\mathbf{p}'_d,S'^2_d,\epsilon'_d\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2,m_l+\delta_l\right\}\cup\left\{\boldsymbol{\alpha}_c,\mathbf{p}_c,S_c^2,\epsilon_c\right\}\right)\\
&\times V\left(\left\{\bar{\boldsymbol{\alpha}}'_{\bar{l}},\bar{\mathbf{p}}'_{\bar{l}},\bar{S}'^2_{\bar{l}},\bar{m}'_{\bar{l}}+\delta_k\right\}\cup\left\{\bar{\boldsymbol{\alpha}}'_{\bar{d}},\bar{\mathbf{p}}'_{\bar{d}},\bar{S}'^2_{\bar{d}},\epsilon'_d\right\},\left\{\bar{\boldsymbol{\alpha}}_{\bar{k}},\bar{\mathbf{p}}_{\bar{k}},\bar{S}^2_{\bar{k}},\bar{m}_{\bar{k}}+\delta'_l\right\}\cup\left\{\bar{\boldsymbol{\alpha}}_{\bar{c}},\bar{\mathbf{p}}_{\bar{c}},\bar{S}^2_{\bar{c}},\epsilon_c\right\}\right)
\end{aligned}$$

with $P_K,P_L$ are partitions of $\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right\}$ and $\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L,M'_L\right\}$:

$$\begin{aligned}
\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L,M'_L\right\} &= \left\{\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l,\bar{m}'_{l'}\right\}\cup\left\{\bar{\boldsymbol{\alpha}}'_{\bar{l}},\bar{\mathbf{p}}'_{\bar{l}},\bar{S}'^2_{\bar{l}},m'_{\bar{l}}\right\}\\
\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right\} &= \left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2,\bar{m}_k\right\}\cup\left\{\bar{\boldsymbol{\alpha}}_{\bar{k}},\bar{\mathbf{p}}_{\bar{k}},\bar{S}^2_{\bar{k}},m_{\bar{k}}\right\}
\end{aligned}$$

We can rewrite (221) by using the fact that the elmnts of $F$ are:

$$\begin{aligned}
&F_{\left(m_1+\delta_1,\ldots,m_n+\delta_n,\epsilon_1,,\ldots,\epsilon_p\right),m'_1+\delta'_1,\ldots,m'_{n'}+\delta'_n,\epsilon'_1,,\ldots,\epsilon'_{p'}}\left(\left\{\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l\right\}\cup\left\{\boldsymbol{\alpha}'_d,\mathbf{p}'_d,S'^2_d\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\cup\left\{\boldsymbol{\alpha}_c,\mathbf{p}_c,S_c^2\right\}\right)\\
=&-\frac{V_{\left(m_1+\delta_1,\ldots,m_n+\delta_n,\epsilon_1,,\ldots,\epsilon_p\right),m'_1+\delta'_1,\ldots,m'_{n'}+\delta'_n,\epsilon'_1,,\ldots,\epsilon'_{p'}}\left(\left\{\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l\right\}\cup\left\{\boldsymbol{\alpha}'_d,\mathbf{p}'_d,S'^2_d\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\cup\left\{\boldsymbol{\alpha}_c,\mathbf{p}_c,S_c^2\right\}\right)}{\sum_{l=1}^{n'}\left(m'_l+\delta'_{l'}\right)\bar{\mathbf{D}}_{S'^2_l}^{\boldsymbol{\alpha}'_l}-\sum_{k=1}^{n}\left(m_k+\delta_k\right)\bar{\mathbf{D}}_{S_k^2}^{\boldsymbol{\alpha}_k}+\sum_{d=1}^{p'}\epsilon'_d\bar{\mathbf{D}}_{S'^2_d}^{\boldsymbol{\alpha}'_d}-\sum_{c=1}^{p}\epsilon_c\bar{\mathbf{D}}_{S_c^2}^{\boldsymbol{\alpha}_c}}
\end{aligned}$$

and this leads to the expression of the matrices elements for commutator (221):

$$\begin{aligned}
&[I,F]_{M_1,\ldots,M_n,M'_1,\ldots,M'_{n'}}\left(\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L\right\},\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right\}\right)\\
=&-\sum_{P_K,P_L}\sum_{\{\epsilon'_d\},\{\epsilon_c\}}\sum_{\substack{\{\delta_k\},\{\delta'_l\}\\ \{\delta_{\bar{k}}\},\{\delta'_{\bar{l}}\}}}\prod\left(\epsilon'_d!\right)^2\left(\epsilon_c!\right)^2\prod_{\bar{k}\bar{l}}\prod_{kl}(-1)^{\delta'_l}C_{m'_l+\delta'_l}^{\delta'_l}C_{m_k+\delta_k}^{\delta_k}\bar{\mathbf{D}}_{S_k^2}^{\boldsymbol{\alpha}_k}\bar{\mathbf{D}}_{S_k^2}^{\boldsymbol{\alpha}'_l}\bar{\mathbf{D}}_{S_c^2}^{\boldsymbol{\alpha}_{0c}}\bar{\mathbf{D}}_{S_d^2}^{\boldsymbol{\alpha}'_{0d}}\\
\times\;&\delta\left(\left(\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right)-\left(\bar{\boldsymbol{\alpha}}'_{\bar{l}},\bar{\mathbf{p}}'_{\bar{l}},\bar{S}'^2_{\bar{l}}\right)\right)\delta\left(\left(\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l\right)-\left(\bar{\boldsymbol{\alpha}}_{\bar{k}},\bar{\mathbf{p}}_{\bar{k}},\bar{S}^2_{\bar{k}}\right)\right)\\
\times\;&\delta\left(\left(\boldsymbol{\alpha}_c,\mathbf{p}_c,S_c^2\right)-\left(\bar{\boldsymbol{\alpha}}'_{\bar{d}},\bar{\mathbf{p}}'_{\bar{d}},\bar{S}'^2_{\bar{d}}\right)\right)\delta\left(\left(\boldsymbol{\alpha}'_d,\mathbf{p}'_d,S'^2_d\right)-\bar{\boldsymbol{\alpha}}_{\bar{c}},\bar{\mathbf{p}}_{\bar{c}},\bar{S}^2_{\bar{c}}\right)\\
&\times V_1\left(\left\{\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l,m'_l+\delta'_l\right\}\cup\left\{\boldsymbol{\alpha}'_d,\mathbf{p}'_d,S'^2_d,\epsilon'_d\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2+\delta_k\right\}\cup\left\{\boldsymbol{\alpha}_c,\mathbf{p}_c,S_c^2,\epsilon_c\right\}\right)\\
&\times V\left(\left\{\bar{\boldsymbol{\alpha}}'_{\bar{l}},\bar{\mathbf{p}}'_{\bar{l}},\bar{S}'^2_{\bar{l}},\bar{m}'_l+\delta_k\right\}\cup\left\{\bar{\boldsymbol{\alpha}}'_{\bar{d}},\bar{\mathbf{p}}'_{\bar{d}},\bar{S}'^2_{\bar{d}},\epsilon_c\right\},\left\{\bar{\boldsymbol{\alpha}}_{\bar{k}},\bar{\mathbf{p}}_{\bar{k}},\bar{S}^2_{\bar{k}},\bar{m}_{\bar{k}}+\delta'_l\right\}\cup\left\{\bar{\boldsymbol{\alpha}}_{\bar{c}},\bar{\mathbf{p}}_{\bar{c}},\bar{S}^2_{\bar{c}},\epsilon'_d\right\}\right)
\end{aligned}$$

