Statistical Field Theory and Neural Structures Dynamics I: Action Functionals, Background States and External Perturbations

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Abstract

This series of papers models the dynamics of a large set of interacting neurons within the framework of statistical field theory. The system is described using a two-field model. The first field represents the neuronal activity, while the second field accounts for the interconnections between cells. This model is derived by translating a probabilistic model involving a large number of interacting cells into a field formalism. The current paper focuses on deriving the background fields of the system, which describe the potential equilibria in terms of interconnected groups. Dynamically, we explore the perturbation of these background fields, leading to processes such as activation, association, and reactivation of groups of cells.

1 Introduction

Bridging the micro and macroscopic behaviors remains largely problematic when dealing with systems characterized by a large number of degrees of freedom. When investigating neural activity, one typically either begins directly with a macroscopic description of the system or initiates a study with a microscopic description that is subsequently treated numerically. In a previous study ([52]), we introduced a statistical field-theoretic approach to establish a connection between the micro and macro levels. For a dynamic system consisting of a large number of interacting spiking neurons distributed within a defined spatial region, also referred to as the "thread," we can associate a field-theoretic framework that encompasses the fundamental microscopic characteristics of the system.

This field-theoretic framework enables the determination of the system's effective action, along with the associated background field, which corresponds to the minimum of the effective action. This background field characterizes the collective state of the system. The field framework facilitates the computation of neurons firing frequencies, i.e., neural activity, at each point within the system in a specific background state. Furthermore, we can derive the propagation of perturbations in neural activity from one point to another. We showed the presence of persistent nonlinear traveling waves along the thread.

Nonetheless, ([52]) considered the connectivity functions between various points in the thread as endogenous functions of the neural activities. We introduced an extension that involved dynamic equations for these functions driven by the activities dynamics. However, even in this extended case,

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the connectivities between the points in the thread were not treated as a dynamically interacting system that could be addressed as a field-theoretic system in its own right.

The current series of papers addresses this issue and takes a significant step towards the field-theoretic representation of a system of interacting neurons. We introduce a two-fields model that characterizes both the dynamics of neural activity and the connectivity between cells within the thread. Furthermore, we investigate the implications of this model in the context of neural and connectivity dynamics.

This field theory results from a two-step process and is grounded in a method initially developed by [47] and subsequently adapted to complex interacting systems in [48][49][50][51]. In the first step, the conventional formalism governing the dynamic equations of a large assembly of interacting neurons, as outlined in ([46]), is extended to encompass a dynamic system that accounts for the dynamic nature of neuronal connectivity. We use the formalism for connectivity functions as presented in ([53]), rewritten in a format suitable to translation into field theory. In the second step, these two sets of dynamic equations are transformed into a second-quantized Euclidean field theory, as elucidated in [48][49][50]. The action functional of this field theory relies on two fields. The first field, akin to the one introduced in ([52]), characterizes the assembly of neurons and their activity. Meanwhile, the second field delineates the dynamics of connectivity between cells. Both fields exhibit self-interaction, portraying interactions across the network, and interact with each other, encapsulating the interdependencies between neural activities and connectivities. This field-based description encompasses both collective and individual aspects of the system. The system with these two fields is described by a field action functional that comprehensively records the interactions at the microscopic level. This action functional encapsulates the dynamics of the entire system.

Our description enables the derivation of background fields for both neural interactions and connectivities, which minimize the action functional. These background fields depict the collective configuration of the system and determine the potential static equilibria for neural activities and connectivities. These equilibria serve as the foundational framework for the system, organizing fluctuations and signal propagation within it. The background fields are contingent upon internal system parameters and external stimuli, and thus, the entire system may undergo transitions in response to variations in these parameters and stimuli.

Throughout the four articles of this series, we investigate the implications of our formalism. We explore the conditions and the possible forms for collective interconnected structures, their interactions, and the mechanisms governing their merger. This study progressively reorients the relative importance of our prior research between neural activity and connectivity dynamics. The structures under examination are defined by sets of interconnected cells, and the activity within the system is determined by the specific configuration of these sets.

We accomplish this by deriving an effective action for the connectivity field alone, with neural activity becoming an endogenous variable dependent on connectivity. Ultimately, based on the findings presented in these papers, the fourth section of this work will introduce an expanded effective model designed to address a substantial or even infinite number of potential collective states. The system will be characterized by fields representing collective states, each of these fields depending on a large set of parameters. These sets of parameters describe potential interconnected structures with various amplituds and frequencies of activity.

From a field theory perspective, this approach is tantamount to engaging in a second-quantized formulation of the initial formalism, that is, to second-quantizing a previously second-quantized formalism. Starting directly with a field formalism for collective states allows us to model how these structures interact, activate or deactivate, and experience transitions in terms of activities amplitudes and activity frequencies.

In this first paper, we expound upon the development of the field model that characterizes the system, encompassing both neural activity and connectivity functions. We present the derivation of an effective action for the connectivities and subsequently compute its quasi-static background field. This background field delineates the average connectivities among elements within the thread, as well as the average activities between these elements. Subsequently, we outline the dynamic implications of these findings by examining the effects of external perturbations that alter the activities between individual cells.

Taking into account that the timescale for connectivities is slower than that of individual cells, we demonstrate how repeated activations at certain points can propagate throughout the network and progressively alter the connectivity functions. In the presence of oscillatory perturbations, the oscillatory response may exhibit interference phenomena. At points where constructive interference occurs, the background state for connectivities and the average connectivities experience modification. These long-term alterations manifest as emerging states characterized by enhanced connectivities between specific points.

These states reflect the impact of external activations and can be regarded as a record of these activations. They exhibit a gradual fading over time but remain capable of reactivation by external perturbations. Furthermore, the association of such emerging states is possible when their activation occurs at closed times. The resulting state is a composite of two distinct states, which can be described as a modification of the initial background state at various points. Activating one of these two states leads to the reactivation of their combined effect. Thus, regardless of the cause of their activation, these states of enhanced connectivity present characteristics akin to interacting partial neuronal assemblies.

The second application focuses on the system of connectivities as an independent self-interacting object. By replacing the individual cells' field as an effective quantity dependent on connectivities, we can derive the effective dynamics for the connectivity fields. This approach engenders internal dynamics that can induce shifts in the static background state,particularly at certain points within the system. Self-interactions triggered by perturbations may initiate internal patterns of connections among specific cells. Depending on internal parameters, we observe the potential for enduring shifts in connectivity background states within certain regions of the network, while other areas remain unaffected. This effective theory can also be applied to study the mechanisms underpinning connectivity reinforcement among multiple cells.

This paper is organised as follows: Section 2 provides a literature review. Part I of the paper develops a theoretical model for a neural system. Section 3 introduces a general method for translating a system with a large number of agents into a field framework. In Section 4, we elucidate the individual dynamics of interacting neurons and its probabilistic interpretation. Subsequently, we transcribe this framework in terms of fields in Section 5.

The second section of the paper applies this model to explore structural aspects of the neural system. We investigate the background states of the system along with the corresponding cellular activities and connectivities. In Section 6, we establish the saddle point equation for the neural field and derive the general form of the equation for activities. Section 7 provides the saddle point equation for the connectivity field. In Section 8, we deduce solutions to these equations, which represent the background fields for connectivities. Additionally, we derive equations for the average values of connectivities within these states.

Section 9 involves the computation of static equilibrium, yielding the possible average connectivities and activities within the background field. Multiple solutions arise, each corresponding to different potential configurations of interconnected states. Section 10 presents a model extension in which n different types of cells interact. The background states for such a model are derived, alongside the connectivities and activities within these background states.

The third part of this work delves into dynamic aspects of the system, investigating how external perturbations can induce transitions between connected equilibrium states. In Section 11, we revisit the dynamics of the neural field and elucidate the dynamic aspects of the neurons'background field. Section 12 subsequently derives the dynamic component of cellular activity.

Within this dynamic framework, stable oscillations in activity emerge, some of which may be driven by external signals. These oscillations, stemming from source terms, are expounded upon in Section 13. Building on these results, Section 14 examines how oscillations in activities can modify the background states of connectivities and the groups of connected states.

The mechanism of interferences between waves of activity, isolating and synchronizing groups of interacting cells, is explored in Section 15. Sections 16, 17, 18 are dedicated to the study of group reactivation by subsequent signals, group associations through signals, the reactivation of associated groups by subsequent signals, and the impact of sequences of signals on state associations, respectively. Section 19 concludes the paper.

2 Litterature review

Several branches of the literature are related to our work. First, at the macroscopic scale, and at the modeling level we are considering, our approach shares common goals with the literature on mean fields or neural fields. Neural fields model large populations of neurons as homogeneous structures, with individual neurons indexed by spatial coordinates. These models are employed to describe various patterns of brain activity. Following the work of Wilson, Cowan, and Amari ([1][2][3][4][5][6][7][8][9]), neural field dynamics are typically investigated in the continuum limit, with neural activity represented by a macroscopic variable—the population-averaged firing rate. Mean field theory has been extended in various ways and has found a wide range of applications.

It permits the existence of traveling wave solutions (see [20][21], and related literature). Incorporating stochastic effects in firing rates allows to model perturbations and diffusion patterns in pulse wave dynamics as well as noisy transitions between various mean field regimes (see [15] for instance). Besides, mean field approaches can be expanded to investigate the influence of neural network topology on spatial arrangements of neural activity, with relevant work found in [17], further developments in [18], and related sources.

Nevertheless, the mean field approach is an effective theoretical framework in which the degrees of freedom of certain underlying processes are aggregated. Despite the convenience and practical applications of mean field formalism, it relies on simplifications to represent the microscopic level, such as the neglect of interaction delays or variations in neuronal connectivity. Furthermore, owing to its aggregated nature, this framework is unable to capture emerging behaviors.

Compared to this approach, our statistical field theory model maintains a detailed account of the individual dynamics and connectivities of interacting neurons while retaining certain features and objectives of neural field dynamics. For example, we assign spatial coordinates to neurons in order to derive continuous dynamic equations for the entire system. However, unlike Mean Field Theory and its extensions, our fields do not directly represent neural activity. Similar to Statistical Field Theory (as seen in [47]), they are rather abstract, complex-valued functionals that encapsulate microscopic information on a larger collective scale. It is only after translating the microscopic model into the language of fields that we reconstruct specific quantities to describe neural activities.

Our approach incorporates features that have been explored in certain extensions of mean field theory. Our results inherently involve stochastic elements: the field accounts for the interactions of neurons subject to dynamic uncertainty. We are able to recover certain patterns of traveling waves.

However, beyond that, our formalism offers several advantages. It sheds light on the influence of internal variables on firing rate dynamics. It can provide a direct approach to phenomena related to phase transitions, such as the impact of collective patterns on individual ones, by examining the system's effective action. Additionally, it allows for a wide range of extensions, as demonstrated in this series of paper, which incorporates the dynamics of connectivity functions within the framework of field theory.

Please note that the Mean Field approach has already been extended using the tools of statistical field theory, albeit in a manner different from ours (see [19][22][23][24][25][26][27]). In these extensions, statistical fields represent neural activity, or spike counts, at each point within the network. An effective action is formulated for neural activity. Because these extensions account for covariances between neural activities at different network points, the perturbation expansion of the effective action goes beyond the mean field approximation. However, these models are constructed based on the mean field model and incorporate some deviations from it, ultimately remaining at the collective level rather than emerging from the network's microscopic features. Closer to our approach are studies such as those in ([28]) and ([29]), which utilize partition functions for the entire neural system, or ([30]), which uses an effective action. Nevertheless, these approaches either make simplified micro-level assumptions or impose *a priori* constraints on the effective action.

At the same scale as the neural field, a specific portion of the literature has focused on the role of connectivities and plasticity in systems of interacting neurons. This literature is situated within the context of neural networks (see ([54]) and its references for network-related studies) or neural field modeling. In ([55]), the author investigated the stability of the resting-state activity within a neural field concerning variations in connectivity with respect to a homogeneous connectivity matrix. This concept was further extended in ([**?**]) in the context of a generic network differential equation (see ([57]) and ([58]) for a comprehensive account) to explore the impact of non-homogeneous connectivities on network properties and neural dynamics. The authors revealed that symmetry breaking in network connectivity, or inhomogeneous connectivities, leads to the emergence of an attractive functional subspace. The states of symmetry breaking share some similarities with the background states considered in this paper, by encompassing the decomposition into background and fluctuations. Nevertheless, in their context, the backgrounds are postulated, and parameters must be fine-tuned to produce stable states.

A more detailed examination of connectivity dynamics and their characteristics is provided in ([59]) (see ([60]) and ([61]) for applications). Neural field theory is employed to define connectivity tensors in terms of bare and dressed propagators, and a diagrammatic analysis, akin to a Feynman graph expansion, is conducted. This approach to connectivities enables to surpass the typical phenomenological approach, allowing for the characterization and exploration of patterns in brain connectivity and activity. The utilization of graph expansion is similar to our approach in the sense that connectivities are comprehended through propagators and series expansion. However, this approach is tailored within the framework of neural fields and lacks the emergence of global or background states. Connectivities are not regarded as a dynamic system on their own.

At the microscopic scale, another branch of literature closely related to our work lies at the intersection of dynamical systems, complex systems, and neural networks. This body of research is primarily concerned with the dynamics and interactions of individual neurons (see [31][32][33], and the references therein).

In literature strands such as cognitive neurodynamics or computational neuroscience, neural processes arise from the interactions of assemblies of individual neurons. This finer-grained approach enables a more detailed account of the interplay between neurons' connectivity and firing rates compared to that of neural fields. Typically, it does not assume spatial indices; neurons are not arranged within a spatial structure, and the model's resolution relies on numerical studies. This approach accommodates neurons' cyclical dynamics, variations in oscillation regimes (for further details, see [31] and the references therein), and, of greater relevance to our work, the emergence of local connectivity and higher-scale phenomena. These phenomena include the binding problem or polychronization (see [36][37][38][39][40][41][42][43][44][45][46]).

However, unlike mean fields, these models lack an analytical treatment of collective effects. Our method aims to bridge the gap between macro-scale modeling in neural field theory and the assembly of interacting neurons. Nonetheless, they provide the foundational elements for developing a microscopic basis for a field-theoretic description. Our work is based on one of these frameworks described in ([53]). Initially designed to address polychronization, we employ it to describe the dynamics of connectivity functions and investigate the emergence of connected patterns across the thread of individual cells.

A third branch of the literature is of relevance to our work. Our research is closely tied to the ongoing debate regarding the existence of specific sets of connected cells responsible for storing information, particularly related to memory, often referred to as 'engrams.' Recent empirical and theoretical studies provide increasing evidence for the mechanisms underlying engram formation and memory linkage. The articles [62] (and its references), [63], [64] (and its references), [65], among others, support the engram hypothesis and its significance in memory storage. Engrams exhibit persistence and can be subject to reactivation. They underscore the non-local aspects of engrams, which may span different regions and types of neurons, as well as the importance of interactions among multiple engram ensembles that result in enhanced memory recall compared to the reactivation of a single engram ensemble: multiple engram ensembles are conferred a greater level of memory recall than reactivation of a single engram ensemble.

In ([66]) engram allocation is linked to the excitability of neurons, offering insights into engram interactions. Engrams are associated with dynamic interactions among connectivity states through co-allocation and overlapping. Some characteristics of these interactions are further examined in studies such as ([67]), ([70]), and ([68]).

These studies provide strong support for the notion that changes in the strength of neuronal connections are stored in the brain. They emphasize the growing consensus regarding the role of interactions between sets of neurons in the formation of recurrent memories and the composition of complex behaviors. This confirms the idea that changes in connectivities and the emergence of associated patterns correspond to the formation of engrams. In other words, the formation of engrams is directly linked to the dynamic aspects of connectivity functions, including their modifications and interactions. Engram states can thus be viewed as dynamic interacting states of connectivities, akin to those studied in our work. Furthermore, the mechanism of engram interaction proposed in ([69]), which involves a competitive process that integrates memories of events occurring closely in time (co-allocating overlapping populations of neurons to both engrams) and separates memories of events occurring at distant times, bears similarity to our mechanism of associating connectivity states. In our model, connectivity states interact and may associate to produce more stable states, with the timing of activation being crucial for the formation of such associated states.

Last but not least, several studies converge on the significance of regulatory mechanisms involving connectivities and interactions between connectivities and neuronal activity. In ([71]), the authors review recent research on the mechanisms that shape engrams and regulate memory functions. They speculate that countervailing forces within local microcircuits contribute to the generation and maintenance of engrams. On the other hand, some studies emphasize the role of homeostatic processes that stabilize neuronal activity, as seen in ([73]), ([74]), ([72]), ([75]), ([76]), ([77]), ([78]), ([79]). The interaction of Hebbian homosynaptic plasticity with rapid non-Hebbian heterosynaptic plasticity, when complemented with slower homeostatic changes and consolidation, is sufficient for the formation of assemblies. This confirms both the role of interactions between connectivities and neural activity in the formation of connectivity states, as well as the importance of certain homeostatic processes that we aim to capture through the concept of activity and connectivity potential.

Part I: Field formalism for large set of neurons activities and connectivity functions

In this initial section, we present the method to transform a dynamic system with a large number of agents into a field-theoretic model. In general, this translation entails the introduction of one field for each type of dynamic variable. Subsequently, we delve into neural dynamics, where we elucidate the conventional dynamic equations governing neuronal activities and connectivities (also referred to as transfer functions in ([52])). These equations are then translated into a formalism based on two fields.

3 Translation of dynamical systems with large number of degrees of freedom into fields models: General method

The formalism we propose transforms a dynamic system with large number of agents into a statistical field model. In the present work, the term agent will refer to individual neurons or the connectivity between cells that are themselves dynamic variables. In classical models, each agent's dynamics is described by an optimal path for some vector variable, say $A_i(t)$, from an initial to a final point, up to some fluctuations.

But this system of agents could also be seen as probabilistic: each agent could be described by a *probability density* centered around the classical optimal path, up to some idiosyncratic uncertainties[1] [2]. In this probabilistic approach, each possible trajectory of the whole set of N agents has a specific probability. The classical model is therefore described by the set of trajectories of the group of N agents, each one being endowed with its own probability, its statistical weight. The statistical weight is therefore a function that associates a probability with each trajectory of the group.

This probabilistic approach can be translated into a more compact *field formalism*[3] that preserves the essential information encoded in the model but implements a change in perspective. A field model is a structure governed by its own intrinsic rules that encapsulate the dynamic model chosen. This field model contains all possible realizations that could arise from the initial economic model, i.e. all the possible global outcomes, or collective state, permitted by the economic model. So that, once constructed, the field model provides a unique advantage over a standard dynamic model: it allows to compute the probabilities of each of the possible outcomes for each collective state of the model. These probabilities are computed indirectly through the *action functional* of the model, a function that assigns a specific value to each realization of the field. Technically, the random N agents' trajectories $\{\mathbf{A}_i(t)\}$ are replaced by a field, a random variable whose realizations are complex-valued functions Ψ of the variables $\mathbf{A}$, and the statistical weight of the N agents'

[1] Because the number of possible paths is infinite, the probability of each individual path is null. We, therefore, use the word "probability density" rather than "probability".

[2] See Gosselin, Lotz and Wambst (2017, 2020, 2021).

[3] Ibid.

trajectories $\{\mathbf{A}_i(t)\}$ in the probabilistic approach is translated into a statistical weight for each realization Ψ. They encapsulate the collective states of the system.

Once the probabilities of each collective state computed, the most probable collective state among all other collective states, can be found. In other words, a field model allows to consider the true global outcome induced by any standard economic model. This is what we will call the *expression* of the field model, more usually called the *background field* of the model.

This most probable realization of the field, the expression or background field of the model, should not be seen as a final outcome resulting from a trajectory, but rather as its most recurring realization. Actually, the probability of the realizations of the model is peaked around the expression of the field. This expression, which is characteristic of the system, will determine the nature of individual trajectories within the structure, in the same way as a biased dice would increase the probability of one event.

3.1 Statistical weight and minimization functions for a classical system

In a dynamic system with a large number of agents, each agent is characterized by one or more stochastic dynamic equations. Some of these equations result from the optimization of one or several objective functions. Deriving the statistical weight from these equations is straightforward: it associates, to each trajectory of the group of agents $\{T_i\}$, a probability that is peaked around the set of optimal trajectories of the system, of the form:

$$W\left(s\left(\{T_i\}\right)\right)=\exp\left(-s\left(\{T_i\}\right)\right) \tag{1}$$

where $s\left(\{T_i\}\right)$ measures the distance between the trajectories $\{T_i\}$ and the optimal ones.

As explained above, this paper studies two types of agents: cells and connectivities between cells. To remain at a general level in this section, we rather consider two arbitrary types of agents characterized by vector-variables $\{\mathbf{A}_i(t)\}_{i=1,\ldots N}$, and $\left\{\hat{\mathbf{A}}_l(t)\right\}_{i=1,\ldots\hat{N}}$ respectively, where N and $\hat{N}$ are the number of agents of each type, with vectors $\mathbf{A}_i(t)$ and $\hat{\mathbf{A}}_l(t)$ of arbitrary dimension. For such a system, the statistical weight writes:

$$W\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)=\exp\left(-s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)\right) \tag{2}$$

The optimal paths for the system are assumed to be described by the sets of equations:

$$\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l\ldots}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)=\epsilon_i,\ i=1\ldots N \tag{3}$$

$$\frac{d\hat{\mathbf{A}}_l(t)}{dt}-\sum_{i,j,k\ldots}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)=\hat{\epsilon}_l,\ i=1\ldots\hat{N} \tag{4}$$

where the ϵ_i and $\hat{\epsilon}_i$ are idiosynchratic random shocks. These equations describe the general dynamics of the two types agents, including their interactions with other agents. They may encompass the dynamics of optimizing agents where interactions act as externalities so that this set of equations is the full description of a system of interacting agents[4][5].

[4] Expectations of agents could be included by replacing $\frac{d\mathbf{A}_i(t)}{dt}$ with $E\frac{d\mathbf{A}_i(t)}{dt}$, where E is the expectation operator. This would amount to double some variables by distinguishing "real variables" and expectations. However, for our purpose, in the context of a large number of agents, at least in this work, we discard as much as possible this possibility.

[5] A generalisation of equations (3) and (4), in which agents interact at different times, and its translation in term of field is presented in appendix 1.

For equations (3) and (4), the quadratic deviation at time t of any trajectory with respect to the optimal one for each type of agent are:

$$\left(\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l...}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \tag{5}$$

and:

$$\left(\frac{d\hat{\mathbf{A}}_l(t)}{dt}-\sum_{i,j,k...}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \tag{6}$$

Since the function (2) involves the deviations for all agents over all trajectories, the function:

$$s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)$$

is obtained by summing (5) and (6) over all agents, and integrate over t. We thus find:

$$\begin{aligned} s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right) &= \int dt\sum_i\left(\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l...}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \\ &+\int dt\sum_l\left(\frac{d\hat{\mathbf{A}}_l(t)}{dt}-\sum_{i,j,k...}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \end{aligned} \tag{7}$$

There is an alternate, more general, form to (7). We can assume that the dynamical system is originally defined by some equations of type (3) and (4), plus some objective functions for agents i and l, and that these agents aim at minimizing respectively:

$$\sum_{j,k,l...}g\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right) \tag{8}$$

and:

$$\sum_{i,j,k..}\hat{g}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right) \tag{9}$$

In the above equations, the objective functions depend on other agents' actions seen as externalities[6]. The functions (8) and (9) could themselves be considered as a measure of the deviation of a trajectory from the optimum. Actually, the higher the distance, the higher (8) and (9).

Thus, rather than describing the systm by a full system of dynamic equations, we can consider some ad-hoc equations of type (3) and (4) and some objective functions (8) and (9) to write the alternate form of (7) as:

$$\begin{aligned} & s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right) \\ =& \int dt\sum_i\left(\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l...}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \\ &+\int dt\sum_l\left(\frac{d\hat{\mathbf{A}}_l(t)}{dt}-\sum_{i,j,k...}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \\ &+\int dt\sum_{i,j,k,l...}\left(g\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)+\hat{g}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right) \end{aligned} \tag{10}$$

[6] We may also assume intertemporal objectives, see ([48]).

In the sequel, we will refer to the various terms arising in equation (10) as the "minimization functions", **i.e**. the functions whose minimization yield the dynamics equations of the system[7].

We have shown in [48][49][50] that the probabilistic description of the system (10) is equivalent to a statistical field formalism. In such a formalism, the system is collectively described by a field that is an element of the Hilbert space of complex functions. The arguments of these functions are the same as those describing an individual neuron and the connectivity function between two cells. A shortcut of the translation of systems similar to (10) in terms of field, is given in [51] . The next paragraph gives an account of this method.

3.2 Translation techniques

Once the statistical weight $W\left(s\left(\{T_i\}\right)\right)$ defined in (1) iscomputed, it can be translated in terms of field. To do so, and for each type α of agent, the sets of trajectories $\{\mathbf{A}_{\alpha i}(t)\}$ are replaced by a field $\Psi_\alpha(\mathbf{A}_\alpha)$, a random variable whose realizations are complex-valued functions Ψ of the variables $\mathbf{A}_\alpha$[8]. The statistical weight for the whole set of fields $\{\Psi_\alpha\}$ has the form $\exp\left(-S\left(\{\Psi_\alpha\}\right)\right)$. The function $S\left(\{\Psi_\alpha\}\right)$ is called the *fields action functional*. It represents the interactions among different types of agents. Ultimately, the expression $\exp\left(-S\left(\{\Psi_\alpha\}\right)\right)$ is the statistical weight for the field[9] that computes the probability of any realization $\{\Psi_\alpha\}$ of the field.

The form of $S\left(\{\Psi_\alpha\}\right)$ is obtained directly from the classical description of our model. For two types of agents, we start with expression (10). The various minimizations functions involved in the definition of $s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)$ will be translated in terms of field and the sum of these translations will produce finally the action functional $S\left(\{\Psi_\alpha\}\right)$. The translation method can itself be divided into two relatively simple processes, but varies slightly depending on the type of terms that appear in the various minimization functions.

3.2.1 Terms without temporal derivative

In equation (10), the terms that involve indexed variables but no temporal derivative terms are the easiest to translate. They are of the form:

$$\sum_i \sum_{j,k,l,m...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right)$$

These terms describe the whole set of interactions both among and between two groups of agents. Here, agents are characterized by their variables $\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t)...$ and $\hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...$ respectively, for instance in our model firms and investors.

In the field translation, agents of type $\mathbf{A}_i(t)$ and $\hat{\mathbf{A}}_l(t)$ are described by a field $\Psi(\mathbf{A})$ and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$, respectively.

In a first step, the variables indexed i such as $\mathbf{A}_i(t)$ are replaced by variables $\mathbf{A}$ in the expression of g. The variables indexed j, k, l, m..., such as $\mathbf{A}_j(t)$, $\mathbf{A}_k(t)$, $\hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...$ are replaced by $\mathbf{A}', \mathbf{A}''$, $\hat{\mathbf{A}}$, $\hat{\mathbf{A}}'$, and so on for all the indices in the function. This yields the expression:

$$\sum_i \sum_{j,k,l,m...} g\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right)$$

[7] A generalisation of equation (10), in which agents interact at different times, and its translation in term of field is presented in appendix 1.

[8] In the following, we will use indifferently the term "field" and the notation Ψ for the random variable or any of its realization Ψ.

[9] In general, one must consider the integral of $\exp\left(-S\left(\{\Psi_\alpha\}\right)\right)$ over the configurations $\{\Psi_\alpha\}$. This integral is the partition function of the system.

In a second step, each sum is replaced by a weighted integration symbol:

$$\begin{aligned}\sum_i &\rightarrow \int |\Psi(\mathbf{A})|^2 d\mathbf{A},\ \sum_j \rightarrow \int |\Psi(\mathbf{A}')|^2 d\mathbf{A}',\ \sum_k \rightarrow \int |\Psi(\mathbf{A}'')|^2 d\mathbf{A}''\\ \sum_l &\rightarrow \int \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 d\hat{\mathbf{A}},\ \sum_m \rightarrow \int \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}}'\end{aligned}$$

which leads to the translation:

$$\begin{aligned}&\sum_i\sum_j\sum_{j,k...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right)\\ \rightarrow &\int g\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right)|\Psi(\mathbf{A})|^2|\Psi(\mathbf{A}')|^2|\Psi(\mathbf{A}'')|^2 \times ...d\mathbf{A}d\mathbf{A}'d\mathbf{A}''...\\ &\times\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2\left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 \times ...d\hat{\mathbf{A}}d\hat{\mathbf{A}}'...\end{aligned} \tag{11}$$

where the dots stand for the products of square fields and integration symbols needed.

3.2.2 Terms with temporal derivative

In equation (10), the terms that involve a variable temporal derivative are of the form:

$$\sum_i\left(\frac{d\mathbf{A}_i^{(\alpha)}(t)}{dt} - \sum_{j,k,l,m...} f^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right)\right)^2 \tag{12}$$

This particular form represents the dynamics of the α-th coordinate of a variable $\mathbf{A}_i(t)$ as a function of the other agents.

The method of translation is similar to the above, but the time derivative adds an additional operation.

In a first step, we translate the terms without derivative inside the parenthesis:

$$\sum_{j,k,l,m...} f^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right) \tag{13}$$

This type of term has already been translated in the previous paragraph, but since there is no sum over i in equation (13), there should be no integral over $\mathbf{A}$, nor factor $|\Psi(\mathbf{A})|^2$.

The translation of equation (13) is therefore, as before:

$$\int f^{(\alpha)}\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right)|\Psi(\mathbf{A}')|^2|\Psi(\mathbf{A}'')|^2 d\mathbf{A}'d\mathbf{A}''\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2\left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}}d\hat{\mathbf{A}}' \tag{14}$$

A free variable $\mathbf{A}$ remains, which will be integrated later, when we account for the external sum $\sum_i$. We will call $\Lambda(\mathbf{A})$ the expression obtained:

$$\Lambda(\mathbf{A}) = \int f^{(\alpha)}\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right)|\Psi(\mathbf{A}')|^2|\Psi(\mathbf{A}'')|^2 d\mathbf{A}'d\mathbf{A}''\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2\left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}}d\hat{\mathbf{A}}' \tag{15}$$

In a second step, we account for the derivative in time by using field gradients. To do so, and as a rule, we replace :

$$\sum_i\left(\frac{d\mathbf{A}_i^{(\alpha)}(t)}{dt} - \sum_j\sum_{j,k...} f^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right)\right)^2 \tag{16}$$

by:

$$\int \Psi^{\dagger}(\mathbf{A})\left(-\nabla_{\mathbf{A}^{(\alpha)}}\left(\frac{\sigma_{\mathbf{A}^{(\alpha)}}^{2}}{2}\nabla_{\mathbf{A}^{(\alpha)}}-\Lambda(\mathbf{A})\right)\right)\Psi(\mathbf{A})\,d\mathbf{A} \tag{17}$$

The variance $\sigma_{\mathbf{A}^{(\alpha)}}^{2}$ reflects the probabilistic nature of the model which is hidden behind the field formalism. This variance represents the characteristic level of uncertainty of the system's dynamics. It is a parameter of the model. Note also that in (17), the integral over $\mathbf{A}$ reappears at the end, along with the square of the field $|\Psi(\mathbf{A})|^{2}$. This square is split into two terms, $\Psi^{\dagger}(\mathbf{A})$ and $\Psi(\mathbf{A})$, with a gradient operator inserted in between.

3.3 Action functional

The field description is ultimately obtained by summing all the terms translated above and introducing a time dependency. This sum is called the action functional. It is the sum of terms of the form (11) and (17), and is denoted $S(\Psi,\Psi^{\dagger})$.

For example, in a system with two types of agents described by two fields $\Psi(\mathbf{A})$and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$, the action functional has the form:

$$\begin{aligned}
S\left(\Psi,\Psi^{\dagger}\right) &= \int \Psi^{\dagger}(\mathbf{A})\left(-\nabla_{\mathbf{A}^{(\alpha)}}\left(\frac{\sigma_{\mathbf{A}^{(\alpha)}}^{2}}{2}\nabla_{\mathbf{A}^{(\alpha)}}-\Lambda_{1}(\mathbf{A})\right)\right)\Psi(\mathbf{A})\,d\mathbf{A} \\
&+\int \hat{\Psi}^{\dagger}\left(\hat{\mathbf{A}}\right)\left(-\nabla_{\hat{\mathbf{A}}^{(\alpha)}}\left(\frac{\sigma_{\hat{\mathbf{A}}^{(\alpha)}}^{2}}{2}\nabla_{\hat{\mathbf{A}}^{(\alpha)}}-\Lambda_{2}(\hat{\mathbf{A}})\right)\right)\hat{\Psi}\left(\hat{\mathbf{A}}\right)d\hat{\mathbf{A}} \\
&+\sum_{m}\int g_{m}\left(\mathbf{A},\mathbf{A}',\mathbf{A}'',\hat{\mathbf{A}},\hat{\mathbf{A}}'...\right)|\Psi(\mathbf{A})|^{2}\,|\Psi(\mathbf{A}')|^{2}\,|\Psi(\mathbf{A}'')|^{2}\times ...d\mathbf{A}d\mathbf{A}'d\mathbf{A}''... \\
&\times\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^{2}\left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^{2}\times ...d\hat{\mathbf{A}}d\hat{\mathbf{A}}'...
\end{aligned} \tag{18}$$

where the sequence of functions g_m describes the various types of interactions in the system.

4 Probabilistic description of large set of cells and connectivity functions

We describe a dynamic system of a large number of neurons ($N >> 1$) and their connectivity functions. We define their individual equations. Then, we write a probability density for the configurations of the whole system over time.

4.1 Cells Individual dynamics

We follow the description of [46] for coupled quadratic integrate-and-fire (QIF) neurons, but use the additional hypothesis that each neuron is characterized by its position in some spatial range.

Each neuron's potential $X_i(t)$ satisfies the differential equation:

$$\dot{X}_i(t)=\gamma X_i^2(t)+J_i(t) \tag{19}$$

for $X_i(t)<X_p$, where X_p denotes the potential level of a spike. When $X=X_p$, the potential is reset to its resting value $X_i(t)=X_r<X_p$. For the sake of simplicity, following ([46]) we have chosen the squared form $\gamma X_i^2(t)$ in (19). However any form $f(X_i(t))$ could be used. The current of signals reaching cell i at time t is written $J_i(t)$.

Our purpose is to find the system dynamics in terms of the spikes' frequencies, that is neural activities. First, we consider the time for the n-th spike of cell i, $\theta_n^{(i)}$. This is written as a function of n, $\theta^{(i)}(n)$. Then, a continuous approximation $n \to t$ allows to write the spike time variable as $\theta^{(i)}(t)$. We thus have replaced:

$$\theta_n^{(i)} \to \theta^{(i)}(n) \to \theta^{(i)}(t)$$

The continuous approximation could be removed, but is convenient and simplifies the notations and computations. We assume now that the timespans between two spikes are relatively small. The time between two spikes for cell i is obtained by writing (19) as:

$$\frac{dX_i(t)}{dt} = \gamma X_i^2(t) + J_i(t)$$

and by inverting this relation to write:

$$dt = \frac{dX_i}{\gamma X_i^2 + J^{(i)}\left(\theta^{(i)}(n-1)\right)}$$

Integrating the potential between two spikes thus yields:

$$\theta^{(i)}(n) - \theta^{(i)}(n-1) \simeq \int_{X_r}^{X_p} \frac{dX}{\gamma X^2 + J^{(i)}\left(\theta^{(i)}(n-1)\right)}$$

Replacing $J^{(i)}\left(\theta^{(i)}(n-1)\right)$ by its average value during the small time period $\theta^{(i)}(n) - \theta^{(i)}(n-1)$, we can consider $J^{(i)}\left(\theta^{(i)}(n-1)\right)$ as constant in first approximation, so that:

$$\begin{aligned}\theta^{(i)}(n) - \theta^{(i)}(n-1) &\simeq \frac{\left[\arctan\left(\sqrt{\frac{\gamma}{J^{(i)}\left(\theta^{(i)}(n-1)\right)}}X\right)\right]_{X_r}^{X_p}}{\sqrt{\gamma J^{(i)}\left(\theta^{(i)}(n-1)\right)}} \\ &= \frac{\arctan\left(\frac{\left(\frac{1}{X_r}-\frac{1}{X_p}\right)\sqrt{\frac{J^{(i)}\left(\theta^{(i)}(n-1)\right)}{\gamma}}}{1+\frac{J^{(n)}\left(\theta^{(n-1)}\right)}{\gamma X_r X_p}}\right)}{\sqrt{\gamma J^{(i)}\left(\theta^{(i)}(n-1)\right)}}\end{aligned} \tag{20}$$

To work at the highest level of generality when possible, we write:

$$\theta^{(i)}(n) - \theta^{(i)}(n-1) \equiv G\left(\theta^{(i)}(n-1)\right)$$

understood that for computations and numerical approximations we will use formula (20) for G.

For $\gamma << 1$, (20) yields:

$$\theta^{(i)}(n) - \theta^{(i)}(n-1) \equiv G\left(\theta^{(i)}(n-1)\right) = \frac{X_p - X_r}{J^{(i)}\left(\theta^{(i)}(n-1)\right)}$$

For $\gamma = O(1)$ and for γ normalized to 1 and $\frac{J^{(n)}\left(\theta^{(n-1)}\right)}{X_r X_p} << 1$, this is:

$$\theta^{(i)}(n) - \theta^{(i)}(n-1) \equiv G\left(\theta^{(i)}(n-1)\right) = \frac{\arctan\left(\left(\frac{1}{X_r}-\frac{1}{X_p}\right)\sqrt{J^{(i)}\left(\theta^{(i)}(n-1)\right)}\right)}{\sqrt{J^{(i)}\left(\theta^{(i)}(n-1)\right)}} \tag{21}$$

The activity or firing rate at t, $\omega_i(t)$, is defined by the inverse time span (21) between two spikes:

$$\begin{aligned}\omega_i(t) &= \frac{1}{G\left(\theta^{(i)}(n-1)\right)} \\ &\equiv F\left(\theta^{(i)}(n-1)\right) = \frac{\sqrt{J^{(i)}\left(\theta^{(i)}(n-1)\right)}}{\arctan\left(\left(\frac{1}{X_r}-\frac{1}{X_p}\right)\sqrt{J^{(i)}\left(\theta^{(i)}(n-1)\right)}\right)}\end{aligned}$$

Since we consider small time intervals between two spikes, we can write:

$$\theta^{(i)}(n) - \theta^{(i)}(n-1) \simeq \frac{d}{dt}\theta^{(i)}(t) - \omega_i^{-1}(t) = \varepsilon_i(t) \tag{22}$$

where the white noise perturbation $\varepsilon_i(t)$ for each period was added to account for any internal uncertainty in the time span $\theta^{(i)}(n) - \theta^{(i)}(n-1)$. This white noise is independent from the instantaneous inverse activity $\omega_i^{-1}(t)$. We assume these $\varepsilon_i(t)$ to have variance σ^2, so that equation (22) writes:

$$\frac{d}{dt}\theta^{(i)}(t) - G\left(\theta^{(i)}(t), J^{(i)}\left(\theta^{(i)}(t)\right)\right) = \varepsilon_i(t) \tag{23}$$

The $\omega_i(t)$ are computed by considering the overall current which, using the discrete time notation, is given by:

$$\hat{J}^{(i)}((n-1)) = J^{(i)}((n-1)) + \frac{\kappa}{N}\sum_{j,m}\frac{\omega_j(m)}{\omega_i(n-1)}\delta\left(\theta^{(i)}(n-1) - \theta^{(j)}(m) - \frac{|Z_i - Z_j|}{c}\right)T_{ij}((n-1, Z_i),(m, Z_j)) \tag{24}$$

The quantity $J^{(i)}((n-1))$ denotes an external current. The term inside the sum is the average current sent to i by neuron j during the short time span $\theta^{(i)}(n) - \theta^{(i)}(n-1)$. The function $T_{ij}((n-1, Z_i),(m, Z_j))$ is the connectivity function between cells j and i. It measures the level of connectivity between i and j. If we consider $T_{ij}((n-1, Z_i),(m, Z_j))$ as exogenous, we may assume that (see [**?**]):

$$T_{ij}((n-1, Z_i),(m, Z_j)) = T((n-1, Z_i),(m, Z_j))$$

so that the connectivity function of Z_j on Z_i only depends on positions and times. It models the connectivity function as an average connectivity between local zones of the thread. this transfer function is typically considered as gaussian or decreasing exponentially with the distance between neurons, so that the closer the cells, the more connected they are.

However, in this paper, the connectivity function is a dynamical object whose dynamic equations are described in the next paragraph.