where we defined:

$$\begin{aligned}
&V_1\left(\left\{\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l,m'_l+\delta'_l\right\}\cup\left\{\boldsymbol{\alpha}'_d,\mathbf{p}'_d,S'^2_d,\epsilon'_d\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2+\delta_k\right\}\cup\left\{\boldsymbol{\alpha}_c,\mathbf{p}_c,S_c^2,\epsilon_c\right\}\right)\\
=&\frac{V_{\left(m_1+\delta_1,\ldots,m_n+\delta_n,\epsilon_1,,\ldots,\epsilon_p\right),m'_1+\delta'_1,\ldots,m'_{n'}+\delta'_n,\epsilon'_1,,\ldots,\epsilon'_{p'}}\left(\left\{\boldsymbol{\alpha}'_l,\mathbf{p}'_l,S'^2_l\right\}\cup\left\{\boldsymbol{\alpha}'_d,\mathbf{p}'_d,S'^2_d\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2\right\}\cup\left\{\boldsymbol{\alpha}_c,\mathbf{p}_c,S_c^2\right\}\right)}{\sum_{l=1}^{n'}\left(m'_l+\delta'_{l'}\right)\bar{\mathbf{D}}_{S'^2_l}^{\boldsymbol{\alpha}'_l}-\sum_{k=1}^{n}\left(m_k+\delta_k\right)\bar{\mathbf{D}}_{S_k^2}^{\boldsymbol{\alpha}_k}+\sum_{d=1}^{p'}\epsilon'_d\bar{\mathbf{D}}_{S'^2_d}^{\boldsymbol{\alpha}'_d}-\sum_{c=1}^{p}\epsilon_c\bar{\mathbf{D}}_{S_c^2}^{\boldsymbol{\alpha}_c}}
\end{aligned}$$

This is the results presented in the text.

## 6.3 Exemple

We present the previous computations for the example given in the text.

### 6.3.1 Computation of $F$

In this case $F$ is defined by:

$$I+[F,S_0]=0$$

and this equation writes in expanded form

$$\begin{aligned}0 &= \left[F,\mathbf{A}^{+}\left(\alpha_i,\mathbf{p}_i,S_i^2\right)\mathbf{A}^{-}\left(\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\right] \\ &\quad+\sum_{i=1,2} I\left(\left(\boldsymbol{\alpha}_i,\mathbf{p}_i\right),\left(\boldsymbol{\alpha}_i',\mathbf{p}_i'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_i^2,S_0^2\right)\mathbf{A}^{+}\left(\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\mathbf{A}^{-}\left(\boldsymbol{\alpha}_i',\mathbf{p}_i',S_i^2\right)\left(\mathbf{A}^{-}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)+\mathbf{A}^{+}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)\right)\end{aligned}$$

which yields the matrices elements of the transformation. Considering the interaction term:

$$I=\sum_i I\left(\left(\boldsymbol{\alpha}_i,\mathbf{p}_i\right),\left(\boldsymbol{\alpha}_i',\mathbf{p}_i'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_i^2,S_0^2\right)\mathbf{A}^{+}\left(\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\mathbf{A}^{-}\left(\boldsymbol{\alpha}_i',\mathbf{p}_i',S_i^2\right)\left(\mathbf{A}^{-}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)+\mathbf{A}^{+}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)\right)$$

it can be written in a compact notation:

$$\begin{aligned}I &= I_{a,b,c,d,e,f}\left(\left(\boldsymbol{\alpha}_1,\mathbf{p}_1\right),\left(\boldsymbol{\alpha}_1',\mathbf{p}_1'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_1^2,S_0^2\right) \\ &\quad\left(\mathbf{A}^{+}\left(\boldsymbol{\alpha}_i,\mathbf{p}_i,S_1^2\right)\right)^a\left(\mathbf{A}^{-}\left(\boldsymbol{\alpha}_i,\mathbf{p}_i,S_1^2\right)\right)^b\left(\mathbf{A}^{+}\left(\boldsymbol{\alpha}_{i'},\mathbf{p}_{i'},S_{i'}^2\right)\right)^c\left(\mathbf{A}^{-}\left(\boldsymbol{\alpha}_{i'},\mathbf{p}_{i'},S_{i'}^2\right)\right)^d \\ &\quad\times\left(\mathbf{A}^{+}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)\right)^e\left(\mathbf{A}^{-}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)\right)^f\end{aligned}$$

The only matrices elements arising in the interaction are:

$$\begin{gathered}I_{1,1,0,0,1,0}\left(\left(\boldsymbol{\alpha}_1,\mathbf{p}_1\right),\left(\boldsymbol{\alpha}_1',\mathbf{p}_1'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_1^2,S_0^2\right) \\ I_{1,1,0,0,0,1}\left(\left(\boldsymbol{\alpha}_1,\mathbf{p}_1\right),\left(\boldsymbol{\alpha}_1',\mathbf{p}_1'\right),\left(\boldsymbol{\alpha}_0',\mathbf{p}_0'\right),S_1^2,S_0^2\right) \\ I_{0,0,1,1,1,0}\left(\left(\boldsymbol{\alpha}_2,\mathbf{p}_2\right),\left(\boldsymbol{\alpha}_2',\mathbf{p}_2'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_2^2,S_0^2\right) \\ I_{0,0,1,1,0,1}\left(\left(\boldsymbol{\alpha}_2,\mathbf{p}_2\right),\left(\boldsymbol{\alpha}_2',\mathbf{p}_2'\right),\left(\boldsymbol{\alpha}_0',\mathbf{p}_0'\right),S_2^2,S_0^2\right)\end{gathered}$$

### 6.3.2 Transformation matrix $F$

With these notations, equation (111):

$$F\left(\left\{\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2,m_l'\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2,m_k\right\}\right)=-\frac{V_{m_1,\ldots,m_n,m_1',\ldots,m_{n'}'}\left(\left\{\boldsymbol{\alpha}_l',\mathbf{p}_l',S_l'^2,m_l'\right\},\left\{\boldsymbol{\alpha}_k,\mathbf{p}_k,S_k^2,m_k\right\}\right)}{\sum_{l=1}^{n'}m_l'\bar{\mathbf{D}}_{S_l'^2}^{\boldsymbol{\alpha}_l'}-\sum_{k=1}^{n}m_k\bar{\mathbf{D}}_{S_k^2}^{\boldsymbol{\alpha}_k}}$$

solves as:

$$\begin{aligned}&F_{1,1,0,0,1,0}\left(\left(\boldsymbol{\alpha}_1,\mathbf{p}_1\right),\left(\boldsymbol{\alpha}_1',\mathbf{p}_1'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_1^2,S_0^2\right) \\ =\;&-\frac{I_{1,1,0,0,1,0}\left(\left(\boldsymbol{\alpha}_1,\mathbf{p}_1\right),\left(\boldsymbol{\alpha}_1',\mathbf{p}_1'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_1^2,S_0^2\right)}{\left\{\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_1,\mathbf{p}_1,S_1^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_1',\mathbf{p}_1',S_1^2}\right)^2}+\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}\right\}} \\ &F_{1,1,0,0,1,0}\left(\left(\boldsymbol{\alpha}_1,\mathbf{p}_1\right),\left(\boldsymbol{\alpha}_1',\mathbf{p}_1'\right),\left(\boldsymbol{\alpha}_0',\mathbf{p}_0'\right),S_1^2,S_0^2\right) \\ =\;&-\frac{I_{1,1,0,0,1,0}\left(\left(\boldsymbol{\alpha}_1,\mathbf{p}_1\right),\left(\boldsymbol{\alpha}_1',\mathbf{p}_1'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_1^2,S_0^2\right)}{\left\{\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_1,\mathbf{p}_1,S_1^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_1',\mathbf{p}_1',S_1^2}\right)^2}+\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}\right\}}\end{aligned}$$