We can justify the other terms arising in (24): given the distance $|Z_i - Z_j|$ between the two cells and the signals' velocity c, signals arrive with a delay $\frac{|Z_i - Z_j|}{c}$. The spike emitted by cell j at time $\theta^{(j)}(m)$ has thus to satisfy:

$$\theta^{(i)}(n-1) < \theta^{(j)}(m) + \frac{|Z_i - Z_j|}{c} < \theta^{(i)}(n)$$

to reach cell i during the timespan $\left[\theta^{(i)}(n-1), \theta^{(i)}(n)\right]$. This relation must be represented by a step function in the current formula. However given our approximations, this can be replaced by:

$$\delta\left(\theta^{(i)}(n-1) - \theta^{(j)}(m) - \frac{|Z_i - Z_j|}{c}\right)$$

as in (24). However, this Dirac function must be weighted by the number of spikes emitted during the rise of the potential. This number is the ratio $\frac{\omega_j(m)}{\omega_i(n-1)}$ that counts the number of spikes emitted by neuron j towards neuron i between the spikes $n-1$ and n of neuron i. Again, this is valid for relatively small timespans between two spikes. For larger timespans, the firing rates should be replaced by their average over this period of time.

The sum over m and i is the overall contribution to the current from any possible spike of the thread, provided it arrives at i during the interval $\theta^{(i)}(n) - \theta^{(i)}(n-1)$ considered. Note that the current (24) is partly an endogenous variable. It depends on signals external to i, but depends also on i through $\omega_i(n-1)$. This is a consequence of the intrication between the system's elements.

In the sequel, we will work in the continuous approximation, so that formula (24) is replaced by:

$$\hat{J}^{(i)}(t) = J^{(i)}(t) + \frac{\kappa}{N} \int \sum_j \frac{\omega_j(s)}{\omega_i(t)} \delta\left(\theta^{(i)}(t) - \theta^{(j)}(s) - \frac{|Z_i - Z_j|}{c}\right) T_{ij}((t, Z_i), (s, Z_j))\, ds \tag{25}$$

Formula (25) shows that the dynamic equation (22) has to be coupled with the activity equation:

$$\begin{aligned}\omega_i(t) &= G\left(\theta^{(i)}(t), \hat{J}\left(\theta^{(i)}(t)\right)\right) + \upsilon_i(t) \\ &= \frac{\sqrt{\hat{J}^{(i)}(t)}}{\arctan\left(\left(\frac{1}{X_r} - \frac{1}{X_p}\right)\sqrt{\hat{J}^{(i)}(t)}\right)} + \upsilon_i(t)\end{aligned} \tag{26}$$

and $J^{(i)}(t)$ is defined by (25). A white noise $\upsilon_i(t)$ accounts for the possible deviations from this relation, due to some internal or external causes for each cell. We assume that the variances of $\upsilon_i(t)$ are constant, and equal to η^2, such that $\eta^2 << \sigma^2$.

4.2 Connectivity functions dynamics

We describe the dynamics for the connectivity functions $T_{ij}((n-1, Z_i), (m, Z_j))$ between cells. To do so we adapt the description of ([53]) to our context. In this work, the connectivity functions depend on some intermediate variables and do not present any space index. The connectivity between neurons i and j satisifies a differential equation:

$$\frac{dT_{ij}}{dt} = -\frac{T_{ij}(t)}{\tau} + \lambda \hat{T}_{ij}(t) \sum_l \delta\left(t - \Delta t_{ij} - t_j^l\right) \tag{27}$$

where $\hat{T}_{ij}(t)$ represents the variation in connectivity, due to the synaptic interactions between the two neurons. The delay Δt_{ij} is the time of arrival at neuton i for a spike of neuron j. The time t_j^l accounts for time of neuron j spikes. The sum:

$$\sum_l \delta\left(t - \Delta t_{ij} - t_j^l\right)$$

counts the number of spikes emitted by neuron j and arriving at time t at neuron i.

The variation in connectivity satisfies itself an equation:

$$\frac{d\hat{T}_{ij}}{dt} = \rho\left(1 - \hat{T}_{ij}(t)\right) C_{ij}(t) \sum_k \delta\left(t - t_i^k\right) - \hat{T}_{ij}(t) D_i(t) \sum_l \delta\left(t - \Delta t_{ij} - t_j^l\right) \tag{28}$$

where $C_{ij}(t)$ and $D_i(t)$ measure the cumulated postsynaptic and presynaptic activity. The sum:

$$\sum_k \delta\left(t - t_i^k\right)$$

counts the number of spikes emitted at time t. Quantities $C_{ij}(t)$ and $D_i(t)$ follow the dynamics:

$$\frac{dC_{ij}}{dt} = -\frac{C_{ij}(t)}{\tau_C} + \alpha_C\left(1 - C_{ij}(t)\right)\sum_l \delta\left(t - \Delta t_{ij} - t_j^l\right) \tag{29}$$

$$\frac{dD_i}{dt} = -\frac{D_i(t)}{\tau_D} + \alpha_C\left(1 - D_i(t)\right)\sum_k \delta\left(t - t_i^k\right) \tag{30}$$

To translate these equations in our set up, we have to consider connectivity functions of the form:

$$T_{ij}\left(\left(n_i, Z_i\right), \left(n_j, Z_j\right)\right)$$

that include the positions of neurons i and j and the parameter n_i and n_j which are our counting variables of neurons spikes. However, equations (27), (28), (29), (30) include a time variable.

In our formalism, the time variable $\theta^{(i)}(n_i)$ is the time at which neuron i produces its n_i-th spike. We should write classical equations depending on these variables.

Moreover, the number of spikes $\sum_l \delta\left(t - \Delta t_{ij} - t_j^l\right)$ emitted by cell j at time t_j^l and the number of spikes $\sum_k \delta\left(t - t_i^k\right)$ emitted by cell i at time t are proportional to $\delta\left(\theta^{(j)}(n_j) - (t - \Delta t_{ij})\right)\omega_j(n_j)$ and $\delta\left(\theta^{(i)}(n_i) - t\right)\omega_j(n_i)$ respectively. Given the introduction of a spatial indices, we have the relation:

$$\Delta t_{ij} = \frac{|Z_i - Z_j|}{c}$$

and the first δ function writes:

$$\delta\left(\theta^{(j)}(n_j) - (t - \Delta t_{ij})\right) = \delta\left(\theta^{(j)}(n_j) - \left(\theta^{(i)}(n_i) - \frac{|Z_i - Z_j|}{c}\right)\right)\delta\left(\theta^{(i)}(n_i) - t\right)$$

As a consequence, we will write first the connectivity functions from i to j as:

$$T\left(\left(Z_i, \theta^{(i)}(n_i), \omega_i(n_i)\right), \left(Z_j, \theta^{(j)}(n_j), \omega_j(n_j)\right)\right)$$

This function, together with the variation in connectivity:

$$\hat{T}\left(\left(Z_i, \theta^{(i)}(n_i), \omega_i(n_i)\right), \left(Z_j, \theta^{(j)}(n_j), \omega_j(n_j)\right)\right)$$

along with the auxiliary variables:

$$C\left(\left(Z_i, \theta^{(i)}(n_i), \omega_i(n_i)\right), \left(Z_j, \theta^{(j)}(n_j), \omega_j(n_j)\right)\right)$$

and:

$$D\left(\left(Z_i, \theta^{(i)}(n_i), \omega_i(n_i)\right)\right)$$

satisfy the following translations of equations (27), (28), (29), (30):

$$\begin{aligned}
& \nabla_{\theta^{(i)}(n_i)} T\left(\left(Z_i, \theta^{(i)}(n_i), \omega_i(n_i)\right), \left(Z_j, \theta^{(j)}(n_j), \omega_j(n_j)\right)\right) \\
= & -\frac{1}{\tau} T\left(\left(Z_i, \theta^{(i)}(n_i), \omega_i(n_i)\right), \left(Z_j, \theta^{(j)}(n_j), \omega_j(n_j)\right)\right) \\
& +\lambda\left(\hat{T}\left(\left(Z_i, \theta^{(i)}(n_i), \omega_i(n_i)\right), \left(Z_j, \theta^{(j)}(n_j), \omega_j(n_j)\right)\right)\right)\delta\left(\theta^{(i)}(n_i) - \theta^{(j)}(n_j) - \frac{|Z_i - Z_j|}{c}\right)
\end{aligned} \tag{31}$$

where $\hat{T}$ measures the variation of T due to the signals send from j to i and the signals emitted by i. It satisfies the following equation:

$$\begin{aligned}
&\nabla_{\theta^{(i)}(n_i)}\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right) \qquad (32)\\
=\;& \rho\delta\left(\theta^{(i)}(n_i)-\theta^{(j)}(n_j)-\frac{|Z_i-Z_j|}{c}\right)\\
&\times\left\{\left(h(Z,Z_1)-\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right)\right)C\left(\theta^{(i)}(n)\right)h_C\left(\omega_i(n_i)\right)\right.\\
&\left.-D\left(\theta^{(i)}(n)\right)\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right)h_D\left(\omega_j(n_j)\right)\right\}
\end{aligned}$$

where h_C and h_D are increasing functions. In the set of equations (27), (28), (29), (30):

$$\begin{aligned}
h_C\left(\omega_i(n_i)\right) &= \omega_i(n_i)\\
h_D\left(\omega_j(n_j)\right) &= \omega_j(n_j)
\end{aligned}$$

We depart slightly from ([53]) by the introduction of the function $h(Z,Z_1)$ (they choose $h(Z,Z_1)=1$), to implement some loss due to the distance. We may choose for example:

$$h(Z,Z_1)=\exp\left(-\frac{|Z_i-Z_j|}{\nu c}\right)$$

where ν is a parameter. Equation (32) involves two dynamic factors $C\left(\theta^{(i)}(n-1)\right)$ and $D\left(\theta_i(n-1)\right)$. The factor $C\left(\theta^{(i)}(n-1)\right)$ describes the accumulation of input spikes. It is solution of the differential equation:

$$\begin{aligned}
\nabla_{\theta^{(i)}(n-1)}C\left(\theta^{(i)}(n-1)\right) &= -\frac{C\left(\theta^{(i)}(n-1)\right)}{\tau_C} \qquad (33)\\
&\quad+\alpha_C\left(1-C\left(\theta^{(i)}(n-1)\right)\right)\omega_j\left(\theta^{(i)}(n-1)-\frac{|Z_i-Z_j|}{c}\right)
\end{aligned}$$

The term $D\left(\theta_i(n-1)\right)$ is proportional to the accumulation of output spikes and is solution of:

$$\nabla_{\theta^{(i)}(n-1)}D\left(\theta^{(i)}(n-1)\right)=-\frac{D\left(\theta^{(i)}(n-1)\right)}{\tau_D}+\alpha_D\left(1-D\left(\theta^{(i)}(n-1)\right)\right)\omega_i(n_i) \qquad (34)$$

For the purpose of field translation, we have to change the variables in the derivatives by the counting variable n_i and replace $\nabla_{\theta^{(i)}(n_i)}\simeq\omega_i(n_i)\nabla_{n_i}$ in the previous dynamics equations. We thus rewrite the dynamic equations in the following form:

For the connectivity T:

$$\begin{aligned}
&\nabla_{n_i}T\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right) \qquad (35)\\
=\;& -\frac{1}{\tau\omega_i(n_i)}T\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right)\\
&+\frac{\lambda}{\omega_i(n_i)}\left(\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right)\right)\delta\left(\theta^{(i)}(n_i)-\theta^{(j)}(n_j)-\frac{|Z_i-Z_j|}{c}\right)
\end{aligned}$$

For the variation in connectivity $\hat{T}$:

$$\begin{aligned}
&\nabla_{n_i}\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right) && (36)\\
=\;& \frac{\rho}{\omega_i(n_i)}\delta\left(\theta^{(i)}(n_i)-\theta^{(j)}(n_j)-\frac{|Z_i-Z_j|}{c}\right)\\
&\times\left\{\left(h(Z,Z_1)-\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right)\right)C\left(\theta^{(i)}(n-1)\right)h_C(\omega_i(n_i))\right.\\
&\left.-D\left(\theta^{(i)}(n-1)\right)\hat{T}\left(\left(Z_i,\theta^{(i)}(n_i),\omega_i(n_i)\right),\left(Z_j,\theta^{(j)}(n_j),\omega_j(n_j)\right)\right)h_D(\omega_j(n_j))\right\}
\end{aligned}$$

and for the auxiliary variables C and D:

$$\begin{aligned}
\nabla_{n_i}C\left(\theta^{(i)}(n-1)\right) &= -\frac{C\left(\theta^{(i)}(n-1)\right)}{\tau_C\omega_i(n_i)} && (37)\\
&+\alpha_C\left(1-C\left(\theta^{(i)}(n-1)\right)\right)\frac{\omega_j\left(Z_j,\theta^{(i)}(n-1)-\frac{|Z_i-Z_j|}{c}\right)}{\omega_i(n_i)}
\end{aligned}$$

$$\nabla_{n_i}D\left(\theta^{(i)}(n-1)\right)=-\frac{D\left(\theta^{(i)}(n-1)\right)}{\tau_D\omega_i(n_i)}+\alpha_D\left(1-D\left(\theta^{(i)}(n-1)\right)\right) \quad (38)$$

Then, to describe the connectivity by a field, we have to describe the connectivity as a set of vectors depending of a set of double indices kl (replacing ij) and interacting with the activities, seen as independent variables indexed by i,j...

We thus describe connectivity by a set of matrices:

$$\left(T_{kl}(n_{kl}),\hat{T}_{kl}(n_{kl}),(Z_{kl}(n_{kl})=(Z_{k,},Z_l)),\theta^{(kl)}(n_{kl}),\omega_k(n_{kl}),\omega_l'(n_{kl}),C_{kl}(n_{kl}),D_k(n_{kl})\right)$$

where n_{kl} is an internal parameter given by the average counting variable for cells or synapses firing simultaneously at point $Z_{k,}$.

Then, we replace the description (35), (36), (37), (38) by a set of equations in which connectivities $T_{kl}(n_{kl})$ interact with all pairs of neurons at points $Z_{k,}$ and Z_l whose average firing rates at time $\theta^{(kl)}(n_{kl})$ and $\theta^{(kl)}(n_{kl})-\frac{|Z_k-Z_l|}{c}$ are given by $\omega_k(n_{kl}),\omega_l'(n_{kl})$ respectively. As a consequence, we replace the notion of connectivity $T_{ij}((n-1,Z_i),(m,Z_j))$ between two specific neurons i and j by the average connectivity between the two sets of neurons with identical activities at each extremity of the segment (Z_i,Z_j) This approximation is justified if we consider that neurons located at the same place and firing at the same rate can be considered as closely connected and in average identical. Alternatively, this can also be justified if we consider one neuron per spatial location and assume each neuron as a complex entity sending several signals simultaneously. Under this hypothesis, the average considered are taken over the multiple activities of the same neuron[10].

Stated mathematically, the variable n_{kl} is replacd by an average $n_{kl}=\bar{n}_i$ at a given time $\theta^{(kl)}$ and we assume that in average, connectivity variable $T_{kl}(n_{kl})$ interacts with all neurons pairs located at $(Z_{k,},Z_l)$ at times $\theta^{(i)}(n_i)=\theta^{(kl)}(n_{kl})$. Writing $\bar{\omega}(Z_i,n_i)$ for the average activity, we impose $\bar{\omega}(Z_i,n_i)=\omega_k(n_{kl})$ and $\bar{\omega}(Z_j,n_j)=\omega_l'(n_{kl})$ and $\theta^{(j)}(n_j)=\theta^{(kl)}(n_{kl})-\frac{|Z_k-Z_l|}{c}$ respectively. The densities $T_{kl}(n_{kl})$ are thus the set of all connections between points $Z_{k,}$ and Z_l between sets of synchronized neurons at Z_k and synchronized neurons at Z_l, i.e. between set of neurons at this points or alternatively between multiple synapses for one or a few number of cells. In this point of view, we replace $\nabla_{\theta^{(i)}(n_i)}\simeq\omega_i(n_i)\nabla_{n_i}$ by:

$$\nabla_{\theta^{(kl)}(n_{kl})}\simeq\frac{\partial n_{kl}}{\partial\theta^{(kl)}(n_{kl})}\nabla_{n_{kl}}=\bar{\omega}(Z_i,n_i)\nabla_{n_{kl}}$$

[10] See section 5.2.1 for more details about these alternative interpretations.

As a consequence, the dynamic equations (35), (36), (37), (38) are replaced by:

$$\nabla_{n_{kl}} T_{kl}(n_{kl}) = \left(-\sum_{i,n_i} \frac{1}{\tau \bar{\omega}(Z_i, n_i)} T_{kl}(n_{kl}) + \frac{\lambda}{\bar{\omega}(Z_i, n_i)} \hat{T}_{kl}(n_{kl})\right) \times \delta\left(\theta^{(i)}(n_i) - \theta^{(kl)}(n_{kl})\right) \delta(Z_k - Z_i) \delta(\omega_k(n_{kl}) - \bar{\omega}(Z_i, n_i)) \tag{39}$$

$$\begin{aligned} &\nabla_{n_{kl}} \hat{T}(n_{kl}) \\ = &\left(\sum_{i,n_i} \left(h(Z_k, Z_l) - \hat{T}(n_{kl})\right) C_{kl}(n_{kl}) h_C(\omega_i(n_i)) - \sum_{j,n_j} D_k(n_{kl}) \hat{T}(n_{kl}) h_D(\omega_j(n_j))\right) \\ &\times \frac{\rho}{\bar{\omega}(Z_i, n_i)} \delta\left(\theta^{(i)}(n_i) - \theta^{(j)}(n_j) - \frac{|Z_i - Z_j|}{c}\right) \delta\left(\theta^{(i)}(n_i) - \theta^{(kl)}(n_{kl})\right) \\ &\times \delta((Z_{k,}, Z_l) - (Z_{i,}, Z_j)) \delta(\omega_k(n_{kl}) - \bar{\omega}(Z_i, n_i)) \delta(\omega_l(n_{kl}) - \bar{\omega}(Z_j, n_j)) \end{aligned} \tag{40}$$

$$\begin{aligned} \nabla_{n_{kl}} C(n_{kl}) = &\left(-\frac{C(n_{kl})}{\tau_C \bar{\omega}(Z_i, n_i)} + \sum_{j,n_j} \alpha_C (1 - C_{kl}(n_{kl})) \frac{\omega_j(n_j)}{\bar{\omega}(Z_i, n_i)}\right) \\ &\times \delta\left(\theta^{(i)}(n_i) - \theta^{(j)}(n_j) - \frac{|Z_i - Z_j|}{c}\right) \delta\left(\theta^{(i)}(n_i) - \theta^{(kl)}(n_{kl})\right) \delta((Z_{k,}, Z_l) - (Z_{i,}, Z_j)) \\ &\times \delta(\omega_k(n_{kl}) - \bar{\omega}(Z_i, n_i)) \delta(\omega_l(n_{kl}) - \bar{\omega}(Z_j, n_j)) \end{aligned} \tag{41}$$

$$\begin{aligned} \nabla_{n_{kl}} D_k(n_{kl}) = &\left(-\frac{D_k(n_{kl})}{\tau_D \bar{\omega}(Z_i, n_i)} + \frac{1}{\bar{\omega}(Z_i, n_i)} \sum_{i,n_i} \alpha_D (1 - D_k(n_{kl})) \omega_i(n_i)\right) \\ &\times \delta\left(\theta^{(i)}(n_i) - \theta^{(kl)}(n_{kl})\right) \delta(Z_k - Z_i) \delta(\omega_k(n_{kl}) - \bar{\omega}(Z_i, n_i)) \delta(\omega_l(n_{kl}) - \bar{\omega}(Z_j, n_j)) \end{aligned} \tag{42}$$

Similarly, note that we can also rewrite the currents equation (24) as:

$$\hat{J}^{(i)}((n-1)) = J^{(i)}((n-1)) + \frac{\kappa}{N} \sum_{j,m} \frac{\omega_j(m)}{\omega_i(n-1)} \delta\left(\theta^{(i)}(n-1) - \theta^{(j)}(m) - \frac{|Z_i - Z_j|}{c}\right) T_{ij}((n-1, Z_i), (m, Z_j))$$

with:

$$T_{ij}((n_i, Z_i), (m_j, Z_j)) = \sum_{kl} T_{kl}(n_{kl}) \delta\left(\theta^{(i)}(n_i) - \theta^{(kl)}(n_{kl})\right) \delta(\omega_k(n_{kl}) - \bar{\omega}(Z_i, n_i)) \delta(\omega_l(n_{kl}) - \bar{\omega}(Z_j, n_j)) \tag{43}$$

4.3 Probability density for the system

4.3.1 Individual neurons

Due to the stochastic nature of equations (23) and (26), the dynamics of a single neuron can be described by the probability density $P\left(\theta^{(i)}(t), \omega_i^{-1}(t)\right)$ for a path $\left(\theta^{(i)}(t), \omega_i^{-1}(t)\right)$ which is given by, up to a normalization factor:

$$P\left(\theta^{(i)}(t), \omega_i^{-1}(t)\right) = \exp(-A_i) \tag{44}$$

where:

$$A_i = \frac{1}{\sigma^2} \int \left(\frac{d}{dt}\theta^{(i)}(t) - \omega_i^{-1}(t) \right)^2 dt + \int \frac{\left(\omega_i^{-1}(t) - G\left(\theta^{(i)}(t), \hat{J}\left(\theta^{(i)}(t) \right) \right) \right)^2}{\eta^2} dt \tag{45}$$

(see [48] and [49]). The integral is taken over a time period that depends on the time scale of the interactions. Actually, the minimization of (45) imposes both (22) and (26), so that the probability density is, as expected, centered around these two conditions, i.e. (22) and (26) are satisfied in mean. A probability density for the whole system of neurons is obtained by summing S_i over all agents. We thus define the statistical weight for the cells:

$$P\left(\left(\theta^{(i)}(t), \omega_i^{-1}(t), Z_i \right)_{i=1\dots N} \right) = \exp(-A) \tag{46}$$

with:

$$A = \sum_i A_i = \sum_i \frac{1}{\sigma^2} \int \left(\frac{d}{dt}\theta^{(i)}(t) - \omega_i^{-1}(t) \right)^2 dt + \int \frac{\left(\omega_i^{-1}(t) - G\left(\theta^{(i)}(t), \hat{J}\left(\theta^{(i)}(t) \right) \right) \right)^2}{\eta^2} dt \tag{47}$$

and (using (43)):

$$\begin{aligned} \hat{J}^{(i)}((n-1)) &= J^{(i)}((n-1)) + \frac{\kappa}{N} \sum_{j,m} \frac{\omega_j(m)}{\omega_i(n-1)} \delta\left(\theta^{(i)}(n-1) - \theta^{(j)}(m) - \frac{|Z_i - Z_j|}{c} \right) T_{ij}((n-1, Z_i), (m, Z_j)) \\ &\times \sum_{kl} T_{kl}(n_{kl}) \delta\left(\theta^{(i)}(n_i) - \theta^{(kl)}(n_{kl}) \right) \delta\left(\omega_k(n_{kl}) - \omega_i(n_i) \right) \delta\left(\omega_l(n_{kl}) - \omega_j(m) \right) \end{aligned}$$

4.3.2 Connectivity functions

The statistical exponent associated to the connectivity functions is obtained as in the previous paragraph. We obtain the statistical weight:

$$\prod_{k,l} P\left(T_{kl}(n_{kl}), \hat{T}_{kl}(n_{kl}), (Z_{kl}(n_{kl}) = (Z_{k,}, Z_l)), \theta^{(kl)}(n_{kl}), \omega_k(n_{kl}), \omega_l'(n_{kl}), C_{kl}(n_{kl}), D_k(n_{kl}) \right) = \exp(-B)$$

where:

$$\begin{aligned} B &= \sum_{kl} \left(\nabla_{n_{kl}} T_{kl}(n_{kl}) - B_{kl}^{(1)} \right)^2 + \left(\nabla_{n_{kl}} \hat{T}(n_{kl}) - B_{kl}^{(2)} \nabla_{n_{kl}} \right)^2 \\ &\quad + \left(C(n_{kl}) - B_{kl}^{(3)} \right)^2 + \left(\nabla_{n_{kl}} D_k(n_{kl}) - B_k \right)^2 \end{aligned} \tag{48}$$

and:

$$\begin{aligned} B_{kl}^{(1)} &= \left(-\sum_{i,n_i} \frac{1}{\tau \bar{\omega}(Z_i, n_i)} T_{kl}(n_{kl}) + \frac{\lambda}{\bar{\omega}(Z_i, n_i)} \hat{T}_{kl}(n_{kl}) \right) \\ &\times \delta\left(\theta^{(i)}(n_i) - \theta^{(kl)}(n_{kl}) \right) \delta(Z_k - Z_i) \delta\left(\omega_k(n_{kl}) - \bar{\omega}(Z_i, n_i) \right) \end{aligned} \tag{49}$$

$$\begin{aligned} B_{kl}^{(2)} &= \left(\sum_{i,n_i} \left(h(Z_k, Z_l) - \hat{T}(n_{kl}) \right) C_{kl}(n_{kl}) h_C(\omega_i(n_i)) - \sum_{j,n_j} D_k(n_{kl}) \hat{T}(n_{kl}) h_D(\omega_j(n_j)) \right) \\ &\times \frac{\rho}{\bar{\omega}(Z_i, n_i)} \delta\left(\theta^{(i)}(n_i) - \theta^{(j)}(n_j) - \frac{|Z_i - Z_j|}{c} \right) \delta\left(\theta^{(i)}(n_i) - \theta^{(kl)}(n_{kl}) \right) \\ &\times \delta\left((Z_{k,}, Z_l) - (Z_{i,}, Z_j) \right) \delta\left(\omega_k(n_{kl}) - \bar{\omega}(Z_i, n_i) \right) \delta\left(\omega_l(n_{kl}) - \bar{\omega}(Z_j, n_j) \right) \end{aligned} \tag{50}$$

$$B_{kl}^{(3)} = \left(-\frac{C(n_{kl})}{\tau_C \bar{\omega}(Z_i,n_i)} + \sum_{j,n_j} \alpha_C (1 - C_{kl}(n_{kl})) \frac{\omega_j(n_j)}{\bar{\omega}(Z_i,n_i)}\right) \tag{51}$$
$$\times\delta\left(\theta^{(i)}(n_i) - \theta^{(j)}(n_j) - \frac{|Z_i - Z_j|}{c}\right)\delta\left(\theta^{(i)}(n_i) - \theta^{(kl)}(n_{kl})\right)\delta\left((Z_k,,Z_l) - (Z_i,,Z_j)\right)$$
$$\times\delta\left(\omega_k(n_{kl}) - \bar{\omega}(Z_i,n_i)\right)\delta\left(\omega_l(n_{kl}) - \bar{\omega}(Z_j,n_j)\right)$$

$$B_k = \left(-\frac{D_k(n_{kl})}{\tau_D \bar{\omega}(Z_i,n_i)} + \frac{1}{\bar{\omega}(Z_i,n_i)} \sum_{i,n_i} \alpha_D (1 - D_k(n_{kl}))\,\omega_i(n_i)\right) \tag{52}$$
$$\times\delta\left(\theta^{(i)}(n_i) - \theta^{(kl)}(n_{kl})\right)\delta(Z_k - Z_i)\,\delta\left(\omega_k(n_{kl}) - \bar{\omega}(Z_i,n_i)\right)\delta\left(\omega_l(n_{kl}) - \bar{\omega}(Z_j,n_j)\right)$$

4.3.3 Probability density for the full system

The probability for the full system is obtained by the product:

$$\prod_{k,l} P\left(T_{kl}(n_{kl}), \hat{T}_{kl}(n_{kl}), (Z_{kl}(n_{kl}) = (Z_k,,Z_l)), \theta^{(kl)}(n_{kl}), \omega_k(n_{kl}), \omega'_l(n_{kl}), C_{kl}(n_{kl}), D_k(n_{kl})\right) \tag{53}$$
$$\times P\left(\left(\theta^{(i)}(t), \omega_i^{-1}(t), Z_i\right)_{i=1\dots N}\right)$$
$$= \exp(-B)\exp(-A)$$

5 Field theoretic description of the system

5.1 Translation of formula (53) in terms of field theory

In our context two fields are necessary. The field representing the set of neurons depends on the three variables (θ, Z, ω), and is denoted $\Psi(\theta, Z, \omega)$. The connectivity functions are characterized by the set of variables $\left(T, \hat{T}, \omega, \omega', \theta, Z, Z', C, D\right)$ and represented by the field $\Gamma\left(T, \hat{T}, \omega, \omega', \theta, Z, Z', C, D\right)$.We provide an interpretation of the various fields at the end of this paragraph.

5.1.1 Translation of (47)

The dynamics of neurons is described by an action functional for the field $\Psi(\theta, Z, \omega)$ and its associated partition function. This partition function captures both collective and individual aspects of the system, enabling the retrieval of correlation functions for number of neurons.

The field theoretic version of (45) is obtained using (47). The correspondence detailed in [48][49]) yields an action $S(\Psi)$ for a field $\Psi(\theta, Z, \omega)$ and a statistical weight $\exp(-(S(\Psi)))$ for each configuration $\Psi(\theta, Z, \omega)$ of this field. The functional $S(\Psi)$ is decomposed in two parts corresponding to the two contributions in (47).

The first term of (47):

$$\frac{1}{\sigma^2}\int\left(\frac{d}{dt}\theta^{(i)}(t) - \omega_i^{-1}(t)\right)^2 dt \tag{54}$$

is a term with temporal derivative. Its form is simple since the function $f^{(\alpha)}$ in (16) depends only on the variable $\mathbf{X}_i(t) = \left(\theta^{(i)}(t), \omega_i^{-1}(t), Z_i\right)$. Actually $f^{(\theta)}(\mathbf{X}_i(t)) = \omega_i^{-1}(t)$. Using (17), the term (54) is thus replaced by the corresponding quadratic functional in field theory:

$$-\frac{1}{2}\Psi^\dagger(\theta, Z, \omega)\nabla\left(\frac{\sigma^2}{2}\nabla - \omega^{-1}\right)\Psi(\theta, Z, \omega) \tag{55}$$

where σ^2 is the variance of the errors ε_i.

The field functional that corresponds to the second term of (45):

$$V=\int\frac{\left(\omega_i^{-1}(t)-G\left(\theta^{(i)}(t),\hat{J}\left(\theta^{(i)}(t)\right)\right)\right)^2}{\eta^2}dt$$

is obtained by expanding the formula (25) for the current induced by all j:

$$\begin{aligned}V &= \frac{1}{2\eta^2}\int dt\sum_i\left(\omega_i^{-1}(t)\right. \qquad (56)\\ &\left.-G\left(J\left(\theta^{(i)}(t),Z_i\right)+\frac{\kappa}{N}\int ds\sum_j\frac{\omega_j(s)T_{ij}((t,Z_i),s,Z_j)}{\omega_i(t)}\delta\left(\theta^{(i)}(t)-\theta^{(j)}(s)-\frac{|Z_i-Z_j|}{c}\right)\right)\right)^2\end{aligned}$$

with $\eta << 1$, which is the constraint (26) imposed stochastically. Its corresponding potential in field theory is obtained straightforwardly by using the translation (11):

$$\frac{1}{2\eta^2}\int|\Psi(\theta,Z,\omega)|^2\left(\omega^{-1}-G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega_1T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega}\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1,\omega_1\right)\right|^2dZ_1d\omega_1\right)\right)^2 \qquad (57)$$

and $T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)$ is obtained by the translation of the term (43):

$$\begin{aligned}&\sum_{kl}T_{kl}(n_{kl})\delta\left(\theta^{(i)}(n_i)-\theta^{(kl)}(n_{kl})\right)\delta(\omega_k(n_{kl})-\omega_i(n_i))\delta(\omega_l(n_{kl})-\omega_j(m))\\ \rightarrow &\int T\left|\Gamma\left(T,\hat{T},\hat{\omega},\hat{\omega}',\hat{\theta},\hat{Z},\hat{Z}',C,D\right)\right|^2\delta\left(\theta-\hat{\theta}\right)\delta(\hat{\omega}-\omega)\delta(\hat{\omega}-\omega_1)\delta\left(\left(\hat{Z},\hat{Z}'\right)-(Z,Z_1)\right)\\ = &\int T\left|\Gamma\left(T,\hat{T},\omega,\omega_1,\theta,Z,Z_1,C,D\right)\right|^2dTd\hat{T}dCdD\equiv T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)\end{aligned}$$

To simplify, we will write in the sequel:

$$T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)=\int T\left|\Gamma\left(T,\hat{T},\omega,\omega_1,\theta,Z,Z_1,C,D\right)\right|^2dTd\hat{T}dCdD\equiv T(Z,\theta,Z_1) \qquad (58)$$

which represents the average connectivity between points Z and Z_1 in state $\Gamma\left(T,\hat{T},\omega,\omega_1,\theta,Z,Z_1,C,D\right)$.

The field action is then the sum of (55) and (57):

$$\begin{aligned}S &= -\frac{1}{2}\Psi^\dagger(\theta,Z,\omega)\nabla\left(\frac{\sigma_\theta^2}{2}\nabla-\omega^{-1}\right)\Psi(\theta,Z,\omega) \qquad (59)\\ &+\frac{1}{2\eta^2}\int|\Psi(\theta,Z,\omega)|^2\left(\omega^{-1}-G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega_1}{\omega}\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1,\omega_1\right)\right|^2T(Z,\theta,Z_1)dZ_1d\omega_1\right)\right)^2\end{aligned}$$

5.1.2 Translation for connectivity dynamics (48)

The translation of the four action terms describing the connectivity dynamics (49), (50), (51) and (52) in (48) is straightforward. We obtain four contributions:

$$S_\Gamma^{(1)}=\int\Gamma^\dagger\left(T,\hat{T},\omega_\Gamma,\omega'_\Gamma,\theta,Z,Z',C,D\right)\nabla_T\left(\frac{\sigma_T^2}{2}\nabla_T+O_T^\omega\right)\Gamma\left(T,\hat{T},\omega_\Gamma,\omega'_\Gamma,\theta,Z,Z',C,D\right) \qquad (60)$$

$$S_{\Gamma}^{(2)} = \int \Gamma^{\dagger}\left(T, \hat{T}, \omega_{\Gamma}, \omega_{\Gamma}', \theta, Z, Z', C, D\right) \nabla_{\hat{T}} \left(\frac{\sigma_{\hat{T}}^2}{2} \nabla_{\hat{T}} + O_{\hat{T}}^{\omega}\right) \Gamma\left(T, \hat{T}, \omega_{\Gamma}, \omega_{\Gamma}', \theta, Z, Z', C, D\right) \tag{61}$$

$$S_{\Gamma}^{(3)} = \int \Gamma^{\dagger}\left(T, \hat{T}, \omega_{\Gamma}, \omega_{\Gamma}', \theta, Z, Z', C, D\right) \nabla_{C} \left(\frac{\sigma_{C}^2}{2} \nabla_{C} + O_{C}^{\omega}\right) \Gamma\left(T, \hat{T}, \omega_{\Gamma}, \omega_{\Gamma}', \theta, Z, Z', C, D\right) \tag{62}$$

$$S_{\Gamma}^{(4)} = \int \Gamma^{\dagger}\left(T, \hat{T}, \omega_{\Gamma}, \omega_{\Gamma}', \theta, Z, Z', C, D\right) \nabla_{D} \left(\frac{\sigma_{D}^2}{2} \nabla_{D} + O_{D}^{\omega}\right) \Gamma\left(T, \hat{T}, \omega_{\Gamma}, \omega_{\Gamma}', \theta, Z, Z', C, D\right) \tag{63}$$

with:

$$O_{C}^{\omega} = \left(\frac{C}{\tau_C \bar{\omega}} - \frac{\alpha_C (1-C) \int \omega' \left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z', \omega'\right)\right|^2 d\omega'}{\bar{\omega}}\right) \delta\left((\omega_{\Gamma}, \omega_{\Gamma}') - (\bar{\omega}, \bar{\omega}')\right) \tag{64}$$

$$\begin{aligned}
O_{D}^{\omega} &= \frac{D}{\tau_D \bar{\omega}} - \frac{\alpha_D (1-D) \int \omega \left|\Psi(\theta, Z, \omega)\right|^2 d\omega}{\bar{\omega}} \delta\left((\omega_{\Gamma}, \omega_{\Gamma}') - (\bar{\omega}, \bar{\omega}')\right) \\
O_{\hat{T}}^{\omega} &= -\frac{\rho}{\bar{\omega}} \left(\left(h(Z, Z') - \hat{T}\right) C \int \left|\Psi(\theta, Z, \omega)\right|^2 h_C(\omega)\, d\omega \right. \\
&\quad \left. - D\hat{T} \int \left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z', \omega'\right)\right|^2 h_D(\omega')\, d\omega' \right) \delta\left((\omega_{\Gamma}, \omega_{\Gamma}') - (\bar{\omega}, \bar{\omega}')\right) \\
O_{T}^{\omega} &= -\left(-\frac{1}{\tau \bar{\omega}} T + \frac{\lambda}{\bar{\omega}} \hat{T}\right) \delta\left((\omega_{\Gamma}, \omega_{\Gamma}') - (\bar{\omega}, \bar{\omega}')\right)
\end{aligned}$$

Here:

$$\begin{aligned}
\bar{\omega} &= \frac{\int \omega \left|\Psi(\theta, Z, \omega)\right|^2 d\omega}{\int \left|\Psi(\theta, Z, \omega)\right|^2 d\omega} \\
\bar{\omega}' &= \frac{\int \omega' \left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z', \omega'\right)\right|^2 d\omega'}{\int \left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z', \omega'\right)\right|^2 d\omega'}
\end{aligned}$$

5.2 Full action for the system

The full action for the system is obtained by gathering the different terms:

$$-\frac{1}{2} \int \Psi^{\dagger}(\theta, Z, \omega) \nabla \left(\frac{\sigma_{\theta}^2}{2} \nabla - \omega^{-1}\right) \Psi(\theta, Z, \omega) + \frac{1}{2\eta^2} \left(S_{\Gamma}^{(1)} + S_{\Gamma}^{(2)} + S_{\Gamma}^{(3)} + S_{\Gamma}^{(4)}\right) \tag{65}$$

with $S_{\Gamma}^{(1)}$, $S_{\Gamma}^{(2)}$, $S_{\Gamma}^{(3)}$, $S_{\Gamma}^{(4)}$ given by (60), (61), (62), (63).

5.2.1 Remark: interpretation of the various field

The action functional depends on two fields: $\Psi(\theta, Z, \omega)$ and $\Gamma\left(T, \hat{T}, \omega_{\Gamma}, \omega_{\Gamma}', \theta, Z, Z', C, D\right)$. These two abstract quantities will enable us to derive the dynamic state of the entire system and subsequently study transitions between different states. However, the squared modulus of the two functions can be interpreted in terms of statistical distribution, depending on the chosen description. If we consider a system of simple cells spread along the thread, the function $\left|\Psi(\theta, Z, \omega)\right|^2$ measures at time θ, the density of active cells at point Z with activity ω. In the perspective of complex cells with multiple axons and dendrites, we can consider that one cell stands at Z, and $\left|\Psi(\theta, Z, \omega)\right|^2$

measures for that cell the density of axons with activity ω. A similar interpretation works for $\Gamma\left(T,\hat{T},\omega_{\Gamma},\omega_{\Gamma}^{\prime},\theta,Z,Z^{\prime},C,D\right)$. In the perspective of system of simple cells "accumulated" in the neighborhood of Z, $\left|\Gamma\left(T,\hat{T},\omega_{\Gamma},\omega_{\Gamma}^{\prime},\theta,Z,Z^{\prime},C,D\right)\right|^{2}$ measures the density of connections of value T (and auxiliary variabls $\hat{T}$, C,D) between the set of cells located at points Z and $Z^{\prime}$ with activity ω_{Γ} and $\omega_{\Gamma}^{\prime}$. In the context of complex cells, it describes the density of connections with strength T between sets of axons nd dendrites of cells with activity $\omega_{\Gamma},\omega_{\Gamma}^{\prime}$.

5.3 Projection on dependent activity states and effective action:

We have shown in ([52]) that some simplifications arise in the action functional. Using the fact that $\eta^{2}<<1$, and noting that in this case, field configurations $\Psi\left(\theta,Z,\omega\right)$ such that:

$$\omega^{-1}-G\left(J\left(\theta,Z\right)+\int\frac{\kappa}{N}\frac{\omega_{1}}{\omega}\left|\Psi\left(\theta-\frac{\left|Z-Z_{1}\right|}{c},Z_{1},\omega_{1}\right)\right|^{2}T\left(Z,\theta,Z_{1}\right)dZ_{1}d\omega_{1}\right)\neq0$$

have negligible statistical weight, we can simplify (59) and restrict the fields to those of the form:

$$\Psi\left(\theta,Z\right)\delta\left(\omega^{-1}-\omega^{-1}\left(J,\theta,Z,\left|\Psi\right|^{2}\right)\right)\tag{66}$$

where $\omega^{-1}\left(J,\theta,Z,\Psi\right)$ satisfies:

$$\begin{aligned}\omega^{-1}\left(J,\theta,Z,\left|\Psi\right|^{2}\right) &= G\left(J\left(\theta,Z\right)+\int\frac{\kappa}{N}\frac{\omega_{1}T\left(Z,\theta,Z_{1},\theta-\frac{\left|Z-Z_{1}\right|}{c}\right)}{\omega\left(J,\theta,Z,\left|\Psi\right|^{2}\right)}\left|\Psi\left(\theta-\frac{\left|Z-Z_{1}\right|}{c},Z_{1},\omega_{1}\right)\right|^{2}dZ_{1}d\omega_{1}\right)\\
&= G\left(J\left(\theta,Z\right)+\int\frac{\kappa}{N}\frac{\omega_{1}T\left(Z,\theta,Z_{1},\theta-\frac{\left|Z-Z_{1}\right|}{c}\right)}{\omega\left(J,\theta,Z,\left|\Psi\right|^{2}\right)}\left|\Psi\left(\theta-\frac{\left|Z-Z_{1}\right|}{c},Z_{1}\right)\right|^{2}\right.\\
&\quad\left.\times\delta\left(\omega_{1}^{-1}-\omega^{-1}\left(J,\theta-\frac{\left|Z-Z_{1}\right|}{c},Z_{1},\left|\Psi\right|^{2}\right)\right)dZ_{1}d\omega_{1}\right)\end{aligned}$$

The last equation simplifies to yield:

$$\begin{aligned}&\omega^{-1}\left(J,\theta,Z,\left|\Psi\right|^{2}\right)\\
=\;&G\left(J\left(\theta,Z\right)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{\left|Z-Z_{1}\right|}{c},Z_{1},\Psi\right)T\left(Z,\theta,Z_{1},\theta-\frac{\left|Z-Z_{1}\right|}{c}\right)}{\omega\left(J,\theta,Z,\left|\Psi\right|^{2}\right)}\left|\Psi\left(\theta-\frac{\left|Z-Z_{1}\right|}{c},Z_{1}\right)\right|^{2}dZ_{1}\right)\end{aligned}\tag{67}$$

The configurations $\Psi\left(\theta,Z,\omega\right)$ that minimize the potential (57) can now be considered: the field $\Psi\left(\theta,Z,\omega\right)$ is projected on the subspace (66) of functions of two variables, and we can therefore replace in (57):

$$\omega\rightarrow\omega\left(J,\theta,Z,\left|\Psi\right|^{2}\right)\tag{68}$$

$$\omega^{\prime}\rightarrow\omega\left(J,\theta-\frac{\left|Z-Z^{\prime}\right|}{c},Z^{\prime},\left|\Psi\right|^{2}\right)\tag{69}$$

The "classical" effective action becomes (see appendix 1):

$$-\frac{1}{2}\Psi^{\dagger}\left(\theta,Z\right)\left(\nabla_{\theta}\left(\frac{\sigma^{2}}{2}\nabla_{\theta}-\omega^{-1}\left(J,\theta,Z,\left|\Psi\right|^{2}\right)\right)\right)\Psi\left(\theta,Z\right)\tag{70}$$

with $\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)$ given by equation (67). As in ([52]) we add to this action a stabilization potential $V(\Psi)$ ensuring an average activity of the system. The precise form of this potential is irrelevant here, but we assume that it has a minimum $\Psi_0(\theta,Z)$.

The projection on dependent activity also applies to connectivity action terms. We can thus replace $\Gamma\left(T,\hat{T},\omega_\Gamma,\omega'_\Gamma,\theta,Z,Z',C,D\right)$ by $\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)$ and the action becomes:

$$\begin{aligned} S_{full} &= -\frac{1}{2}\Psi^\dagger(\theta,Z,\omega)\nabla\left(\frac{\sigma_\theta^2}{2}\nabla-\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right)\Psi(\theta,Z)+V(\Psi) \\ &\quad+\frac{1}{2\eta^2}\left(S_\Gamma^{(0)}+S_\Gamma^{(1)}+S_\Gamma^{(2)}+S_\Gamma^{(3)}+S_\Gamma^{(4)}\right)+U\left(\left\{|\Gamma(\theta,Z,Z',C,D)|^2\right\}\right) \end{aligned} \tag{71}$$

with $S_\Gamma^{(1)}$, $S_\Gamma^{(2)}$, $S_\Gamma^{(3)}$, $S_\Gamma^{(4)}$ now given by:

$$S_\Gamma^{(1)}=\int\Gamma^\dagger\left(T,\hat{T},\theta,Z,Z',C,D\right)\nabla_T\left(\frac{\sigma_T^2}{2}\nabla_T+O_T\right)\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right) \tag{72}$$

$$S_\Gamma^{(2)}=\int\Gamma^\dagger\left(T,\hat{T},\theta,Z,Z',C,D\right)\nabla_{\hat{T}}\left(\frac{\sigma_{\hat{T}}^2}{2}\nabla_{\hat{T}}+O_{\hat{T}}\right)\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right) \tag{73}$$

$$S_\Gamma^{(3)}=\Gamma^\dagger\left(T,\hat{T},\theta,Z,Z',C,D\right)\nabla_C\left(\frac{\sigma_C^2}{2}\nabla_C+O_C\right)\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right) \tag{74}$$

$$S_\Gamma^{(4)}=\Gamma^\dagger\left(T,\hat{T},\theta,Z,Z',C,D\right)\nabla_D\left(\frac{\sigma_D^2}{2}\nabla_D+O_D\right)\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right) \tag{75}$$

where:

$$\begin{aligned} O_C &= \frac{C}{\tau_C\omega\left(J,\theta,Z,|\Psi|^2\right)}-\frac{\alpha_C(1-C)\,\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}{\omega\left(J,\theta,Z,|\Psi|^2\right)} \\ O_D &= \frac{D}{\tau_D\omega\left(J,\theta,Z,|\Psi|^2\right)}-\alpha_D(1-D)\,|\Psi(\theta,Z)|^2 \\ O_{\hat{T}} &= -\frac{\rho}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\left(\left(h(Z,Z')-\hat{T}\right)C\,|\Psi(\theta,Z)|^2\,h_C\left(\omega\left(J,\theta,Z,|\Psi|^2\right)\right)\right. \\ &\quad\left.-D\hat{T}\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2 h_D\left(\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)\right)\right) \\ O_T &= -\left(-\frac{1}{\tau\omega\left(J,\theta,Z,|\Psi|^2\right)}T+\frac{\lambda}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\hat{T}\right) \end{aligned} \tag{76}$$

In these equations, the averages $\bar{\omega}$ and $\bar{\omega}'$ have been replaced by $\omega\left(J,\theta,Z,|\Psi|^2\right)$ and $\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)$ as a consequence of the projection.

In (71), we added a potential:

$$U\left(\left\{|\Gamma(\theta,Z,Z',C,D)|^2\right\}\right)=U\left(\int T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2 dTd\hat{T}\right) \tag{77}$$

that models the constraint about the number of active connections in the system.

Part II. Structural aspects of the system. Background fields, averages

The following sections concentrate on solving the saddle-point equations for (71). Considering that the time scale for cell activity is shorter than that for connectivities, we solve in first approximation the saddle-point equation for the neuron field action, given the connectivity field. Subsequently, we calculate the background field for connectivities. This ultimately leads to the equilibrium equations for average connectivity variables and cell activities.

6 Background states equations for neuron field and activities depending on connectivty functions

Our goal is to find the possible background states of action (71). In principle we should minimize S_{full} both over the neurons field $\left(\Psi,\Psi^{\dagger}\right)$ and the connectivity field $\left(\Gamma,\Gamma^{\dagger}\right)$. However, the time scale of the neuron field is lower than that of the connectivity field. To obtain an effective action for $\left(\Gamma,\Gamma^{\dagger}\right)$, we intend to integrate over the degrees of freedom for the field Ψ in the partition function defined by S_{full} in (71). In first approximation, this corresponds to set $\left(\Psi,\Psi^{\dagger}\right)$ to its backround obtained through the minimization of the effective action, written $S_{\Psi}\left(\Psi,\Psi^{\dagger}\right)$. The series expansion of the effective action has been computed in ([52]). At the lowest order in perturbation, this corresponds to modify the action by a translation in $|\Psi|^2$:

$$S_{\Psi}\left(\Psi,\Psi^{\dagger}\right)=-\frac{1}{2}\Psi^{\dagger}\left(\theta,Z,\omega\right)\nabla\left(\frac{\sigma_{\theta}^{2}}{2}\nabla-\omega^{-1}\left(J,\theta,Z,\mathcal{G}_{0}+|\Psi|^{2}\right)\right)\Psi\left(\theta,Z\right)+V\left(\Psi\right)$$

where $\mathcal{G}_0$ is computed in ([52]), it is given by:

$$\mathcal{G}_{0}=\mathcal{G}_{0}\left(Z,Z\right)$$

where $\mathcal{G}_0\left(Z,Z\right)$ is the static Green function for the field Ψ. At the first order corrections, the activities defined classically by formula (67) are now defined by the equation:

$$\begin{aligned}&\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)\\=\;&G\left(J\left(\theta,Z\right)+\int\frac{\kappa}{N}\frac{\omega'T\left(Z,\theta,Z_{1},\theta-\frac{|Z-Z_{1}|}{c}\right)}{\omega}\left(\mathcal{G}_{0}+\left|\Psi\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1}\right)\right|^{2}\right)dZ_{1}\right)\end{aligned}\tag{78}$$

This equation depends on both on the external currents $J\left(\theta,Z\right)$ and the fields Ψ and Γ through the expression (58) of $T\left(Z,\theta,Z_{1},\theta-\frac{|Z-Z_{1}|}{c}\right)$. At the scale of activities, $T\left(Z,\theta,Z_{1},\theta-\frac{|Z-Z_{1}|}{c}\right)$ can be considered quasi static, allowing us to find quasi-static equilibria for $\omega^{-1}\left(J,\theta,Z,|\Psi|^{2}\right)$ and $\left|\Psi\left(\theta-\frac{|Z-Z_{1}|}{c},Z_{1}\right)\right|^{2}$ as functions of $T\left(Z,\theta,Z_{1}\right)$. This result will be reintroduced in the dynamics for the background field Γ.

The minimization equation for the background field becomes:

$$-\nabla\left(\frac{\sigma_{\theta}^{2}}{2}\nabla-\omega^{-1}\left(J,\theta,Z,\mathcal{G}_{0}+|\Psi|^{2}\right)\right)+\frac{\delta V\left(\Psi\right)}{\delta\Psi\left(\theta,Z\right)}=0\tag{79}$$

and the solutions of (79) are functions of the connectivities.

7 Background state equations for Γ under simplifying assumptions

The minimization equation of (71) for the connectivity functions background states have the form:

$$0=\frac{\delta}{\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)}\left(S_{\Gamma}^{(1)}+S_{\Gamma}^{(2)}+S_{\Gamma}^{(3)}+S_{\Gamma}^{(4)}+U\left(\left\{\left|\Gamma\left(\theta,Z,Z',C,D\right)\right|^{2}\right\}\right)+S_{\Psi}\left(\Psi,\Psi^{\dagger}\right)\right) \quad (80)$$

and:

$$0=\frac{\delta}{\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)}\left(S_{\Gamma}^{(1)}+S_{\Gamma}^{(2)}+S_{\Gamma}^{(3)}+S_{\Gamma}^{(4)}+U\left(\left\{\left|\Gamma\left(\theta,Z,Z',C,D\right)\right|^{2}\right\}\right)+S_{\Psi}\left(\Psi,\Psi^{\dagger}\right)\right) \quad (81)$$

where $\left|\Psi\left(\theta,Z\right)\right|^{2}$ is defined by (79).

To solve these equations, we introduce some simplifying assumptions. A more formal treatment of these equations is given in appendix 2.

7.1 Use of background neuron field

The first simplification uses equation (79) to set the background field $\Psi\left(\theta,Z\right)$ to its average. As a consequence, we will replace in the sequel $\left|\Psi\left(\theta,Z\right)\right|^{2}$ by its average $\left\langle\left|\Psi\left(\theta,Z\right)\right|^{2}\right\rangle$. To simplify the notations, we will write:

$$\left\langle\left|\Psi\left(\theta,Z\right)\right|^{2}\right\rangle\rightarrow\left|\Psi\left(\theta,Z\right)\right|^{2}$$

The activities are thus estimated for these averages. Moreover, using (58), (78) and (79), the averages depend on:

$$\left\{T\left(Z,\theta,Z_{1},\theta-\frac{\left|Z-Z_{1}\right|}{c}\right)\right\}_{(Z,\theta,Z_{1})} \quad (82)$$

where:

$$T\left(Z,\theta,Z_{1},\theta-\frac{\left|Z-Z_{1}\right|}{c}\right)=\int T\left|\Gamma\left(T,\hat{T},\omega,\omega_{1},\theta,Z,Z_{1},C,D\right)\right|^{2}dTd\hat{T}dCdD\equiv T\left(Z,\theta,Z_{1}\right) \quad (83)$$

7.2 Neglecting the derivatives of $S_{\Psi}\left(\Psi,\Psi^{\dagger}\right)$

Given that the neuron fields $\left(\Psi,\Psi^{\dagger}\right)$ are set to their background field values, the derivatives od $S_{\Psi}\left(\Psi,\Psi^{\dagger}\right)$ simplify. Actually in this case:

$$\frac{\delta S_{\Psi}\left(\Psi,\Psi^{\dagger}\right)}{\delta\Psi}=\frac{\delta S_{\Psi}\left(\Psi,\Psi^{\dagger}\right)}{\delta\Psi^{\dagger}}=0$$

and the derivatives with respect to connectivity fields reduce to partial derivatives:

$$\begin{aligned}\frac{\delta S_{\Psi}\left(\Psi,\Psi^{\dagger}\right)}{\delta\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)}&=\frac{\partial S_{\Psi}\left(\Psi,\Psi^{\dagger}\right)}{\partial\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)}\\ \frac{\delta}{\delta\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)}&=\frac{\partial S_{\Psi}\left(\Psi,\Psi^{\dagger}\right)}{\partial\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)}\end{aligned}$$

These partial derivatives involve the derivatives of G in (67) with respect to the connectivity field. However, due to the disparity in time scales between activities and connectivities, a modification of Γ and $\Gamma^{\dagger}$ initially modifies the density of active axons/dentrite, subsequently influencing the level of activity. As a result, the partial derivatives can thus be neglected in first approximation[11].

[11] Including these derivatives in the saddle point equations for connectivities would modify these equation by a quasi linear contribution $\frac{1}{2}\frac{\kappa}{N}\Psi^{\dagger}\left(\theta,Z,\omega\right)\nabla G'\left(G^{-1}\left(\omega^{-1}\left(J,\theta,Z,\left|\Psi\right|^{2}\right)\right)\right)\Psi\left(\theta,Z\right)T\Gamma$. This contribution shifts slightly the average connectivities. In a quasi static approximation, it can be neglected.

7.3 Neglecting backreaction contributions

equation (80) (there is a similar treatment for (81)) becomes:

$$\begin{aligned} 0 &= \left(\nabla_C\left(\frac{\sigma_C^2}{2}\nabla_C+O_C\right)+\nabla_D\left(\frac{\sigma_C^2}{2}\nabla_D+O_D\right)+\nabla_T\left(\frac{\sigma_C^2}{2}\nabla_T+O_T\right)+\nabla_{\hat{T}}\left(\frac{\sigma_C^2}{2}\nabla_{\hat{T}}+O_{\hat{T}}\right)\right. \\ &\left.+K\left(\theta,Z,Z',\|\Psi\|^2,\|\Gamma\|^2\right)T+\frac{\delta U\left(\left\{\left|\Gamma\left(\theta,Z,Z',C,D\right)\right|^2\right\}\right)}{\delta\Gamma^\dagger\left(T,\hat{T},\theta,Z,Z',C,D\right)}\right)\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right) \end{aligned} \tag{84}$$

and K is given by:

$$\begin{aligned} &K\left(\theta,Z,Z',\|\Psi\|^2,\|\Gamma\|^2\right)=\int\Gamma^\dagger\left(T_1,\hat{T}_1,\theta_1,Z_1,Z_1',C_1,D_1\right) \\ &\frac{\delta W\left(T_1,\hat{T}_1,\theta_1,Z_1,Z_1',C_1,D_1\right)}{\delta T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2}\Gamma\left(T_1,\hat{T}_1,\theta_1,Z_1,Z_1',C_1,D_1\right)d\left(T_1,\hat{T}_1,\theta_1,Z_1,Z_1',C_1,D_1\right) \end{aligned} \tag{85}$$

with:

$$W\left(T,\hat{T},\theta,Z,Z',C,D\right)=\nabla_C O_C+\nabla_D O_D+\nabla_T O_T+\nabla_{\hat{T}}O_{\hat{T}}$$

The last term in (84) arises from the dependency of averages in $\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)$ as given in (82). It represents the backreaction of the system as a whole when a variation in $\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)$ occurs. This term can be considered as a correction and will be disregarded in the sequel. Appendix 2 includes this contributions and computes its impact on the background state.

Note that neglecting $W\left(T,\hat{T},\theta,Z,Z',C,D\right)$ amounts to consider in (92) that the action:

$$S_\Gamma^{(1)}+S_\Gamma^{(2)}+S_\Gamma^{(3)}+S_\Gamma^{(4)}$$

is quadratic in $\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)$.

7.4 Separating variables in the connectivty field

The third simplification arises from the fact that in (92) the dependency of the field in the variables C and D is independent from the dependency in T and $\hat{T}$. We thus start by the minimization of $S_\Gamma^{(3)}+S_\Gamma^{(4)}$. We assume the existence of non-trivial minima $\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)$, and we will determine, in the end, a condition for the existence of such states.

Furthermore, the solutions for Γ will depend on $|\Psi(\theta,Z)|^2$ and $\omega\left(J,\theta,Z,|\Psi|^2\right)$. These fields, in turn, depend as functionals on the entire collection $\left\{\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right\}_{(Z,Z')}$, or, in first approximation, on the norm $\|\Gamma\|^2$. This implies that $\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)$ wil satisfy some compatibility conditions that will define the equilibrium of the system. This equilibrium will be computed in the next section.

8 Solutions for the background state equations

8.1 Principle

As a consequence of our simplifying assumptions, we can factor the solutions of the minimization equations as:

$$\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)=\Gamma_1\left(Z,Z',C\right)\Gamma_2\left(Z,Z',D\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$$

and minimize first:

$$S_{\Gamma}^{(3)}+S_{\Gamma}^{(4)}$$

to find $\Gamma_1(Z,Z',C)$ and $\Gamma_2(Z,Z',D)$ along with the average values of C and D in these states.When these functions are determined, we substitute their expression in:

$$S_{\Gamma}^{(1)}+S_{\Gamma}^{(2)}+S_{\Gamma}^{(3)}+S_{\Gamma}^{(4)}$$

and minimize this action with respect to $\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$. The equations for these functions allow to find the consistency equations for the average values of $\left(T,\hat{T}\right)$ in the state $\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$. Given the threshold to creating connections, several possible solutions arise. These values are then used to derive the final form of the possible states $\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$.

8.2 Background state for C and D

We first transform the terms involving the gradients ∇_C and ∇_D in $S_{\Gamma}^{(3)}+S_{\Gamma}^{(4)}$ by changing the variables. Starting with:

$$\begin{aligned}S_{\Gamma}^{(3)}+S_{\Gamma}^{(4)}&=\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)\\&\times\left(\nabla_C\left(\frac{\sigma_C^2}{2}\nabla_C+O_C\right)+\nabla_D\left(\frac{\sigma_D^2}{2}\nabla_D+O_D\right)\right)\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\end{aligned}\tag{86}$$

where O_C and O_D are defined in (76).

We considerthechange of variables:

$$\begin{aligned}&\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\\\rightarrow\ &\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\exp\left(\int\frac{1}{\sigma_D^2}O_D dD\right)\exp\left(\frac{1}{\sigma_C^2}\int O_C dC\right)\end{aligned}$$

and:

$$\begin{aligned}&\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)\\\rightarrow\ &\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)\exp\left(-\int\frac{1}{\sigma_D^2}O_D dD\right)\exp\left(-\frac{1}{\sigma_C^2}\int O_C dC\right)\end{aligned}$$

As a consequence, the terms involving the gradients ∇_C and ∇_D in (71) rewrite:

$$\begin{aligned}&\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)\left(\frac{\sigma_C^2}{2}\nabla_C^2-\frac{1}{2\sigma_C^2}O_C^2\right.\\&\left.+\frac{1}{2}\left(\frac{1}{\tau_C\omega}+\alpha_C\frac{\omega'\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z',\omega'\right)\right|^2}{\omega}\right)\right)\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\end{aligned}$$

and:

$$\begin{aligned}&\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)\\&\times\left(\frac{\sigma_D^2}{2}\nabla_D^2-\frac{1}{2\sigma_D^2}O_D^2+\frac{1}{2}\left(\frac{1}{\tau_D\omega}+\alpha_D\left|\Psi(\theta,Z)\right|^2\right)\right)\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\end{aligned}$$

where we use the notation:

$$\begin{aligned}\omega &\equiv \omega\left(J,\theta,Z,|\Psi|^2\right)\\ \omega' &\equiv \omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)\end{aligned}$$

and ω and ω' are defined by equation (67):

$$\begin{aligned}&\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\\ &= G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left|\Gamma\left(T,\hat{T},\theta,Z,Z_1\right)\right|^2}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2 dZ_1\right)\end{aligned}$$

The field $\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)$ can be written as a product:

$$\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)=\Gamma_1(Z,Z',C)\,\Gamma_2(Z,Z',D)\,\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$$

and since we are looking for non trivial background states, i.e. states with positive norm, we can constrain $\Gamma_1(Z,Z',C)$ and $\Gamma_2(Z,Z',D)$ to have a norm equal to 1, so that $\Gamma_1(Z,Z',C)$ and $\Gamma_2(Z,Z',D)$ satisfy:

$$\left(\frac{\sigma_C^2}{2}\nabla_C^2-\frac{1}{2\sigma_C^2}O_C^2-\frac{1}{2}a_C(Z)+\lambda_1(Z)\right)\Gamma_1(Z,Z',C)=0 \tag{87}$$

$$\left(\frac{\sigma_D^2}{2}\nabla_D^2-\frac{1}{2\sigma_D^2}O_D^2-\frac{1}{2}a_D(Z)+\lambda_2(Z)\right)\Gamma_2(Z,Z',D)=0 \tag{88}$$

with:

$$\begin{aligned}a_C(Z) &= \frac{1}{\tau_C\omega}+\alpha_C\frac{\omega'\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z',\omega'\right)\right|^2}{\omega}\\ a_D(Z) &= \frac{1}{\tau_D\omega}+\alpha_D|\Psi(\theta,Z)|^2\end{aligned} \tag{89}$$

These equations can be rewritten by defining the averages $\langle C(\theta)\rangle$ and $\langle D(\theta)\rangle$:

$$C \rightarrow \langle C(\theta)\rangle=\frac{\alpha_C\frac{\omega'\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}{\omega}}{\frac{1}{\tau_C\omega}+\alpha_C\frac{\omega'\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}{\omega}}=\frac{\alpha_C\omega'\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}{\frac{1}{\tau_C}+\alpha_C\omega'\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}\equiv C(\theta) \tag{90}$$

$$D \rightarrow \langle D(\theta)\rangle=\frac{\alpha_D\omega|\Psi(\theta,Z)|^2}{\frac{1}{\tau_D}+\alpha_D\omega|\Psi(\theta,Z)|^2}\equiv D(\theta) \tag{91}$$

so that:

$$\left(\frac{\sigma_C^2}{2}\nabla_C^2-\frac{1}{2\sigma_C^2}\left(a_C(Z)(C-C(\theta))\right)^2-\frac{1}{2}a_C(Z)+\lambda_1(Z)\right)\Gamma_1(Z,Z',C)=0 \tag{92}$$

and:

$$\left(\frac{\sigma_D^2}{2}\nabla_D^2-\frac{1}{2\sigma_D^2}\left(a_D(Z)(D-D(\theta))\right)^2-\frac{1}{2}a_D(Z)+\lambda_2(Z,Z')\right)\Gamma_2(Z,Z',D)=0 \tag{93}$$

Note that we can consider $C(\theta)$ and $D(\theta)$ as slowly varying, given that $\frac{1}{\tau_C} << 1$, $\frac{1}{\tau_D} << 1$. Moreover, the value $|\Psi(\theta, Z)|^2$ may also be considered as slowly varying in time, as this background field represents the average activity of the cells at point Z. Consequently, $\Gamma_1(Z, Z', C)$ and $\Gamma_2(Z, Z', D)$ are slowly varying as required for the connectivity background field.

The solutions of equations (92) and (93) are parabolic cylinder function at each pair of points (Z, Z'). Imposing a unit norm to these functions for C and D varying over $\mathbb{R}$ yields the fundamental state for $\Gamma_1(Z, Z', C)$ and $\Gamma_2(Z, Z', D)$ which implies the condition:

$$a_C(Z) - \lambda_1(Z, Z') = -\frac{1}{2}\left(\frac{1}{\tau_C \omega} + \alpha_C \frac{\omega' \left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right)\right|^2}{\omega}\right) = -\frac{1}{2} a_C(Z) \tag{94}$$

$$a_D(Z) - \lambda_2(Z, Z') = -\frac{1}{2}\left(\frac{1}{\tau_D \omega} + \alpha_D |\Psi(\theta, Z)|^2\right) = -\frac{1}{2} a_D(Z) \tag{95}$$

The possibility to adjust the value of $\lambda_1(Z, Z')$ translates that the determination of C and D represents local activity, with the constraint on global activity being borne by the variable T (see below).

Practically, constraint (94) corresponds to the fundamental state of a harmonic oscillator whose lowest eigenvalue is $\frac{1}{2}$ times the fundamental frequency. If we relax the simplifying hypothesis that C and D vary over $\mathbb{R}$, the eigenvalue constraint would not hold anymore. Actually, we should also impose $\Gamma_1(Z, Z', C) = \Gamma_2(Z, Z', C) = 0$ for $C < 0$ and for $D < 0$. However, the appearance of $C(\theta)$ and $D(\theta)$ imply that for $|\Psi(\theta, Z)|^2 > 1$, these constraints are satisfied in first approximation. Moreover, assuming relatively low variances in the variables, and given that the variables $C - \langle C\rangle$ and $D - \langle D\rangle$ are centered justifies our assumption.

As a consequence, introducing normalization factors $\mathcal{N}_i$, $i = 1, 2$, we find:

$$\Gamma_1(Z, Z', C) = \mathcal{N}_1 \exp\left(-\frac{a_C(Z)}{8\sigma_C^2}(C - C(\theta))^2\right) \tag{96}$$
$$\Gamma_2(Z, Z', D) = \mathcal{N}_2 \exp\left(-\frac{a_D(Z)}{8\sigma_D^2}(D - D(\theta))^2\right)$$

and $\Gamma\left(T, \hat{T}, \theta, Z, Z', C, D\right)$ factors as:

$$\Gamma\left(T, \hat{T}, \theta, Z, Z', C, D\right)$$
$$\simeq \Gamma\left(T, \hat{T}, \theta, Z, Z'\right) \exp\left(-\frac{1}{8\sigma_C^2} a_C(Z)\left((C - C(\theta))\right)^2\right) \exp\left(-\frac{a_D(Z)}{8\sigma_D^2}(D - D(\theta))^2\right)$$

These solutions yield the following action for the field:

$$\Gamma_1^{\dagger}(Z, Z', C)\Gamma_2^{\dagger}(Z, Z', D)\left(\frac{1}{2} + \frac{1}{2}a_C(Z) + \frac{1}{2} + \frac{1}{2}a_D(Z)\right)\Gamma_1(Z, Z', C)\Gamma_2(Z, Z', D)$$
$$= \left(1 + \frac{1}{2}(a_C(Z) + a_D(Z))\right)$$

Iintroducing these expressions in the full action for $\Gamma\left(T, \hat{T}, \theta, Z, Z', C, D\right)$ yields the contribution:

$$S_{\Gamma}^{(3)} + S_{\Gamma}^{(4)} = \Gamma^{\dagger}\left(T, \hat{T}, \theta, Z, Z'\right)(a_C(Z) + a_D(Z))\Gamma\left(T, \hat{T}, \theta, Z, Z'\right) \tag{97}$$

Given our approximations, this is a constant that will not affect the minimisation of the action for $\Gamma\left(T, \hat{T}, \theta, Z, Z'\right)$. However it will impact the condition of existence of a state with $\left\|\Gamma\left(T, \hat{T}, \theta, Z, Z', C, D\right)\right\|^2 > 0$.

8.3 Minimization equation for $T, \hat{T}$

8.3.1 Action for $T, \hat{T}$

Given the previous projection on the background states for C and D, and introducing the additional term

$$K\left(\theta, Z, Z', \|\Psi\|^2, \|\Gamma\|^2\right) T\Gamma\left(T, \hat{T}, \theta, Z, Z', C, D\right)$$

up to contribution (97), the effective action for $\Gamma\left(T, \hat{T}, \theta, Z, Z'\right)$ reduces to two terms:

$$\begin{aligned} & S_{\Gamma}^{(1)} + S_{\Gamma}^{(2)} \\ = \; & \Gamma^{\dagger}\left(T, \hat{T}, \theta, Z, Z'\right)\left(\nabla_T\left(\frac{\sigma_T^2}{2}\nabla_T + O_T\right) + K\left(\theta, Z, Z', \|\Psi\|^2, \|\Gamma\|^2\right) T\right)\Gamma\left(T, \hat{T}, \theta, Z, Z'\right) \\ & +\Gamma^{\dagger}\left(T, \hat{T}, \theta, Z, Z'\right)\nabla_{\hat{T}}\left(\frac{\sigma_{\hat{T}}^2}{2}\nabla_{\hat{T}} + \bar{O}_{\hat{T}}\right)\Gamma\left(T, \hat{T}, \theta, Z, Z'\right) \end{aligned} \tag{98}$$

with:

$$\begin{aligned} O_T &= -\left(-\frac{1}{\tau\omega}T + \frac{\lambda}{\omega}\hat{T}\right) \\ \bar{O}_{\hat{T}} &= -\frac{\rho}{\omega\left(J, \theta, Z, |\Psi|^2\right)}\left(\left(h\left(Z, Z'\right) - \hat{T}\right) C\left(\theta\right)\left|\Psi\left(\theta, Z\right)\right|^2 h_C\left(\omega\left(\theta, Z, |\Psi|^2\right)\right) - \eta H\left(\delta - T\right)\right. \\ & \left. -D\left(\theta\right)\hat{T}\left|\Psi\left(\theta - \frac{|Z - Z'|}{c}, Z'\right)\right|^2 h_D\left(\omega\left(\theta - \frac{|Z - Z'|}{c}, Z', |\Psi|^2\right)\right)\right) \\ &= \bar{O}_{\hat{T}} + \frac{\rho\eta H\left(\delta - T\right)}{\omega\left(J, \theta, Z, |\Psi|^2\right)} \end{aligned}$$

and $C(\theta)$ and $D(\theta)$ defined in (90) and (91). We will consider them as relatively constant while finding the background state. This amounts to projecting onto states of fields $\Gamma\left(T, \hat{T}, \theta, Z, Z'\right)$ and $\Gamma^{\dagger}\left(T, \hat{T}, \theta, Z, Z'\right)$ that are slowly varying. As for C and D, we will neglect the condition $T < 0$ holds. However, as long that $\langle T \rangle$ is significant enough, this case is irrelevant.

8.3.2 Change of variable

We perform the following change of variable on the fields:

$$\begin{aligned} & \Gamma\left(T, \hat{T}, \theta, Z, Z'\right) \\ \rightarrow \; & \exp\left(\int \frac{\bar{O}_{\hat{T}}}{\sigma_{\hat{T}}^2\omega\left(\theta, Z, |\Psi|^2\right)} d\hat{T}\right)\exp\left(\int \frac{O_T}{\sigma_T^2} dT\right)\Gamma\left(T, \hat{T}, \theta, Z, Z'\right) \end{aligned} \tag{99}$$

$$\begin{aligned} & \Gamma^{\dagger}\left(T, \hat{T}, \theta, Z, Z'\right) \\ \rightarrow \; & \exp\left(-\int \frac{\bar{O}_{\hat{T}}}{\sigma_{\hat{T}}^2\omega\left(\theta, Z, |\Psi|^2\right)} d\hat{T}\right)\exp\left(-\int \frac{O_T}{\sigma_T^2} dT\right)\Gamma^{\dagger}\left(T, \hat{T}, \theta, Z, Z'\right) \end{aligned}$$

and $S_{\Gamma}^{(1)}$ and $S_{\Gamma}^{(2)}$ (98) write:

$$S_{\Gamma}^{(1)}=\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\frac{\sigma_{T}^{2}}{2}\nabla_{T}^{2}-\frac{O_{T}^{2}}{2\sigma_{T}^{2}}+K\left(\theta,Z,Z',\left\|\Psi\right\|^{2},\left\|\Gamma\right\|^{2}\right)T\right.$$
$$\left.-\frac{\sigma_{\hat{T}}^{2}}{\sigma_{T}^{2}}\left(\nabla_{T}\frac{\lambda}{\omega}\left(\hat{T}-\lambda\tau\hat{T}\right)\right)-\frac{1}{2\tau\omega\left(Z\right)}\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right) \tag{100}$$

$$\begin{aligned}S_{\Gamma}^{(2)} &= \Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\frac{\sigma_{\hat{T}}^{2}}{2}\nabla_{\hat{T}}^{2}-\frac{1}{2\sigma_{\hat{T}}^{2}}\bar{O}_{\hat{T}}^{2}\right.\\ &\left.-\frac{\rho\left(C\left(\theta\right)\left|\Psi\left(\theta,Z\right)\right|^{2}h_{C}-\eta H\left(\delta-T\right)+D\left(\theta\right)\left|\Psi\left(\theta-\frac{\left|Z-Z'\right|}{c},Z'\right)\right|^{2}h_{D}\right)}{2\omega\left(\theta,Z,\left|\Psi\right|^{2}\right)}\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\end{aligned} \tag{101}$$

with:

$$\begin{aligned}h_{C}\left(\omega\left(\theta,Z,\left|\Psi\right|^{2}\right)\right) &\equiv h_{C}\\ h_{D}\left(\omega\left(\theta-\frac{\left|Z-Z'\right|}{c},Z',\left|\Psi\right|^{2}\right)\right) &\equiv h_{D}\end{aligned}$$

We aim at factoring:

$$\begin{aligned}\Gamma\left(T,\hat{T},\theta,Z,Z'\right) &= \Gamma\left(T,\theta,Z,Z'\right)\Gamma\left(\hat{T},\theta,Z,Z'\right)\\ \Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right) &= \Gamma^{\dagger}\left(T,\theta,Z,Z'\right)\Gamma^{\dagger}\left(\hat{T},\theta,Z,Z'\right)\end{aligned} \tag{102}$$

but the presence of the term:

$$\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\frac{\sigma_{\hat{T}}^{2}}{\sigma_{T}^{2}}\nabla_{T}\frac{\lambda}{\omega}\left(\hat{T}-\lambda\tau\hat{T}\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$$

in (100) prevents this factorization since this contribution mixes both variables $T,\hat{T}$. However, we may assume the variance of T is larger than the variance of $\hat{T}$. Actually, the value of connectivity may depend on exogenous factors, whereas $\hat{T}$ is a variable counting the output and input spikes. Thus, we can consider that $\frac{\sigma_{\hat{T}}^{2}}{\sigma_{T}^{2}}<<1$ and that

$$\left|\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\frac{\sigma_{\hat{T}}^{2}}{\sigma_{T}^{2}}\nabla_{T}\frac{\lambda}{\omega}\left(\hat{T}-\lambda\tau\hat{T}\right)\left|\Psi\left(\theta,Z\right)\right|^{2}\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right|<<1$$

As a consequence the factorization (102) holds in first approximation, so that we can minimize (100) in two parts, as we did for C and D.

In Appendix 2, we show how to remove the hypothesis concerning $\hat{T}-\lambda\tau\hat{T}$. It amounts to solve directly the background state of (98) both for $\hat{T}$ and T , but the approximation we made above is sufficient to understand the properties of $\Gamma_{0}\left(T,\theta,Z,Z'\right)$.

8.4 Average values for T and $\hat{T}$

8.4.1 General case

Given that $C(\theta)$ and $D(\theta)$ given by (90) and (91).can be considered as slowly varying, we can look for average values for T and $\hat{T}$. They are obtained by setting the quadratic potentials in the effective actions (100) and (101) to 0. They are thus defined by:

$$-\frac{1}{\tau\omega(\theta,Z)}\langle T(Z,Z')\rangle+\frac{\lambda}{\omega(\theta,Z)}\left\langle\hat{T}(Z,Z')\right\rangle=0$$

and:

$$\begin{aligned}0 &= \left(h(Z,Z')-\left\langle\hat{T}(Z,Z')\right\rangle\right)C(\theta)\left|\Psi(\theta,Z)\right|^2 h_C(\omega(Z))-\eta H(\delta-\langle T(Z,Z')\rangle)\\ &\quad-\left\langle\hat{T}(Z,Z')\right\rangle D(\theta)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2 h_D\left(\omega\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right)\end{aligned}$$

given by:

$$\langle T(Z,Z')\rangle=\lambda\tau\left\langle\hat{T}(Z,Z')\right\rangle=\frac{\lambda\tau\left(h(Z,Z')C_{Z,Z'}(\theta)h_C(\omega(\theta,Z))\left|\Psi(\theta,Z)\right|^2-\eta H(\delta-\langle T(Z,Z')\rangle)\right)}{h_C(\omega(\theta,Z))\left|\bar{\Psi}(\theta,Z,Z')\right|^2}$$

where:

$$\left|\bar{\Psi}(\theta,Z,Z')\right|^2=\frac{C_{Z,Z'}(\theta)\left|\Psi(\theta,Z)\right|^2 h_C(\omega(\theta,Z))+D_{Z,Z'}(\theta)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2 h_D\left(\omega\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right)}{h_C(\omega(\theta,Z))} \tag{103}$$

is a weighted sum of the values of field $\left|\Psi(\theta,Z)\right|^2$ and $\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2$.

The threshold δ implies three possibilities.

First, if:

$$\frac{\lambda\tau\left(h(Z,Z')C_{Z,Z'}(\theta)h_C(\omega(\theta,Z))\left|\Psi(\theta,Z)\right|^2\right)}{h_C(\omega(\theta,Z))\left|\bar{\Psi}(\theta,Z,Z')\right|^2}<\delta$$

then the average connectivity is given by:

$$\langle T(Z,Z')\rangle=\sup\left(\frac{\lambda\tau\left(h(Z,Z')C_{Z,Z'}(\theta)h_C(\omega(\theta,Z))\left|\Psi(\theta,Z)\right|^2-\eta\right)}{h_C(\omega(\theta,Z))\left|\bar{\Psi}(\theta,Z,Z')\right|^2},0\right)$$

Second, if on the contrary:

$$\frac{\lambda\tau\left(\left(h(Z,Z')C_{Z,Z'}(\theta)h_C(\omega(\theta,Z))\left|\Psi(\theta,Z)\right|^2\right)-\eta\right)}{h_C(\omega(\theta,Z))\left|\bar{\Psi}(\theta,Z,Z')\right|^2}>\delta$$

then the average connectivity is:

$$\langle T(Z,Z')\rangle=\frac{\lambda\tau\left(h(Z,Z')C_{Z,Z'}(\theta)h_C(\omega(\theta,Z))\left|\Psi(\theta,Z)\right|^2-\eta\right)}{h_C(\omega(\theta,Z))\left|\bar{\Psi}(\theta,Z,Z')\right|^2}$$

Third, if:

$$\begin{aligned}
\frac{\lambda\tau\left(h\left(Z,Z'\right)C_{Z,Z'}\left(\theta\right)h_C\left(\omega\left(\theta,Z\right)\right)\left|\Psi\left(\theta,Z\right)\right|^2\right)}{h_C\left(\omega\left(\theta,Z\right)\right)\left|\bar{\Psi}\left(\theta,Z,Z'\right)\right|^2} &> \delta \\
\frac{\lambda\tau\left(h\left(Z,Z'\right)C_{Z,Z'}\left(\theta\right)h_C\left(\omega\left(\theta,Z\right)\right)\left|\Psi\left(\theta,Z\right)\right|^2-\eta\right)}{h_C\left(\omega\left(\theta,Z\right)\right)\left|\bar{\Psi}\left(\theta,Z,Z'\right)\right|^2} &< \delta
\end{aligned}$$

both solutions:

$$\begin{aligned}
\langle T\rangle &= \sup\left(\frac{\lambda\tau\left(h\left(Z,Z'\right)C_{Z,Z'}\left(\theta\right)h_C\left(\omega\left(\theta,Z\right)\right)\left|\Psi\left(\theta,Z\right)\right|^2-\eta\right)}{h_C\left(\omega\left(\theta,Z\right)\right)\left|\bar{\Psi}\left(\theta,Z,Z'\right)\right|^2},0\right) \\
\langle T\rangle &= \frac{\lambda\tau h\left(Z,Z'\right)C_{Z,Z'}\left(\theta\right)h_C\left(\omega\left(\theta,Z\right)\right)\left|\Psi\left(\theta,Z\right)\right|^2}{h_C\left(\omega\left(\theta,Z\right)\right)\left|\bar{\Psi}\left(\theta,Z,Z'\right)\right|^2}
\end{aligned}$$

are possible.