and:

$$
\begin{aligned}
& F_{0,0,1,1,1,0}\left(\left(\boldsymbol{\alpha}_2,\mathbf{p}_2\right),\left(\boldsymbol{\alpha}_2',\mathbf{p}_2'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_2^2,S_0^2\right)\\
= & -\frac{I_{0,0,1,1,1,0}\left(\left(\boldsymbol{\alpha}_2,\mathbf{p}_2\right),\left(\boldsymbol{\alpha}_2',\mathbf{p}_2'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_2^2,S_0^2\right)}{\left\{\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_1,\mathbf{p}_1,S_1^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_1',\mathbf{p}_1',S_1^2}\right)^2}+\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}\right\}}\\
& F_{0,0,1,1,1,0}\left(\left(\boldsymbol{\alpha}_2,\mathbf{p}_2\right),\left(\boldsymbol{\alpha}_2',\mathbf{p}_2'\right),\left(\boldsymbol{\alpha}_0',\mathbf{p}_0'\right),S_2^2,S_0^2\right)\\
= & -\frac{I_{0,0,1,1,1,0}\left(\left(\boldsymbol{\alpha}_2,\mathbf{p}_2\right),\left(\boldsymbol{\alpha}_2',\mathbf{p}_2'\right),\left(\boldsymbol{\alpha}_0',\mathbf{p}_0'\right),S_2^2,S_0^2\right)}{\left\{\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_1,\mathbf{p}_1,S_1^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_1',\mathbf{p}_1',S_1^2}\right)^2}+\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}\right\}}
\end{aligned}
$$

Written in term of operator, this is:

$$
\begin{aligned}
F=- & \frac{I\left(\left(\boldsymbol{\alpha}_i,\mathbf{p}_i\right),\left(\boldsymbol{\alpha}_i',\mathbf{p}_i'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_i^2,S_0^2\right)}{\left(\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i',\mathbf{p}_i',S_i^2}\right)^2}\right)^2-\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2} \\
& \times\left(\left(\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i',\mathbf{p}_i',S_i^2}\right)^2}+\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}\right)\mathbf{A}^{-}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)\right.\\
& \left.+\left(\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i',\mathbf{p}_i',S_i^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}\right)\mathbf{A}^{+-}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)\right)
\end{aligned}
\tag{222}
$$

### 6.3.3 Remark

Note that formula (222) could have been retrieved also by postulating the form:

$$
\begin{aligned}
F = & \sum_i F\left(\left(\boldsymbol{\alpha}_i,\mathbf{p}_i\right),\left(\boldsymbol{\alpha}_i',\mathbf{p}_i'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_i^2,S_0^2\right)\mathbf{A}^{+}\left(\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\mathbf{A}^{-}\left(\boldsymbol{\alpha}_i',\mathbf{p}_i',S_i^2\right)\\
& \times\left(\varepsilon\mathbf{A}^{-}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)+\mathbf{A}^{+-}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)\right)
\end{aligned}
$$

with $\varepsilon$ to be determined to ensure:

$$I+\left[F,S_0\right]=0$$

As a consequence:

$$
\begin{aligned}
& \left[F,S_0\right]\\
= & \left[F\left(\left(\boldsymbol{\alpha}_i,\mathbf{p}_i\right),\left(\boldsymbol{\alpha}_i',\mathbf{p}_i'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_i^2,S_0^2\right)\mathbf{A}^{+}\left(\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\mathbf{A}^{-}\left(\boldsymbol{\alpha}_i',\mathbf{p}_i',S_i^2\right)\left(\mathbf{A}^{-}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)+\mathbf{A}^{+-}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)\right),S_0\right]\\
= & \left(\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i',\mathbf{p}_i',S_i^2}\right)^2}\right)\\
& \times F\left(\left(\boldsymbol{\alpha}_i,\mathbf{p}_i\right),\left(\boldsymbol{\alpha}_i',\mathbf{p}_i'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_i^2,S_0^2\right)\mathbf{A}^{+}\left(\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\mathbf{A}^{-}\left(\boldsymbol{\alpha}_i',\mathbf{p}_i',S_i^2\right)\left(\varepsilon\mathbf{A}^{-}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)+\mathbf{A}^{+-}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)\right)\\
& +\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}F\left(\left(\boldsymbol{\alpha}_i,\mathbf{p}_i\right),\left(\boldsymbol{\alpha}_i',\mathbf{p}_i'\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_i^2,S_0^2\right)\mathbf{A}^{+}\left(\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2\right)\mathbf{A}^{-}\left(\boldsymbol{\alpha}_i',\mathbf{p}_i',S_i^2\right)\\
& \times\left(-\varepsilon\mathbf{A}^{-}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)+\mathbf{A}^{+-}\left(\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2\right)\right)
\end{aligned}
$$

which leads to:

$$
\left(\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i',\mathbf{p}_i',S_i^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}\right)\varepsilon=\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i',\mathbf{p}_i',S_i^2}\right)^2}+\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}
$$

and:

$$\left(\left(\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_i,\mathbf{p}_i,S_i^2}\right)^2}-\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}'_i,\mathbf{p}'_i,S_i^2}\right)^2}\right)+\sqrt{\left(\bar{\mathbf{D}}_{\boldsymbol{\alpha}_0,\mathbf{p}_0,S_0^2}\right)^2}\right)F\left(\left(\boldsymbol{\alpha}_i,\mathbf{p}_i\right),\left(\boldsymbol{\alpha}'_i,\mathbf{p}'_i\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_i^2,S_0^2\right)$$
$$= -F\left(\left(\boldsymbol{\alpha}_i,\mathbf{p}_i\right),\left(\boldsymbol{\alpha}'_i,\mathbf{p}'_i\right),\left(\boldsymbol{\alpha}_0,\mathbf{p}_0\right),S_i^2,S_0^2\right)$$

These two equations lead to (222).

## 6.4 Bounded states

We compute the eigenstates of $\left(S^{(O)}\right)$ that are the stable collective states resulting from the effective interactions.

As stated in the text, those states are series expansion:

$$\left|\prod_K\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\right)\right\rangle=\sum_{(M_K)}A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\right)\prod_{K=1}^{n}\prod_{s=1}^{M_K}\mathbf{A}^+\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\prod_K|Vac\rangle_K \tag{223}$$

Our aim is first to find the lowest gnstts which have the form:

$$\left|\prod_K\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right\rangle=\sum_K A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right)\mathbf{A}^+\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\prod_K|Vac\rangle_K \tag{224}$$

and then to generalize the result to obtain the full series expansion.