8.4.2 Average values for T and $\hat{T}$ for sharp threshold

The previous averages simplify for $\delta << 1$. If:

$$\lambda\tau\left(h\left(Z,Z'\right)C_{Z,Z'}\left(\theta\right)h_C\left(\omega\left(\theta,Z\right)\right)\left|\Psi\left(\theta,Z\right)\right|^2-\eta\right)>0$$

then:

$$\left\langle T\left(Z,Z'\right)\right\rangle=\frac{\lambda\tau h\left(Z,Z'\right)C_{Z,Z'}\left(\theta\right)\left|\Psi\left(\theta,Z\right)\right|^2}{\left|\bar{\Psi}\left(\theta,Z,Z'\right)\right|^2} \tag{104}$$

and if:

$$\lambda\tau\left(h\left(Z,Z'\right)C_{Z,Z'}\left(\theta\right)h_C\left(\omega\left(\theta,Z\right)\right)\left|\Psi\left(\theta,Z\right)\right|^2-\eta\right)<0$$

$$\left\langle T\left(Z,Z'\right)\right\rangle=0$$

otherwise.

8.5 Background state for T and $\hat{T}$

Once the average values for $\left\langle T\left(Z,Z'\right)\right\rangle$ are obtained, we can close the resolution for the background $\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$.We will consider a sharp threshold only and consider two cases separately.

8.5.1 First case, cleared threshold $T\left(Z,Z'\right)>0$

For points such that:

$$h\left(Z,Z'\right)\left\langle C_{Z,Z'}\left(\theta\right)h_C\left(\omega\left(\theta,Z\right)\right)\left|\Psi\left(\theta,Z\right)\right|^2\right\rangle-\eta>0$$

then the averages are:

$$\left\langle T\left(Z,Z'\right)\right\rangle=\lambda\tau\left\langle\hat{T}\left(Z,Z'\right)\right\rangle=\frac{\lambda\tau h\left(Z,Z'\right)\left\langle C_{Z,Z'}\left(\theta\right)\left|\Psi\left(\theta,Z\right)\right|^2\right\rangle}{\left|\bar{\Psi}\left(\theta,Z,Z'\right)\right|^2} \tag{105}$$

Whereas, for points such that:

$$h\left(Z,Z'\right)\left\langle C_{Z,Z'}\left(\theta\right)h_C\left(\omega\left(\theta,Z\right)\right)\left|\Psi\left(\theta,Z\right)\right|^2\right\rangle-\eta<0$$

then:

$$\left\langle T\left(Z,Z'\right)\right\rangle=0$$

$$\left\langle \hat{T}\left(Z,Z'\right)\right\rangle=\frac{h\left(Z,Z'\right)\left\langle C_{Z,Z'}\left(\theta\right)h_C\left(\omega\left(\theta,Z\right)\right)\left|\Psi\left(\theta,Z\right)\right|^2\right\rangle-\eta}{\left\langle h_C\left(\omega\left(\theta,Z\right)\right)\left|\bar{\Psi}\left(\theta,Z,Z'\right)\right|^2\right\rangle}<0 \tag{106}$$

To compute the background state, we proceed in two steps as for C and D.

We consider the two cases separately. If $T\left(Z,Z'\right)$ is given by (105), rxpression (101) is:

$$\begin{aligned} S_{\Gamma}^{(2)} &= \Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\frac{\sigma_{\hat{T}}^2}{2}\nabla_{\hat{T}}^2-\frac{1}{2\sigma_{\hat{T}}^2}\left(\frac{\rho\left(h_C\left(\omega\left(\theta,Z\right)\right)\left|\bar{\Psi}\left(\theta,Z,Z'\right)\right|^2\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)\right)}{\omega\left(\theta,Z,\left|\Psi\right|^2\right)}\right)^2\right. \\ &\left.-\frac{\rho h_C\left(\omega\left(\theta,Z\right)\right)\left|\bar{\Psi}\left(\theta,Z,Z'\right)\right|^2}{2\omega\left(\theta,Z,\left|\Psi\right|^2\right)}\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right) \end{aligned} \tag{107}$$

with $\left\langle\hat{T}\right\rangle$ given by (105).

As explained in the previous paragraph, given the form (100) of $S_{\Gamma}^{(1)}$ and (107) of $S_{\Gamma}^{(2)}$, $\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$ factors in first approximation as:

$$\Gamma\left(T,\hat{T},\theta,Z,Z'\right)=\Gamma_0\left(T,\theta,Z,Z'\right)\Gamma_0\left(\hat{T},\theta,Z,Z'\right) \tag{108}$$

Introducing a constraint normalizing $\left\|\Gamma_0\left(\hat{T},\theta,Z,Z'\right)\right\|^2$ to 1 allows to proceed as for the background state for C and D and minimizing (107) under the constraint allows to project $\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$ on the background state:

$$\begin{aligned} &\Gamma_0\left(T,\hat{T},\theta,Z,Z'\right) \\ =\ &\Gamma_0\left(T,\theta,Z,Z'\right)\Gamma_0\left(\hat{T},\theta,Z,Z'\right) \\ =\ &\Gamma_0\left(T,\hat{T},\theta,Z\right)\exp\left(-\frac{\rho h_C\left(\omega\left(\theta,Z\right)\right)\left|\bar{\Psi}\left(\theta,Z,Z'\right)\right|^2}{4\sigma_{\hat{T}}^2\omega\left(\theta,Z,\left|\Psi\right|^2\right)}\left(\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)\right)^2\right) \end{aligned} \tag{109}$$

and in this state, (107) becomes:

$$S_{\Gamma}^{(2)}=\Gamma_0^{\dagger}\left(T,\theta,Z,Z'\right)\left(\frac{\rho h_C\left(\omega\left(\theta,Z\right)\right)\left|\bar{\Psi}\left(\theta,Z,Z'\right)\right|^2}{\omega\left(\theta,Z,\left|\Psi\right|^2\right)}\right)\Gamma_0\left(T,\theta,Z,Z'\right) \tag{110}$$

Ultimately, using (100) and (154) we can rewrite $S_{\Gamma}^{(1)}$:

$$S_{\Gamma}^{(1)}=\Gamma_0^{\dagger}\left(T,\theta,Z,Z'\right)\left(\frac{\sigma_T^2}{2}\nabla_T^2-\frac{1}{2\sigma_T^2}\left(\left(\frac{1}{\tau\omega}\left(T-\left\langle T\right\rangle\right)\right)\right)^2-\frac{1}{2\tau\omega\left(Z\right)}\right)\Gamma_0\left(T,\theta,Z,Z'\right)$$

To obtain the complete action we have to add the contributions (97) and (110). We find ultimately the action for $\Gamma_0(T,\theta,Z,Z')$:

$$\Gamma_0^{\dagger}(T,\theta,Z,Z')\left(\frac{\sigma_T^2}{2}\nabla_T^2-\frac{1}{2\sigma_T^2}\left(\left(\frac{1}{\tau\omega}(T-\langle T\rangle)\right)|\Psi(\theta,Z)|^2\right)^2-\frac{1}{2\tau\omega(Z)}\right)\Gamma_0(T,\theta,Z,Z') \quad (111)$$
$$-\Gamma_0^{\dagger}(T,\theta,Z,Z')\left(a_C(Z)+a_D(Z)+\frac{\rho h_C(\omega(\theta,Z))\left|\bar{\Psi}(\theta,Z,Z')\right|^2}{\omega\left(\theta,Z,|\Psi|^2\right)}\right)\Gamma_0(T,\theta,Z,Z')$$

This action results from the successive projections of partial background states and becomes a function of T only.

At this point we aim at minimizing (111), as we did for the other parts of the background state. However, a difference appears. The potential:

$$U\left(\left\{\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2\right\}\right) \quad (112)$$

introduced in the full action imposes an overall constraint for the whole set of connections in the system. We assume for the sake of simplicity that U depends only on the connectivities at points (Z,Z'), i.e. on:

$$\left\{\int\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2 d\left(T,\hat{T},\theta,C,D\right)\right\}=\left\{\int\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2 d\left(T,\hat{T},\theta,C,D\right)\right\}$$

Given our assumption about the norms of the fields arising in the decomposition of $\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)$, we have:

$$\int\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2 d\left(T,\hat{T},C,D\right)=\int\|\Gamma_0(T,\theta,Z,Z')\|^2 dT=\|\Gamma_0(Z,Z')\|^2$$

and the potential becomes:

$$U\left(\left\{\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2\right\}\right)=U\left(\left\{\|\Gamma_0(Z,Z')\|^2\right\}\right)$$

Including the potential in (111) leads thus to the saddle point equation:

$$\left(\frac{\sigma_T^2}{2}\nabla_T^2-\frac{1}{2\sigma_T^2}\left(\left(\frac{1}{\tau\omega}(T-\langle T\rangle)\right)|\Psi(\theta,Z)|^2\right)^2-\frac{1}{\tau\omega(Z)}-2a_C(Z)-2a_D(Z)\right. \quad (113)$$
$$\left.-\frac{\rho h_C(\omega(\theta,Z))\left|\bar{\Psi}(\theta,Z,Z')\right|^2}{\omega\left(\theta,Z,|\Psi|^2\right)}-\frac{\delta U\left(\left\{\|\Gamma_0(\theta,Z,Z')\|^2\right\}\right)}{\delta\|\Gamma_0(\theta,Z,Z')\|^2}\right)\Gamma_0(T,\theta,Z,Z')$$

This saddle point equation has a minimum at (Z,Z') for:

$$\begin{aligned}0 &= \frac{1}{\tau\omega(Z)}+\frac{1}{2}a_C(Z)+\frac{1}{2}a_D(Z) \\ &\quad+\frac{\rho h_C(\omega(\theta,Z))\left|\bar{\Psi}(\theta,Z,Z')\right|^2}{2\omega\left(\theta,Z,|\Psi|^2\right)}+\frac{\delta U\left(\left\{\|\Gamma_0(\theta,Z,Z')\|^2\right\}\right)}{\delta\|\Gamma_0(\theta,Z,Z')\|^2}\end{aligned} \quad (114)$$

and this set of equations yields the norm $\|\Gamma_0(\theta,Z,Z')\|^2$ at each point.

The background $\Gamma_0(T,\theta,Z,Z')$ is given by:

$$\Gamma_0(T,\theta,Z,Z') = \|\Gamma_0(\theta,Z,Z')\| \exp\left(-\frac{1}{4\sigma_T^2\tau\omega}\left((T-\langle T\rangle)\right)^2\right) \tag{115}$$

Using (114), the sum of (111) and (112) at point (Z,Z') for this state writes:

$$S\left(\|\Gamma_0(Z,Z')\|^2\right) = U\left(\left\{\|\Gamma_0(Z,Z')\|^2\right\}\right) - \frac{\delta U\left(\left\{\|\Gamma_0(\theta,Z,Z')\|^2\right\}\right)}{\delta\|\Gamma_0(\theta,Z,Z')\|^2}\|\Gamma_0(Z,Z')\|^2 \tag{116}$$

This expression yields the condition for the existence of a non trivial minimum for the action at (Z,Z').

If $S\left(\|\Gamma_0(Z,Z')\|^2\right)<0$, the state:

$$\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right) = \Gamma_1(Z,Z',C)\,\Gamma_2(Z,Z',D)\,\Gamma_0(T,\theta,Z,Z')\,\Gamma_0\left(\hat{T},\theta,Z,Z'\right)$$

with the various contributions defined by (96), (109) and (115), is a non trivial minimum. Otherwise the minimum of the action for the connectivity field is:

$$\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)=0$$

As a consequence, given the parameters arising in (114), that is, the average activity at some points, the density of active neurons or axons $|\Psi(\theta,Z)|^2$, and depending on the potential, the non-trivial state (115) $\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)$ may be a minimum of (116). Since the parameters describing the activity vary across space defined by the points (Z,Z'), we can expect some islands of connectivity. Submanifolds satisfying $S\left(\|\Gamma_0(Z,Z')\|^2\right)<0$ present connectivity patterns, whereas those satisfying $S\left(\|\Gamma_0(Z,Z')\|^2\right)>0$ present low levels of connectivity.

8.5.2 Second case: $T(Z,Z')=0$

In the second case, i.e. $T(Z,Z')=0$ we find the $\hat{T}$ dependency of the background state as in the first case, and we obtain:

$$\begin{aligned}
&\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\\
=\;&\Gamma_0\left(T,\hat{T},\theta,Z\right)\\
&\times\exp\left(-\left(\frac{\rho h_C(\omega(\theta,Z))\left|\bar{\Psi}(\theta,Z,Z')\right|^2\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)}{4\sigma_{\hat{T}}^2\omega\left(\theta,Z,|\Psi|^2\right)}\right)^2\right)
\end{aligned}$$

but now, $\left\langle\hat{T}\right\rangle$ is given by formula (106).

The difference with the first case arises when looking for the dependency in $\langle T\rangle$. When $\langle T(Z,Z')\rangle=0$, the threshold makes the background state is peaked at $T=0$. Only a significative change in activities and field $|\Psi|^2$, i.e. a change in background state, allows to depart from $T=0$. We thus have:

$$\Gamma_0\left(T,\hat{T},\theta,Z\right)=\delta(T)$$

8.5.3 Extension global constraint

In addition to the potential $U\left(\left\{\|\Gamma_0(Z,Z')\|^2\right\}\right)$, a global constraint on the overall level of connectivity could be introduced, either with a potential $U\left(\left\{\|\Gamma_0\|^2\right\}\right)$ with $\|\Gamma_0\|^2=\int\|\Gamma_0(Z,Z')\|^2 d(Z,Z')$, or with a constraint of the form:

$$\|\Gamma_0\|^2=\overline{\|\Gamma\|}^2$$

This constraint may be introduced through a Lagrange multiplier α_0 in (113). The resolution is identical as in the previous paragraph and amounts to add a term α_0 in (114). This implies that the solution of this equation becomes a function of α_0 $\|\Gamma_0(\theta,Z,Z',\alpha_0)\|^2$, the value of α_0 being obtained by implementing:

$$\int\|\Gamma_0(\theta,Z,Z',\alpha_0)\|^2=\overline{\|\Gamma\|}^2$$

Again depending on the sign of the action:

$$S\left(\|\Gamma_0(Z,Z')\|^2\right)=U\left(\left\{\|\Gamma_0(Z,Z')\|^2\right\}\right)+\left(\alpha_0-\frac{\delta U\left(\left\{\|\Gamma_0(\theta,Z,Z')\|^2\right\}\right)}{\delta\|\Gamma_0(\theta,Z,Z')\|^2}\right)\|\Gamma_0(Z,Z')\|^2 \tag{117}$$

i.e. for $S\left(\|\Gamma_0(Z,Z')\|^2\right)<0$, a nontrivial state for $\Gamma_0\left(T,\hat{T},\theta,Z\right)$ will exist.

9 Full background state for connectivity and static averages

In the previous section, we determined the form of the background connectivity state between two points, Z and Z'. The full background state is, therefore, the tensor product of such states for every pair (Z,Z'). However, to complete the description, the average values of connectivity in these states need to be derived. The entire background state is depicted by a set of equations for these averages.

In this section, by performing the integrations arising from the various changes of variables, we can consolidate the previous results by providing the background state of the system and the equations for the averages. Then, as an example, we will solve these equations under some simplifying assumptions.

9.1 Fields

To write the background state of the system in a compact form, we define:

$$\begin{aligned}
\Gamma\left(T,\hat{T},\theta,C,D\right) &= \left\{\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right\}_{(Z,Z')}\\
\Gamma^{\dagger}\left(T,\hat{T},\theta,C,D\right) &= \left\{\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)\right\}_{(Z,Z')}\\
\left|\Gamma\left(T,\hat{T},\theta,C,D\right)\right|^2 &= \left\{\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2\right\}_{(Z,Z')}
\end{aligned}$$

These fields decompose as:

$$\begin{aligned}
\Gamma\left(T,\hat{T},\theta,C,D\right) &= \left(\Gamma_a\left(T,\hat{T},\theta,C,D\right),\Gamma_u\left(T,\hat{T},\theta,C,D\right)\right)\\
\Gamma^{\dagger}\left(T,\hat{T},\theta,C,D\right) &= \left(\Gamma_a^{\dagger}\left(T,\hat{T},\theta,C,D\right),\Gamma_u^{\dagger}\left(T,\hat{T},\theta,C,D\right)\right)\\
|\Gamma|^2\left(T,\hat{T},\theta,C,D\right) &= \left(|\Gamma|_a^2\left(T,\hat{T},\theta,C,D\right),|\Gamma|_u^2\left(T,\hat{T},\theta,C,D\right)\right)
\end{aligned}$$

where

$$\Gamma_{a/u}\left(T,\hat{T},\theta,C,D\right) = \left\{\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right\}_{(Z,Z'),\langle T(Z,Z')\rangle\neq 0/\langle T(Z,Z')\rangle=0}$$
$$\Gamma^{\dagger}_{a/u}\left(T,\hat{T},\theta,C,D\right) = \left\{\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z',C,D\right)\right\}_{(Z,Z'),\langle T(Z,Z')\rangle\neq 0/\langle T(Z,Z')\rangle=0}$$
$$\left|\Gamma_{a/u}\left(T,\hat{T},\theta,C,D\right)\right|^2 = \left\{\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2\right\}_{(Z,Z'),\langle T(Z,Z')\rangle\neq 0/\langle T(Z,Z')\rangle=0}$$

The subscripts refer to active or unactive doublet (Z,Z'). The expression for Γ_a,Γ_u and their conjugates are:

$$\begin{aligned}&\Gamma_a\left(T,\hat{T},\theta,C,D\right)\\ \simeq\ &\left\{\mathcal{N}\exp\left(-\frac{1}{2\sigma_C^2}\left(\frac{1}{\tau_C\omega}+\alpha_C\frac{\omega'\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}{\omega}\right)(C-C(\theta))^2\right)\right.\\ &\times\exp\left(-\frac{\left(\frac{1}{\tau_D\omega}+\alpha_D\left|\Psi(\theta,Z)\right|^2\right)}{2\sigma_D^2}(D-D(\theta))^2\right)\\ &\times\exp\left(-\frac{\rho h_C(\omega(\theta,Z))\left|\bar{\Psi}(\theta,Z,Z')\right|^2}{2\sigma_{\hat{T}}^2\omega\left(\theta,Z,|\Psi|^2\right)}\left(\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)\right)^2\right)\\ &\left.\times\left\|\Gamma_0(\theta,Z,Z')\right\|\exp\left(-\frac{\left|\Psi(\theta,Z)\right|^2}{2\sigma_T^2\tau\omega}\left((T-\langle T\rangle)\right)^2\right)\right\}_{(Z,Z'),\langle T(Z,Z')\rangle\neq 0}\end{aligned} \tag{118}$$

$$\begin{aligned}&\Gamma_u\left(T,\hat{T},\theta,C,D\right)\\ \simeq\ &\left\{\mathcal{N}\exp\left(-\frac{1}{2}a_C(Z)(C-C(\theta))^2\right)\exp\left(-\frac{a_D(Z)}{2}(D-\langle D\rangle)^2\right)\right.\\ &\left.\times\exp\left(-\frac{\rho h_C(\omega(\theta,Z))\left|\bar{\Psi}(\theta,Z,Z')\right|^2}{2\omega\left(\theta,Z,|\Psi|^2\right)}\left(\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)\right)^2\right)\delta(T)\right\}_{(Z,Z'),\langle T(Z,Z')\rangle=0}\end{aligned} \tag{119}$$

with (see (89)):

$$\begin{aligned}a_C(Z) &= \frac{1}{\tau_C\omega}+\alpha_C\frac{\omega'\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z',\omega'\right)\right|^2}{\omega}\\ a_D(Z) &= \frac{1}{\tau_D\omega}+\alpha_D\left|\Psi(\theta,Z)\right|^2\end{aligned} \tag{120}$$

and where $\mathcal{N}$ is a normalization factor ensuring that the constraint over the number of connections is satisfied.

$$\Gamma^{\dagger}_a\left(T,\hat{T},\theta,C,D\right)\simeq\{1\}_{(Z,Z'),\langle T(Z,Z')\rangle\neq 0}$$

$$\Gamma^{\dagger}_u\left(T,\hat{T},\theta,C,D\right)\simeq\{\delta(T)\}_{(Z,Z'),\langle T(Z,Z')\rangle=0}$$

The modulus of these fields define the density of the various variables in the state defined by Γ_a or Γ_u.

$$\begin{aligned}
& |\Gamma|_a^2 \left(T, \hat{T}, \theta, C, D\right) \\
\simeq & \left\{\mathcal{N} \exp\left(-\frac{1}{2} a_C(Z)(C - C(\theta))^2\right) \exp\left(-\frac{a_D(Z)}{2}(D - \langle D\rangle)^2\right)\right. \\
& \times \exp\left(-\frac{\rho h_C(\omega(\theta, Z)) \left|\bar{\Psi}(\theta, Z, Z')\right|^2}{2\omega\left(\theta, Z, |\Psi|^2\right)}\left(\hat{T} - \left\langle\hat{T}\right\rangle\right)^2\right) \\
& \left.\times \left\|\Gamma_0(\theta, Z, Z')\right\| \exp\left(-\frac{|\Psi(\theta, Z)|^2}{2\tau\omega}(T - \langle T\rangle)^2\right)\right\}_{(Z,Z'),\langle T(Z,Z')\rangle \neq 0}
\end{aligned} \tag{121}$$

$$\begin{aligned}
& |\Gamma|_u^2 \left(T, \hat{T}, \theta, C, D\right) \\
\simeq & \left\{\mathcal{N} \exp\left(-\frac{1}{2} a_C(Z)(C - C(\theta))^2\right) \exp\left(-\frac{a_D(Z)}{2}(D - \langle D\rangle)^2\right)\right. \\
& \left.\times \exp\left(-\frac{\rho h_C(\omega(\theta, Z)) \left|\bar{\Psi}(\theta, Z, Z')\right|^2}{2\omega\left(\theta, Z, |\Psi|^2\right)}\left(\hat{T} - \left\langle\hat{T}\right\rangle\right)^2\right) \times \delta(T)\right\}_{(Z,Z'),\langle T(Z,Z')\rangle \neq 0}
\end{aligned} \tag{122}$$

9.2 Average values for connctivities

The average values in this background states are given by:

$$\begin{aligned}
C_{Z,Z'} &= \frac{\alpha_C \omega' \left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right)\right|^2}{\frac{1}{\tau_C} + \alpha_C \omega' \left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right)\right|^2} \\
D_{Z,Z'} &= \frac{\alpha_D \omega |\Psi(\theta, Z)|^2}{\frac{1}{\tau_D} + \alpha_D \omega |\Psi(\theta, Z)|^2}
\end{aligned}$$

$$\begin{aligned}
\langle T(Z, Z')\rangle &= \lambda\tau \left\langle \hat{T}(Z, Z')\right\rangle \\
&= \frac{\lambda\tau h(Z, Z') \left\langle C_{Z,Z'}(\theta) |\Psi(\theta, Z)|^2 \right\rangle}{\left|\bar{\Psi}(\theta, Z, Z')\right|^2}
\end{aligned} \tag{123}$$

for (Z, Z') an "a" (active) doublet, and:

$$\langle T(Z, Z')\rangle = 0$$

$$\left\langle \hat{T}(Z, Z')\right\rangle = \frac{h(Z, Z') \left\langle C_{Z,Z'}(\theta) h_C(\omega(\theta, Z)) |\Psi(\theta, Z)|^2 \right\rangle - \eta}{\left\langle h_C(\omega(\theta, Z)) \left|\bar{\Psi}(\theta, Z, Z')\right|^2 \right\rangle} < 0 \tag{124}$$

for an "u" (unactive) doublet.

Choosing an exponential form for the function $h(Z, Z')$:

$$h(Z, Z') = \exp\left(-\frac{|Z - Z'|}{\nu c}\right)$$

the average simplifies as:

$$\langle T(Z,Z')\rangle = \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{1+\frac{\alpha_D\omega h_D}{\alpha_C\omega' h_C}\frac{\frac{1}{\tau_C}+\alpha_C\omega'\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}{\frac{1}{\tau_D}+\alpha_D\omega|\Psi(\theta,Z)|^2}} \tag{125}$$

in an a doublet.

Note that the background state obtained above is quasi-static. As explained in the previous paragraph, the variations in θ of the background field are slow. Some modifications in the parameters may induce a switch at some point from an 'a' to a 'u' doublet or from a 'u' to an 'a' doublet. To consider a static background, the quantities involved in the definition of (118), (119) and (121) can be averaged over time.

9.2.1 Interpretation of the background states

Formulas (121) and (122) for the densities of connectivities may be interpreted as follows: Regardless of the system's interpretation, whether as a description of groups of simple cells or a single complex cell at each point, the stable backgrounds are not defined with a given value of connectivity. On the contrary, the background states are described by a normal distribution around some average value. That is, the cells or groups of axons/dendrites are connected with connectivities that are spread around this average.

10 System's background states averages

The full system background state average is given by several equations defining the averages connectivits, the neural background stt and the neurons' activities. Equations (123) and (125)

$$\langle T(Z,Z')\rangle = \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{1+\frac{\alpha_D\omega h_D}{\alpha_C\omega' h_C}\frac{\frac{1}{\tau_C}+\alpha_C\omega'\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}{\frac{1}{\tau_D}+\alpha_D\omega|\Psi(\theta,Z)|^2}} = \lambda\tau\left\langle \hat{T}(Z,Z')\right\rangle$$

are considered together with (67) determining the activit at the lowest order in perturbation:

$$\begin{aligned}&\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right) &(126)\\ =\;& G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2 dZ_1\right)\end{aligned}$$

and the minimizing equation for the field $|\Psi(\theta,Z)|^2$

$$\begin{aligned}0 &= \frac{\delta}{\delta|\Psi(\theta,Z)|^2}\left[\int\Psi^{\dagger}(\theta,Z)\left(-\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\frac{1}{\hat{G}\left(\left(T(Z,Z_1)\right)_{Z_1},|\Psi(Z)|^2\right)}\right)\right)\Psi(\theta,Z)\right]\\ &\quad+\frac{\delta}{\delta|\Psi(\theta,Z)|^2}\left[\int U\left(|\Psi(\theta,Z)|^2\right)\right]\end{aligned}$$

where we will assume that the potential has the specific form:

$$U\left(|\Psi(\theta,Z)|^2\right) = V\left(|\Psi(\theta,Z)|^2-\int T(Z',Z_1)|\Psi_0(Z)|^2 dZ_1\right) \tag{127}$$

This potential encompasses the fact that the field is constrained by a potential limiting the activity around some average[12] $|\Psi_0(Z)|^2$. It also implies that the activity at some point depends on the average activity in the neighborhood of the point. We will look for static solutions of these equations under some simplifying assumptions in the next section. However, given the form of the various equations, we can draw some general patterns for the solutions. Equation (123) shows that groups of mutually connected states arise in localized region, i.e. for $|Z-Z'|<1$, and bewteen cells such that:

$$\omega \simeq \omega'$$

and:

$$|\Psi(\theta, Z)|^2 \simeq \left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right)\right|^2$$

Given (127) and (126), this is also realized in localized regions. As a consequence, assuming an implicit global potential limiting the overall activity, we may expect groups of connected points, with relatively close activities. We inspect this possibility more precisely for the static case in the next section.

11 Static background state for the system

We look for a static background state for the whole system. It corresponds to averaging over time in the background fields (118) (119) and (121). The equilibrium we look for is obtained as consistency conditions for (118). Actually, the background Γ depends on $\omega(\theta, Z)$ and $|\Psi(\theta, Z)|^2$ that depend themselves functionally on Γ, through (78) (79).

11.1 General equations

An approximate static solution of (78) can be found for the constant background and a constant current, i.e. $J = \bar{J}$. We also set:

$$T(Z, Z_1) = \bar{T}\left(Z, \theta, Z_1, \theta - \frac{|Z-Z_1|}{c}\right)$$

From now on, the quantity $T(Z, Z_1)$ refers to the average of the connectivity function at points (Z, Z_1), in the background state defined above, i.e. $T(Z, Z_1)$ refers to $\langle T(Z, Z_1)\rangle$ defined as:

$$\langle T(Z, Z_1)\rangle = \int T \left|\Gamma\left(T, \hat{T}, \theta, Z, Z_1\right)\right|^2 dT$$

For points such that $T(Z, Z') \neq 0$, it is defined by the set of equations (67) or (78), (104) (125) and the minimizatn equatn (79) if:

$$h(Z, Z') C_{Z,Z'}(\theta) h_C(\omega(\theta, Z)) |\Psi(\theta, Z)|^2 > 0$$

We choose, as in ([53]), $h_C(\omega) = \omega$ and $h_C(\omega') = \omega'$. The static equilibrium for $\omega(Z)$, $T(Z, Z_1)$ and $|\Psi(Z)|^2$ is found in three steps.

[12] This characteristic in line with the litterature about homeostatic activit quoted in the literature review.

11.1.1 First step: finding $\omega(Z)$

We first solve for $\omega(Z)$ as a function of other variables using (125) that allows to replace $\omega' |\Psi(Z')|^2$:

$$\omega' |\Psi(Z')|^2 = \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right) - T(Z,Z')}{T(Z,Z')} \left(\frac{1}{\alpha_D\tau_D} + \omega |\Psi(Z)|^2\right) - \frac{1}{\alpha_C\tau_C}$$

We show in appendix 3 that this allows to rewrite (78) as an equation for $\omega^{-1}(Z)$:

$$\begin{aligned} & \omega(Z) \\ \simeq\ & G\left(\int \frac{\kappa}{N}\left(\left(\lambda\tau\exp\left(-\frac{|Z-Z_1|}{\nu c}\right) - T(Z,Z_1)\right)\left(\left(\frac{1}{\alpha_D\tau_D} - \frac{T(Z,Z_1)}{\alpha_C\tau_C}\right)\omega^{-1} + |\Psi(Z)|^2\right)\right) dZ_1\right) \end{aligned} \tag{128}$$

with the solution defined by a function:

$$\omega(Z) = \hat{G}\left(T(Z), |\Psi(Z)|^2\right)$$

where:

$$\frac{1}{V}T(Z) = \frac{1}{V}\int T(Z,Z_1)\, dZ_1$$

11.1.2 Second step: finding $T(Z)$ and $T(Z,Z')$

In a second step, once $\omega(Z)$ found, we use (104) (125) to obtain $T(Z)$ and $T(Z,Z')$ as a function of $\hat{G}\left(T(Z'), |\Psi(Z')|^2\right) |\Psi(Z,\omega)|^2$. We show in appendix 3 that:

$$T(Z) = \frac{\lambda\tau\nu c}{1 + \frac{\frac{1}{\tau_C\alpha_C} + \Omega}{\frac{1}{\tau_D\alpha_D} + \hat{G}\left(T(Z), |\Psi(Z)|^2\right)|\Psi(Z)|^2}} = \frac{\lambda\tau\nu c\left(\frac{1}{\tau_D\alpha_D} + \hat{G}\left(T(Z), |\Psi(Z)|^2\right)|\Psi(Z)|^2\right)}{\frac{1}{\tau_D\alpha_D} + \frac{1}{\tau_C\alpha_C} + \Omega + \hat{G}\left(T(Z), |\Psi(Z)|^2\right)|\Psi(Z)|^2} \tag{129}$$

where:

$$\Omega = \frac{1}{V}\int \hat{G}\left(T(Z'), |\Psi(Z')|^2\right)|\Psi(Z')|^2 dZ' \tag{130}$$

and this leads to the following formula for $T(Z,Z')$:

$$T(Z,Z') = \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{1 + \frac{\frac{1}{\tau_C\alpha_C} + \omega'|\Psi(Z',\omega')|^2}{\frac{1}{\tau_D\alpha_D} + \omega|\Psi(Z,\omega)|^2}} = \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{1 + \frac{\frac{1}{\tau_C\alpha_C} + \hat{G}\left(T(Z'), |\Psi(Z')|^2\right)|\Psi(Z',\omega')|^2}{\frac{1}{\tau_D\alpha_D} + \hat{G}\left(T(Z), |\Psi(Z)|^2\right)|\Psi(Z,\omega)|^2}} \tag{131}$$

11.1.3 Third step: finding $\Psi(\theta, Z)$

In a third step, the system is closed by minimizing the action for the field $\Psi(\theta, Z)$:

$$\begin{aligned} & \int \Psi^{\dagger}(\theta,Z)\left(-\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \frac{1}{\hat{G}\left((T(Z,Z_1))_{Z_1}, |\Psi(Z)|^2\right)}\right)\right)\Psi(\theta,Z) \\ & + \int V\left(|\Psi(\theta,Z)|^2 - \int T(Z',Z_1)|\Psi_0(Z)|^2 dZ_1\right) \end{aligned}$$

Where we have assumed that the field is constrained by a potential limiting the activit around some average $|\Psi_0(Z)|^2$. We choose:

$$V = \frac{1}{2}\left(|\Psi(Z)|^2 - \int T(Z',Z_1)|\Psi_0(Z)|^2 dZ_1\right)^2$$

We show in Appendix 3 that an equilibrium with static $|\Psi(Z)|^2$ exist and minimizes:

$$\int\left(\frac{1}{\hat{G}\left((T(Z,Z_1))_{Z_1},|\Psi(Z)|^2\right)}\right)^2|\Psi(Z)|^2+\int V\left(|\Psi(\theta,Z)|^2-\int T(Z',Z_1)|\Psi_0(Z)|^2 dZ_1\right)$$

with saddle point equation:

$$0\simeq\left(\left(\frac{1}{\hat{G}\left((T(Z,Z_1))_{Z_1},|\Psi(Z)|^2\right)}\right)^2+\left(|\Psi(Z)|^2-\int T(Z',Z_1)|\Psi_0(Z)|^2 dZ_1\right)\right)\Psi(\theta,Z)$$

with solutions:

$$\Psi(\theta,Z)=0$$

or approximately:

$$\begin{aligned}\left(\frac{1}{\hat{G}\left((T(Z,Z_1))_{Z_1},|\Psi(Z)|^2\right)}\right)^2+|\Psi(Z)|^2 &\simeq T(Z)\frac{\int|\Psi_0(Z')|^2 k(Z,Z')\,dZ'}{V} \\ &\equiv T(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z\end{aligned}\tag{132}$$

Remark that $\left\langle|\Psi_0(Z')|^2\right\rangle_Z$ is a function of Z given by an average over the points Z' surrounding Z. It is defind in appendix 3 and represents the average activity in the neighbourhood of Z.

The system is now reduced to two variables $T(Z)$ and $|\Psi(Z)|^2$ together with equations (129) and (132). The average connectivity being then retrieved by (131).

11.2 Solving for $|\Psi(Z)|^2$ and $T(Z)$

Appendix 3 solves the system for $T(Z)$ and $|\Psi(Z)|^2$. We show that:

$$|\Psi(Z)|^2=\frac{2T(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z}{\left(1+\sqrt{1+4\left(\frac{\lambda\tau\nu c-T(Z)}{\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega\right)T(Z)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}\right)^2 T(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z}\right)}\tag{133}$$

Ultimately, inserting this result in (241) yields the following equation for $T(Z)$.

$$\begin{aligned}\frac{\hat{\Omega}T(Z)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}{\lambda\tau\nu c-T(Z)} &= \hat{G}\left(T(Z),\frac{2T(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z}{\left(1+\sqrt{1+4\left(\frac{\lambda\tau\nu c-T(Z)}{\hat{\Omega}T(Z)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}\right)^2 T(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z}\right)}\right) \\ &\times\frac{2T(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z}{\left(1+\sqrt{1+4\left(\frac{\lambda\tau\nu c-T(Z)}{\hat{\Omega}T(Z)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}\right)^2 T(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z}\right)}\end{aligned}\tag{134}$$

with:

$$\Omega = \frac{1}{V}\int \hat{G}\left(T(Z), |\Psi(Z)|^2\right)|\Psi(Z)|^2$$
$$\hat{\Omega} = \left(\frac{1}{\tau_D\alpha_D} + \frac{1}{\tau_C\alpha_C} + \Omega\right)$$

Equation (134) has in general several solutions (see below) corresponding to several regime of activity, depending on the point. Once $T(Z)$ is found, one can obtain $|\Psi(Z)|^2$ using (133). To obtain more precise formula for these solutions, we will detail a particular case below. The system is ultimately determined by finding Ω. Details are given in appendix 3.

11.3 Particular case

For G an increasing function of the form $G(x) \simeq b_0 x$ for $x < 1$, we can solve the system. In our preliminary set up, this corresponds, for $b_0 = 1$, to consider $\gamma \simeq 0$, that is, the activity of a cell depends only on the external currents. The resolution starts by finding Ω, then ultimately $T(Z)$ and $T(Z, Z')$.

Setting $b = b_0\frac{\kappa V}{N}$, $\bar{T} = \frac{\lambda\tau\nu c}{2}$ and $\left\langle|\Psi_0|^2\right\rangle = \left\langle\left\langle|\Psi_0(Z')|^2\right\rangle_Z\right\rangle$, the average of $\left\langle|\Psi_0(Z')|^2\right\rangle_Z$, we find in appendix 3 the equation for Ω:

$$1 = \frac{b\bar{T}\Omega^3}{\left(b\bar{T}^2\left\langle|\Psi_0|^2\right\rangle\Omega - 1\right)^2} \tag{135}$$

Several cases arise:

For $d = \left(b\bar{T}\right)^2\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^3 < \frac{27}{4}$ there is one solution:

$$\Omega \simeq \left(b\bar{T}\right)^{-\frac{1}{3}} << 1 \tag{136}$$

For $d = \left(b\bar{T}\right)^2\left\langle\bar{T}|\Psi_0|^2\right\rangle^3 > \frac{27}{4}$ there are three solutions. The first one is:

$$\Omega \simeq b\bar{T}\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^2 \tag{137}$$

The two other solutions are centered around $\frac{1}{b\bar{T}^2\left\langle|\Psi_0|^2\right\rangle}$. They are given by:

$$\Omega = \frac{1 \pm \sqrt{\frac{1}{\left(b\bar{T}\right)^2\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^3}}}{b\bar{T}^2\left\langle|\Psi_0|^2\right\rangle} \tag{138}$$

Solution (136) corresponds to relatively low activity, i.e. $\left\langle|\Psi_0|^2\right\rangle << 1$, so that we only consider solutions (137) and (138) in the sequel. Moreover, since the solutions of (138) are centered around $\frac{1}{b\left\langle|\Psi_0|^2\right\rangle}$, we can consider only the two solutions:

$$\Omega_\pm = (\Omega_+, \Omega_-)$$

with:

$$\Omega_+ = b\bar{T}\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^2, \Omega_- = \frac{1}{b\bar{T}^2\left\langle|\Psi_0|^2\right\rangle} \tag{139}$$

To these solutions for Ω, we associate the Z dependent parameters, which are at the lwst rdr[13]:

$$Y_{+}(Z) \simeq b\bar{T}\left(\bar{T}\left\langle\left|\Psi_0(Z')\right|^2\right\rangle_Z^2\right)^2 \tag{140}$$

$$Y_{-}(Z) = \frac{1}{b\bar{T}^2\left\langle\left|\Psi_0(Z')\right|^2\right\rangle_Z} \tag{141}$$

where $Y_{\pm}$ gathers the possibilities Y_{+} and Y. Appendix 3 shows that $T(Z)$ is a function of these parameters:

$$T(Z_{\pm}) = \frac{\lambda\tau\nu c Y_{\pm}(Z) + \lambda\tau\nu c\frac{1}{\tau_D\alpha_D}}{Y_{\pm}(Z) + \frac{1}{\tau_D\alpha_D} + \frac{1}{\alpha_C\tau_C} + \Omega_{\pm}}$$

and takes the values:

$$\begin{aligned}
T_{+}(Z) &\simeq \bar{T} - \frac{3\bar{T}\left(\left\langle\left|\Psi_0\right|^2\right\rangle^2 - \left\langle\left|\Psi_0(Z')\right|^2\right\rangle_Z^2\right)}{2\left\langle\left|\Psi_0(Z')\right|^2\right\rangle_Z} \\
T_{-}(Z) &\simeq \bar{T} + \frac{5}{2}\sqrt{\frac{1}{\bar{T}^2 b^2\left\langle\bar{T}\left|\Psi_0(Z')\right|^2\right\rangle_Z^5}}\,\bar{T}\left(\left\langle\left|\Psi_0\right|^2\right\rangle^2 - \left\langle\left|\Psi_0(Z')\right|^2\right\rangle_Z^2\right)
\end{aligned}$$

Ultimately, we show that there are four possibilities for the connectivity functions that are written:

$$T\left(Z_{\pm}, Z'_{\pm}\right) = \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left(\frac{1}{\tau_D\alpha_D} + Y_{\pm}(Z)\right)}{\frac{1}{\tau_D\alpha_D} + Y_{\pm}(Z) + \frac{1}{\alpha_C\tau_C} + Y_{\pm}(Z')}$$

Given our assumptions on d, in most cases[14]:

$$Y_{+}(Z) >> Y_{-}(Z)$$

Moreover:

$$\frac{1}{\tau_D\alpha_D} << 1,\ \frac{1}{\alpha_C\tau_C} << 1$$

[13] First order corrections are given in appendix 3.