### 6.4.1 Lowest eigenvalues

As in the text, we start with the action of $\left(S^{(O)}\right)$ on (224):

$$\begin{aligned}&\left(S^{(O)}\right)\left|\prod_K\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right\rangle \qquad (225)\\ =&\sum_K A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right)\left(\sum_K\bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}}+2\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}\right)\mathbf{A}^+\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\prod_K|Vac\rangle_K\\ &+\sum_{K,L}A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right)[I,F]_{1,1,0,\dots,0}\left(\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S_L'^2\right\},\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right\},\left(\boldsymbol{\alpha}_P,\mathbf{p}_P,S_P^2\right)\right)\mathbf{A}^+\left(\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S_L'^2\right)\prod_K|Vac\rangle_K\end{aligned}$$

The condition for $\left|\prod_K\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right\rangle$ to be an eignstate for $\eta$ writes:

$$\begin{aligned}\eta A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right)=&\;A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right)\left(\sum_{K'}\bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}}+2\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}\right)\\ &+\sum_L A\left(\left(\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S_L'^2\right)\right)[I,F]_{1,1,0,\dots,0}\left(\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right\},\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S_L'^2\right\},\left(\boldsymbol{\alpha}_P,\mathbf{p}_P,S_P^2\right)\right)\end{aligned} \tag{226}$$

or, which is equivalent, in terms of determinant:

$$\begin{aligned}0=&\;\det\left\{[I,F]_{1,1}\left(\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S_L'^2\right\},\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right\}\right)\right.\\ &\left.+\left(\sum_{K'}\bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}}+2\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}-\eta\right)\delta\left(\left(\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S_L'^2\right)-\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right)\right\}\end{aligned} \tag{227}$$

that is:

$$\prod_K\left(\sum_{K'}\bar{\mathbf{D}}^{\boldsymbol{\alpha}_{K'}}_{S^2_{K'}}+2\bar{\mathbf{D}}^{\boldsymbol{\alpha}_K}_{S^2_K}-\eta\right)\det\left\{1+\frac{[I,F]_{1,1}\left(\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L\right\},\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right\}\right)}{\sum_{K'}\bar{\mathbf{D}}^{\boldsymbol{\alpha}_{K'}}_{S^2_{K'}}+2\bar{\mathbf{D}}^{\boldsymbol{\alpha}_K}_{S^2_K}-\eta}\right\}=0$$

For relatively small interactions $[I,F]_{(N_{K'}),M_K-M_{K'}+N_{K'}}<<1$ and the determinant equation is in first approximation:

$$0=\prod_K\left(\sum_{K'}\bar{\mathbf{D}}^{\boldsymbol{\alpha}_{K'}}_{S^2_{K'}}+2\bar{\mathbf{D}}^{\boldsymbol{\alpha}_K}_{S^2_K}-\eta\right)\tag{228}$$

$$\left(1-\sum_{K,K'}\frac{[I,F]_{1,1}\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S^2_{K'}\right)\right)[I,F]_{1,1}\left(\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S^2_{K'}\right),\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)\right)}{\left(\bar{\mathbf{D}}^{\boldsymbol{\alpha}_K}_{S^2_K}+2\bar{\mathbf{D}}^{\boldsymbol{\alpha}_K}_{S^2_K}-\eta\right)\left(\bar{\mathbf{D}}^{\boldsymbol{\alpha}_{K'}}_{S^2_{K'}}+2\bar{\mathbf{D}}^{\boldsymbol{\alpha}_{K'}}_{S^2_{K'}}-\eta\right)}\right)$$

writing the eigenvalues solving (228):

$$\eta\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)=\sum_{K'}\bar{\mathbf{D}}^{\boldsymbol{\alpha}_{K'}}_{S^2_{K'}}+2\bar{\mathbf{D}}^{\boldsymbol{\alpha}_K}_{S^2_K}-\varepsilon\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)\tag{229}$$

and inserting this expression in (228) yields the value of $\varepsilon\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)$:

$$\varepsilon\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)=\sum_{K'}\frac{[I,F]_{1,1}\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S^2_{K'}\right)\right)[I,F]_{1,1}\left(\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S^2_{K'}\right),\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)\right)}{\left(\bar{\mathbf{D}}^{\boldsymbol{\alpha}_{K'}}_{S^2_{K'}}+2\bar{\mathbf{D}}^{\boldsymbol{\alpha}_{K'}}_{S^2_{K'}}-\eta\right)}$$

Inserting the eigenvalue (229) in the eigenstate equation (228) becomes:

$$0=\varepsilon\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)A\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)+\sum_{K'}P\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S^2_{K'}\right)\right)A\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S^2_{K'}\right)$$

f $L\neq K$:

$$0\simeq\left(\bar{\mathbf{D}}^{\boldsymbol{\alpha}_L}_{S^2_L}+2\bar{\mathbf{D}}^{\boldsymbol{\alpha}_L}_{S^2_L}\right)A\left(\boldsymbol{\alpha}_L,\mathbf{p}_L,S^2_L\right)+\sum_{K'M_{K'}}[I,F]_{1,1}\left(\left(\boldsymbol{\alpha}_L,\mathbf{p}_L,S^2_L\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S^2_{K'}\right)\right)A\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S^2_{K'}\right)$$

with solution:

$$A\left(\boldsymbol{\alpha}_L,\mathbf{p}_L,S^2_L\right)\simeq-\frac{[I,F]_{1,1}\left(\left(\boldsymbol{\alpha}_L,\mathbf{p}_L,S^2_L\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S^2_{K'}\right)\right)}{\bar{\mathbf{D}}^{\boldsymbol{\alpha}_L}_{S^2_L}+2\bar{\mathbf{D}}^{\boldsymbol{\alpha}_L}_{S^2_L}}A\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)$$

In first approximation, using:

$$\eta_K=\sum_{K'}\bar{\mathbf{D}}^{\boldsymbol{\alpha}_{K'}}_{S^2_{K'}}+2\bar{\mathbf{D}}^{\boldsymbol{\alpha}_K}_{S^2_K}+\varepsilon_K$$

the $\varepsilon_K$ are solutions of:

$$\det\left\{[I,F]_{1,1}\left(\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L\right\},\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right\}\right)-\varepsilon_K\delta\left(\left(\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L\right)-\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)\right)\right\}=0$$

So that the $\varepsilon_K$ are eigenvalues f $[I,F]_{1,1}$ and the coefficients $A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)\right)$ are eigenvectors:

$$\varepsilon_K A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)\right)=\sum_L A\left(\left(\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L\right)\right)[I,F]_{1,1,0,\ldots,0}\left(\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right\},\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S'^2_L\right\},\left(\boldsymbol{\alpha}_P,\mathbf{p}_P,S^2_P\right)\right)$$

As a consequence, the states arr:

$$\left|\prod_K\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)\right\rangle=\sum_K A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)\right)\mathbf{A}^+\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S^2_K\right)\prod_K|Vac\rangle_K$$

### 6.4.3 Arbitrary eigenvalues and eigenstates. Full series expansion

To find the more general eigenstates, we consider a series expansion:

$$\left|\prod_K \left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right\rangle = \sum_{(M_K)} A\left(\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2, M_K\right)\right) \prod_{K=1}^{n} \prod_{s=1}^{M_K} \mathbf{A}^+\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right) \prod_K |Vac\rangle_K \tag{230}$$

We first need to compute the correction to $\left(S^{(O)}\right)\left|\prod_K \left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right\rangle$ due to $\frac{1}{2}[I,F]$, we thus consider the action of the full operator:

$$\left(S^{(O)}\right)\left|\prod_K \left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right\rangle \to \left(S^{(O)}\right)\left|\prod_K \left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right\rangle + \frac{1}{2}[I,F]\left|\prod_K \left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right\rangle$$

### 6.4.4 Correction to eigenstates due to $\frac{1}{2}[I,F]$

This correction is computed by first considering the commutator:

$$\begin{aligned} C &= \left[\exp\left(\sum t_k \mathbf{A}^-\left(\boldsymbol{\alpha}_k, \mathbf{p}_k, S_k^2\right)\right), \exp\left(\sum t_l' \mathbf{A}^+\left(\boldsymbol{\alpha}_l, \mathbf{p}_l, S_l^2\right)\right)\right] \\ &= \exp\left(\sum t_l' \mathbf{A}^+\left(\boldsymbol{\alpha}_l, \mathbf{p}_l, S_l^2\right)\right)\exp\left(\sum t_k \mathbf{A}^-\left(\boldsymbol{\alpha}_k, \mathbf{p}_k, S_k^2\right)\right)\exp\left(\sum t_k t_l' \delta\left(\left(\boldsymbol{\alpha}_k, \mathbf{p}_k, S_k^2\right) - \left(\boldsymbol{\alpha}_l, \mathbf{p}_l, S_l^2\right)\right)\right) \end{aligned} \tag{231}$$

From formula (231), we deduce that:

$$\begin{aligned} &\prod_{k=1}^{n}\prod_{s=1}^{m_k} \mathbf{A}^-\left(\boldsymbol{\alpha}_k, \mathbf{p}_k, S_k^2\right) \prod_{q=1}^{n_1'}\prod_{s=1}^{m_q'} \mathbf{A}^+\left(\boldsymbol{\alpha}_q', \mathbf{p}_q', S_q'^2\right) \prod_K |Vac\rangle_K \\ = &\left[\prod_{k=1}^{n}\prod_{s=1}^{m_k} \mathbf{A}^-\left(\boldsymbol{\alpha}_k, \mathbf{p}_k, S_k^2\right), \prod_{q=1}^{n_1'}\prod_{s=1}^{m_q'} \mathbf{A}^+\left(\boldsymbol{\alpha}_q', \mathbf{p}_q', S_q'^2\right)\right] \prod_K |Vac\rangle_K \end{aligned}$$

This expression is different from 0 if $\left\{\left(\boldsymbol{\alpha}_k, \mathbf{p}_k, S_k^2\right)\right\} \subset \left\{\left(\boldsymbol{\alpha}_q', \mathbf{p}_q', S_q'^2\right)\right\}$ and $m_k \leqslant m_k'$. In this case, let:

$$\left\{\left(\boldsymbol{\alpha}_p', \mathbf{p}_p', S_p'^2, m_p'\right)\right\} = \left\{\left(\boldsymbol{\alpha}_q', \mathbf{p}_q', S_q'^2\right)\right\} \setminus \left\{\left(\boldsymbol{\alpha}_k, \mathbf{p}_k, S_k^2\right)\right\}$$

with $m_p' = m_{q=p}' - m_{k=p}$ and we have:

$$\begin{aligned} &\left[\prod_{k=1}^{n}\prod_{s=1}^{m_k} \mathbf{A}^-\left(\boldsymbol{\alpha}_k, \mathbf{p}_k, S_k^2\right), \prod_{q=1}^{n_1'}\prod_{s=1}^{m_q'} \mathbf{A}^+\left(\boldsymbol{\alpha}_q', \mathbf{p}_q', S_q'^2\right)\right] \prod_K |Vac\rangle_K \\ = &\; m_{q=p}'! C_{m_{q=p}'}^{m_{k=p}} \prod_p \prod_{s=1}^{m_p'} \mathbf{A}^+\left(\boldsymbol{\alpha}_q', \mathbf{p}_q', S_q'^2\right) \prod_K |Vac\rangle_K \end{aligned}$$

As a consequence, using (230), we have:

$$\begin{aligned} &\frac{1}{2}[I,F]\left|\prod_K \left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right\rangle \\ = &\sum_{(M_K)} A\left(\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2, M_K\right)\right) \prod_{L=1}^{n'}\prod_{s'=1}^{M_L'} \mathbf{A}^+\left(\boldsymbol{\alpha}_L', \mathbf{p}_L', S_L'^2\right) \\ &\times \sum [I,F]_{N_1,\ldots,N_n,M_1',\ldots,M_{n'}'}\left(\left\{\boldsymbol{\alpha}_L', \mathbf{p}_L', S_L'^2\right\}, \left\{\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right\}\right) \\ &\times \left[\prod_{K=1}^{n}\prod_{s=1}^{N_K \leqslant M_K} \mathbf{A}^-\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right), \prod_{K=1}^{n}\prod_{s=1}^{M_K} \mathbf{A}^+\left(\boldsymbol{\alpha}_K, \mathbf{p}_K, S_K^2\right)\right] \prod_K |Vac\rangle_K \end{aligned}$$

and this is equal to:

$$\begin{aligned}
&\frac{1}{2}\left[I,F\right]\left|\prod_K\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right\rangle \\
=&\sum_{(M_K)}A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\right)\prod_{L=1}^{n'}\prod_{s'=1}^{M'_L}\mathbf{A}^+\left(\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S_L'^2\right) \\
&\times\sum\left[I,F\right]_{N_1,\ldots,N_n,M'_1,\ldots,M'_{n'}}\left(\left\{\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S_L'^2\right\},\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right\}\right) \\
&\times\prod_{L=1}^{n'}\prod_{s'=1}^{M'_L}\mathbf{A}^+\left(\boldsymbol{\alpha}'_L,\mathbf{p}'_L,S_L'^2\right)\left(\prod_{K=1}^{n}N_K!C_{M_K}^{N_K}\right)\prod_{K=1}^{n}\prod_{s=1}^{M_K-N_K}\mathbf{A}^+\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\prod_K\left|Vac\right\rangle_K
\end{aligned}$$

This expression can be reorganized as:

$$\frac{1}{2}\left[I,F\right]\left|\prod_K\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right\rangle=\sum_{(M_K)}\hat{A}\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\right)\prod_{K=1}^{n}\prod_{s=1}^{M_K}\mathbf{A}^+\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\prod_K\left|Vac\right\rangle_K$$

where:

$$\begin{aligned}
&\hat{A}\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\right) \\
=&\sum_{\mathcal{P}}\sum_{\left(N_K^{(2)}\right)}\left(\prod_{K=1}^{n_2}N_K^{(2)}!C_{M_K^{(2)}+N_K^{(2)}}^{N_K^{(2)}}\right)A\left(\left(\boldsymbol{\alpha}_K^{(2)},\mathbf{p}_K^{(2)},\left(S_K^2\right)^{(2)},M_K^{(2)}+N_K^{(2)}\right)\right) \\
&\times\left[I,F\right]_{N_1^{(2)},\ldots,N_{n_1}^{(2)},M_1^{(1)},\ldots,M_{n_2}^{(1)}}\left(\left\{\boldsymbol{\alpha}_K^{(1)},\mathbf{p}_K^{(1)},\left(S_K^2\right)^{(1)},M_K^{(1)}\right\},\left\{\boldsymbol{\alpha}_K^{(2)},\mathbf{p}_K^{(2)},\left(S_K^2\right)^{(2)},N_K^{(2)}\right\}\right)
\end{aligned}\tag{232}$$

and the sum is over partitions:

$$\mathcal{P}=\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)=\left(\boldsymbol{\alpha}_K^{(1)},\mathbf{p}_K^{(1)},\left(S_K^2\right)^{(1)},M_K^{(1)}\right)\cup\left(\boldsymbol{\alpha}_K^{(2)},\mathbf{p}_K^{(2)},\left(S_K^2\right)^{(2)},M_K^{(2)}\right)$$

and where $M_K=M_K^{(1)}+M_K^{(2)}$.