[14] In general $\bar{T} = \frac{\lambda\tau\nu c}{2} >> 1$

so that due to the threshold in connectivity, we can write:

$$T\left(Z_{-},Z_{+}'\right) = \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{b\bar{T}^2\left\langle|\Psi_0(Z')|^2\right\rangle_Z}\right)}{\frac{1}{\tau_D\alpha_D}+\frac{1}{\alpha_C\tau_C}+\frac{1}{b\bar{T}^2\left\langle|\Psi_0(Z')|^2\right\rangle_Z}+b\bar{T}\left(\bar{T}\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_{Z'}^2\right)^2}\simeq 0 \tag{142}$$

$$T\left(Z_{+},Z_{+}'\right) = \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left(\frac{1}{\tau_D\alpha_D}+b\bar{T}\left(\bar{T}\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_{Z}^2\right)^2\right)}{\frac{1}{\tau_D\alpha_D}+\frac{1}{\alpha_C\tau_C}+b\bar{T}\left(\bar{T}\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_{Z}^2\right)^2+b\bar{T}\left(\bar{T}\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_{Z'}^2\right)^2}\simeq\frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{2}$$

$$T\left(Z_{+},Z_{-}'\right) = \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left(\frac{1}{\tau_D\alpha_D}+b\bar{T}\left(\bar{T}\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_{Z}^2\right)^2\right)}{\frac{1}{\tau_D\alpha_D}+\frac{1}{\alpha_C\tau_C}+b\bar{T}\left(\bar{T}\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_{Z}^2\right)^2+\frac{1}{b\bar{T}^2\left\langle|\Psi_0(Z')|^2\right\rangle_{Z'}}}\simeq\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)$$

$$T\left(Z_{-},Z_{-}'\right) \simeq \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)+\frac{1}{b\bar{T}^2\left\langle|\Psi_0(Z')|^2\right\rangle_Z}}{1+\frac{\tau_D\alpha_D}{\alpha_C\tau_C}+\frac{1}{b\bar{T}^2\left\langle|\Psi_0(Z')|^2\right\rangle_Z}+\frac{1}{b\bar{T}^2\left\langle|\Psi_0(Z')|^2\right\rangle_{Z'}}}\simeq\frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{2}$$

Given equation (130) and (139), points Z_{+},Z_{+}' correspond to cells with high activity while Z_{-},Z_{-}' describe points with low activity. This is confirmed in appendix 3, where we provide the following formula for activities:

$$\begin{aligned}\omega_{+}\left(Z\right) &\simeq b_0\frac{\kappa}{N}\bar{T}\left(1-\frac{3\left(\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z^2-\left\langle\left|\Psi_0\right|^2\right\rangle^2\right)}{2\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z^2}\right)\left|\Psi_{+}\left(Z\right)\right|^2\\ \omega_{-}\left(Z\right) &\simeq b_0\frac{\kappa}{N}\bar{T}\left(\left(1+\frac{5}{2}\sqrt{\frac{1}{\bar{T}^2b^2\left\langle\bar{T}\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z^5}}\left(\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z^2-\left\langle\left|\Psi_0\right|^2\right\rangle^2\right)\right)\left|\Psi_{-}\left(Z\right)\right|^2\right)\end{aligned}$$

and given our assumptions, we show that:

$$\begin{aligned}\left|\Psi_{+}\left(Z\right)\right|^2 &\simeq 2\bar{T}\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z^2\\ \left|\Psi_{-}\left(Z\right)\right|^2 &< <1\end{aligned}$$

so that:

$$\omega_{-}\left(Z\right)<<\omega_{+}\left(Z\right)$$

As a consequence, equation (142) shows that cells with high activity bind together, as do points with low activity, provided that the distance between them is moderate. However if low activity cells connect to high activity neighbors, the high activity cells do not bind to low activity neighbors. If we assume an overall bounded activity (which could be included as a global potential in the action, or equivalently, as a constraint), this may favour islands of connected cells with high activity, relatively independent from the rest of the thread. Actually, the activity is defined by regions. Either in some zone, activity and the level of connectivity are high, or activity is low, with the possibility of bonded groups. Given that connectivity decreases with distance, groups of high activity are of bounded extension, and given that global activity is limited, a discrete set of such groups exists.

12 Generalization: Background state for n interacting fields

The previous description may be generalized to describe n interacting types of cells, with arbitrary interactions. Each type of cells is caracterized by its activity $i = 1, ..., n$, and interacts either positively or negatively with each other. Each type is defined by a field Ψ_i and activities $\omega_i(\theta, Z)$. The general version of (71), that includes (77), becomes:

$$\begin{aligned} S_{full} &= -\frac{1}{2}\sum_i \Psi_i^{\dagger}(\theta, Z)\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega_i^{-1}\left(J, \theta, Z, [\Psi_j]_{j=1,...,n}\right)\right)\Psi_i(\theta, Z) &(143)\\ &+\frac{1}{2\eta^2}\left(S_\Gamma^{(0)} + S_\Gamma^{(1)} + S_\Gamma^{(2)} + S_\Gamma^{(3)} + S_\Gamma^{(4)}\right) + U\left(\left\{\left|\Gamma_{ij}(\theta, Z, Z', C, D)\right|^2\right\}\right) &(144)\end{aligned}$$

and equations for activities are defined by:

$$\begin{aligned}\omega_i(\theta, Z) &= F_i\left(J(\theta) + \frac{\kappa}{N}\int \frac{\omega_j\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)}{\omega_i(\theta, Z)}G^{ij}\right. &(145)\\ &\left.\times T_{ij}(Z, Z_1)\left(\bar{\mathcal{G}}_{0j}(0, Z_1) + \left|\Psi_j\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)\right|^2\right)dZ_1\right)\end{aligned}$$

The $n \times n$ matrix G has coefficients in the interval $[-1, 1]$. In the sequel, the sum over index j is implicit. For instance, if $n = 2$, the matrix g:

$$G = \begin{pmatrix} 1 & -g \\ -g & 1 \end{pmatrix}$$

represents inhibitory interactions between the two populations of cells.

As in the one field case, we define:

$$T_{ij}(Z, Z_1) = \int T_{ij}\left|\Gamma_{ij}\left(T_{ij}, \hat{T}_{ij}, \theta, Z, Z', C_{ij}, D_{ij}\right)\right|^2$$

and $S_\Gamma^{(1)}$, $S_\Gamma^{(2)}$, $S_\Gamma^{(3)}$, $S_\Gamma^{(4)}$ are given by:

$$\begin{aligned} S_\Gamma^{(1)} &= \int \Gamma_{ij}^{\dagger}\left(T_{ij}, \hat{T}_{ij}, \theta, Z, Z', C_{ij}, D_{ij}\right)\nabla_{T_{ij}}\left(\frac{\sigma_T^2}{2}\nabla_{T_{ij}} - \left(-\frac{1}{\tau\omega_i}T_{ij} + \frac{\lambda}{\omega_i}\hat{T}_{ij}\right)\left|\Psi_i(\theta, Z)\right|^2\right) &(146)\\ &\times\Gamma_{ij}\left(T_{ij}, \hat{T}_{ij}, \theta, Z, Z', C_{ij}, D_{ij}\right)\end{aligned}$$

$$\begin{aligned} S_\Gamma^{(2)} &= \int \Gamma_{ij}^{\dagger}\left(T_{ij}, \hat{T}_{ij}, \theta, Z, Z', C_{ij}, D_{ij}\right) &(147)\\ &\times\nabla_{\hat{T}_{ij}}\left(\frac{\sigma_{\hat{T}_{ij}}^2}{2}\nabla_{\hat{T}_{ij}} - \frac{\rho}{\omega_i}\left(\left(h_{ij}(Z, Z') - \hat{T}_{ij}\right)C_{ij}\left|\Psi_i(\theta, Z)\right|^2 h_C(\omega_i)\right.\right.\\ &\left.\left.-D_{ij}\hat{T}_{ij}\left|\Psi_j\left(\theta - \frac{|Z-Z'|}{c}, Z'\right)\right|^2 h_D(\omega_j')\right)\right)\Gamma_{ij}\left(T_{ij}, \hat{T}_{ij}, \theta, Z, Z', C_{ij}, D_{ij}\right)\end{aligned}$$

$$S_{\Gamma}^{(3)} = \Gamma_{ij}^{\dagger}\left(T,\hat{T},\theta,Z,Z',C_{ij},D_{ij}\right) \tag{148}$$
$$\times\nabla_{C_{ij}}\left(\frac{\sigma_{C_{ij}}^{2}}{2}\nabla_{C_{ij}}+\left(\frac{C_{ij}}{\tau_{C}\omega_{i}}-\alpha_{C}\left(1-C_{ij}\right)\frac{\omega_{j}'\left|\Psi_{j}\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}}{\omega_{i}}\right)\left|\Psi_{i}\left(\theta,Z\right)\right|^{2}\right)$$
$$\times\Gamma_{ij}\left(T,\hat{T},\theta,Z,Z',C_{ij},D_{ij}\right)$$

$$S_{\Gamma}^{(4)} = \Gamma_{ij}^{\dagger}\left(T,\hat{T},\theta,Z,Z',C_{ij},D_{ij}\right)\nabla_{D_{ij}}\left(\frac{\sigma_{D}^{2}}{2}\nabla_{D_{ij}}+\left(\frac{D_{ij}}{\tau_{D}\omega_{i}}-\alpha_{D_{ij}}\left(1-D_{ij}\right)\left|\Psi_{i}\left(\theta,Z\right)\right|^{2}\right)\right) \tag{149}$$
$$\times\Gamma_{ij}\left(T,\hat{T},\theta,Z,Z',C_{ij},D_{ij}\right)$$

where;

$$\omega_{i} = \omega_{i}\left(J,\theta,Z,|\Psi|^{2}\right)$$
$$\omega_{j}' = \omega_{j}\left(J,\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^{2}\right)$$

In (71), we added a potential:

$$U\left(\left\{\left|\Gamma\left(\theta,Z,Z',C,D\right)\right|^{2}\right\}\right)=U\left(\int T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^{2}dTd\hat{T}\right)$$

that models the constraint about the number of active connections in the system. The resolution for the bckground fld follows several stps.

12.1 Equilibrium activities

The equilibrium activities (145) can be rewritten by replacing $T_{ij}(Z,Z_1)$ with its average values depending on the connectivity field:

$$\omega_{i}(Z)=G\left(\sum_{j}\int\frac{\kappa}{N}\frac{G^{ij}T_{ij}\left|\Gamma_{ij}\left(T,\hat{T},Z,Z_{1}\right)\right|^{2}\omega_{j}\left(J,Z_{1}\right)}{\omega_{i}(Z)}\left(\mathcal{G}_{j0}+\left|\Psi_{j}\left(Z_{1}\right)\right|^{2}\right)dZ_{1}\right)$$

Considering the interactions between the different types of fields as relatively weak, we can perform an expansion of this equation around the non-interacting firing rates:

$$\omega_{i}(Z) \simeq G\left(\int\frac{\kappa}{N}\frac{G^{ii}T_{ii}\left|\Gamma_{ii}\left(T,\hat{T},Z,Z_{1}\right)\right|^{2}\omega_{i}\left(Z_{1}\right)}{\omega_{i}(Z)}\left(\mathcal{G}_{i0}+\left|\Psi_{i}\left(Z_{1}\right)\right|^{2}\right)dZ_{1}\right)$$
$$+\sum_{j}G'\left(\int\frac{\kappa}{N}\frac{G^{ii}T_{ii}\left|\Gamma_{ii}\left(T,\hat{T},Z,Z_{1}\right)\right|^{2}\omega_{i}\left(Z_{1}\right)}{\omega_{i}(Z)}\left(\mathcal{G}_{i0}+\left|\Psi_{i}\left(Z_{1}\right)\right|^{2}\right)dZ_{1}\right)$$
$$\times\sum_{j\neq i}\int\frac{\kappa}{N}\frac{G^{ij}T_{ij}\left|\Gamma_{ij}\left(T,\hat{T},Z,Z_{1}\right)\right|^{2}\omega_{j}\left(J,Z_{1}\right)}{\omega_{i}(Z)}\left(\mathcal{G}_{j0}+\left|\Psi_{j}\left(Z_{1}\right)\right|^{2}\right)dZ_{1}$$

Given that:

$$\omega_{0i}(Z)=G\left(\int\frac{\kappa}{N}\frac{G^{ii}T_{ii}\left|\Gamma_{ii}\left(T,\hat{T},Z,Z_1\right)\right|^2\omega_i(Z_1)}{\omega_i(Z)}\left(\mathcal{G}_{i0}+|\Psi_i(Z_1)|^2\right)dZ_1\right)$$

and that:

$$G'\left(\int\frac{\kappa}{N}\frac{G^{ii}T_{ii}\left|\Gamma_{ii}\left(T,\hat{T},Z,Z_1\right)\right|^2\omega_i(Z_1)}{\omega_i(Z)}\left(\mathcal{G}_{i0}+|\Psi_i(Z_1)|^2\right)dZ_1\right)=G'\left(G^{-1}\left(\omega_{0i}(Z)\right)\right)$$

we find the solution for $\omega_i(Z)$ at the first order:

$$\begin{aligned}\omega_i(Z)=&\sum_j\left(1-G'\left(G^{-1}\left(\omega_{0i}(Z)\right)\right)\right.\\&\times\left.\left(\int\frac{\kappa}{N}\frac{G^{ij}T_{ij}\left|\Gamma_{ij}\left(T,\hat{T},Z,Z_1\right)\right|^2\omega_j(J,Z_1)}{\omega_{0i}(Z)}\left(\mathcal{G}_{j0}+|\Psi_j(Z_1)|^2\right)dZ_1\right)_{j\neq i}\right)_{ij}^{-1}\omega_{0j}(Z)\end{aligned}$$

for inhibitory interactions $\omega_i(Z)<\omega_{0i}(Z)$.

12.2 Background fields for connectivity functions

The formula are similar to the case of one type of cells. We find:

$$\begin{aligned}&\Gamma_{aij}\left(T_{ij},\hat{T}_{ij},\theta,C_{ij},D_i\right)\\\simeq&\left\{\mathcal{N}\exp\left(-\frac{1}{2\sigma_C^2}\left(\frac{1}{\tau_C\omega_i}+\alpha_C\frac{\omega_j'\left|\Psi_j\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}{\omega_i}\right)|\Psi_i(\theta,Z)|^2\left(C_{ij}-C_{ij}(\theta)\right)^2\right)\right.\\&\times\exp\left(-\frac{\left(\frac{1}{\tau_D\omega_i}+\alpha_D|\Psi_i(\theta,Z)|^2\right)}{2\sigma_D^2}\left(D_i-D_i(\theta)\right)^2\right)\exp\left(-\frac{\rho\left|\bar{\Psi}_{ij}(\theta,Z,Z')\right|^2}{2\sigma_{\hat{T}}^2\omega_i}\left(\hat{T}_{ij}-\left\langle\hat{T}_{ij}\right\rangle\right)^2\right)\\&\times\left.\|\Gamma_{0ij}(\theta,Z,Z')\|\exp\left(-\frac{|\Psi_i(\theta,Z)|^2}{2\sigma_T^2\tau\omega_i}\left(T_{ij}-\langle T_{ij}\rangle\right)^2\right)\right\}_{(Z,Z'),\left\langle T_{ij}(Z,Z')\right\rangle\neq 0}\end{aligned}\tag{150}$$

$$\begin{aligned}&\Gamma_u\left(T_{ij},\hat{T}_{ij},\theta,C_{ij},D_i\right)\\\simeq&\left\{\mathcal{N}\exp\left(-\frac{1}{2\sigma_C^2}\left(\frac{1}{\tau_C\omega_i}+\alpha_C\frac{\omega_j'\left|\Psi_j\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}{\omega_i}\right)|\Psi_i(\theta,Z)|^2\left(C_{ij}-C_{ij}(\theta)\right)^2\right)\right.\\&\times\exp\left(-\frac{\left(\frac{1}{\tau_D\omega_i}+\alpha_D|\Psi_i(\theta,Z)|^2\right)}{2\sigma_D^2}\left(D_i-\langle D_i\rangle\right)^2\right)\\&\left.\exp\left(-\frac{\rho\left|\bar{\Psi}_{ij}(\theta,Z,Z')\right|^2}{2\sigma_{\hat{T}}^2}\left(\hat{T}_{ij}-\left\langle\hat{T}_{ij}\right\rangle\right)^2\right)\delta(T_{ij})\right\}_{(Z,Z'),\left\langle T_{ij}(Z,Z')\right\rangle=0}\end{aligned}\tag{151}$$

with:

$$\left|\bar{\Psi}_{ij}(\theta,Z,Z')\right|^2=\frac{\left(C_{ij}(\theta)\left|\Psi_i(\theta,Z)\right|^2h_C(\omega_i(\theta,Z))+D_i(\theta)\left|\Psi_j\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2h_D\left(\omega_j\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right)\right)}{2\omega_i(\theta,Z)}$$

and where $\mathcal{N}$ is a normalization factor ensuring that the constraint over the number of connections is satisfied. For the conjugate fields, we find:

$$\Gamma_a^{\dagger}\left(T,\hat{T},\theta,C,D\right)\simeq\{1\}_{(Z,Z'),\langle T(Z,Z')\rangle\neq0}$$

$$\Gamma_u^{\dagger}\left(T,\hat{T},\theta,C,D\right)\simeq\{\delta(T)\}_{(Z,Z'),\langle T(Z,Z')\rangle=0}$$

12.3 averages for the background field

To complete the resolution of the minimization equation, we also need the averages of the dynamic variables in the state defined by the background field. The generalization from the one-field case is straightforward:

$$C_{ij}\rightarrow\langle C(\theta)\rangle=\frac{\alpha_C\frac{\omega_j'\left|\Psi_j\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}{\omega_i}}{\frac{1}{\tau_C\omega_i}+\alpha_C\frac{\omega_j'\left|\Psi_j\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}{\omega_i}}=\frac{\alpha_C\omega_j'\left|\Psi_j\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}{\frac{1}{\tau_C}+\alpha_C\omega_j'\left|\Psi_j\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2}\equiv C_j(\theta,Z,Z')\tag{152}$$

$$D_i\rightarrow\langle D(\theta)\rangle=\frac{\alpha_D\omega_i\left|\Psi(\theta,Z)\right|^2}{\frac{1}{\tau_D}+\alpha_D\omega_j'\left|\Psi(\theta,Z)\right|^2}\equiv D(\theta)\tag{153}$$

$$\langle T_{ij}(Z,Z')\rangle=\lambda\tau\left\langle\hat{T}_{ij}(Z,Z')\right\rangle=\frac{\lambda\tau h(Z,Z')\left\langle C_j(\theta,Z,Z')h_C(\omega_i(\theta,Z))\left|\Psi_i(\theta,Z)\right|^2\right\rangle}{C_j(\theta,Z,Z')\left|\Psi_i(\theta,Z)\right|^2h_C(\omega_i(\theta,Z))+D_i(\theta,Z,Z')\left|\Psi_i\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2h_D\left(\omega_j\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right)}\tag{154}$$

We look for a static background state for the whole system. In the static case, we assume that the static background field $\Psi_{0j}(Z)$ is the minimum of $V(\Psi_j)$.

12.4 General equations and methdod of resolution

Derivation of equations for connectivity functions is similar to one field case. As before, we first solve for the activities, then for the connectivities and the neurons fields.

12.4.1 Expression of $\omega(Z)$

We use (125) to replace $\omega_j'\left|\Psi_j(Z')\right|^2$:

$$\omega_j(Z')\left|\Psi_j(Z')\right|^2=\frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)-T_{ij}(Z,Z')}{T_{ij}(Z,Z')}\left(\frac{1}{\alpha_D\tau_D}+\omega_i\left|\Psi_i(Z)\right|^2\right)-\frac{1}{\alpha_C\tau_C}$$

which yilds:

$$\begin{aligned}
\omega_i(Z) &= G\left(\sum_j \int \frac{\kappa}{N} \frac{G^{ij} T_{ij} \left|\Gamma_{ij}\left(T, \hat{T}, Z, Z_1\right)\right|^2 \omega_j(J, Z_1)}{\omega_i(Z)} \left(\mathcal{G}_{j0} + |\Psi_j(Z_1)|^2\right) dZ_1\right) \\
&\rightarrow G\left(\sum_j G^{ij} \int \frac{\kappa}{N} \left(\frac{\lambda\tau \exp\left(-\frac{|Z-Z_1|}{\nu c}\right) - T_{ij}(Z, Z_1)}{T_{ij}(Z, Z_1)} \left(\frac{1}{\alpha_D \tau_D} + \omega_i |\Psi_i(Z)|^2\right) - \frac{1}{\alpha_C \tau_C}\right)\right. \\
&\quad \left. \times \frac{T_{ij} \left|\Gamma_{ij}\left(T, \hat{T}, Z, Z_1\right)\right|^2}{\omega_i(Z)} dZ_1\right) \\
&\simeq G\left(\sum_j G^{ij} \int \frac{\kappa}{N} \left(\left(\lambda\tau \exp\left(-\frac{|Z-Z_1|}{\nu c}\right) - T_{ij}(Z, Z_1)\right) \left(\left(\frac{1}{\alpha_D \tau_D} - \frac{T_{ij}(Z, Z_1)}{\alpha_C \tau_C}\right) \omega_i^{-1} + |\Psi_i(Z)|^2\right)\right) dZ_1\right)
\end{aligned}$$

We can replace $T(Z, Z_1)$ in the integral by its average:

$$\frac{1}{V} T_{ij}(Z) = \frac{1}{V} \int T_{ij}(Z, Z_1)\, dZ_1$$

so that:

$$\omega_i(Z) = G\left(\sum_j G^{ij} \int \frac{\kappa}{N} \left((\lambda\tau\nu c - T_{ij}(Z)) \left(\left(\frac{1}{\alpha_D \tau_D} - \frac{T_{ij}(Z)}{V \alpha_C \tau_C}\right) \omega_i^{-1} + |\Psi_i(Z)|^2\right)\right)\right)$$

The solution for the firing rates is defined by a function:

$$\omega_i(Z) = \hat{G}\left(\left(G^{ij}\right), (T_{ij}(Z)), |\Psi_i(Z)|^2\right) = \hat{G}_i\left((T_{ij}(Z)), |\Psi_i(Z)|^2\right)$$

Here, $(T_{ij}(Z))$ denotes the set of all the $T_{ij}(Z)$ with j running over the whole space.

To find $\omega_i(Z)$, we thus have to determine $T_{ij}(Z)$ and $|\Psi_i(Z)|^2$.

12.4.2 Equations for $T_{ij}(Z)$, $T_{ij}(Z, Z')$ and $|\Psi_i(Z)|^2$

As in the one field case, the average connectivities at Z are given by:

$$T_{ij}(Z) = \frac{\lambda\tau\nu c}{1 + \frac{\frac{1}{\tau_C \alpha_C} + \Omega_j}{\frac{1}{\tau_D \alpha_D} + \hat{G}\left((T_{ij}(Z)), |\Psi_i(Z)|^2\right) |\Psi_i(Z)|^2}} = \frac{\lambda\tau\nu c \left(\frac{1}{\tau_D \alpha_D} + \hat{G}_i\left((T_{ij}(Z)), |\Psi_i(Z)|^2\right) |\Psi_i(Z)|^2\right)}{\frac{1}{\tau_D \alpha_D} + \frac{1}{\tau_C \alpha_C} + \Omega_j + \hat{G}_j\left((T_{ij}(Z)), |\Psi_i(Z)|^2\right) |\Psi_i(Z)|^2} \tag{155}$$

where:

$$\Omega_j = \frac{1}{V} \int \hat{G}_j\left((T_{jk}(Z')), |\Psi_j(Z')|^2\right) |\Psi_j(Z')|^2\, dZ' \tag{156}$$

and this leads to the following formula for $T_{ij}(Z, Z')$:

$$T_{ij}(Z, Z') = \frac{\lambda\tau \exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{1 + \frac{\frac{1}{\tau_C \alpha_C} + \omega'_j |\Psi_j(Z')|^2}{\frac{1}{\tau_D \alpha_D} + \omega_i |\Psi_i(Z)|^2}} = \frac{\lambda\tau \exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{1 + \frac{\frac{1}{\tau_C \alpha_C} + \hat{G}_j\left((T_{jk}(Z)), |\Psi_j(Z')|^2\right) |\Psi_j(Z')|^2}{\frac{1}{\tau_D \alpha_D} + \hat{G}_i\left((T_{ij}(Z)), |\Psi_i(Z)|^2\right) |\Psi_i(Z)|^2}} \tag{157}$$

On the other hand, the minimization equation for the field Ψ_i is:

$$0\simeq\left(\left(\frac{1}{\hat{G}_i\left(\left(T_{ij}\left(Z,Z_1\right)\right)_{Z_1},\left|\Psi_i\left(Z\right)\right|^2\right)}\right)^2+\left(\left|\Psi_i\left(Z\right)\right|^2-\int T_{ij}\left(Z',Z_1\right)\left|\Psi_{j0}\left(Z\right)\right|^2dZ_1\right)\right)\Psi_i\left(\theta,Z\right)$$

with solutions:

$$\Psi_i\left(\theta,Z\right)=0$$

or $\left|\Psi_i\left(Z\right)\right|^2$ satisfying:

$$\left(\frac{1}{\hat{G}_i\left(\left(T_{ij}\left(Z,Z_1\right)\right)_{Z_1},\left|\Psi_i\left(Z\right)\right|^2\right)}\right)^2+\left|\Psi_i\left(Z\right)\right|^2\simeq\sum_j\int T_{ij}\left(Z,Z'\right)\left|\Psi_{j0}\left(Z'\right)\right|^2k_{ij}\left(Z,Z'\right)dZ'$$

This equation can be approximated by:

$$\begin{aligned}\left(\frac{1}{\hat{G}_i\left(\left(T_{ij}\left(Z,Z_1\right)\right)_{Z_1},\left|\Psi_i\left(Z\right)\right|^2\right)}\right)^2+\left|\Psi_i\left(Z\right)\right|^2&\simeq\sum_j T_{ij}\left(Z\right)\frac{\int\left|\Psi_{j0}\left(Z'\right)\right|^2k_{ij}\left(Z,Z'\right)dZ'}{V}\\&\equiv\sum_j T_{ij}\left(Z\right)\left\langle\left|\Psi_{j0}\left(Z'\right)\right|^2\right\rangle_Z\end{aligned}\tag{158}$$

12.4.3 Expression of $\left|\Psi_i\left(Z\right)\right|^2$ and $T_{ij}\left(Z\right)$ as functions of average values

To solve (155) and (158) for $T_{ij}\left(Z\right)$ and $\left|\Psi_i\left(Z\right)\right|^2$, we first use (155) to express $\hat{G}\left(\left(T_{ij}\left(Z\right)\right),\left|\Psi_i\left(Z\right)\right|^2\right)\left|\Psi_i\left(Z\right)\right|^2$ as a function of $T_{ij}\left(Z\right)$:

$$\hat{G}_i\left(\left(T_{ij}\left(Z\right)\right),\left|\Psi_i\left(Z\right)\right|^2\right)\left|\Psi_i\left(Z\right)\right|^2=\frac{\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega_j\right)T_{ij}\left(Z\right)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}{\lambda\tau\nu c-T_{ij}\left(Z\right)}\tag{159}$$

Inserting this result in (240) leads to thefollowng equation for $\left|\Psi\left(Z\right)\right|^2$:

$$\left(\frac{\lambda\tau\nu c-T_{ij}\left(Z\right)}{\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega_j\right)T_{ij}\left(Z\right)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}\left|\Psi_i\left(Z\right)\right|^2\right)^2+\left|\Psi_i\left(Z\right)\right|^2\simeq\sum_j T_{ij}\left(Z\right)\left\langle\left|\Psi_{j0}\left(Z'\right)\right|^2\right\rangle_Z\tag{160}$$

with solution:

$$\left|\Psi_i\left(Z\right)\right|^2=\frac{2\sum_j T_{ij}\left(Z\right)\left\langle\left|\Psi_{j0}\left(Z'\right)\right|^2\right\rangle_Z}{\left(1+\sqrt{1+4\left(\frac{\lambda\tau\nu c-T_{ij}(Z)}{\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega_j\right)T_{ij}(Z)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}\right)^2\sum_j T_{ij}\left(Z\right)\left\langle\left|\Psi_{j0}\left(Z'\right)\right|^2\right\rangle_Z}\right)}\tag{161}$$

Ultimately, inserting this result in (159) writes $\hat{G}_i$ as a function of $\left(T_{ij}\left(Z\right)\right)$, $\left(\hat{\Omega}_j\right)$:

$$\hat{G}_i\left(\left(T_{ij}\left(Z\right)\right),\left|\Psi_i\left(Z\right)\right|^2\right)\left|\Psi_i\left(Z\right)\right|^2=\hat{G}_i\left(\left(T_{ij}\left(Z\right)\right),\left|\Psi_i\left(Z,\left(\hat{\Omega}_j\right),T_{ij}\left(Z\right)\right)\right|^2\right)\tag{162}$$

with:

$$\hat{\Omega}_j=\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega_j\right)$$

Averaging this relation will yield a consistency condtn for the $\hat{\Omega}_j$.

12.4.4 Identification of Ω_j and $\bar{T}_{ij}(Z)$

The resolution is finalized by using (162) to identify the constant Ω_j. Averagng (162) and using (156) yields:

$$\begin{aligned}\Omega_i &= \hat{G}_i\left(\left(\bar{T}_{ij}\right), \frac{2\sum_j \bar{T}_{ij}(Z)\left\langle\left|\Psi_{j0}(Z')\right|^2\right\rangle_Z}{\left(1+\sqrt{1+\frac{4}{\Omega_i^2}\sum_j \bar{T}_{ij}(Z)\left\langle\left|\Psi_{j0}(Z')\right|^2\right\rangle_Z}\right)}\right) \\ &\times \frac{2\sum_j \bar{T}_{ij}(Z)\left\langle\left|\Psi_{j0}(Z')\right|^2\right\rangle_Z}{\left(1+\sqrt{1+\frac{4}{\Omega_i^2}\sum_j \bar{T}_{ij}(Z)\left\langle\left|\Psi_{j0}(Z')\right|^2\right\rangle_Z}\right)}\end{aligned} \tag{163}$$

The $\bar{T}_{ij}(Z)$ can be replaced using (159) and (156), and computing averages:

$$\begin{aligned}\Omega_j &\simeq \frac{1}{V}\int \hat{G}\left(\left(T_{jk}(Z)\right), \left|\Psi_j(Z)\right|^2\right)\left|\Psi_j(Z)\right|^2 \\ &= \frac{1}{V}\int \frac{\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega_k\right)T_{jk}(Z)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}{\lambda\tau\nu c-T_{jk}(Z)} \\ &\simeq \frac{\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega_k\right)\bar{T}_{jk}-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}{\lambda\tau\nu c-\bar{T}_{jk}}\end{aligned} \tag{164}$$

where:

$$\bar{T}_{ij} = \frac{1}{V}\int T_{ij}(Z)\,dZ \tag{165}$$

is the average cnnctvt of the system. This can be solved for $\bar{T}_{ij}$:

$$\bar{T}_{jk} = \frac{\lambda\tau\nu c\Omega_j+\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}{\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega_j+\Omega_k\right)} \simeq \frac{\lambda\tau\nu c\Omega_j}{\Omega_j+\Omega_k} = \frac{2\bar{T}\Omega_j}{\Omega_j+\Omega_k} \tag{166}$$

Inserted in (163), this yields an equation for the Ω_j. Once found (166) yields $\bar{T}_{ij}$.

We can assume that the several types have approximatively the constant ration of activt, so that $\frac{\Omega_j}{\Omega_j+\Omega_k} \simeq \frac{1}{2}g_{jk}$ so that:

$$\bar{T}_{jk} \simeq g_{jk}\bar{T}$$

so that equation (163) bcms:

$$\Omega_i = \hat{G}_i\left(\bar{T}, \frac{2\bar{T}\sum_j g_{ij}\left\langle\left|\Psi_{j0}(Z')\right|^2\right\rangle_Z}{\left(1+\sqrt{1+\frac{4\bar{T}}{\Omega_i^2}\sum_j g_{ij}\left\langle\left|\Psi_{j0}(Z')\right|^2\right\rangle_Z}\right)}\right)\frac{2\bar{T}\sum_j g_{ij}\left\langle\left|\Psi_{j0}(Z')\right|^2\right\rangle_Z}{\left(1+\sqrt{1+\frac{4\bar{T}}{\Omega_i^2}\sum_j g_{ij}\left\langle\left|\Psi_{j0}(Z')\right|^2\right\rangle_Z}\right)} \tag{167}$$

and solvng ths equation ylds Ω_i.

12.4.5 Deriving $T_{ij}(Z)$

Once the Ω_j is determined,, we can use (161) to substitute $\left|\Psi_i(Z)\right|^2$ in (155). This results in an equation for $T_{ij}(Z)$. Substituting the various expressions in (157), we obtain the connectivity $T_{ij}(Z, Z')$.

12.5 Particular case

We can solve the system for G an increasing function of the form $G_i(x_j) = G\left(\sum_j G_{ij} x_j\right) \simeq b_0 G_{ij} x$ for $x_j < 1$. We start with the derivation of Ω:

$$\begin{aligned} \omega_i(Z) &= G\left(\sum_j G^{ij} \int \frac{\kappa}{N}\left((\lambda\tau\nu c - T_{ij}(Z))\left(\left(\frac{1}{\alpha_D\tau_D} - \frac{T_{ij}(Z)}{V\alpha_C\tau_C}\right)\omega_i^{-1} + |\Psi_i(Z)|^2\right)\right)\right) \\ &\simeq G\left(\sum_j V G^{ij} \frac{\kappa}{N}\left(\left(\lambda\tau\nu c - \bar{T}_{ij}\right)\left(\left(\frac{1}{\alpha_D\tau_D} - \frac{\bar{T}_{ij}}{V\alpha_C\tau_C}\right)\omega_i^{-1} + |\Psi_i(Z)|^2\right)\right)\right) \end{aligned}$$

In first approximation, we thus find for the activity:

$$\begin{aligned} \omega_i(Z) &\simeq G\left(\sum_j V G^{ij} \frac{\kappa}{N}\left(\left(\lambda\tau\nu c - \bar{T}_{ij}\right)\left(|\Psi_i(Z)|^2\right)\right)\right) \\ &\simeq b_0\left(\sum_j V G^{ij} \frac{\kappa}{N}\left(\left(\lambda\tau\nu c - \bar{T}_{ij}\right)\left(|\Psi_i(Z)|^2\right)\right)\right) \end{aligned}$$

Given that the average $\bar{T}_{ij}$ is given by (164):

$$\bar{T}_{ij} = \frac{\lambda\tau\nu c\Omega_i + \frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}{\frac{1}{\tau_D\alpha_D} + \frac{1}{\tau_C\alpha_C} + \Omega_j + \Omega_i} \simeq \frac{\lambda\tau\nu c\Omega_i}{\Omega_j + \Omega_i} \simeq g_{ij}\bar{T}$$

we obtain:

$$\omega_i(Z) \simeq b G^i \bar{T} |\Psi_i(Z)|^2$$

with $G^i = \sum_j G^{ij}(1 - g_{ij})$ and $b = b_0 \frac{\kappa}{N} V$

12.5.1 Derivation of the Ω_i

In this particular case, equation (247) writes:

$$\Omega_i = G^i b\bar{T}\left(\frac{2\sum_j \bar{T}_{ij}\left\langle|\Psi_{j0}(Z')|^2\right\rangle_Z}{\left(1 + \sqrt{1 + \frac{4}{\Omega_i^2}\sum_j \bar{T}_{ij}\left\langle|\Psi_{j0}(Z')|^2\right\rangle_Z}\right)}\right)^2$$

that can be rewritten as:

$$1 \simeq \frac{bG^i\bar{T}\Omega_i^3}{\left(bG^i\bar{T}\left(\sum_j \bar{T}_{ij}\left\langle|\Psi_{j0}(Z')|^2\right\rangle_Z\right)\Omega_i - 1\right)^2}$$

Defining:

$$\bar{T}\left\langle\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z = \sum_j \bar{T}_{ij}\left\langle|\Psi_{j0}(Z')|^2\right\rangle_Z$$

and its average:

$$\bar{T}\left\langle\left|\bar{\Psi}_{i0}\right|^2\right\rangle = \left\langle\sum_j \bar{T}_{ij}\left\langle|\Psi_{j0}(Z')|^2\right\rangle_Z\right\rangle$$

we find:

For $d=\left(bG^{i}\bar{T}\right)^{2}\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}\right|^{2}\right\rangle\right)^{3}<\frac{27}{4}$ there is one solution:

$$\Omega_{i}\simeq\left(bG^{i}\bar{T}\right)^{-\frac{1}{3}}<<1 \tag{168}$$

For $d=\left(bG^{i}\bar{T}\right)^{2}\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}\right|^{2}\right\rangle\right)^{3}>\frac{27}{4}$ there are three solutions. The first one is:

$$\Omega_{i}\simeq\left(bG^{i}\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}\right|^{2}\right\rangle_{Z}\right)^{2} \tag{169}$$

The two other solutions are centered around $\frac{1}{bG^{i}\bar{T}\left\langle\bar{T}\left|\bar{\Psi}_{i0}\right|^{2}\right\rangle}$. We set:

$$\Omega_{i}=\frac{1\pm\delta}{bG^{i}\bar{T}\left\langle\bar{T}\left|\bar{\Psi}_{i0}\right|^{2}\right\rangle}$$

and (**??**) becomes:

$$\left(bG^{i}\bar{T}\right)^{2}\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}\right|^{2}\right\rangle\right)^{3}\simeq\frac{1}{\delta^{2}}$$

so that:

$$\Omega_{i}=\frac{1\pm\sqrt{\frac{1}{\left(bG^{i}\bar{T}\right)^{2}\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}\right|^{2}\right\rangle\right)^{3}}}}{bG^{i}\bar{T}\left\langle\bar{T}\left|\bar{\Psi}_{i0}\right|^{2}\right\rangle} \tag{170}$$

Solution (168) corresponds to relatively low activity, i.e. $\left\langle\left|\bar{\Psi}_{i0}\right|^{2}\right\rangle<<1$, so we only consider solutions (169) and (170) in the sequel. For $\lambda\tau\nu cb^{2}>>1$ the solutions of (170) are both approximatively given by:

$$\Omega_{i}=\frac{1}{bG^{i}\bar{T}\left\langle\bar{T}\left|\bar{\Psi}_{i0}\right|^{2}\right\rangle}$$

Writing Ω_{i-} this solution and Ω_{i+} the solution (169), we gather them and write $\Omega_{i\pm}$. To these solutions for Ω, we associate the Z dependent parameters:

$$Y_{i+}\left(Z\right)\simeq\frac{\left(bG^{i}\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}\left(Z'\right)\right|^{2}\right\rangle_{Z}\right)^{2}}{2} \tag{171}$$

$$Y_{i-}\left(Z\right)=\frac{1}{bG^{i}\bar{T}\left\langle\bar{T}\left|\bar{\Psi}_{i0}\left(Z'\right)\right|^{2}\right\rangle_{Z}} \tag{172}$$

where $Y_{i\pm}$ gathers the possibilities Y_{i+} and Y_{i-}. Similarly to the derivation in Appendix 3 of $T\left(Z\right)$, $T_{ij}\left(Z_{\pm}\right)$ is a function of these parameters and can take the values:

$$T_{ij}\left(Z_{\pm}\right)=\frac{\lambda\tau\nu cY_{i\pm}\left(Z\right)+\frac{1}{\tau_{D}\alpha_{D}}\lambda\tau\nu c}{\frac{1}{\tau_{D}\alpha_{D}}+\frac{1}{\tau_{C}\alpha_{C}}+\Omega_{j\pm}+Y_{i\pm}\left(Z\right)}$$

Ultimately, we show that there are four possibilities for the connectivity functions that are written:

$$T_{ij}\left(Z_{\pm},Z'_{\pm}\right)=\frac{\lambda\tau\exp\left(-\frac{\left|Z-Z'\right|}{\nu c}\right)\left(\frac{1}{\tau_{D}\alpha_{D}}+Y_{i\pm}\left(Z\right)\right)}{\frac{1}{\tau_{D}\alpha_{D}}+Y_{i\pm}\left(Z\right)+\frac{1}{\alpha_{C}\tau_{C}}+Y_{j\pm}\left(Z'\right)}$$

Given our assumptions on d, in most cases:

$$Y_{+}(Z) >> Y_{-}(Z)$$

Moreover:

$$\frac{1}{\tau_D \alpha_D} << 1, \ \frac{1}{\alpha_C \tau_C} << 1$$

so that due to the threshold in connectivity, we have:

$$\begin{aligned}
T_{ij}\left(Z_{-}, Z'_{+}\right) &= \frac{\lambda\tau \exp\left(-\frac{\left|Z-Z'\right|}{\nu c}\right)\left(\frac{1}{\tau_D\alpha_D} + \frac{1}{bG^i\bar{T}\left\langle \bar{T}\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z}\right)}{\frac{1}{\tau_D\alpha_D} + \frac{1}{\alpha_C\tau_C} + \frac{1}{bG^i\bar{T}\left\langle \bar{T}\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z} + \frac{\left(bG^j\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{j0}(Z')\right|^2\right\rangle_{Z'}\right)^2}{2}} \simeq 0 \qquad (173) \\
T_{ij}\left(Z_{+}, Z'_{+}\right) &= \frac{\lambda\tau \exp\left(-\frac{\left|Z-Z'\right|}{\nu c}\right)\left(\frac{1}{\tau_D\alpha_D} + \frac{\left(bG^i\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z\right)^2}{2}\right)}{\frac{1}{\tau_D\alpha_D} + \frac{1}{\alpha_C\tau_C} + \frac{\left(bG^i\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z\right)^2}{2} + \frac{\left(bG^j\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{j0}(Z')\right|^2\right\rangle_{Z'}\right)^2}{2}} \simeq \frac{G^i\lambda\tau\exp\left(-\frac{\left|Z-Z'\right|}{\nu c}\right)}{\left(G^i+G^j\right)} \\
T_{ij}\left(Z_{+}, Z'_{-}\right) &= \frac{\lambda\tau \exp\left(-\frac{\left|Z-Z'\right|}{\nu c}\right)\left(\frac{1}{\tau_D\alpha_D} + \frac{\left(bG^i\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z\right)^2}{2}\right)}{\frac{1}{\tau_D\alpha_D} + \frac{1}{\alpha_C\tau_C} + \frac{\left(bG^i\bar{T}\right)\left(\bar{T}\left\langle\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z\right)^2}{2} + \frac{1}{bG^j\bar{T}\left\langle \bar{T}\left|\bar{\Psi}_{j0}(Z')\right|^2\right\rangle_Z}} \simeq \lambda\tau\exp\left(-\frac{\left|Z-Z'\right|}{\nu c}\right) \\
T\left(Z_{-}, Z'_{-}\right) &\simeq \frac{\lambda\tau \exp\left(-\frac{\left|Z-Z'\right|}{\nu c}\right) + \frac{1}{bG^i\bar{T}\left\langle \bar{T}\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z}}{1 + \frac{\tau_D\alpha_D}{\alpha_C\tau_C} + \frac{1}{bG^i\bar{T}\left\langle \bar{T}\left|\bar{\Psi}_{i0}(Z')\right|^2\right\rangle_Z} + \frac{1}{bG^j\bar{T}\left\langle \bar{T}\left|\bar{\Psi}_{j0}(Z')\right|^2\right\rangle_Z}} \simeq \frac{G^j\lambda\tau\exp\left(-\frac{\left|Z-Z'\right|}{\nu c}\right)}{\left(G^i+G^j\right)}
\end{aligned}$$

Part III Dynamic aspects of the system: modifications of background fields due to external sources.