Note that (232) can be rewritten matricially:

$$\hat{A}\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)=\sum_{K',(M_{K'})}P\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right)A\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)$$

with:

$$\begin{aligned}
&P\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right) \\
=&\sum_{N_{K'}\leqslant M_{K'}}\left(\prod_{K'=1}^{n}N_{K'}!C_{M_{K'}}^{N_{K'}}\right)A\left(\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},\left(S_{K'}^2\right),M_{K'}\right)\right)\left[I,F\right]_{(N_{K'}),M_K-M_{K'}+N_{K'}}\left(\left\{\boldsymbol{\alpha}_K,\mathbf{p}_K,\left(S_K^2\right),M_K\right\}\right)
\end{aligned}\tag{233}$$

#### 13.0.1 6.4.4 Eigenvalues and eigenstates

The action of $\left(S^{(O)}\right)$ on $\left|\prod_K\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right\rangle$ writes:

$$\begin{aligned}
&\left(S^{(O)}\right)\left|\prod_K\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right\rangle \\
=&\sum_{(M_K)}A\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\right)\left(\sum_K\left(\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}+2M_K\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}\right)\right)\prod_{K=1}^{n}\prod_{s=1}^{M_K}\mathbf{A}^+\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\prod_K\left|Vac\right\rangle_K \\
&+\sum_{(M_K)}\hat{A}\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\right)\prod_{K=1}^{n}\prod_{s=1}^{M_K}\mathbf{A}^+\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\prod_K\left|Vac\right\rangle_K
\end{aligned}$$

and we look for eigenstates:

$$\left(S^{(O)}\right)\left|\prod_K\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right\rangle=\eta_{(M_K)}\left|\prod_K\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2\right)\right\rangle$$

and consider eigenvalues of the form:

$$\eta_{(M_K)}=\sum_K\left(\bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}}+2M_K\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}\right)+\varepsilon_{(M_K)}$$

corresponding to perturbations of eigenstates without interactions.

The eigenvalues equation becomes:

$$\begin{aligned}0 \;=\; & \det\left(\left(\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}+2M_K\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}-\eta\right)\delta\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)-\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right.\\ & \left.+P\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right)\right)\end{aligned}\tag{234}$$

where the matrix $P$ is defined in (233). We factor the determinant arising in (234) as:

$$\begin{aligned}& \det\left(\left(\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}+2M_K\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}-\eta\right)\delta\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)-\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right.\\ & \left.+P\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right)\right)\\ =\; & \det\left(\left(\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}+2M_K\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}-\eta\right)\delta\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)-\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right)\\ & \times\det\left(1+\frac{P\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right)}{\left(\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}+2M_K\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}-\eta\right)}\right)\end{aligned}\tag{235}$$

For relatively small interactions:

$$[I,F]_{(N_{K'}),M_K-M_{K'}+N_{K'}}<<1$$

and in first approximation the determinant (235) write:

$$\begin{aligned}& \det\left(\left(\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}+2M_K\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}-\eta\right)\delta\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)-\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right)\\ & \times\left(1-\sum_{K,K'}\frac{P\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right)P\left(\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right),\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right),\right)}{\left(\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}+2M_K\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}-\eta\right)\left(\bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}}+2M_{K'}\bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}}-\eta\right)}\right)\end{aligned}$$

As in the previous paragraph, we replace the eigenvalue by:

$$\eta\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)=\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}+2M_K\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}-\varepsilon\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\tag{236}$$

where $\varepsilon\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)$ is given in first approximation by equation:

$$1-\sum_{K'}\frac{P\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right)P\left(\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right),\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right),\right)}{\varepsilon\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\left(\bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}}+2M_{K'}\bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}}\right)}\simeq 0$$

with solution:

$$\begin{aligned}& \varepsilon\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\\ \simeq\; & \sum_{K'}\frac{P\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right)P\left(\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right),\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right),\right)}{\left(\bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}}+2M_{K'}\bar{\mathbf{D}}_{S_{K'}^2}^{\boldsymbol{\alpha}_{K'}}\right)}\end{aligned}$$

Once the eigenvalue is found, we can come back to the eigenstate equation:

$$\begin{aligned} 0 &= \left(\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}+2M_K\bar{\mathbf{D}}_{S_K^2}^{\boldsymbol{\alpha}_K}-\eta\right)A\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right) \\ &+\sum_{K'M_{K'}}P\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right)A\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\end{aligned}$$

for:

$$\eta=\eta\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)$$

Using (236) this eigenstate equation is:

$$\begin{aligned} 0 &= \varepsilon\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)A\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_{K'}\right) \\ &+\sum_{K'M_{K'}}P\left(\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right)A\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\end{aligned}$$

Now, if $L\neq K$ it writes:

$$0\simeq\left(\bar{\mathbf{D}}_{S_L^2}^{\boldsymbol{\alpha}_L}+2M_L\bar{\mathbf{D}}_{S_L^2}^{\boldsymbol{\alpha}_L}\right)A\left(\boldsymbol{\alpha}_L,\mathbf{p}_L,S_L^2,M_L\right)+\sum_{K'M_{K'}}P\left(\left(\boldsymbol{\alpha}_L,\mathbf{p}_L,S_L^2,M_L\right),\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)\right)A\left(\boldsymbol{\alpha}_{K'},\mathbf{p}_{K'},S_{K'}^2,M_{K'}\right)$$

and leads to:

$$A\left(\boldsymbol{\alpha}_L,\mathbf{p}_L,S_L^2,M_L\right)\simeq-\frac{P\left(\left(\boldsymbol{\alpha}_L,\mathbf{p}_L,S_L^2,M_L\right),\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)\right)}{\bar{\mathbf{D}}_{S_L^2}^{\boldsymbol{\alpha}_L}+2M_L\bar{\mathbf{D}}_{S_L^2}^{\boldsymbol{\alpha}_L}}A\left(\boldsymbol{\alpha}_K,\mathbf{p}_K,S_K^2,M_K\right)$$

# Appendix 6 variable localisation

## 6. 1 Interaction term

To compute the interaction term:

$$I=\sum_{\delta S^2}\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha}+\delta\boldsymbol{\alpha},\mathbf{p}+\delta\mathbf{p},S^2+\delta S^2\right)U\left(S^2,S^2+\delta S^2\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$$

we choose the following form for the potential $U\left(S^2,S^2+\delta S^2\right)$:

$$\begin{aligned} U\left(S^2,S^2+\delta S^2\right) &= \int_{\delta S^2}d^{+/-}\left(Z,Z'\right)U\left(\delta\bar{\mathbf{T}}_p^{\alpha}\left(Z,Z'\right)\right) \\ &= \int_{S^2+\delta S^2/S^2}d\left(Z,Z'\right)U\left(\delta\bar{\mathbf{T}}_p^{\alpha}\left(Z,Z'\right)\right)-\int_{S^2/S^2+\delta S^2}d\left(Z,Z'\right)U\left(\delta\bar{\mathbf{T}}_p^{\alpha}\left(Z,Z'\right)\right)\end{aligned}$$

where:

$$\begin{aligned} S^2+\delta S^2/S^2 &= \left(S^2+\delta S^2\right)\backslash S^2\cap\left(S^2+\delta S^2\right) \\ S^2/S^2+\delta S^2 &= \left(S^2\right)\backslash S^2\cap\left(S^2+\delta S^2\right)\end{aligned}$$

and $\delta\bar{\mathbf{T}}_p^{\alpha}\left(Z,Z'\right)$ is the variation defined by:

$$\delta\bar{\mathbf{T}}_p^{\alpha}\left(Z,Z'\right)=\bar{\mathbf{T}}_p^{\alpha}\left(\left(Z,Z'\right)+\delta\left(Z,Z'\right)\right)-\bar{\mathbf{T}}_p^{\alpha}\left(Z,Z'\right)$$

In a continuous approximation, this corresponds to:

$$U\left(S^2,S^2+\delta S^2\right)=\int_{\delta S^2}U\left(\delta\bar{\mathbf{T}}_p^{\alpha}\left(Z,Z'\right)\right)\left(n\left(\left(Z,Z'\right)\right).\frac{\delta\left(Z,Z'\right)}{\left\|\delta\left(Z,Z'\right)\right\|}\right)d\left(Z,Z'\right)$$

with $n((Z,Z'))$ the normal to $S^2$.

A second order expansion rewrites the potential:

$$\begin{aligned}I &= \sum_{\delta S^2}\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha}+\delta\boldsymbol{\alpha},\mathbf{p}+\delta\mathbf{p},S^2+\delta S^2\right)U\left(S^2,S^2+\delta S^2\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right) \\ &= \sum_{\delta S^2}\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)U\left(S^2,S^2+\delta S^2\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right) \\ &\quad -\sum_{\delta S^2}\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)U\left(S^2,S^2+\delta S^2\right)\delta S^2\frac{\delta}{\delta S^2}\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\end{aligned} \tag{237}$$

Here, $\frac{\delta\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)}{\delta S^2}$ should be considered as a full derivatives, but, to focus on nonlocalization, we neglect variations in $\boldsymbol{\alpha},\mathbf{p}$. We have defined:

$$\delta S^2\frac{\delta}{\delta S^2}\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)=\int_{S^2}\delta\left(Z,Z'\right)\nabla_{(Z,Z')}\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$$

In (237), the first term can be neglected. Actually:

$$\begin{aligned}&\sum_{\delta S^2}U\left(S^2,S^2+\delta S^2\right) \\ =&\sum_{\delta S^2}\left(\int d\left(S^2+\delta S^2/S^2\right)U\left(\bar{\mathbf{T}}_p^{\alpha}\left(Z,Z'\right)\right)-\int d\left(S^2/S^2+\delta S^2\right)U\left(\bar{\mathbf{T}}_p^{\alpha}\left(Z,Z'\right)\right)\right)\rightarrow 0\end{aligned}$$

The second term in (237) can be rewriten as:

$$\sum_{\delta S^2}U\left(S^2,S^2+\delta S^2\right)\int_{S^2}\delta\left(Z,Z'\right)\nabla_{(Z,Z')}\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)=\int_{S^2}\bar{U}\left(S^2,Z,Z'\right)\nabla_{(Z,Z')}\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$$

where:

$$\bar{U}\left(S^2,Z,Z'\right)=\sum_{\delta S^2}U\left(S^2,S^2+\delta S^2\right)\delta\left(Z,Z'\right)$$

As a consequence, the potential (119) writes:

$$\begin{aligned}I &= \sum_{\delta S^2}\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha}+\delta\boldsymbol{\alpha},\mathbf{p}+\delta\mathbf{p},S^2+\delta S^2\right)U\left(S^2,S^2+\delta S^2\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right) \\ &= \int_{S^2}\bar{U}\left(S^2,Z,Z'\right)\nabla_{(Z,Z')}\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right) \\ &\equiv -\bar{U}\left(S^2\right)\frac{\delta}{\delta S^2}\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\end{aligned}$$

### 6.2 Computation of the Green function

The Green function between two states $\left|\mathbf{T}_p^{\alpha},S^2\right\rangle$ and $\left|\mathbf{T}_p'^{\alpha},S'^2\right\rangle$ is computed by changing the variables in the action:

$$\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\left(-\frac{1}{2}\frac{\delta^2}{\delta\left(S^2\right)^2}-\bar{U}\left(S^2\right)\frac{\delta}{\delta S^2}-\frac{1}{2}\nabla_{(\mathbf{T})_{S^2}}^2+\frac{1}{2}\left(\Delta\mathbf{T}_p^{\alpha}\right)^t\left(\mathbf{A}_p^{\alpha}\right)_{S^2}\Delta\mathbf{T}_p^{\alpha}+\mathbf{C}\left(S^2\right)\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)$$

Replacing:

$$\begin{aligned}\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right) &\rightarrow \exp\left(\int^{S^2}\bar{U}\left(S^2\right)dS^2\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right) \\ \underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right) &\rightarrow \exp\left(-\int^{S^2}\bar{U}\left(S^2\right)dS^2\right)\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^2\right)\end{aligned}$$

the action becomes:

$$\underline{\Gamma}^{\dagger}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right) \tag{238}$$
$$\times\left(-\frac{1}{2}\frac{\delta^{2}}{\delta\left(S^{2}\right)^{2}}+\frac{1}{2}\bar{U}^{2}\left(S^{2}\right)+\frac{1}{2}\frac{\delta}{\delta S^{2}}\bar{U}\left(S^{2}\right)-\frac{1}{2}\nabla_{(\mathbf{T})_{S^{2}}}^{2}+\frac{1}{2}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)^{t}\left(\mathbf{A}_{p}^{\alpha}\right)_{S^{2}}\Delta\mathbf{T}_{p}^{\alpha}+\mathbf{C}\left(S^{2}\right)\right)\underline{\Gamma}\left(\mathbf{T},\boldsymbol{\alpha},\mathbf{p},S^{2}\right)$$

with:

$$\frac{\delta}{\delta S^{2}}\bar{U}\left(S^{2}\right)=\int\nabla_{(Z,Z')}\bar{U}\left(S^{2},Z,Z'\right)d\left(Z,Z'\right)$$

For $\bar{U}^{2}\left(S^{2},Z,Z'\right)$ quadratic in $(Z,Z')$:

$$\begin{aligned}\frac{1}{2}\bar{U}\left(S^{2},Z,Z'\right)+\frac{1}{2}\frac{\delta}{\delta S^{2}}\bar{U}\left(S^{2}\right) &\simeq \int_{S^{2}}\frac{1}{2}A\left(Z,Z'\right)\left(Z^{2}+Z'^{2}\right)\\ &\simeq \int_{S^{2}}\frac{1}{2}\langle A\rangle\left(Z^{2}+Z'^{2}\right)\end{aligned}$$

with $\langle A\rangle$ is the average of $A\left(Z,Z'\right)$ on $S^{2}$.