Having derived the possible forms for background, i.e. equilibrium states, we turn to the study of the dynamical aspects of the system. External sources may induce fluctuations around static values. We first present the results of ([52]) concerning flutuations in cell background field and activities: we derive the dynamic corrections for the neurons background field $\Psi(\theta, Z)$ and obtain a formula for these corrections $\delta\Psi(\theta, Z)$ as a function of the corrections in the activities $\omega(J(\theta, Z), \theta, Z)$. We then derive a wave equation for these frequencies, and show that stable oscillations may occur. We also inspect the interferences of waves of activities induced by several oscillating sources and their effect on the connectivities. Some patterns of bound cells arise from these interferences so that, the propagation of periodic perturbations may change the static background field for connectivity functions in the long run.

We also consider the association of signals through synchronized stimuli and, ultimately, the recovery of a combined state by the reactivation of part of the full state.

The approach in this section is local and relies only partly on the field formalism. We show in the next papers how to derive more generally the presented result via a field-theoretic based description.

13 Source induced fluctuations of neurons background field

We first study dynamic fluctuations in cells background field $\Psi(\theta, Z)$ and activities $\omega\left(J(\theta, Z), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)$ around some static background state. These fluctuations may be induced by some time-dependent current $J(\theta, Z)$. The connectivity function are assumed to remain constant, since the time scale of their modification is slower. We follow the derivations of the fluctuating background state $\Psi(\theta, Z)$ in ([52]).

13.1 Minimization equation for dynamic fields

In ([52]) we show that the background field $\Psi(\theta, Z)$ can be decomposed in a static part and a fluctuation part:

$$\begin{aligned}
\Psi(\theta, Z) &\simeq \Psi_0(Z) + \delta\Psi(\theta, Z) \\
\Psi^{\dagger}(\theta, Z) &\simeq \Psi_0^{\dagger}(Z)
\end{aligned}$$

where:

$$\left|\delta\Psi\left(\theta^{(j)}, Z_j\right)\right| < |\Psi_0(Z_j)|$$

The static part $\Psi_0(Z)$ is the minimum of $V(\Psi)$. To simplify, we could consider $|\Psi(Z)|^2$ as exogenous and minimizing a stabilizing potential with minimum $\Psi_0(\theta, Z) = X_0$.

Expanding the potential around $\Psi_0(\theta, Z)$ and setting $V = 1$, yields at the second order the effective action:

$$\begin{aligned}
S_{\Psi}\left(\Psi, \Psi^{\dagger}\right) = &-\frac{1}{2}\int \delta\Psi^{\dagger}(\theta, Z)\left(\nabla_{\theta}\left(\frac{\sigma_{\theta}^2}{2}\nabla_{\theta} - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right) X_0 \\
&-\frac{1}{2}\int \delta\Psi^{\dagger}(\theta, Z)\left(\nabla_{\theta}\left(\frac{\sigma_{\theta}^2}{2}\nabla_{\theta} - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right)\delta\Psi(\theta, Z) \\
&+\frac{1}{2}\int \delta\Psi^{\dagger}(\theta, Z)\, U''(X_0)\, \delta\Psi(\theta, Z)
\end{aligned}$$

with $|\Psi|^2 = X_0 + \sqrt{X_0}\left(\delta\left(\Psi^{\dagger} + \delta\Psi\right)\right)$. This leads to the first order condition for $\delta\Psi(\theta_1, Z_1)$:

$$\begin{aligned}
0 = &\frac{1}{2}\delta\Psi^{\dagger}(\theta, Z)\left(-\nabla_{\theta}\left(\frac{\sigma_{\theta}^2}{2}\nabla_{\theta} - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + X_0\right)\right) + U''(X_0)\right) \\
&-\frac{1}{2}\int \delta\Psi^{\dagger}(\theta_1, Z_1)\sqrt{X_0}\left(\nabla_{\theta}\frac{\delta\omega^{-1}\left(J(\theta_1), \theta_1, Z_1, \mathcal{G}_0 + X_0\right)}{\delta|\Psi(\theta, Z)|^2}\right) X_0 d\theta_1 dZ_1
\end{aligned}$$

with solution $\delta\Psi^{\dagger}(\theta, Z) = 0$. This implies that the first order condition for $\delta\Psi^{\dagger}(\theta, Z)$ becomes:

$$\begin{aligned}
0 = &-\frac{1}{2}\left(\nabla_{\theta}\left(\frac{\sigma_{\theta}^2}{2}\nabla_{\theta} - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right) X_0 \\
&-\frac{1}{2}\left(\nabla_{\theta}\left(\frac{\sigma_{\theta}^2}{2}\nabla_{\theta} - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right)\delta\Psi(\theta, Z) \\
&+\frac{1}{2}U''(X_0)\,\delta\Psi(\theta, Z)
\end{aligned} \tag{174}$$

In first approximation, for $U''(X_0) >> 1$ and $\sigma_\theta^2 << 1$, this yields[15]:

$$\delta\Psi(\theta, Z) \simeq -\frac{\nabla_\theta \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)}{U''(X_0) + \nabla_\theta \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)} X_0 \tag{176}$$

$$\simeq -\frac{\nabla_\theta \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)}{U''(X_0)} X_0$$

Equation (174) also rewrites:

$$\left(-\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right) + U''(X_0)\right)(\delta\Psi(\theta, Z) + X_0) = U''(X_0) X_0 \tag{177}$$

Equation (177) can be used to write $\delta\Psi(\theta, Z)$ as a function of $\omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)$:

$$\delta\Psi(\theta, Z) = \left(\frac{\left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right)}{U''(X_0) - \left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right)}\right) X_0 \tag{178}$$

$$= -\frac{\nabla_\theta\left(\omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{U''(X_0) - \left(\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\right)} X_0$$

$$\simeq \frac{\nabla_\theta\left(\omega\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{\left(\omega\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)^2 U''(X_0)} X_0$$

leading to system of equation for activities and field:

$$\omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)$$

$$= G\left(J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega\left(J, \theta - \frac{|Z-Z_1|}{c}, Z_1, \Psi\right) T\left(Z, \theta, Z_1, \theta - \frac{|Z-Z_1|}{c}\right)}{\omega\left(J, \theta, Z, |\Psi|^2\right)} \left(\mathcal{G}_0 + \left|\Psi\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)\right|^2\right) dZ_1\right)$$

and:

$$\delta\Psi(\theta, Z) = \frac{\nabla_\theta\left(\omega\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)}{\left(\omega\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)^2 U''(X_0)} X_0 \tag{179}$$

Relation (178) is sufficient to derive the next section's activities equations, but can however be used to find $\delta\Psi(\theta, Z)$, at our order of approximation (see appendix 4). In the local approximation and for slowly varying currents, we show that the fluctuation $\delta\Psi(\theta, Z)$ (179) is equal to:

$$\delta\Psi(\theta, Z) = \left(G^{-1}\left(-\frac{U''(X_0)}{X_0}\exp\left(H^{-1}\left(\frac{\theta}{\Gamma\left(\mathcal{G}_0(Z_1) + \sqrt{X_0}\right)} + d\right)\right)\right) - J(\theta, Z)\right) \tag{180}$$

$$\times \exp\left(H^{-1}\left(\frac{\theta}{\Gamma\left(\mathcal{G}_0(Z_1) + \sqrt{X_0}\right)} + d\right)\right)$$

[15] Note that for a slowly background field $\Psi_0(\theta, Z)$, equation (176) remains valid and becomes:

$$\delta\Psi(\theta, Z) \simeq -\frac{\nabla_\theta \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)}{U''(X_0)} \Psi_0(\theta, Z) \tag{175}$$

with:

$$H(Y)=\int\frac{dY}{G^{-1}\left(-\frac{U''(X_0)}{X_0}\exp Y\right)-J(\theta,Z)}$$

and:

$$\Gamma=\int\frac{\kappa}{N}\frac{|Z-Z_1|}{c}T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)dZ_1$$

The constant d is chosen so that $\lim_{\theta\rightarrow\infty}\delta\Psi(\theta,Z)=0$.

The field $\Psi\left(\theta^{(j)},Z_j\right)$ is the - phase-dependent - background field. It is null in the trivial phase, so that the effective action is the "classical" one. In a non-trivial phase, $\Psi\left(\theta^{(j)},Z_j\right)$ is not null and may be time-dependent. It describes the accumulation of currents or signals that shapes the long-term dynamics of activities. Incidentally, we note that a non-trivial minimum that depends on the system parameters should allow for phase transition in the system of activities.

14 Dynamic wave equation for activities

This section studies the dynamic solutions of (67). We use relation (178) to replace the non-static part of the field Ψ as a function of the activities and then deduce a wave equation for the activities.

14.1 Differential equation for activities in the local approximation

A local approximation of (78) around some position-independent static equilibrium can be derived for non static activities. Assuming a static background field Ψ_0, we derived above the relation between $\delta\Psi(\theta,Z)$ and $\omega\left(J,Z,|\Psi|^2\right)$ (see (179)):

$$\delta\Psi(\theta,Z)\simeq\frac{\nabla_\theta\omega\left(J,Z,|\Psi|^2\right)}{V''(\Psi_0(Z))\,\omega_0^2\left(J,Z,|\Psi|^2\right)}\Psi_0 \tag{181}$$

where $\omega\left(J,Z,|\Psi|^2\right)$ is the time-dependent firing rate, or activity.

We can find a local approximation of (67) if we expand $\omega\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)$ to the second-order in $Z-Z_1$, and consider the other terms in the right-hand side of (67) as corrections. The equation for $\omega\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)$ is:

$$\begin{aligned}F^{-1}(\omega(J(\theta),\theta)) &= J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)} \\ &\quad\times\left(\left|\Psi_0+\delta\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\end{aligned} \tag{182}$$

where $F=\frac{1}{G}$ and F^{-1} the reciprocal function of F. We then expand $\omega\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)$ around $\omega(\theta,Z)$ to the second-order in $Z-Z_1$ and compute the integrals, which yields for the right-hand side of (182):

$$\begin{aligned} & J(\theta)+\int\frac{\kappa}{N}\frac{\omega\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega(\theta,Z)}T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right) \\ & \times\left|\Psi_0(Z_1)+\delta\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2 dZ_1 \\ \simeq\; & J(\theta)+\frac{TW(1)}{\bar{\Lambda}}+\frac{\hat{f}_1\nabla_\theta\omega(\theta,Z)}{\omega(\theta,Z)}+\frac{\hat{f}_3\nabla_\theta^2\omega(\theta,Z)}{\omega(\theta,Z)}+c^2\frac{\hat{f}_3\nabla_Z^2\omega(\theta,Z)}{\omega(\theta,Z)}+T\Psi_0\delta\Psi(\theta,Z) \end{aligned} \tag{183}$$

where we defined:

$$\begin{aligned} \hat{f}_1 &= -\frac{\Gamma_1}{c},\ \hat{f}_3=\frac{\Gamma_2}{c^2} \\ \Gamma_1 &= \frac{\kappa}{NX_r}\int|Z-Z_1|\,T(Z,Z_1)\,|\Psi_0(Z_1)|^2\,dZ_1 \\ \Gamma_2 &= \frac{\kappa}{2NX_r}\int(Z-Z_1)^2\,T(Z,Z_1)\,|\Psi_0(Z_1)|^2\,dZ_1 \end{aligned} \tag{184}$$

and:

$$T\Psi_0\delta\Psi(\theta,Z)=\int\frac{\kappa T(Z,Z_1)}{N}\Psi_0(Z_1)\,\delta\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)dZ_1 \tag{185}$$

Using (181) we can rewrite (185) as:

$$\begin{aligned} T\delta\Psi(\theta,Z) &\simeq \delta\Psi(\theta,Z)-\Gamma_1\nabla_\theta\delta\Psi(\theta,Z) \\ &\simeq N_1\nabla_\theta\omega_0\left(J,Z,|\Psi_0|^2\right)-N_2\nabla_\theta\omega_0\left(J,Z,|\Psi_0|^2\right) \end{aligned} \tag{186}$$

with:

$$\begin{aligned} N_1 &= \frac{\Psi_0(Z)}{U''(X_0)\,\omega^2\left(J,Z,|\Psi_0|^2\right)} \\ N_2 &= \frac{\Gamma_1\Psi_0(Z)}{U''(X_0)\,\omega^2\left(J,Z,|\Psi_0|^2\right)} \end{aligned}$$

Then, replacing (186) in (183), equation (182) becomes:

$$\begin{aligned} & F^{-1}(\omega(J(\theta),\theta))-F^{-1}(\omega_0) \\ =\; & J(\theta,Z)+\left(\frac{\hat{f}_1}{\omega(\theta,Z)}+N_1\right)\nabla_\theta\omega(\theta,Z)+\left(\frac{\hat{f}_3}{\omega(\theta,Z)}-N_2\right)\nabla_\theta^2\omega(\theta,Z)+c^2\hat{f}_3\frac{\nabla_Z^2\omega(\theta,Z)}{\omega(\theta,Z)} \end{aligned} \tag{187}$$

To obtain an equation for the fluctuations of activities around the background state values, we assume that F^{-1} is slowly varying, so that:

$$F^{-1}(\omega(J(\theta),\theta))-F^{-1}(\omega_0)\simeq\Gamma_0(\omega(J(\theta),\theta)-\omega_0)$$

with[16]:

$$f=\left(F^{-1}\right)'\left(\frac{\kappa}{N}\int T(Z,Z_1)\,W(1)\,dZ_1\bar{\mathcal{G}}_0(0,Z_1)\right)$$

[16] Given our assumption that F is an increasing function, $f>0$.

and define:

$$\Omega(\theta, Z) = \omega(\theta, Z) - \omega_0$$

As a result, the expansion of (187) for a non-static current is then:

$$f\Omega(\theta, Z) = J(\theta, Z) + \left(\frac{\hat{f}_1}{\omega(\theta, Z)} + N_1\right)\nabla_\theta \Omega(\theta, Z) + \left(\frac{\hat{f}_3}{\omega(\theta, Z)} - N_2\right)\nabla_\theta^2 \Omega(\theta, Z) + \frac{c^2 \hat{f}_3}{\omega(\theta, Z)}\nabla_Z^2 \Omega(\theta, Z) \tag{188}$$

A careful study of this equation is performed in ([52]). We show that this equation has non sinusoidal stable traveling wave solutions and that in first approximation it can be replaced by a usual wave equation:

$$f\Omega(\theta, Z) - \left(\frac{\hat{f}_3}{\omega_0} - N_2\right)\nabla_\theta^2 \Omega(\theta, Z) - \frac{c^2 \hat{f}_3}{\omega_0}\nabla_Z^2 \Omega(\theta, Z) = J(\theta, Z) \tag{189}$$

where ω_0 is the average of the static activity.

14.2 Perturbative corrections to the local frequency equations

The perturbative expansion of the path integral for the field action (71) local modifies the activities equation. We computed this effective action, written $\Gamma(\Psi^\dagger, \Psi)$, in ([52]). In the local approximation it is given by:

$$\Gamma(\Psi^\dagger, \Psi) \simeq \int \Psi^\dagger(\theta, Z)\left(-\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta - \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right)\right)\delta(\theta_f - \theta_i) + \Omega(\theta, Z)\right)\Psi(\theta, Z) \tag{190}$$

where $\Omega(\theta, Z)$ is a corrective term depending on the successive derivatives of the field. The term $\mathcal{G}_0$ is a function of Z and represents a two points free Green function (see ([52])).

The previous equation (190) defines an effective activity that can be identified as:

$$\omega_e^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) = \omega^{-1}\left(J(\theta), \theta, Z, \mathcal{G}_0 + |\Psi|^2\right) + \int^\theta \Omega(\theta, Z) \tag{191}$$

where $\omega\left(J(\theta), \theta, Z, \bar{\mathcal{G}}_0 + |\Psi|^2\right)$ is the solution of:

$$\begin{aligned}\omega^{-1}\left(J, \theta, Z, |\Psi|^2\right) &= G\left(J(\theta, Z) + \int \frac{\kappa}{N}\frac{\omega\left(J, \theta - \frac{|Z-Z_1|}{c}, Z_1, \Psi\right)T\left(Z, \theta, Z_1, \theta - \frac{|Z-Z_1|}{c}\right)}{\omega\left(J, \theta, Z, |\Psi|^2\right)}\right.\\ &\quad\left.\times\left(\bar{\mathcal{G}}_0(0, Z_1) + \left|\Psi\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)\right|^2\right)dZ_1\right)\end{aligned}$$

Wich is the classical activity equation, up to the inclusion of the Green function $\bar{\mathcal{G}}_0(0, Z_1)$.

The second term $\int^\theta \Omega(\theta, Z)$ in (191) represents corrections due to the interactions. Using (190), we can find its expression as a series expansion in terms of activities and field. The computations of these corrections to the classical equation are presented in ([52]) and confirm the possibility of traveling wave solutions. To sum up, the perturbative corrections account for interactions between the classical solutions and the whole thread and these interactions stabilize the traveling waves.

15 Sources induced activities, interferences.

15.1 Local approximation

In the perspective of this work, we are looking at the solutions of (189) induced by some ponctual sources. Assume several signals arising at some points $(Z_1,\theta_1),...,(Z_N,\theta_N)$.

The solution to (189) are then:

$$\Omega(Z,\theta)=\sum_{i=1}^{N}G((Z,\theta),(Z_i,\theta_i))J(Z_i,\theta_i) \tag{192}$$

Where $G((Z,\theta),(Z_i,\theta_i))$ is the Green function of the operator involvd in (189):

$$f-\left(\frac{\hat{f}_3}{\omega_0}-N_2\right)\nabla_\theta^2-\frac{c^2\hat{f}_3}{\omega_0}\nabla_Z^2$$

Solutions of (192) present interference phenomena. When the number of sources is large, we may expect that solutions of (192) locate mainly at some maxima depending both on the connectivity field $\left|\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right|^2$ and neuron field. In the sequel, we will write:

$$Z_M^{(\varepsilon)}\left(|\Gamma|^2,|\Psi_0|^2\right)$$

the location of these maxima, with $\varepsilon=1,...$ indexing these maxima. We will also assume that at these maxima, the activities are all equal to some value:

$$\omega\simeq\omega'\simeq\omega_M$$

so that:

$$\begin{aligned}h_C(\omega)&\simeq h_C(\omega_M)\\h_D(\omega)&\simeq h_D(\omega_M)\end{aligned}$$

The precise derivation of the interference phenomenom will be presented in a field theoretic context in part II. It is sufficient for the rest of this article to build on the previous qualititative argument.

15.2 Non local propagation and interferences

Note that more generally, we can go farther than the local equation (189) by considering the non local equation (182) in presence of sources, this equation has solutions:

$$\Omega(\theta,Z)\simeq\sum_{i=1}^{N}G_T((Z,\theta),(Z_i,\theta_i))J(Z_i,\theta_i) \tag{193}$$

where $G_T((Z,\theta),(Z_i,\theta_i))$ is defined by:

$$G_T((Z,\theta),(Z_i,\theta_i))=\left(\frac{1}{1-G_T}\right)((Z,\theta),(Z_i,\theta_i))$$

and G_T is an operator whose kernel $G_{T,\Psi_0}\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)$ depending on T and Ψ_0 will be studied in the second article of this series. Remark that formula (193) is the non-local version of (192). Both solutions present interference phenomena with maxima located at specifc points $Z_M^{(\varepsilon)}\left(|\Gamma|^2,|\Psi_0|^2\right)$.

16 Effective action and background state for given sources perturbations

We have seen that for a given external source state, interferences occur in cells activity, leading to localized activities at specific points:

$$Z_M^{(\varepsilon)}\left(|\Gamma|^2, |\Psi_0|^2\right)$$

with $\varepsilon = 1, \ldots$ indexing these points. To derive the conntctivity background field associated to these source induced interferences, we have to proceed as we did in section 8 and start with the level of activity at each points of the system. We will assume that the effect of activity is large at the points of positive interferences and neglect at these points the level of static equilibrium level $\omega_0(Z)$ in the static background field. This corresponds to study states with source induced additional connectivity and activation. The background field obtained will thus described this modification "above" the static background field, similar to an activated state above some vacuum.

We will also assume, for the sake of simplicity, that that at these maxima $Z_M^{(\varepsilon)}\left(|\Gamma|^2, |\Psi_0|^2\right)$, the activities are all equal to some value:

$$\omega \simeq \omega' \simeq \omega_M \tag{194}$$

so that:

$$\begin{aligned} h_C(\omega) &\simeq h_C(\omega_M) \\ h_D(\omega) &\simeq h_D(\omega_M) \end{aligned}$$

Assuming that functions $h_C(\omega)$ and $h_D(\omega)$ are proportional to some positive power of ω implies that outside the set of points

$$U_M = \left\{Z_M^{(\varepsilon)}\left(|\Gamma|^2, |\Psi_0|^2\right)\right\}$$

the functions $h_C(\omega)$ and $h_D(\omega)$ can be considered as nul. We will write Z the generic points of the complementary set of U_M, written CU_M.

We compute average connectivity between points of U_M, between points of CU_M, and between points of U_M and CU_M.

16.1 Connectivity between points of U_M

Using (194), we can compute the average connectivities for points of U_M, the points with constructive interferences. We use that the background state at points $\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right) \subset U_M$ is:

$$\begin{aligned} &\Gamma\left(T, \hat{T}, \theta, Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right) \\ = \; &\exp\left(-\left(\left(-\frac{1}{\tau\omega_M}T + \frac{\lambda}{\omega_M}\left\langle\hat{T}\right\rangle\right)\left|\Psi\left(\theta, Z_M^{(\varepsilon_1)}\right)\right|^2\right)^2\right) \\ &\times \exp\left(-\left(\frac{\rho}{\omega_M}H\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right)\left|\Psi\left(\theta, Z_M^{(\varepsilon_1)}\right)\right|^2\left|\Psi\left(\theta - \frac{\left|Z_M^{(\varepsilon_1)} - Z_M^{(\varepsilon_2)}\right|}{c}, Z_M^{(\varepsilon_2)}\right)\right|^2\right)^2\right) \end{aligned}$$

where:

$$H\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right) = \left(\left(h\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right) - \hat{T}\right)C(\theta)\,h_C - D(\theta)\,\hat{T}h_D\right)$$

and the average values of C, D, $\hat{T}$ and Tin this background states are:

$$C_{Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}} = \frac{\alpha_C\omega_M\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2}{\frac{1}{\tau_C}+\alpha_C\omega_M\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2}$$

$$D_{Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}} = \frac{\alpha_D\omega_M}{\frac{1}{\tau_D}+\alpha_D\omega_M\left|\Psi\left(\theta,Z_M^{(\varepsilon_1)}\right)\right|^2}$$

$$T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)=\lambda\tau\hat{T}\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)=\lambda\tau\frac{h\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)C_{Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}}(\theta)h_C}{C_{Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}}(\theta)h_C+D_{Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}}(\theta)h_D}$$

Then assuming an exponential decreasing dependency in distance for the connectivities:

$$h\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)\simeq\exp\left(-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{\nu c}\right)$$

we obtain:

$$\begin{aligned}&T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right) \qquad (195)\\ =&\frac{\lambda\tau\exp\left(-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{\nu c}\right)\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2h_C}{\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2h_C+\left(\frac{1}{\alpha_C\tau_C}+\omega_M\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2\right)\frac{\alpha_Dh_D}{\frac{1}{\tau_D}+\alpha_D\omega_M\left|\Psi\left(\theta,Z_M^{(\varepsilon_1)}\right)\right|^2}}\end{aligned}$$

In a long run static perspective, this formula reduces to:

$$T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)=\frac{\lambda\tau\exp\left(-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{\nu c}\right)\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2h_C}{\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2h_C+\left(\frac{1}{\alpha_C\tau_C}+\omega_M\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2\right)\frac{\alpha_Dh_D}{\frac{1}{\tau_D}+\alpha_D\omega_M\left|\Psi_0\left(Z_M^{(\varepsilon_1)}\right)\right|^2}}$$

The decription of the set U_M is achieved by adding the long term determination of activities ω_M:

$$\begin{aligned}\omega_M^{-1}\left(Z_M^{(\varepsilon_1)},|\Psi|^2\right)&\simeq G\left(\frac{\kappa}{N}\sum_{Z_M^{(\varepsilon_2)}}T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2\right)\\&\simeq G\left(C\frac{\left|\Psi_0\left(Z_M\right)\right|^4h_C}{\left|\Psi_0\left(Z_M\right)\right|^2h_C+\left(\frac{1}{\alpha_C\tau_C}+\omega_M\left|\Psi_0\left(Z_M\right)\right|^2\right)\frac{\alpha_Dh_D}{\frac{1}{\tau_D}+\alpha_D\omega_M\left|\Psi_0(Z_M)\right|^2}}\right)\end{aligned}$$

whr:

$$C=\frac{\kappa\lambda\tau}{N\left(\sharp\left\{Z_M^{(\varepsilon_1)}\right\}\right)}\sum_{Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}}\exp\left(-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{\nu c}\right)$$

nd $|\Psi_0(Z_M)|^2$ is the average of $\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2$ over $\left\{Z_M^{(\varepsilon_2)}\right\}$. The value of $|\Psi_0(Z_M)|^2$ can be approximated in the following way.

In the field-theoretic approach to interferences, we will see that the signals modify the potential for $|\Psi_0(Z)|^2$ but that in first approximation, this modification can be neglected. Thus, the value of $|\Psi_0(Z)|^2$ after interferences may be computed by the background field before interferences. This is formula (133):

$$|\Psi(Z)|^2=\frac{2T(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z}{\left(1+\sqrt{1+4\left(\frac{\lambda\tau\nu c-T(Z)}{\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega\right)T(Z)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}\right)^2T(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z}\right)}$$

where all quantities are computed in the initial background state. The system emrging from the interferences thus depends on the whole initial structure.

16.2 Connectivity between points of CU_M

The connectivity function for two points in CU_M is obtained by setting $\omega << 1$ and $\omega' << 1$:

$$T(Z,Z')\simeq\frac{\alpha_C\omega\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2h_C}{\alpha_C\omega\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2h_C+\left(\frac{\omega'}{\tau_C}\right)\frac{\alpha_Dh_D}{\frac{|\Psi(\theta,Z)|^2}{\tau_D}}}$$

and these values are identitical to those computed for the static background state in the previous section (see (173)), up to some global modifications of the system by the interfering signals. These modifications are encompassed in the values of the constants Ω, $\bar{\Omega}$... in (173). These modifications are negligible in general.

16.3 Connectivity between points of U_M and points of CU_M

Two cases arise. We have ton consider both:

$$T\left(Z_M^{(\varepsilon)},Z'\right)$$

describing the connectivity of points of CU_M towards point of U_M, measuring the strength of signals send from CU_M to U_M, and:

$$T\left(Z,Z_M^{(\varepsilon)}\right)$$

computing the connectivity of points of U_M towards point of CU_M.

The connectivity function $T\left(Z_M^{(\varepsilon)},Z'\right)$ is obtained by setting $\omega=\omega_M$ and $\omega'<<1$ or $\omega<<1$ and $\omega'=\omega_M$. We find

$$T\left(Z_M^{(\varepsilon)},Z'\right)\simeq\frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2h_C}{\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2h_C+\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\frac{\alpha_D\omega_Mh_D}{\frac{|\Psi(\theta,Z)|^2}{\tau_D}+\alpha_D\omega_M}}\tag{196}$$

The connectivity function $T\left(Z_M^{(\varepsilon)},Z'\right)$ is derived by setting $\omega<<1$ and $\omega'=\omega_M$:

$$T\left(Z,Z_M^{(\varepsilon)}\right)\simeq\frac{\alpha_C\omega\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2h_C}{\alpha_C\omega\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2h_C+\frac{\omega_M}{\tau_C}\frac{\alpha_Dh_D\tau_D}{|\Psi(\theta,Z)|^2}}<<1\tag{197}$$

As a consequence, points of the set U do not connect with elements of CU. On the contrary, elements of CU send signals and connect to elements of U but their firing rate being low, they do not influence the whole set that remains unaffected.

16.4 Equations defining the set of connected cells

The results of the previous paragraph where derived considering a given set U_M. However, there is a priori no guarantee for the unicity of this set. Actually, the set of constructive interferences points $Z_M^{(\varepsilon)}$ are by definition, dependent on both fields $\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$ and $\Psi(\theta,Z)$ through $T(Z,Z')$, so that the points $Z_M^{(\varepsilon)}$ are in fact themselves endogeneous:

$$\begin{aligned}
Z_M^{(\varepsilon_1)} &\equiv Z_M^{(\varepsilon_1)}\left(T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right),\left|\Psi\left(\theta,Z_M^{(\varepsilon_1)}\right)\right|^2,\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2\right) \\
Z_M^{(\varepsilon_2)} &\equiv Z_M^{(\varepsilon_2)}\left(T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right),\left|\Psi\left(\theta,Z_M^{(\varepsilon_1)}\right)\right|^2,\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2\right)
\end{aligned}$$

As a consequence, equation (195) is a self consistent functionnal non linear equation for $\left|\Psi\left(\theta,Z_M^{(\varepsilon_1)}\right)\right|^2$, $\left|\Psi\left(\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c},Z_M^{(\varepsilon_2)}\right)\right|^2$ and $T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)$. This implies multiple possible states of constructive interferences. The full environment impacts the interference pattern, and is itself modified by these interferences, leading possibly to multiple equilibria. These solutions depend on Ψ. If thereisa threshold for connectivity to be effective, given the form of $h\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)$, connections arise along lines or branched lines. This may induce some effective form of engrams, asset of branched lines, described by activities and connectivity at nodes.

17 Medium term state reactivation

In this section, we focus on the possible reactivation of a state defined by the points $Z_M^{(\varepsilon)}$. When the sources are off, we have $J(Z)=0$ for $\theta>\theta_t$, so that in the medium run:

$$\begin{aligned}
\omega\left(Z_M^{(\varepsilon_1)}\right) &\simeq \omega_{J=0}<<\omega_M \\
T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right) &\simeq T\exp\left(-\lambda\left(\theta-\theta_t\right)\right)
\end{aligned}$$

The activation at some point $Z_M^{(\varepsilon_2)}$, $\omega\left(Z_M^{(\varepsilon_1)}\right)=\omega_M$ leads to the dynamics between points $\left\{Z_M^{(\varepsilon_1)}\right\}$ with $T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)=T$. Set U do not connect with set CU. Writing the equation for $\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)$:

$$\begin{aligned}
&\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right) \\
&= G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left|\Gamma\left(T,\hat{T},\theta,Z,Z_1\right)\right|^2}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2dZ_1\right)
\end{aligned}$$

and using the condition about the average connectivit:

$$\int T\left|\Gamma\left(T,\hat{T},\theta,Z,Z_1\right)\right|^2dT=T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)$$

that connects only the points $Z_M^{(\varepsilon)}$ in first approximation yields:

$$\omega_M^{-1}\left(Z_M^{(\varepsilon_1)},|\Psi|^2,\theta\right)\simeq G\left(\frac{\kappa}{N}\sum_{Z_M^{(\varepsilon_2)}}\frac{\omega_M\left(Z_M^{(\varepsilon_2)},|\Psi|^2,\theta-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{c}\right)}{\omega_M\left(Z_M^{(\varepsilon_1)},|\Psi|^2,\theta\right)}T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2\right)$$

with the static lmt:

$$\begin{aligned}\omega_M^{-1}\left(Z_M^{(\varepsilon_1)},|\Psi|^2\right) &\simeq G\left(\frac{\kappa}{N}\sum_{Z_M^{(\varepsilon_2)}}T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2\right)\\ &\simeq G\left(C\frac{|\Psi_0(Z_M)|^4 h_C}{|\Psi_0(Z_M)|^2 h_C+\left(\frac{1}{\alpha_C\tau_C}+\omega_M|\Psi_0(Z_M)|^2\right)\frac{\alpha_D h_D}{\frac{1}{\tau_D}+\alpha_D\omega_M|\Psi_0(Z_M)|^2}}\right)\end{aligned}$$

and th st is reactivated as a whole, with lower level activity compared to the initial level.

18 Background state for associated signals

We can can consider that several disconnected state become associated by an external source. When two states U_M and U'_M activated simultaneously their average connectivities are:

$$T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right)=\frac{\lambda\tau\exp\left(-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{\nu c}\right)\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2 h_C}{\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2 h_C+\left(\frac{1}{\alpha_C\tau_C}+\omega_M\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2\right)\frac{\alpha_D h_D}{\frac{1}{\tau_D}+\alpha_D\omega_M\left|\Psi_0\left(Z_M^{(\varepsilon_1)}\right)\right|^2}}$$

$$T\left(Z_M'^{(\varepsilon_1)},Z_M'^{(\varepsilon_2)}\right)=\frac{\lambda\tau\exp\left(-\frac{\left|Z_M'^{(\varepsilon_1)}-Z_M'^{(\varepsilon_2)}\right|}{\nu c}\right)\left|\Psi_0\left(Z_M'^{(\varepsilon_2)}\right)\right|^2 h_C}{\left|\Psi_0\left(Z_M'^{(\varepsilon_2)}\right)\right|^2 h_C+\left(\frac{1}{\alpha_C\tau_C}+\omega'_M\left|\Psi_0\left(Z_M'^{(\varepsilon_2)}\right)\right|^2\right)\frac{\alpha_D h_D}{\frac{1}{\tau_D}+\alpha_D\omega'_M\left|\Psi_0\left(Z_M'^{(\varepsilon_1)}\right)\right|^2}}$$

But crssd connectivities have also to be considered:

$$T\left(Z_M^{(\varepsilon_1)},Z_M'^{(\varepsilon_2)}\right)=\frac{\lambda\tau\exp\left(-\frac{\left|Z_M^{(\varepsilon_1)}-Z_M'^{(\varepsilon_2)}\right|}{\nu c}\right)\left|\Psi_0\left(Z_M'^{(\varepsilon_2)}\right)\right|^2 h_C}{\left|\Psi_0\left(Z_M'^{(\varepsilon_2)}\right)\right|^2 h_C+\left(\frac{1}{\alpha_C\tau_C}+\omega'_M\left|\Psi_0\left(Z_M'^{(\varepsilon_2)}\right)\right|^2\right)\frac{\alpha_D h_D}{\frac{1}{\tau_D}+\alpha_D\omega_M\left|\Psi_0\left(Z_M^{(\varepsilon_1)}\right)\right|^2}}$$

$$T\left(Z_M'^{(\varepsilon_2)},Z_M^{(\varepsilon_1)}\right)\simeq\frac{\lambda\tau\exp\left(-\frac{\left|Z_M'^{(\varepsilon_1)}-Z_M^{(\varepsilon_2)}\right|}{\nu c}\right)\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2 h_C}{\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2 h_C+\left(\frac{1}{\alpha_C\tau_C}+\omega_M\left|\Psi_0\left(Z_M^{(\varepsilon_2)}\right)\right|^2\right)\frac{\alpha_D h_D}{\frac{1}{\tau_D}+\alpha_D\omega'_M\left|\Psi_0\left(Z_M'^{(\varepsilon_1)}\right)\right|^2}}$$

along with the associated activities:

$$\omega_M^{-1}\left(\hat{Z}_M^{(\varepsilon_1)},|\Psi|^2,\theta\right)\simeq G\left(\frac{\kappa}{N}\sum_{\hat{Z}_M^{(\varepsilon_2)}}\frac{\omega_M\left(\hat{Z}_M^{(\varepsilon_2)},|\Psi|^2,\theta-\frac{\left|\hat{Z}_M^{(\varepsilon_1)}-\hat{Z}_M^{(\varepsilon_2)}\right|}{c}\right)}{\omega_M\left(\hat{Z}_M^{(\varepsilon_1)},|\Psi|^2,\theta\right)}T\left(\hat{Z}_M^{(\varepsilon_1)},\hat{Z}_M^{(\varepsilon_2)}\right)\left|\Psi_0\left(\hat{Z}_M^{(\varepsilon_2)}\right)\right|^2\right)$$

with:

$$\hat{Z}_M^{(\varepsilon)}\in\left\{Z_M^{(\varepsilon_1)},Z_M^{\prime(\varepsilon_2)}\right\}$$

nd sttc lmt:

$$\omega_M^{-1}\left(\hat{Z}_M^{(\varepsilon_1)}\right)\simeq G\left(\frac{\kappa}{N}\sum_{\hat{Z}_M^{(\varepsilon_2)}}\frac{\omega_M\left(\hat{Z}_M^{(\varepsilon_2)},|\Psi|^2\right)}{\omega_M\left(\hat{Z}_M^{(\varepsilon_1)},|\Psi|^2\right)}T\left(\hat{Z}_M^{(\varepsilon_1)},\hat{Z}_M^{(\varepsilon_2)}\right)\left|\Psi_0\left(\hat{Z}_M^{(\varepsilon_2)}\right)\right|^2\right)$$

$$\begin{aligned}\omega_M &= \omega_M\left(Z_M^{(\varepsilon)}\right)\\ \omega'_M &= \omega_M\left(Z_M^{\prime(\varepsilon)}\right)\end{aligned}$$

19 Reactivation of associated signals

As before, activation at one point for constant connectivities yields, dynamic system:

$$\omega_M^{-1}\left(\hat{Z}_M^{(\varepsilon_1)},|\Psi|^2,\theta\right)\simeq G\left(\frac{\kappa}{N}\sum_{\hat{Z}_M^{(\varepsilon_2)}}\frac{\omega_M\left(\hat{Z}_M^{(\varepsilon_2)},|\Psi|^2,\theta-\frac{\left|\hat{Z}_M^{(\varepsilon_1)}-\hat{Z}_M^{(\varepsilon_2)}\right|}{c}\right)}{\omega_M\left(\hat{Z}_M^{(\varepsilon_1)},|\Psi|^2,\theta\right)}T\left(\hat{Z}_M^{(\varepsilon_1)},\hat{Z}_M^{(\varepsilon_2)}\right)\left|\Psi_0\left(\hat{Z}_M^{(\varepsilon_2)}\right)\right|^2\right)$$

cnvrgng twrd qlbrm:

$$\omega_M^{-1}\left(\hat{Z}_M^{(\varepsilon_1)},|\Psi|^2,\theta\right)\simeq G\left(\frac{\kappa}{N}\sum_{\hat{Z}_M^{(\varepsilon_2)}}\frac{\omega_M\left(\hat{Z}_M^{(\varepsilon_2)},|\Psi|^2,\theta-\frac{\left|\hat{Z}_M^{(\varepsilon_1)}-\hat{Z}_M^{(\varepsilon_2)}\right|}{c}\right)}{\omega_M\left(\hat{Z}_M^{(\varepsilon_1)},|\Psi|^2,\theta\right)}T\left(\hat{Z}_M^{(\varepsilon_1)},\hat{Z}_M^{(\varepsilon_2)}\right)\left|\Psi_0\left(\hat{Z}_M^{(\varepsilon_2)}\right)\right|^2\right)$$

20 Background state for sequence of signals

The previous section allows to consider now the combined effects of two different sources modifying the system at different moments. We will consider two possibilities, distant and subsequent activations. We present these two possibilities qualitatively and provide some details while developping the field formalism dynamic for connectivities.

20.1 Distant activation

We assume that the system is in a static background state, as computed in the previous section, and described as $\{\omega_0,T_0\}$. The effect of two distant signals in time can be described in the sequence:

$$\begin{aligned}\{\omega_0,T_0\} &\rightarrow \left\{T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right),\omega_M\right\}\rightarrow\left\{T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right),\omega_0\right\}\\ &\rightarrow \left\{T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right),\omega_0\right\}+\left\{T\left(Z_M^{\prime(\varepsilon_1)},Z_M^{\prime(\varepsilon_2)}\right),\omega'_M\right\}\\ &\rightarrow \left\{T\left(Z_M^{(\varepsilon_1)},Z_M^{(\varepsilon_2)}\right),\omega_0\right\}+\left\{T\left(Z_M^{\prime(\varepsilon_1)},Z_M^{\prime(\varepsilon_2)}\right),\omega_0\right\}\end{aligned}$$

where $\left\{T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \omega_M\right\}$ describes a modified state where the connectivities are modified at points $\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right)$ with values $T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right)$ and ω_M are the modified activities. The sequence describe the modification by a first signal. This implies modified connectivities and modified activities. As the signals ends, and time increase, the activities, whose time scale can be considered smaller than the connectivities time scale, come back to their equilibrium, while the modifications $T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right)$ are persistent. In a second step an other signal modifies the system at some points, which induces a new set of modification in connectivities. As time increases the frequencies come back to some equilibrium values, but connection remain active. The main feature of the modification is that the two sets:

$$\left\{T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \omega_0\right\} + \left\{T\left(Z_M^{\prime(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), \omega_0\right\}$$

are not connected a priori. In the activation, the connections between activated points increase, but connections with other points decrease as seen from (196) and (197). The two set remain independent, and in a reactivation of one set, the following sequence applies (we consider here,the reactivation of ω_M'):

$$\left\{T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \omega_0\right\} + \left\{T\left(Z_M^{\prime(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), \omega_0\right\} \rightarrow \left\{T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \omega_0\right\} + \left\{T\left(Z_M^{\prime(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), \omega_M'\right\}$$

While ω_M' is activated at some points of $\left(Z_M^{\prime(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right)$, the activated points will send signals to the points of the same set to which they are connected. It will lead to reactivate the activities at these points. However, the other state $\left\{T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \omega_0\right\}$ being independent, as a consequenc of (196) and (197), will remain silent. More about the mechanism of diffusion of activities will be given in the dynamic field theoretic approach.

20.2 Subsequent activation

When the activation are subsequent, i.e. when the two perturbations are closed in time, the following sequence applies:

$$\begin{aligned}
\{\omega_0, T_0\} &\rightarrow \left\{T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \omega_M\right\} \\
&\rightarrow \left\{T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \omega_M, T\left(Z_M^{\prime(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), \omega_M', T\left(Z_M^{(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), T\left(Z_M^{\prime(\varepsilon_2)}, Z_M^{(\varepsilon_1)}\right)\right\} \\
&\rightarrow \left\{T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \omega_0, T\left(Z_M^{\prime(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), \omega_0, T\left(Z_M^{(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), T\left(Z_M^{\prime(\varepsilon_2)}, Z_M^{(\varepsilon_1)}\right)\right\}
\end{aligned}$$

While activating the system with second signal, the track of the first one is still active and thus both sets $\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right)$ and $\left(Z_M^{\prime(\varepsilon_2)}, Z_M^{(\varepsilon_1)}\right)$ are connected. The activated system is thus made of a set of connected points $\left\{\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \left(Z_M^{\prime(\varepsilon_2)}, Z_M^{(\varepsilon_1)}\right)\right\}$. This is translated by the notation:

$$\left\{T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \omega_M, T\left(Z_M^{\prime(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), \omega_M', T\left(Z_M^{(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), T\left(Z_M^{\prime(\varepsilon_2)}, Z_M^{(\varepsilon_1)}\right)\right\}$$

When both signals fade away, we are left with:

$$\left\{T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \omega_0, T\left(Z_M^{\prime(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), \omega_0, T\left(Z_M^{(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), T\left(Z_M^{\prime(\varepsilon_2)}, Z_M^{(\varepsilon_1)}\right)\right\}$$

When one set is reactivated (here choose ω_M'), we can write:

$$\begin{aligned}
&\left\{T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \omega_0, T\left(Z_M^{\prime(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), \omega_0', T\left(Z_M^{(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), T\left(Z_M^{\prime(\varepsilon_2)}, Z_M^{(\varepsilon_1)}\right)\right\} \\
\rightarrow &\left\{T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \omega_0, T\left(Z_M^{\prime(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), \omega_M', T\left(Z_M^{(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), T\left(Z_M^{\prime(\varepsilon_2)}, Z_M^{(\varepsilon_1)}\right)\right\} \\
\rightarrow &\left\{T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right), \omega_M, T\left(Z_M^{\prime(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), \omega_M', T\left(Z_M^{(\varepsilon_1)}, Z_M^{\prime(\varepsilon_2)}\right), T\left(Z_M^{\prime(\varepsilon_2)}, Z_M^{(\varepsilon_1)}\right)\right\}
\end{aligned}$$

Actually, while ω'_M is activated at some point, the whole system defined by the points $\left(Z_M'^{(\varepsilon_2)}, Z_M^{(\varepsilon_1)}\right)$ and their connections $T\left(Z_M'^{(\varepsilon_2)}, Z_M^{(\varepsilon_1)}\right)$ will be activated, as in the previous paragraph. However, given that both sets $T\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right)$ are connected, the diffucion of the signal will reactive the activity between the points $\left(Z_M^{(\varepsilon_1)}, Z_M^{(\varepsilon_2)}\right)$. The two structures are synchronically activated.

21 Conclusion

Our results reveal that the primary entities emerging from our model are sets of interconnected cells. The activity levels of these cells are jointly defined with their interconnections within the set. We observed that dynamically, such sets interact with each other and can engage in associations, deactivations, and reactivations. While our results were obtained in a qualitative manner, the subsequent article will provide a more technical derivation, emphasizing the role of the field theoretic framework. Nevertheless, our study already offers insights into two key characteristics of the formalism.

Firstly, the interactions among interconnected sets imply the need to develop a dynamic effective formalism for connectivity functions. Integrating the degrees of freedom for neuronal activity fields should yield such a formalism, directly describing activations, associations, and other group-related phenomena. This effective formalism is expounded upon in the third article of this series.

Secondly, the existence of additional states that may be activated due to external signals suggests the necessity for a formalism to describe interconnected groups. The coexistence, interaction, creation, or deactivation of several groups prompts the consideration of a field formalism for these groups. This is the objective of the fourth article in this series, which will develop such an expanded model.

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Appendix 1 Effective action for $\Psi(\theta, Z)$

1.1 Projection on activities states

When we restrict the fields to those of the form:

$$\Psi(\theta, Z)\,\delta\left(\omega^{-1} - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right) \tag{198}$$

where $\omega^{-1}(J, \theta, Z, \Psi)$ satisfies:

$$\begin{aligned}
&\omega^{-1}\left(J, \theta, Z, |\Psi|^2\right) \qquad (199)\\
= \;& G\left(J(\theta, Z) + \int \frac{\kappa}{N} \frac{\omega\left(J, \theta - \frac{|Z-Z_1|}{c}, Z_1, \Psi\right) T\left(Z, \theta, Z_1, \theta - \frac{|Z-Z_1|}{c}\right)}{\omega\left(J, \theta, Z, |\Psi|^2\right)} \left|\Psi\left(\theta - \frac{|Z-Z_1|}{c}, Z_1\right)\right|^2 dZ_1\right)
\end{aligned}$$

The classical effective action writes:

$$-\frac{1}{2}\int \Psi^{\dagger}(\theta, Z)\,\Psi(\theta, Z)\,\delta\left(\omega^{-1} - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right)\left(\left(\frac{\sigma^2}{2}\nabla_\theta - \omega^{-1}\right)\nabla_\theta\right)\Psi(\theta, Z)\,\delta\left(\omega^{-1} - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right) \tag{200a}$$

We can replace the first δ function by 1 to normalize the projection on the activity dependent states.. The action of ∇_θ on $\Psi(\theta, Z)\,\delta\left(\omega^{-1} - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right)$ yields:

$$\begin{aligned}
&\nabla_\theta\left(\Psi(\theta, Z)\,\delta\left(\omega^{-1} - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right)\right) \qquad (201)\\
= \;& \left(\nabla_\theta \Psi(\theta, Z)\right)\delta\left(\omega^{-1} - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right)\\
& - \left(\nabla_\theta \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right)\Psi(\theta, Z)\,\delta'\left(\omega^{-1} - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right)
\end{aligned}$$

Inserting the result (201) in (200a) leads to:

$$\begin{aligned}
& -\frac{1}{2}\int \Psi^{\dagger}(\theta, Z)\,\Psi(\theta, Z)\left(\left(\frac{\sigma^2}{2}\nabla_\theta - \omega^{-1}\right)\right)\left(\nabla_\theta \Psi(\theta, Z)\right)\delta\left(\omega^{-1} - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right)\\
& +\frac{1}{2}\int \Psi^{\dagger}(\theta, Z)\,\Psi(\theta, Z)\left(\left(\frac{\sigma^2}{2}\nabla_\theta - \omega^{-1}\right)\right)\Psi(\theta, Z)\,\delta'\left(\omega^{-1} - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right)\\
= & -\frac{1}{2}\int \Psi^{\dagger}(\theta, Z)\,\Psi(\theta, Z)\left(\left(\frac{\sigma^2}{2}\nabla_\theta - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right)\right)\nabla_\theta \Psi(\theta, Z)\\
& -\frac{1}{2}\int \Psi^{\dagger}(\theta, Z)\,\Psi(\theta, Z)\left(\left(\frac{\sigma^2}{2}\nabla_\theta - \nabla_\theta \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right)\right)\Psi(\theta, Z)
\end{aligned}$$

and the sum of the two last terms is, as in the text:

$$-\frac{1}{2}\int \Psi^{\dagger}(\theta, Z)\left(\nabla_\theta\left(\frac{\sigma^2}{2}\nabla_\theta - \omega^{-1}\left(J, \theta, Z, |\Psi|^2\right)\right)\right)\Psi(\theta, Z)$$

Appendix 1.2 Effective action for $\Psi(\theta, Z)$ at the lowest order

To find the effective action for field Ψ at lowest order, we start with the two points Green function and prove (**??**). To do so, we will expand the action functional in series of the field Ψ. The two points Green functions will be computed by using the "free" action's propagator, obtained by replacing $\omega^{-1}(J, \theta, Z, \Psi)$ with $\omega^{-1}(J, \theta, Z, 0)$ in (**??**). The free action is:

$$S_0 = -\frac{1}{2}\Psi^{\dagger}(\theta, Z)\,\nabla_\theta\left(\frac{\sigma^2}{2}\nabla_\theta - \omega^{-1}(J, \theta, Z, 0)\right)\Psi(\theta, Z) \tag{202}$$

and the series in field of (**??**) will be considered, as usual, as a perturbation expansion.

1.2.1 "Free" action propagator

Now, we compute the propagator associated to (202). We decompose the external current into a static and a time dependent parts $\bar{J}+J(\theta)$ where $\bar{J}$ can be thought as the time average of the current. We will consider that $\left|\bar{J}(Z)\right|>|J(\theta,Z)|$. At zeroth order in current $J(\theta)$, the function $\omega^{-1}(J,\theta,Z,0)$ satisfies:

$$\begin{aligned}\omega^{-1}(J,\theta,Z,0) &= G\left(\bar{J}+J(\theta)\right) \\ &\simeq G\left(\bar{J}(Z)\right)=\frac{\arctan\left(\left(\frac{1}{X_r}-\frac{1}{X_p}\right)\sqrt{\bar{J}(Z)}\right)}{\sqrt{\bar{J}(Z)}}=\frac{1}{\bar{X}_r(Z)}\equiv\frac{1}{\bar{X}_r}\end{aligned} \tag{203}$$

where the dependence in Z of $\bar{X}_r$ will be understood. As a consequence $\omega(\theta,Z)$ is thus approximatively equal to $\bar{X}_r$. Under this approximation:

$$S_0=-\Psi^{\dagger}(\theta,Z)\nabla_{\theta}\left(\frac{\sigma^2}{2}\nabla_{\theta}-\frac{1}{\bar{X}_r}\right)\Psi(\theta,Z)$$

and the Green function of the operator $\nabla_{\theta}\left(\frac{\sigma^2}{2}\nabla_{\theta}-\frac{1}{\bar{X}_r}\right)$ is computed as:

$$\left\langle\Psi^{\dagger}(\theta,Z)\Psi\left(\theta',Z\right)\right\rangle\equiv\mathcal{G}_0\left((\theta,Z),\left(\theta',Z'\right)\right)\equiv\mathcal{G}_0\left(\theta,\theta',Z\right)=\delta\left(Z-Z'\right)\int\frac{\exp\left(ik\left(\theta-\theta'\right)\right)}{\frac{\sigma^2}{2}k^2+ik\frac{1}{\bar{X}_r}+\alpha}dk \tag{204}$$

The right hand side of (204) can be computed as:

$$\begin{aligned}\int\frac{\exp\left(ik\left(\theta-\theta'\right)\right)}{\frac{\sigma^2}{2}k^2+ik\frac{1}{\bar{X}_r}+\alpha}dk &= \exp\left(\frac{\theta-\theta'}{\sigma^2\bar{X}_r}\right)\int\frac{\exp\left(ik\left(\theta-\theta'\right)\right)}{\frac{\sigma^2}{2}k^2+\frac{1}{2}\left(\frac{1}{\sigma\bar{X}_r}\right)^2+\alpha}dk \\ &= \frac{1}{\sqrt{\frac{\pi}{2}}}\frac{\exp\left(-\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}\left|\theta-\theta'\right|\right)}{\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}}\exp\left(\frac{\theta-\theta'}{\sigma^2\bar{X}_r}\right)\end{aligned} \tag{205}$$

and this is quickly suppressed for $\theta-\theta'<0$. This is the direct consequence of non-hermiticity of operator. In the sequel, for $\sigma^2\bar{X}_r<<1$, we can thus consider that:

$$\mathcal{G}_0\left(\theta,\theta',Z\right)=\delta\left(Z-Z'\right)\frac{1}{\sqrt{\frac{\pi}{2}}}\frac{\exp\left(-\left(\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}-\frac{1}{\sigma^2\bar{X}_r}\right)\left(\theta-\theta'\right)\right)}{\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}}H\left(\theta-\theta'\right) \tag{206}$$

where H is the Heaviside function:

$$\begin{aligned}H\left(\theta-\theta'\right) &= 0\text{ for }\theta-\theta'<0 \\ &= 1\text{ for }\theta-\theta'>0\end{aligned}$$

Formula (206) for the propagator is sufficient to compute the graphs expansion in the next paragraphs. We can check that the corrections due to a non-static current do not modify the result at a good level of approximation. Considering the following form for $G(J(\theta,Z))$:

$$G(J(\theta,Z))=\frac{\arctan\left(\left(\frac{1}{X_r}-\frac{1}{X_p}\right)\sqrt{J(\theta,Z)}\right)}{\sqrt{J(\theta,Z)}}$$

For relatively high frequency firing rates, i.e., small periods of time between two spikes, we can write in first approximation:

$$\begin{aligned} G\left(\bar{J}+J(\theta,Z)\right) &\simeq G\left(\bar{J}\right)+J(\theta,Z)\,G'\left(\bar{J}\right) \\ &= \frac{1}{\bar{X}_r}+J(\theta,Z)\,G'\left(\bar{J}\right) \end{aligned}$$

and replace (204) by the Green function of:

$$\nabla_\theta\left(\frac{\sigma^2}{2}\nabla_\theta-G\left(J(\theta,Z)\right)\right)\simeq\nabla_\theta\left(\frac{\sigma^2}{2}\nabla_\theta-\frac{1}{\bar{X}_r}-J(\theta,Z)\,G'\left(\bar{J}\right)\right)$$

As a consequence, the inverse activity $\mathcal{G}_0\left(\theta,\theta',Z\right)$ defined in (206) is replaced by:

$$\begin{aligned} \mathcal{G}_0\left((\theta,Z),(\theta',Z')\right) &= \delta\left(Z-Z'\right)\frac{1}{\sqrt{\frac{\pi}{2}}}\frac{\exp\left(-\left(\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}-\frac{1}{\sigma^2\bar{X}_r}\right)\left(\theta-\theta'\right)\right)}{\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}}H\left(\theta-\theta'\right) \\ &\times\left(1-\frac{1}{\sqrt{\frac{\pi}{2}}}\frac{G'\left(\bar{J}\right)}{\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}}\int_\theta^{\theta'}J\left(\theta'',Z\right)d\theta''\right) \end{aligned}$$

Since $J(\theta,Z)$ is a deviation around the static part $\bar{J}$, the corrective term:

$$-\frac{1}{\sqrt{\frac{\pi}{2}}}\frac{G'\left(\bar{J}\right)}{\sqrt{\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\alpha}{\sigma^2}}}\int_\theta^{\theta'}J\left(\theta'',Z\right)d\theta''$$

vanishes quickly as $\theta-\theta'$ increases, which justifies approximation (206).

1.2.2 perturbation expansion and the two points Green function

Formula (206) allows to compute higher order contributions to the Green function of action (**??**) by using a graph expansion. Actually, writing $\omega^{-1}(\theta,Z)$ for $\omega^{-1}(J,\theta,Z,\Psi)$ when no ambiguity is possible, the higher order contribution for the series expansion of $\omega^{-1}(\theta,Z)$ in fields are obtained by solving recursively:

$$\omega^{-1}(J,\theta,Z)=G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega(J,\theta,Z)}\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2T\left(Z,\theta,Z_1\right)dZ_1d\omega_1\right) \tag{207}$$

This will be done precisely in the next paragraph. For now, it is enough to note that given (207), the recursive expansion in $\omega^{-1}(J,\theta,Z)$ of the potential term in (**??**):

$$\frac{1}{2}\Psi^\dagger(\theta,Z)\nabla\left(G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1\right)}{\omega(J,\theta,Z)}\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2T\left(Z,Z_1\right)dZ_1\right)\right)\Psi(\theta,Z) \tag{208}$$

induces the presence of products in the series expansion of the two points Green function:

$$\prod_{i=1}^{m}\int\Psi^{\dagger}\left(\theta^{(i)},Z_i\right)\nabla_{\theta^{(i)}}\prod_{k=1}^{k_i}\left(\prod_{l=1}^{l_k}\prod_{\alpha(l)=1}^{n(\alpha(l))}\int\left|\Psi\left(\theta^{(i)}-\frac{\left|Z_i-Z_{\alpha(l)}^{(1)}\right|+...+\left|Z_{\alpha(l)}^{(l-1)}-Z_{\alpha(l)}^{(l)}\right|}{c},Z_{\alpha(l)}^{(l)}\right)\right|^2\right)$$
$$\times dZ_{\alpha(l)}^{(1)}...dZ_{\alpha(l)}^{(l_k)}\Psi\left(\theta^{(i)},Z_i\right)d\theta^{(i)}dZ_i \tag{209}$$

with $n(\alpha(l))\geqslant n(\alpha(l'))$ for $l>l'$ and $m\in\mathbb{N}$. The function $\delta(Z-Z')$ in (204) and the use of Wick's theorem imply that all subgraphs drawn from this product reduce to a product of free Green functions (206) of the following form (the gradient terms and the indices $\alpha(l)$ are not included and do not impact the reasoning):

$$\begin{aligned}&\int\prod_i\mathcal{G}_0\left(\theta^{(i)}-\sum_{l\leqslant n_i}\frac{\left|Z_i-Z_i^{(l)}\right|}{c},\theta^{(i+1)}-\sum_{k\leqslant n_{i+1}}\frac{\left|Z_{i+1}-Z_{i+1}^{(k)}\right|}{c},Z_i^{(n_i)},Z_i^{(n_{i+1})}\right)\\&\times\delta\left(Z_1-Z_i^{(n_i)}\right)\delta\left(Z_1-Z_{i+1}^{(n_{i+1})}\right)dZ_i^{(n_i)}dZ_{i+1}^{(n_{i+1})}\prod_i d\theta^{(i)}\\=&\int\prod_i\mathcal{G}_0\left(\theta^{(i)}-\sum_{l\leqslant n}\frac{\left|Z_i-Z_1^{(l)}\right|}{c},\theta^{(i+1)}-\sum_{k\leqslant m}\frac{\left|Z_{i+1}-Z_1^{(k)}\right|}{c},Z_1\right)\prod_i d\theta^{(i)}\\=&\int\prod_i\mathcal{G}_0\left(\theta^{(i)},\theta^{(i+1)},Z_1\right)\prod_i d\theta^{(i)}\end{aligned} \tag{210}$$

by change of variable in the successive integrations. Moreover, the cancelation of $\mathcal{G}_0\left(\theta,\theta',Z\right)$ for $\theta<\theta'$ implies that this product is different from zero only for $\theta^{(i)}<\theta^{(i+1)}$. As a consequence, for all closed loops $\theta_1<...<\theta^{(i)}<\theta^{(i+1)}<...\theta_n=\theta_1$, the contribution (210) for loop graphs reduces to:

$$\prod_i\mathcal{G}_0\left(\theta_1,\theta_1,Z_1\right)=\prod_i\mathcal{G}_0\left(0,Z_1\right)$$

with (see (206)):

$$\mathcal{G}_0\left(0,Z\right)=\frac{1}{\sqrt{\frac{\pi}{2}\left(\frac{1}{\sigma^2\bar{X}_r}\right)^2+\frac{2\pi\alpha}{\sigma^2}}}$$

As a consequence, the contribution of (209) to the two points Green function between an initial and final state:

$$\begin{aligned}&\left\langle\Psi^{\dagger}\left(\theta_{in},Z_{in}\right)\int\prod_{i=1}^{m}\Psi^{\dagger}\left(\theta^{(i)},Z_i\right)\right.\\&\times\nabla_{\theta^{(i)}}\prod_{k=1}^{k_i}\left(\left(\prod_{l=1}^{l_k}\int\left|\Psi\left(\theta^{(i)}-\frac{\left|Z_i-Z^{(1)}\right|+...+\left|Z^{(l-1)}-Z^{(l)}\right|}{c},Z^{(l)}\right)\right|^2dZ^{(1)}...dZ^{(l_k)}\right)\right)\\&\left.\times\ \Psi\left(\theta^{(i)},Z_i\right)d\theta^{(i)}dZ_i\Psi\left(\theta_{fn},Z_{fn}\right)\right\rangle\end{aligned} \tag{211}$$

reduces to sums and integrals of the type:

$$\delta\left(Z_{in}-Z_{fn}\right)\sum_p\mathcal{G}_0\left(\theta_{in},\theta_1,Z_{in}\right)\mathcal{G}_0\left(\theta_1,\theta_2,Z_{in}\right)...\mathcal{G}_0\left(\theta_p,\theta_{fn},Z_{in}\right) \tag{212}$$
$$\times\left(\sum_n\sum_{\left\{L_1^{(p)},...,L_n^{(p)}\right\}}\prod_{m=1}^{n}\left(\mathcal{G}_0\left(0,0,Z_m\right)\right)^{l\left(L_m^{(p)}\right)}\right)$$

where $\left\{L_1^{(p)},...,L_n^{(p)}\right\}$ is the set of all n-uplet of possible closed loops that can be drawn from the remaining variables in (211) once p variables have been chosen.

The result (212) is the same as if in (208) the potential had been expanded to the second order in Ψ and in all terms of higher order, $|\Psi(\theta,Z)|^2$ had been replaced by $\mathcal{G}_0(0,Z)$.

Now, writing $\omega\left(J,\theta,Z,|\Psi|^2\right)$ for ω and $\omega(0)=\omega(J,\theta,Z,0)$ (i.e. when we set $\Psi\equiv 0$), this means that the 2 points Green functions are computed using the free action:

$$\begin{aligned}
& -\frac{1}{2}\Psi^{\dagger}(\theta,Z)\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}(0)\right)\Psi(\theta,Z) \qquad (213)\\
& +\frac{1}{2}\Psi^{\dagger}(\theta,Z)\sum_{n>0}\frac{\nabla_\theta\left(\omega^{-1}\right)^{([n])}(0)}{[n]!}\left(\mathcal{G}_0(0,Z)\right)^n\Psi(\theta,Z)\\
& +\sum_{n>0}\left(\nabla_\theta\frac{\left(\omega^{-1}\right)^{([n-1])}(0)\,|\Psi|^2}{[n-1]!}\left(\mathcal{G}_0(0,Z)\right)^{n-1}\mathcal{G}_0\left(\theta,\theta',Z\right)\right)_{\theta'=\theta}\\
= & -\frac{1}{2}\Psi^{\dagger}(\theta,Z)\nabla_\theta\left(\frac{\sigma_\theta^2}{2}\nabla_\theta-\omega^{-1}(0)\right)\Psi(\theta,Z)+\frac{1}{2}\Psi^{\dagger}(\theta,Z)\sum_{n>0}\nabla_\theta\left(\left(\omega^{-1}\right)\left(\mathcal{G}_0(0,Z)\right)-\omega^{-1}(0)\right)\Psi(\theta,Z)\\
& +\Psi^{\dagger}(\theta,Z)\left(\nabla_{\theta'}\left(\left(\omega^{-1}\right)^{([1])}\left(\mathcal{G}_0(0,Z)\right)\Psi\left(\theta',Z\right)\mathcal{G}_0\left(\theta,\theta',Z\right)\right)\right)_{\theta'=\theta}\\
\equiv & -\frac{1}{2}\Psi^{\dagger}(\theta,Z)\left(\nabla_\theta\frac{\sigma_\theta^2}{2}\nabla_\theta\right)\Psi(\theta,Z)+\frac{1}{2}|\Psi|^2\left[\frac{\delta\left[\Psi^{\dagger}\left(\theta',Z\right)\nabla_\theta\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\Psi(\theta,Z)\right]}{\delta|\Psi|^2}\right]_{|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,Z)}
\end{aligned}$$

where $\frac{\left(\omega^{-1}\right)^{([n])}(0)}{[n]!}$ is a short notation for:

$$\sum_{l_i}\int\prod_{i=1}^{n}dZ_{l_i}^{(1)}...dZ_{l_i}^{(l_i)}\left(\frac{\delta^n\left[\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right]}{\prod_{i=1}^{n}\delta\left(\left|\Psi\left(\theta-\frac{\left|Z-Z_{l_i}^{(1)}\right|+...+\left|Z^{l_i(l-1)}-Z_{l_i}^{(l_i)}\right|}{c},Z_{l_i}^{(l_i)}\right)\right|^2\right)}\right)_{|\Psi|=0}$$

and $\frac{\left(\omega^{-1}\right)^{([n-1])}(0)|\Psi|^2}{[n-1]!}$ stands for:

$$\sum_{l_i}\int\prod_{i=1}^{n-1}dZ_{l_i}^{(1)}...dZ_{l_i}^{(l_i)}\left(\frac{\delta^{n-1}\left[\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\right]}{\prod_{i}\delta\left(\left|\Psi\left(\theta-\frac{\left|Z-Z_{l_i}^{(1)}\right|+...+\left|Z_{l_i}^{(l-1)}-Z_{l_i}^{(l_i)}\right|}{c},Z_{l_i}^{(l_i)}\right)\right|^2\right)^{k_{l_i}}}\right)_{|\Psi|=0}$$

$$\times\sum_{j=1}^{n-1}\left|\Psi\left(\theta-\frac{\left|Z-Z_{l_j}^{(1)}\right|+...+\left|Z_{l_i}^{(l-1)}-Z_{l_j}^{(l_j)}\right|}{c},Z_{l_j}^{(l_j)}\right)\right|^2$$

Similar notation is valid for $\frac{\left(\omega^{-1}\right)^{([n])}\left(\mathcal{G}_0(0,0,Z)\right)|\Psi|^2}{[n-1]!}$, the derivatives are evaluated at $|\Psi(\theta,Z)|^2=\mathcal{G}_0(0,0,Z)$.

We have also used $|\Psi|^2\left[\frac{\delta}{\delta|\Psi|^2}\right]$ as a shorthand for:

$$\sum_l\int\left(\frac{dZ_l^{(1)}...dZ_l^{(l)}}{(k_l)!}\right)\left|\Psi\left(\theta-\frac{\left|Z-Z_l^{(1)}\right|+...+\left|Z_l^{(l-1)}-Z_l^{(l_j)}\right|}{c},Z_l^{(l)}\right)\right|^2 \tag{214}$$
$$\times\frac{\delta}{\delta\left(\left|\Psi\left(\theta-\frac{\left|Z-Z_l^{(1)}\right|+...+\left|Z_l^{(l-1)}-Z_l^{(l)}\right|}{c},Z_l^{(l)}\right)\right|^2\right)}$$

Ultimately, the computation of the Green function involves the series expansion of the potential $V(\Psi)$. As shown earlier (see equation(212)) the graphs generated by this expansion are equivalent to those that would result if, in equation (208) the potential had been expanded to the second order in Ψ and if, in all terms of higher order, $|\Psi(\theta,Z)|^2$ had been replaced by $\mathcal{G}_0(0,Z)$. As a consequence, the second order Green functions are computed with the action:

$$-\frac{1}{2}\Psi^{\dagger}(\theta,Z)\left(\nabla_\theta\frac{\sigma_\theta^2}{2}\nabla_\theta\right)\Psi(\theta,Z)$$
$$+\frac{1}{2}|\Psi|^2\left[\frac{\delta\left[\Psi^{\dagger}(\theta',Z)\nabla_\theta\left(\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\Psi(\theta,Z)\right)\right]}{\delta|\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}+|\Psi|^2\left[\frac{\delta[V(\Psi)]}{\delta|\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}$$

Equivalently, this means that the 2 points Green functions are the inverse of the operator:

$$-\frac{1}{2}\nabla_\theta\frac{\sigma_\theta^2}{2}\nabla_\theta+\frac{1}{2}\left[\frac{\delta\left[\Psi^{\dagger}(\theta',Z)\nabla_\theta\left(\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)\Psi(\theta,Z)\right)\right]}{\delta|\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}+\left[\frac{\delta[V(\Psi)]}{\delta|\Psi|^2}\right]_{\substack{|\Psi(\theta,Z)|^2\\=\mathcal{G}_0(0,Z)}}$$

and, at the lowest order in $|\Psi(\theta,Z)|^2$, this corresponds to the effective action of the text.

Appendix 2

Corrections to background field

Corrections to saddle point

To compute the corrections to the background due to:

$$K=K\left(\theta,Z,Z',\|\Psi\|^2,\|\Gamma\|^2\right)$$

we can ultimately rewrite (100) by including the term KT so that the squared term writes:

$$-\frac{\left(\left(\frac{|\Psi(\theta,Z)|^2}{\tau\omega}\left(T-\lambda\tau\hat{T}\right)\right)\right)^2}{2\sigma_T^2}+KT$$
$$=\quad-\frac{1}{2\sigma_T^2}\left(\frac{|\Psi(\theta,Z)|^2}{\tau\omega}\left(T-\lambda\tau\hat{T}-K\sigma_T^2\left(\frac{\tau\omega}{|\Psi(\theta,Z)|^2}\right)^2\right)\right)^2+\frac{1}{2}K^2\sigma_T^2\left(\frac{\tau\omega}{|\Psi(\theta,Z)|^2}\right)^2+K\lambda\tau\hat{T}$$

the last term can be included in $S_{\Gamma}^{(2)}$, and defining:

$$\begin{aligned} h_C &= h_C\left(\omega\left(J,\theta,Z,|\Psi|^2\right)\right) \\ h_D &= h_D\left(\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)\right) \end{aligned}$$

we find ultimately:

$$\begin{aligned} S_{\Gamma}^{(1)} &= \Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\frac{\sigma_T^2}{2}\nabla_T^2-\frac{1}{2\sigma_T^2}\left(\frac{|\Psi(\theta,Z)|^2}{\tau\omega}\left(T-\lambda\tau\hat{T}-K\sigma_T^2\left(\frac{\tau\omega}{|\Psi(\theta,Z)|^2}\right)^2\right)\right)^2\right. \\ &\left.+\frac{1}{2\tau\omega(Z)}+\frac{1}{2}K^2\sigma_T^2\left(\frac{\tau\omega}{|\Psi(\theta,Z)|^2}\right)^2\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right) \end{aligned} \tag{215}$$

$$\begin{aligned} S_{\Gamma}^{(2)} &= \Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\frac{\sigma_{\hat{T}}^2}{2}\nabla_{\hat{T}}^2-\frac{\left(\rho\left(C(\theta)|\Psi(\theta,Z)|^2h_C+D(\theta)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2h_D\right)\left(\hat{T}-\left\langle\hat{T}\right\rangle_0\right)\right)^2}{2\sigma_{\hat{T}}^2\omega^2\left(\theta,Z,|\Psi|^2\right)}\right. \\ &\left.+\frac{\rho\left(C(\theta)|\Psi(\theta,Z)|^2h_C-\eta H(\delta-T)+D(\theta)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2h_D\right)}{2\omega\left(\theta,Z,|\Psi|^2\right)}\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right) \\ &+\frac{1}{2}\sigma_{\hat{T}}^2\left(\frac{\omega\left(\theta,Z,|\Psi|^2\right)}{\rho\left(C(\theta)|\Psi(\theta,Z)|^2h_C+D(\theta)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2h_D\right)}\right)^2(K\lambda\tau)^2+K\lambda\tau\left\langle\hat{T}\right\rangle_0 \end{aligned} \tag{216}$$

with:

$$\begin{aligned} \left\langle\hat{T}(Z,Z)\right\rangle_0 &= \frac{\left(h(Z,Z')C_{Z,Z'}(\theta)h_C|\Psi(\theta,Z)|^2-\eta H(\delta-T(Z,Z'))\right)}{C_{Z,Z'}(\theta)|\Psi(\theta,Z)|^2h_C+D_{Z,Z'}(\theta)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2h_D} \\ &+\sigma_{\hat{T}}^2\left(\frac{\omega\left(\theta,Z,|\Psi|^2\right)}{\rho\left(C(\theta)|\Psi(\theta,Z)|^2h_C+D(\theta)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2h_D\right)}\right)^2K\lambda\tau \end{aligned}$$

The effective potentials are thus modified by shifts proportional to σ_T^2 and $\sigma_{\hat{T}}^2$ respectively. As a consequence for $\frac{\sigma_{\hat{T}}^2}{\sigma_T^2}<<1$, $\sigma_T^2<<1$ these corrections can be treated purturbatively and neglected in first approximation as quoted in the text.

Corrections to the averages

The average equations are modified with terms proportional to $\langle K\rangle$:

$$\left\langle \hat{T}\left(Z,Z'\right)\right\rangle = \left\langle \hat{T}\right\rangle_0 = \frac{\left(\left\langle h\left(Z,Z'\right)C_{Z,Z'}\left(\theta\right)h_C\left|\Psi\left(\theta,Z\right)\right|^2\right\rangle - \eta H\left(\delta-\left\langle T\left(Z,Z'\right)\right\rangle\right)\right)}{\left\langle C_{Z,Z'}\left(\theta\right)\left|\Psi\left(\theta,Z\right)\right|^2 h_C\right\rangle + \left\langle D_{Z,Z'}\left(\theta\right)\left|\Psi\left(\theta-\frac{\left|Z-Z'\right|}{c},Z'\right)\right|^2 h_D\right\rangle}$$
$$+\sigma_{\hat{T}}^2\left(\frac{\omega\left(\theta,Z,\left|\Psi\right|^2\right)}{\rho\left(C\left(\theta\right)\left|\Psi\left(\theta,Z\right)\right|^2 h_C + D\left(\theta\right)\left|\Psi\left(\theta-\frac{\left|Z-Z'\right|}{c},Z'\right)\right|^2 h_D\right)}\right)^2\left\langle K\right\rangle\lambda\tau$$

and:

$$\left\langle T\left(Z,Z'\right)\right\rangle = \lambda\tau\left\langle \hat{T}\left(Z,Z'\right)\right\rangle + \left\langle K\right\rangle\sigma_T^2\left(\frac{\tau\omega}{\left|\Psi\left(\theta,Z\right)\right|^2}\right)^2$$

Given the definition of K, it is a function of the collection $\left\{\left\langle T\left(Z,Z'\right)\right\rangle\right\}_{(Z,Z')}$. The corrections to $\left\langle T\left(Z,Z'\right)\right\rangle$ and $\left\langle \hat{T}\left(Z,Z'\right)\right\rangle$ are obtained purturbatively by replacing $K\left(\left\{\left\langle T\left(Z,Z'\right)\right\rangle\right\}_{(Z,Z')}\right)$ with $\left\langle T\left(Z,Z'\right)\right\rangle$ computed for $K=0$. As a first approximation, it is possible to replace $\left\langle T\left(Z,Z'\right)\right\rangle$ by its space average, so that:

$$K\left(\left\{\left\langle T\left(Z,Z'\right)\right\rangle\right\}_{(Z,Z')}\right) \simeq K\left(\left\langle T\right\rangle\right)$$

where:

$$\left\langle T\right\rangle \simeq \frac{\lambda\tau}{V}\int\frac{\left(\left\langle h\left(Z,Z'\right)C_{Z,Z'}\left(\theta\right)h_C\left|\Psi\left(\theta,Z\right)\right|^2\right\rangle - \eta H\left(\delta-\left\langle T\left(Z,Z'\right)\right\rangle\right)\right)}{\left\langle C_{Z,Z'}\left(\theta\right)\left|\Psi\left(\theta,Z\right)\right|^2 h_C\right\rangle + \left\langle D_{Z,Z'}\left(\theta\right)\left|\Psi\left(\theta-\frac{\left|Z-Z'\right|}{c},Z'\right)\right|^2 h_D\right\rangle}d\left(Z,Z'\right)$$

and V is the thread's volume.