We also expand $\mathbf{C}\left(S^{2}\right)$ with respect to some $\mathbf{C}\left(S_{0}^{2}\right)$. In first approximation:

$$\mathbf{C}\left(S^{2}\right)\simeq\mathbf{C}\left(S_{0}^{2}\right)+\int_{S^{2}}\left(\left(Z-Z_{0}\right)+\left(Z'-Z_{0}'\right)\right)\bar{\mathbf{C}}\left(S^{2}\right)$$

so that, up to come constant:

$$\frac{1}{2}\bar{U}\left(S^{2},Z,Z'\right)+\frac{1}{2}\frac{\delta}{\delta S^{2}}\bar{U}\left(S^{2}\right)\simeq\frac{1}{2}\int_{S^{2}}\langle A\rangle\left(\left(Z-\langle A\rangle^{-1}\bar{\mathbf{C}}\left(S^{2}\right)\right)^{2}+\left(Z'-\langle A\rangle^{-1}\bar{\mathbf{C}}\left(S^{2}\right)\right)^{2}\right)$$

The Green function of:

$$\begin{aligned}O &= -\frac{1}{2}\frac{\delta^{2}}{\delta\left(S^{2}\right)^{2}}+\frac{1}{2}\int\langle A\rangle\left(\left(Z-\langle A\rangle^{-1}\bar{\mathbf{C}}\left(S^{2}\right)\right)^{2}+\left(Z'-\langle A\rangle^{-1}\bar{\mathbf{C}}\left(S^{2}\right)\right)^{2}\right)\\ &\quad -\frac{1}{2}\nabla_{(\mathbf{T})_{S^{2}}}^{2}+\frac{1}{2}\left(\Delta\mathbf{T}_{p}^{\alpha}\right)^{t}\left(\mathbf{A}_{p}^{\alpha}\right)_{S^{2}}\Delta\mathbf{T}_{p}^{\alpha}+\mathbf{C}\left(S^{2}\right)\end{aligned}$$

is written:

$$G\left(S'^{2},\mathbf{T}_{p}'^{\alpha},S^{2},\mathbf{T}_{p}^{\alpha}\right)$$

between two close structures $S^{2}$ and $S^{2}+\delta S^{2}$ is approximatively:

$$\begin{aligned}G\left(S'^{2},\mathbf{T}_{p}'^{\alpha},S^{2},\mathbf{T}_{p}^{\alpha}\right) &\simeq \frac{\exp\left(-\left(\mathbf{X}'\right)^{t}\langle\mathbf{A}\rangle_{S^{2}+\delta S^{2}}\mathbf{X}'-\left(\mathbf{X}\right)^{t}\langle\mathbf{A}\rangle_{S^{2}}\mathbf{X}-\left(\mathbf{C}\left(S'^{2}\right)-\mathbf{C}\left(S^{2}\right)\right)\right)}{\sqrt{\det\langle\mathbf{A}\rangle}}\\ &\quad\times\frac{\exp\left(-\left(\mathbf{T}_{p}'^{\alpha}\right)^{t}\mathbf{A}_{S^{2}}^{\alpha}\mathbf{T}_{p}'^{\alpha}-\left(\mathbf{T}_{p}^{\alpha}\right)^{t}\mathbf{A}_{S^{2}}^{\alpha}\mathbf{T}_{p}^{\alpha}\right)}{\sqrt{\det\left(\mathbf{A}_{S^{2}}^{\alpha}\right)}}\end{aligned} \tag{239}$$

where

$$\mathbf{X}'=\left(Z-\langle A\rangle^{-1}\bar{\mathbf{C}}\left(S^{2}\right),Z'-\langle A\rangle^{-1}\bar{\mathbf{C}}\left(S^{2}\right)\right)_{(Z,Z')\in S'^{2}}$$

and:

$$\mathbf{X}=\left(Z-\langle A\rangle^{-1}\bar{\mathbf{C}}\left(S^{2}\right),Z'-\langle A\rangle^{-1}\bar{\mathbf{C}}\left(S^{2}\right)\right)_{(Z,Z')\in S^{2}}$$

We have set $\langle\mathbf{A}\rangle_{S^{2}}$ for the average $A\left(Z,Z'\right)$ in $S^{2}$ and $\langle\mathbf{A}\rangle_{S^{2}+\delta S^{2}}$ for the average $A\left(Z,Z'\right)$ in $S^{2}+\delta S^{2}$. As a consequence, successive convolutions yield in first approximation the Green function for the system defined in (238):

$$G\left(S'^2,\mathbf{T}'^{\alpha}_p,S^2,\mathbf{T}^{\alpha}_p\right) \simeq \frac{\exp\left(-\left(\mathbf{X}'\right)^t\left\langle\mathbf{A}\right\rangle_{S'^2}\mathbf{X}'-\left(\mathbf{X}\right)^t\left\langle\mathbf{A}\right\rangle_{S^2}\mathbf{X}-\left(\mathbf{C}\left(S'^2\right)-\mathbf{C}\left(S^2\right)\right)\right)}{\left(\left\langle\mathbf{A}\right\rangle_{S'^2}\left\langle\mathbf{A}\right\rangle_{S^2}\right)^{\frac{1}{4}}} \times\frac{\exp\left(-\left(\mathbf{T}'^{\alpha}_p\right)^t\mathbf{A}^{\alpha}_{S'^2}\mathbf{T}'^{\alpha}_p-\left(\mathbf{T}^{\alpha}_p\right)^t\mathbf{A}^{\alpha}_{S^2}\mathbf{T}^{\alpha}_p\right)}{\sqrt{\det\left(\mathbf{A}^{\alpha}_{S^2}\right)}} \tag{240}$$

Expanding $\mathbf{X}$ and $\mathbf{X}'$ leads to:

$$\begin{aligned} & G\left(S'^2,\mathbf{T}'^{\alpha}_p,S^2,\mathbf{T}^{\alpha}_p\right) \\ \simeq\; & \frac{\exp\left(-\left(\left(\mathbf{Z},\mathbf{Z}'\right)'\right)^t\left\langle\mathbf{A}\right\rangle_{S'^2}\left(\mathbf{Z},\mathbf{Z}'\right)'-\left(\mathbf{Z},\mathbf{Z}'\right)^t\left\langle\mathbf{A}\right\rangle_{S^2}\left(\mathbf{Z},\mathbf{Z}'\right)'-\left(\mathbf{C}\left(S'^2\right)-\mathbf{C}\left(S^2\right)\right)\right)}{\sqrt{\det\left\langle\mathbf{A}\right\rangle}} \\ & \times\frac{\exp\left(-\left(\mathbf{T}'^{\alpha}_p\right)^t\mathbf{A}^{\alpha}_{S'^2}\mathbf{T}'^{\alpha}_p-\left(\mathbf{T}^{\alpha}_p\right)^t\mathbf{A}^{\alpha}_{S^2}\mathbf{T}^{\alpha}_p\right)}{\sqrt{\det\left(\mathbf{A}^{\alpha}_{S^2}\right)}} \end{aligned} \tag{241}$$

where $\left(\mathbf{Z},\mathbf{Z}'\right)'=S'^2$ and $\left(\mathbf{Z},\mathbf{Z}'\right)=S^2$.

Coming back to the initial variable amounts to add a factor in (240):

$$\exp\left(\frac{\delta}{\delta S^2}\bar{U}\left(S^2\right)-\frac{\delta}{\delta S'^2}\bar{U}\left(S'^2\right)\right) \tag{242}$$

which yields the result in the text.