The correction terms proportional to K in (215) and (216) modify the condition for minima (114) that writes:

$$2 = \frac{1}{2\tau\omega\left(Z\right)} + a_C\left(Z\right) + a_D\left(Z\right) + a_K\left(Z\right) \tag{217}$$
$$+\frac{\rho\left(C\left(\theta\right)\left|\Psi\left(\theta,Z\right)\right|^2 h_C + D\left(\theta\right)\left|\Psi\left(\theta-\frac{\left|Z-Z'\right|}{c},Z'\right)\right|^2 h_D\right)}{2\omega\left(\theta,Z,\left|\Psi\right|^2\right)} - \frac{\delta U\left(\left\{\left\|\Gamma_0\left(\theta,Z,Z'\right)\right\|^2\right\}\right)}{\delta\left\|\Gamma_0\left(\theta,Z,Z'\right)\right\|^2}$$

with:

$$a_K\left(Z\right) = \frac{1}{2}K^2\sigma_T^2\left(\frac{\tau\omega}{\left|\Psi\left(\theta,Z\right)\right|^2}\right)^2$$
$$+\frac{1}{2}\sigma_{\hat{T}}^2\left(\frac{\omega\left(\theta,Z,\left|\Psi\right|^2\right)}{\rho\left(C\left(\theta\right)\left|\Psi\left(\theta,Z\right)\right|^2 h_C + D\left(\theta\right)\left|\Psi\left(\theta-\frac{\left|Z-Z'\right|}{c},Z'\right)\right|^2 h_D\right)}\right)^2\left(K\lambda\tau\right)^2 + K\lambda\tau\left\langle \hat{T}\right\rangle_0$$

and this set of equations shifts the norm $\left\|\Gamma_0\left(\theta,Z,Z'\right)\right\|^2$ at each point.

As a consequence, the condition (**??**) for non trivial background state at (Z, Z'):

$$S\left(\|\Gamma_0(Z,Z')\|^2\right) = U\left(\left\{\|\Gamma_0(Z,Z')\|^2\right\}\right) - \frac{\delta U\left(\left\{\|\Gamma_0(\theta,Z,Z')\|^2\right\}\right)}{\delta\|\Gamma_0(\theta,Z,Z')\|^2}\|\Gamma_0(Z,Z')\|^2 < 0 \tag{218}$$

is modified by the shift of the norm. The correction due to the backreaction modifies the set of points with non trivial background.

The expression for $K\left(\theta, Z, Z', \|\Psi\|^2, \|\Gamma\|^2\right)$ can be obtained by using (85):

$$\begin{aligned} & K\left(\theta, Z, Z', \|\Psi\|^2, \|\Gamma\|^2\right) \\ = & \int \Gamma^{\dagger}\left(T_1, \hat{T}_1, \theta_1, Z_1, Z_1', C_1, D_1\right) \frac{\delta W\left(T_1, \hat{T}_1, \theta_1, Z_1, Z_1', C_1, D_1\right)}{\delta T\left|\Gamma\left(T, \hat{T}, \theta, Z, Z', C, D\right)\right|^2} \\ & \times\Gamma\left(T_1, \hat{T}_1, \theta_1, Z_1, Z_1', C_1, D_1\right) d\left(T_1, \hat{T}_1, \theta_1, Z_1, Z_1', C_1, D_1\right) \end{aligned} \tag{219}$$

where:

$$\begin{aligned} & W\left(T, \hat{T}, \theta, Z, Z', C, D\right) \\ = & \nabla_C\left(\frac{C}{\tau_C\omega\left(J,\theta,Z,|\Psi|^2\right)} - \frac{\alpha_C(1-C)\,\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z',\omega'\right)\right|^2}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\right)|\Psi(\theta,Z)|^2 \\ & +\nabla_D\left(\frac{D}{\tau_D\omega\left(J,\theta,Z,|\Psi|^2\right)} - \alpha_D(1-D)|\Psi(\theta,Z)|^2\right) \\ & -\nabla_{\hat{T}}\frac{\rho\left(\left(h(Z,Z')-\hat{T}\right)C|\Psi(\theta,Z)|^2 h_C - D\hat{T}\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2 h_D\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)} \\ & -\nabla_T\left(-\frac{1}{\tau\omega}T + \frac{\lambda}{\omega}\hat{T}\right)|\Psi(\theta,Z,\omega)|^2 \end{aligned}$$

Given that $\tau_C >> 1$ and $\tau_D >> 1$ and that C and D are close to 1, the two first contributions are negligibles. Given that $|\Psi(\theta,Z,\omega)|^2$ depends on the potential $V\left(|\Psi(\theta,Z,\omega)|^2\right)$, we can assume that:

$$\left|\frac{\delta|\Psi(\theta,Z,\omega)|^2}{\delta\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2}\right| << \left|\frac{\delta\omega\left(J,\theta,Z,|\Psi|^2\right)}{\delta\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2}\right|$$

the last contribution can be neglected, and we can consider that $K\left(\theta, Z, Z', \|\Psi\|^2, \|\Gamma\|^2\right)$ can be computed with:

$$\begin{aligned} & W\left(T, \hat{T}, \theta, Z, Z', C, D\right) \\ = & -\nabla_{\hat{T}}\frac{\rho\left(\left(h(Z,Z')-\hat{T}\right)C|\Psi(\theta,Z)|^2 h_C - D\hat{T}\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2 h_D\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)} \end{aligned} \tag{220}$$

In the computation of (219), the contribution proportional to:

$$\int \Gamma^{\dagger}\left(T_1,\hat{T}_1,\theta_1,Z_1,Z_1',C_1,D_1\right)\nabla_{\hat{T}}\Gamma\left(T_1,\hat{T}_1,\theta_1,Z_1,Z_1',C_1,D_1\right)d\left(T_1,\hat{T}_1,\theta_1,Z_1,Z_1',C_1,D_1\right)$$

is equal to 0, since Γ is gaussian. As a consequence, the gradient $\nabla_{\hat{T}}$ in (220) acts on the function at its right in (220).

We thus replace:

$$\begin{aligned}
&W\left(T,\hat{T},\theta,Z,Z',C,D\right)\rightarrow \\
&= \frac{\rho\left(C\left|\Psi\left(\theta,Z\right)\right|^2 h_C + D\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2 h_D\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)} \\
&\simeq \rho C\left|\Psi\left(\theta,Z\right)\right|^2 + \frac{\rho D\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}
\end{aligned}\tag{221}$$

for $h_C\left(\omega\left(J,\theta,Z,|\Psi|^2\right)\right)\simeq\omega\left(J,\theta,Z,|\Psi|^2\right)$ and $h_D\left(\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)\right)\simeq\left(\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',|\Psi|^2\right)\right)$.

Given our approximations, we then deduce that (219) reduces to:

$$\begin{aligned}
&K\left(\theta,Z,Z',\|\Psi\|^2,\|\Gamma\|^2\right) \\
&= \int\rho D\left|\Psi\left(\theta-\frac{|Z_1-Z_1'|}{c},Z_1'\right)\right|^2\frac{\delta\frac{\omega\left(J,\theta-\frac{|Z_1-Z_1'|}{c},Z_1',|\Psi|^2\right)}{\omega\left(J,\theta,Z_1,|\Psi|^2\right)}}{\delta T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2} \\
&\quad\times\left|\Gamma\left(T_1,\hat{T}_1,\theta_1,Z_1,Z_1',C_1,D_1\right)\right|^2 d\left(T_1,\hat{T}_1,\theta_1,Z_1,Z_1',C_1,D_1\right)
\end{aligned}\tag{222}$$

where we approximate:

$$\delta\frac{\omega\left(J,\theta-\frac{|Z_1-Z_1'|}{c},Z_1',|\Psi|^2\right)}{\omega\left(J,\theta,Z_1,|\Psi|^2\right)}\simeq\delta\frac{|Z_1-Z_1'|^2\nabla_{Z_1}^2\omega\left(J,\theta,Z_1,|\Psi|^2\right)}{\omega\left(J,\theta,Z_1,|\Psi|^2\right)}$$

The last expression being obtained in the limit of slowly varying activities. In the same approximation:

$$\begin{aligned}
\frac{\delta\frac{|Z_1-Z_1'|^2\nabla_{Z_1}^2\omega\left(J,\theta,Z_1,|\Psi|^2\right)}{\omega\left(J,\theta,Z_1,|\Psi|^2\right)}}{\delta T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2} &= \frac{\delta\left(|Z_1-Z_1'|^2\left(\nabla_{Z_1}^2\ln\left(\omega\left(J,\theta,Z_1,|\Psi|^2\right)\right)-\left(\nabla_{Z_1}\ln\left(\omega\left(J,\theta,Z_1,|\Psi|^2\right)\right)\right)^2\right)\right)}{\delta T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2} \\
&\simeq |Z_1-Z_1'|^2\nabla_{Z_1}^2\frac{\delta\left(\ln\left(\omega\left(J,\theta,Z_1,|\Psi|^2\right)\right)\right)}{\delta T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2}
\end{aligned}$$

We show in appendix 5 that the derivatives:

$$\frac{\delta\omega\left(J,\theta,Z_1,|\Psi|^2\right)}{\delta T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2}$$

are proportional to some exponential:

$$\exp\left(-a\left|Z_1-Z\right|\right)$$

so that, by averaging over the entire space, we find:

$$\begin{aligned}
& K\left(\theta,Z,Z',\|\Psi\|^2,\|\Gamma\|^2\right) \\
\simeq\ & \frac{1}{a}\rho D\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2|Z-Z'|^2\nabla_Z^2\frac{\delta\left(\ln\left(\omega\left(J,\theta,Z,|\Psi|^2\right)\right)\right)}{\delta T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2}\|\Gamma\|^2
\end{aligned} \tag{223}$$

Given (78):

$$\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)=G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\left(\mathcal{G}_0+\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)d\right. \tag{224}$$

and using:

$$\begin{aligned}
& \frac{\delta\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}}{\delta T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2} \\
\simeq\ & \frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',\Psi\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}+\frac{\kappa}{N}\int\frac{\delta\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}}{\delta T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2}T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right) \\
\simeq\ & \frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',\Psi\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}+\frac{1}{a}\frac{\delta\frac{\kappa}{N}\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',\Psi\right)}{\delta T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2}T\left(Z,\theta,Z',\theta-\frac{|Z-Z'|}{c}\right) \\
\simeq\ & \frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z'|}{c},Z',\Psi\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}
\end{aligned}$$

the differentiation of (224) is:

$$\begin{aligned}
& \frac{\delta\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right)}{\delta T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^2} \\
=\ & \frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\left(\mathcal{G}_0+\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^2\right) \\
& G'\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\left(\mathcal{G}_0+\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\right)
\end{aligned}$$

so that:

$$\frac{\delta\omega\left(J,\theta,Z,|\Psi|^{2}\right)}{\delta T\left|\Gamma\left(T,\hat{T},\theta,Z,Z',C,D\right)\right|^{2}} = \frac{\kappa}{N}\omega\left(J,\theta-\frac{|Z-Z_{1}|}{c},Z_{1},\Psi\right)\omega\left(J,\theta,Z,|\Psi|^{2}\right)\left(\mathcal{G}_{0}+\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}\right)$$
$$G'\left(G^{-1}\left(\omega\left(J,\theta,Z,|\Psi|^{2}\right)\right)\right)$$
$$\simeq \frac{\kappa}{N}\omega^{2}\left(J,\theta,Z,|\Psi|^{2}\right)G'\left(G^{-1}\left(\omega\left(J,\theta,Z,|\Psi|^{2}\right)\right)\right)\left(\mathcal{G}_{0}+\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}\right)$$

and (223) writes:

$$K\left(\theta,Z,Z',\|\Psi\|^{2},\|\Gamma\|^{2}\right) \tag{225}$$
$$\simeq \frac{1}{a}\rho D\frac{\kappa}{N}\|\Gamma\|^{2}\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}|Z-Z'|^{2}$$
$$\times\nabla_{Z}^{2}\left(\omega\left(J,\theta,Z,|\Psi|^{2}\right)G'\left(G^{-1}\left(\omega\left(J,\theta,Z,|\Psi|^{2}\right)\right)\right)\left(\mathcal{G}_{0}+\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}\right)\right)$$

Resolution of saddle point equations without approximation

To study the background state for T and $\hat{T}$, we start from the effective action (98)

$$\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\nabla_{T}\left(\nabla_{T}-\left(-\frac{1}{\tau\omega}T+\frac{\lambda}{\omega}\hat{T}\right)|\Psi(\theta,Z)|^{2}\right)\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right) \tag{226}$$
$$+\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\nabla_{\hat{T}}\left(\nabla_{\hat{T}}-\frac{\rho}{\omega\left(J,\theta,Z,|\Psi|^{2}\right)}\right.\right.$$
$$\left.\left.\times\left(\left(h(Z,Z')-\hat{T}\right)C(\theta)|\Psi(\theta,Z)|^{2}h_{C}-\eta H(\delta-T)-D(\theta)\hat{T}\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}h_{D}\right)\right)\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$$

As in the text, we consider points such that:

$$h(Z,Z')\left\langle C_{Z,Z'}(\theta)h_{C}(\omega(\theta,Z))|\Psi(\theta,Z)|^{2}\right\rangle-\eta>0$$

for which the averages are:

$$\langle T(Z,Z')\rangle=\lambda\tau\left\langle\hat{T}(Z,Z')\right\rangle=\frac{\lambda\tau h(Z,Z')\left\langle C_{Z,Z'}(\theta)h_{C}|\Psi(\theta,Z)|^{2}\right\rangle}{C_{Z,Z'}(\theta)|\Psi(\theta,Z)|^{2}h_{C}+D_{Z,Z'}(\theta)\left|\Psi\left(\theta-\frac{|Z-Z'|}{c},Z'\right)\right|^{2}h_{D}} \tag{227}$$

which allows to rewrite the effective action as:

$$\Gamma^{\dagger}\left(T,\hat{T},\theta,Z,Z'\right)\left(\nabla_{T}\left(\nabla_{T}+u(T-\langle T\rangle)+s\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)\right)+\nabla_{\hat{T}}\left(\nabla_{\hat{T}}+v\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)\right)\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right) \tag{228}$$

with:

$$u = \frac{|\Psi_{0}(Z)|^{2}}{\tau\omega_{0}(Z)}$$
$$v = \rho C\frac{|\Psi_{0}(Z)|^{2}h_{C}(\omega_{0}(Z))}{\omega_{0}(Z)}+\rho D\frac{|\Psi_{0}(Z')|^{2}h_{D}(\omega_{0}(Z'))}{\omega_{0}(Z)}$$
$$s = -\frac{\lambda|\Psi_{0}(Z)|^{2}}{\omega_{0}(Z)}$$

Imposing the constraint that the norm of $\Gamma\left(T,\hat{T},\theta,Z,Z'\right)$ is a given number determined by some average number of connections all over the thread, the saddle point equations becomes:

$$\left(\nabla_T\left(\nabla_T+u\left(T-\langle T\rangle\right)+s\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)\right)+\nabla_{\hat{T}}\left(\nabla_{\hat{T}}+v\left(\hat{T}-\left\langle\hat{T}\right\rangle\right)\right)+\alpha\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right)=0$$

with α the Lagrange multiplier.This also rewrite matricially:

$$\left(\boldsymbol{\nabla}^2+\left(\boldsymbol{\nabla}\right)^t\gamma\mathbf{x}+\alpha\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right)=0 \tag{229}$$

with:

$$\gamma=\begin{pmatrix} u & s \\ 0 & v \end{pmatrix}$$

Solution for Fourier transform

This is solved by considering the Fourier transform of this equation:

$$\left(-\mathbf{k}^2-\left(\mathbf{k}\right)^t\gamma\boldsymbol{\nabla}_{\mathbf{k}}+\alpha\right)\Gamma\left(\mathbf{k},\theta,Z,Z'\right)=0 \tag{230}$$

We write the solution:

$$\Gamma\left(\mathbf{k},\theta,Z,Z'\right)=\exp\left(-\frac{1}{2}\mathbf{k}^tN\mathbf{k}\right)\hat{\Gamma}\left(\mathbf{k},\theta,Z,Z'\right)$$

where the matrix N satisifies:

$$-\mathbf{k}^2+\left(\mathbf{k}\right)^t\gamma N\mathbf{k}=0$$

and:

$$\left(-\left(\mathbf{k}\right)^t\gamma\boldsymbol{\nabla}_{\mathbf{k}}+\alpha\right)\hat{\Gamma}\left(\mathbf{k},\theta,Z,Z'\right)=0$$

Equation for N writes:

$$\frac{1}{2}\left(\gamma N+N\gamma^t\right)=I$$

where I is the identity matrix. The solution is:

$$N=\begin{pmatrix} \frac{1}{u}\left(1+\frac{s^2}{v(u+v)}\right) & -\frac{s}{v(u+v)} \\ -\frac{s}{v(u+v)} & \frac{1}{v} \end{pmatrix}$$

The equation for $\hat{\Gamma}\left(\mathbf{k},\theta,Z,Z'\right)$ becomes:

$$\left(-\left(\mathbf{k}\right)^t\gamma\boldsymbol{\nabla}_{\mathbf{k}}+\alpha\right)\hat{\Gamma}\left(\mathbf{k},\theta,Z,Z'\right)=0 \tag{231}$$

If we diagonalize γ:

$$\gamma=PDP^{-1}$$

with:

$$\begin{aligned} D &= \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \\ P &= \begin{pmatrix} 1 & 1 \\ 0 & \frac{v-u}{s} \end{pmatrix} \end{aligned}$$

and define:

$$\hat{\mathbf{k}}=P^t\mathbf{k}$$

equation (231) is:

$$\left(-\left(\hat{\mathbf{k}}\right)^{t} D\boldsymbol{\nabla}_{\hat{\mathbf{k}}}+\alpha\right)\hat{\Gamma}\left(\mathbf{k},\theta,Z,Z'\right)=0$$

with solution:

$$\hat{\Gamma}\left(\mathbf{k},\theta,Z,Z'\right)=\hat{k}_1^{\frac{\alpha\delta}{u}}\hat{k}_2^{\frac{(1-\delta)\alpha}{v}}$$

where $\hat{k}_1$ and $\hat{k}_2$ are the component of $\hat{\mathbf{k}}$:

$$\begin{aligned}\hat{k}_1 &= k_1\\ \hat{k}_1 &= k_1+\frac{v-u}{s}k_2\end{aligned}$$

and ultimately:

$$\Gamma_\delta\left(\mathbf{k},\theta,Z,Z'\right)=\exp\left(-\frac{1}{2}\mathbf{k}^t N\mathbf{k}\right)k_1^{\frac{\alpha\delta}{u}}\left(k_1+\frac{v-u}{s}k_2\right)^{\frac{(1-\delta)\alpha}{v}}$$

The parameter δ implies the possibility of several local global minima for each (Z,Z'). As explained in the text, this parameter is not free if we aim at obtaining a minimum at all points (Z,Z').

Background field

To come back to the background field we compute the inverse Fourier transform:

$$\Gamma_\delta\left(T,\hat{T},\theta,Z,Z'\right)=\int\exp\left(-\frac{1}{2}\mathbf{k}^t N\mathbf{k}-i\mathbf{k}\boldsymbol{\Delta}\mathbf{T}\right)k_1^{\frac{\alpha\delta}{u}}\left(k_1+\frac{v-u}{s}k_2\right)^{\frac{(1-\delta)\alpha}{v}}\frac{d\mathbf{k}}{2\pi}$$

where:

$$\boldsymbol{\Delta}\mathbf{T}=\begin{pmatrix}T-\langle T\rangle\\ \hat{T}-\left\langle\hat{T}\right\rangle\end{pmatrix}$$

To estimate the integral, we diagonalize N:

$$N=\begin{pmatrix}\frac{1}{u}\left(1+\frac{s^2}{v(u+v)}\right) & -\frac{s}{v(u+v)}\\ -\frac{s}{v(u+v)} & \frac{1}{v}\end{pmatrix}=PDP^{-1}$$

with:

$$D=\begin{pmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix}$$

and the eigenvalues:

$$\lambda_\pm=\frac{\frac{1}{u}\left(1+\frac{s^2}{v(u+v)}\right)+\frac{1}{v}}{2}\pm\sqrt{\left(\frac{\frac{1}{u}\left(1+\frac{s^2}{v(u+v)}\right)-\frac{1}{v}}{2}\right)^2+\left(\frac{s}{v\left(u+v\right)}\right)^2}$$

The matrix P is orthogonal:

$$P=\begin{pmatrix}\cos x & -\sin x\\ \sin x & \cos x\end{pmatrix}$$

$$P^{-1}=P^t$$

and x satifies:

$$\begin{aligned} \left(\frac{\lambda_+ - \lambda_-}{2}\right)\sin 2x &= -\frac{s}{v\,(u+v)} \\ \left(\frac{\lambda_+ + \lambda_-}{2}\right) + \left(\frac{\lambda_+ - \lambda_-}{2}\right)(\cos 2x) &= \frac{1}{u}\left(1+\frac{s^2}{v\,(u+v)}\right) \end{aligned}$$

with:

$$\frac{\lambda_+ + \lambda_-}{2} = \frac{\frac{1}{u}\left(1+\frac{s^2}{v(u+v)}\right)+\frac{1}{v}}{2}$$

As a consequence, we obtain:

$$\tan 2x = \frac{-\frac{2s}{v(u+v)}}{\frac{\frac{1}{u}\left(1+\frac{s^2}{v(u+v)}\right)-\frac{1}{v}}{2}} = -\frac{4su}{v^2-u^2+s^2}$$

It thus implies that $\Gamma_\delta\left(T,\hat{T},\theta,Z,Z'\right)$ is given by:

$$\begin{aligned} \Gamma_\delta\left(T,\hat{T},\theta,Z,Z'\right) &= \int \exp\left(-\frac{1}{2}\mathbf{k}^t D\mathbf{k} - i\mathbf{k}\mathbf{\Delta T}'\right)\times \\ &\times\left(k_1\cos x - k_2\sin x\right)^{\frac{\alpha\delta}{u}}\left(k_1\left(\cos x+\frac{v-u}{s}\sin x\right)+\left(\frac{v-u}{s}\cos x-\sin x\right)k_2\right)^{\frac{(1-\delta)\alpha}{v}}\frac{d\mathbf{k}}{2\pi} \end{aligned}$$

with:

$$\mathbf{\Delta T}' = P^t\mathbf{\Delta T}$$

In the approximation given in the text, we have $s << 1$ and:

$$\begin{aligned} &\Gamma_\delta\left(T,\hat{T},\theta,Z,Z'\right) \\ \simeq &\int \exp\left(-\frac{1}{2}\mathbf{k}^t D\mathbf{k} - i\mathbf{k}\mathbf{\Delta T}'\right)\left(k_1 - xk_2\right)^{\frac{\alpha\delta}{u}}\left(\frac{v-u}{s}k_2+k_1\right)^{\frac{(1-\delta)\alpha}{v}}\frac{d\mathbf{k}}{2\pi} \\ \simeq &\int \exp\left(-\frac{1}{2}\mathbf{k}^t D\mathbf{k} - i\mathbf{k}\mathbf{\Delta T}'\right)\left(k_1\right)^{\frac{\alpha\delta}{u}}\left(\frac{v-u}{s}k_2\right)^{\frac{(1-\delta)\alpha}{v}}\left(1-x\frac{\alpha\delta}{u}\frac{k_2}{k_1}\right)\left(1+\frac{s}{u-v}\frac{(1-\delta)\,\alpha}{v}\frac{k_1}{k_2}\right)\frac{d\mathbf{k}}{2\pi} \\ \simeq &\left(\frac{v-u}{s}\right)^{\frac{(1-\delta)\alpha}{v}}\int \exp\left(-\frac{1}{2}\mathbf{k}^t D\mathbf{k} - i\mathbf{k}\mathbf{\Delta T}'\right)\left(k_1\right)^{\frac{\alpha\delta}{u}}\left(k_2\right)^{\frac{(1-\delta)\alpha}{v}}\left(1-x\frac{\alpha\delta}{u}\frac{k_2}{k_1}+\frac{s\,(1-\delta)\,\alpha}{v\,(u-v)}\frac{k_1}{k_2}\right)\frac{d\mathbf{k}}{2\pi} \end{aligned}$$

These integrals are sums of products of parabolic cylinder functions:

$$\begin{aligned} &\Gamma_\delta\left(T,\hat{T},\theta,Z,Z'\right) \\ \simeq &\left(\frac{v-u}{s}\right)^{\frac{(1-\delta)\alpha}{v}} 2^{\frac{1}{\alpha u}+1}\prod_{i=1}^{2}\exp\left(-\left(\left(\frac{D^{-\frac{1}{2}}P^t\mathbf{\Delta T}}{2}\right)_i\right)^2\right) \\ &\times\left\{\prod_{i=1}^{2}D_{p_i}\left(\left(\frac{D^{-\frac{1}{2}}P^t\mathbf{\Delta T}}{4}\right)_i\right) - x\alpha\frac{\delta\prod_{i=1}^{2}D_{p_i^{(1)}}\left(\left(\frac{D^{-\frac{1}{2}}P^t\mathbf{\Delta T}}{4}\right)_i\right)}{u}+\frac{s\,(1-\delta)\,\alpha\prod_{i=1}^{2}D_{p_i^{(2)}}\left(\left(\frac{D^{-\frac{1}{2}}P^t\mathbf{\Delta T}}{4}\right)_i\right)}{v\,(u-v)}\right\} \end{aligned}$$

where:

$$\begin{aligned}
p_1 &= \frac{\alpha\delta}{u}, p_2 = \frac{(1-\delta)\alpha}{v} \\
p_1^{(1)} &= \frac{\alpha\delta}{u} - 1, p_2^{(1)} = \frac{(1-\delta)\alpha}{v} + 1 \\
p_1^{(1)} &= \frac{\alpha\delta}{u} + 1, p_2^{(1)} = \frac{(1-\delta)\alpha}{v} - 1
\end{aligned}$$

The approximation made in the text, i.e. $s << 1$ corresponds to the first terms:

$$\Gamma_\delta\left(T,\hat{T},\theta,Z,Z'\right) \simeq \mathcal{N}\prod_{i=1}^{2}\exp\left(-\left(\left(\frac{D^{-\frac{1}{2}}P^t\mathbf{\Delta T}}{2}\right)_i\right)^2\right)\prod_{i=1}^{2}D_{p_i}\left(\left(\frac{D^{-\frac{1}{2}}P^t\mathbf{\Delta T}}{4}\right)_i\right)$$

where $\mathcal{N}$ is a normalization factor.

Condition for minima

The conditions for finding a minimum with $\mathcal{N}\neq 0$ is similar to those presented in the text. Action (98) for $\eta = 0$:

$$\begin{aligned}
&\Gamma^\dagger\left(T,\hat{T},\theta,Z,Z'\right)\left(\nabla_T\left(\nabla_T - \left(-\frac{1}{\tau\omega}T + \frac{\lambda}{\omega}\hat{T}\right)|\Psi(\theta,Z)|^2\right)\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right) \\
&+\Gamma^\dagger\left(T,\hat{T},\theta,Z,Z'\right)\left(\nabla_{\hat{T}}\left(\nabla_{\hat{T}} - \frac{\rho}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\right.\right. \\
&\times\left.\left.\left(\left(h(Z,Z') - \hat{T}\right)C(\theta)|\Psi(\theta,Z)|^2 h_C - D(\theta)\hat{T}\left|\Psi\left(\theta - \frac{|Z-Z'|}{c}, Z'\right)\right|^2 h_D\right)\right)\right)\Gamma\left(T,\hat{T},\theta,Z,Z'\right)
\end{aligned} \tag{232}$$

reduces, after using the saddle point equation (229), to:

$$\alpha\int|\Gamma(Z,Z')|^2 \tag{233}$$

where:

$$|\Gamma(Z,Z')|^2 = \int\left|\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right|^2 d\left(T,\hat{T}\right)$$

Using $p_1 = \frac{\alpha\delta}{u}, p_2 = \frac{(1-\delta)\alpha}{v}$ allows to find α and δ. As in the text, if we ensures that the action has a minimum with $|\Gamma(Z,Z')|^2 > 0$ at each point (Z,Z') for $\hat{T}$, we find $p_2 = \frac{1}{2}$ and thus $\delta = 1 - \frac{v}{2\alpha}$. The value of $p_1 = \frac{\alpha - \frac{v}{2}}{u}$.

Then, if we include the potential for $\left|\Gamma\left(T,\hat{T},\theta,Z,Z'\right)\right|^2$, equation (229) is modified by shifting:

$$\alpha \to \alpha - U'\left(\left|\Gamma_{p_1,p_2}\left(T,\hat{T},\theta,Z,Z'\right)\right|^2, Z, Z'\right)$$

so that the action (233) becomes:

$$\alpha\int|\Gamma_{p_1,p_2}(Z,Z')|^2 + \hat{U}\left(\left|\Gamma_{p_1,p_2}\left(T,\hat{T},\theta,Z,Z'\right)\right|^2, Z, Z'\right)$$

with:

$$\hat{U}\left(\left|\Gamma_{p_1,p_2}\left(T,\hat{T},\theta,Z,Z'\right)\right|^2, Z, Z'\right) - \left|\Gamma_{p_1,p_2}\left(T,\hat{T},\theta,Z,Z'\right)\right|^2\hat{U}'\left(\left|\Gamma_{p_1,p_2}\left(T,\hat{T},\theta,Z,Z'\right)\right|^2, Z, Z'\right)$$

The action is minimal for:

$$\alpha + \hat{U}'\left(\left|\Gamma_{p_1,p_2}\left(T,\hat{T},\theta,Z,Z'\right)\right|^2\right) = 0$$

and the norm of $\Gamma_{p_1,p_2}\left(T,\hat{T},\theta,Z,Z'\right)$ is given by:

$$\left|\Gamma_{p_1,p_2}\left(T,\hat{T},\theta,Z,Z'\right)\right|^2 = \hat{U}'\left(-\alpha,Z,Z'\right)$$

If:

$$\alpha\int U'\left(-\alpha\right)+\hat{U}\left(U'\left(-\alpha\right),Z,Z'\right)<0 \tag{234}$$

there is a non trivial state at (Z,Z'). Otherwise $\left|\Gamma_{p_1,p_2}\left(T,\hat{T},\theta,Z,Z'\right)\right|^2 = 0$. In the text we have assumed that (234) is satisfied at each point. The value of α is determined by the condition:

$$\int \hat{U}'\left(-\alpha,Z,Z'\right)d\left(Z,Z'\right) = \left\|\Gamma_{p_1,p_2}\right\|^2$$

where $\left\|\Gamma_{p_1,p_2}\right\|^2$ is the norm of Γ_{p_1,p_2}.

Appendix 3 Static background state for the system

We look for a static background state for the whole system. In the static case, we assume that the static background field $\Psi_0\left(Z\right)$ is the minimum of $V\left(\Psi\right)$.

General equations

An approximate static solution of (78) can be found for the constant background and a constant current, i.e. $J=\bar{J}$. We also set:

$$T\left(Z,Z_1\right) = \bar{T}\left(Z,\theta,Z_1,\theta-\frac{\left|Z-Z_1\right|}{c}\right)$$

From now on, the quantity $T\left(Z,Z_1\right)$ refers to the average of the connectivity function at points (Z,Z_1), in the background state defined above, i.e. $T\left(Z,Z_1\right)$ refers to $\left\langle T\left(Z,Z_1\right)\right\rangle$ defined as:

$$\left\langle T\left(Z,Z_1\right)\right\rangle = \int T\left|\Gamma\left(T,\hat{T},\theta,Z,Z_1\right)\right|^2 dT$$

For points such that $T\left(Z,Z'\right)\neq 0$, it is defined by the set of equations (67) or (78), (104) (125) if:

$$h\left(Z,Z'\right)C_{Z,Z'}\left(\theta\right)h_C\left(\omega\left(\theta,Z\right)\right)\left|\Psi\left(\theta,Z\right)\right|^2>0$$

We choose $h_C\left(\omega\right)=\omega$ and $h_C\left(\omega'\right)=\omega'$. As explained in the text, the resolution is in three steps.

We solve first for $\omega\left(Z\right)$.

Expression of $\omega(Z)$

We use (125) to replace $\omega' |\Psi(Z')|^2$:

$$\omega' |\Psi(Z')|^2 = \frac{\lambda\tau \exp\left(-\frac{|Z-Z'|}{\nu c}\right) - T(Z,Z')}{T(Z,Z')}\left(\frac{1}{\alpha_D\tau_D} + \omega |\Psi(Z)|^2\right) - \frac{1}{\alpha_C\tau_C}$$

This allows to rewrite (78) as an equation for $\omega^{-1}(Z)$:

$$\begin{aligned}
\omega(Z) &= G\left(\int \frac{\kappa}{N} \frac{\omega(J,Z_1) T \left|\Gamma\left(T,\hat{T},Z,Z_1\right)\right|^2}{\omega(Z)} \left(\mathcal{G}_0 + |\Psi(Z_1)|^2\right) dZ_1\right) \qquad (235)\\
&\rightarrow G\left(\int \frac{\kappa}{N}\left(\frac{\lambda\tau\exp\left(-\frac{|Z-Z_1|}{\nu c}\right) - T(Z,Z_1)}{T(Z,Z_1)}\left(\frac{1}{\alpha_D\tau_D} + \omega|\Psi(Z)|^2\right) - \frac{1}{\alpha_C\tau_C}\right)\frac{T\left|\Gamma\left(T,\hat{T},\theta,Z,Z_1\right)\right|^2}{\omega\left(J,\theta,Z,|\Psi|^2\right)} dT dZ_1\right)\\
&\simeq G\left(\int \frac{\kappa}{N}\left(\left(\lambda\tau\exp\left(-\frac{|Z-Z_1|}{\nu c}\right) - T(Z,Z_1)\right)\left(\left(\frac{1}{\alpha_D\tau_D} - \frac{T(Z,Z_1)}{\alpha_C\tau_C}\right)\omega^{-1} + |\Psi(Z)|^2\right)\right) dZ_1\right)
\end{aligned}$$

We can replace $T(Z,Z_1)$ in the integral by its average:

$$\frac{1}{V} T(Z) = \frac{1}{V}\int T(Z,Z_1)\, dZ_1$$

so that:

$$\omega(Z) = G\left(\frac{\kappa}{N}\left((\lambda\tau\nu c - VT(Z))\left(\left(\frac{1}{\alpha_D\tau_D} - \frac{T(Z)}{V\alpha_C\tau_C}\right)\omega^{-1} + |\Psi(Z)|^2\right)\right)\right) \qquad (236)$$

with solution defined by a function:

$$\omega(Z) = \hat{G}\left(T(Z), |\Psi(Z)|^2\right)$$

Finding $T(Z)$ and $T(Z,Z')$

In a second step, we can insert the solution for ω in the expression for $T(Z,Z')$ that rewrites:

$$\begin{aligned}
T(Z,Z') &= \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{1 + \frac{\alpha_D}{\alpha_C}\frac{\frac{1}{\tau_C} + \alpha_C\omega'|\Psi(Z')|^2}{\frac{1}{\tau_D} + \alpha_D\omega|\Psi(Z)|^2}}\\
&= \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{1 + \frac{\alpha_D}{\alpha_C}\frac{\frac{1}{\tau_C} + \alpha_C\hat{G}\left(T(Z'),|\Psi(Z')|^2\right)|\Psi(Z')|^2}{\frac{1}{\tau_D} + \alpha_D\hat{G}\left(T(Z),|\Psi(Z)|^2\right)|\Psi(Z)|^2}}
\end{aligned}$$

Then integrating over Z' and replacing:

$$\hat{G}\left(T(Z'), |\Psi(Z')|^2\right)|\Psi(Z')|^2$$

by its average over the volume V leads to:

$$T(Z) = \frac{\lambda\tau\nu c}{1 + \frac{\frac{1}{\tau_C\alpha_C} + \Omega}{\frac{1}{\tau_D\alpha_D} + \hat{G}\left(T(Z),|\Psi(Z)|^2\right)|\Psi(Z)|^2}} = \frac{\lambda\tau\nu c\left(\frac{1}{\tau_D\alpha_D} + \hat{G}\left(T(Z),|\Psi(Z)|^2\right)|\Psi(Z)|^2\right)}{\frac{1}{\tau_D\alpha_D} + \frac{1}{\tau_C\alpha_C} + \Omega + \hat{G}\left(T(Z),|\Psi(Z)|^2\right)|\Psi(Z)|^2} \qquad (237)$$

where:

$$\Omega = \frac{1}{V} \int \hat{G}\left(T\left(Z'\right), \left|\Psi\left(Z'\right)\right|^{2}\right) \left|\Psi\left(Z'\right)\right|^{2} dZ' \tag{238}$$

and this leads to the following formula for $T(Z, Z')$:

$$T\left(Z, Z'\right) = \frac{\lambda\tau \exp\left(-\frac{\left|Z-Z'\right|}{\nu c}\right)}{1 + \frac{\frac{1}{\tau_C \alpha_C} + \omega' \left|\Psi\left(Z', \omega'\right)\right|^2}{\frac{1}{\tau_D \alpha_D} + \omega \left|\Psi\left(Z, \omega\right)\right|^2}} = \frac{\lambda\tau \exp\left(-\frac{\left|Z-Z'\right|}{\nu c}\right)}{1 + \frac{\frac{1}{\tau_C \alpha_C} + \hat{G}\left(T\left(Z'\right), \left|\Psi\left(Z'\right)\right|^2\right) \left|\Psi\left(Z', \omega'\right)\right|^2}{\frac{1}{\tau_D \alpha_D} + \hat{G}\left(T\left(Z\right), \left|\Psi\left(Z\right)\right|^2\right) \left|\Psi\left(Z, \omega\right)\right|^2}} \tag{239}$$

Closing the system with minimization equations

The system is then closed by minimizing the action for the field $\Psi(\theta, Z)$:

$$\int \Psi^{\dagger}(\theta, Z) \left(-\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta} - \frac{1}{\hat{G}\left(\left(T\left(Z, Z_1\right)\right)_{Z_1}, \left|\Psi\left(Z\right)\right|^{2}\right)}\right)\right) \Psi(\theta, Z)$$
$$+V\left(\left|\Psi(\theta, Z)\right|^{2} - \int T\left(Z', Z_1\right) \left|\Psi_0\left(Z\right)\right|^{2} dZ_1\right)$$

We have assumed that the field is constrained by a potential limiting the activit around some average $\left|\Psi_0(Z)\right|^2$. We choose:

$$V = \frac{1}{2}\left(\left|\Psi(Z)\right|^{2} - \int T\left(Z', Z_1\right) \left|\Psi_0\left(Z\right)\right|^{2} dZ_1\right)$$

Performing the changement of variable:

$$\Psi(\theta, Z) \rightarrow \exp\left(-\int \frac{1}{G\left(\left|\Psi(\theta, Z)\right|^{2} \frac{\kappa}{N}\left(\lambda\tau\nu c - \int T\left(Z', Z_1\right) dZ_1\right)\right)} d\theta\right) \Psi(\theta, Z)$$

rewrites the action as:

$$\int \Psi^{\dagger}(\theta, Z) \left(-\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}^{2} + \left(\frac{1}{\hat{G}\left(\left(T\left(Z, Z_1\right)\right)_{Z_1}, \left|\Psi\left(Z\right)\right|^{2}\right)}\right)^{2}\right) \Psi(\theta, Z)$$
$$-\frac{1}{2}\int \nabla_{\theta}\left(\frac{1}{\hat{G}\left(\left(T\left(Z, Z_1\right)\right)_{Z_1}, \left|\Psi\left(Z\right)\right|^{2}\right)}\right) \left|\Psi(\theta, Z)\right|^{2} + V\left(\left|\Psi(\theta, Z)\right|^{2} - \int T\left(Z, Z_1\right) \left|\Psi_0\left(Z\right)\right|^{2} dZ_1\right)$$

In the perspective of a static equilibrium, we aim thus at minimizing:

$$\int \left(\frac{1}{\hat{G}\left(\left(T\left(Z, Z_1\right)\right)_{Z_1}, \left|\Psi\left(Z\right)\right|^{2}\right)}\right)^{2} \left|\Psi(\theta, Z)\right|^{2} + V\left(\left|\Psi(\theta, Z)\right|^{2} - \int T\left(Z', Z_1\right) \left|\Psi_0\left(Z\right)\right|^{2} dZ_1\right)$$

with saddle point equation:

$$\begin{aligned}
0 &= \left(\frac{1}{\hat{G}\left(\left(T\left(Z,Z_1\right)\right)_{Z_1},\left|\Psi\left(Z\right)\right|^2\right)}\right)^2\Psi\left(\theta,Z\right)\\
&\quad -2\frac{G'_{\left|\Psi\left(\theta,Z\right)\right|^2}\left(\left|\Psi\left(\theta,Z\right)\right|^2\frac{\kappa}{N}\left(\lambda\tau\nu c-T\left(Z\right)\right)\right)}{\hat{G}^3\left(\left(T\left(Z,Z_1\right)\right)_{Z_1},\left|\Psi\left(Z\right)\right|^2\right)}\left|\Psi\left(\theta,Z\right)\right|^2\Psi\left(\theta,Z\right)\\
&\quad +\left(\left|\Psi\left(Z\right)\right|^2-\int T\left(Z',Z_1\right)\left|\Psi_0\left(Z\right)\right|^2dZ_1\right)\Psi\left(\theta,Z\right)\\
&\simeq \left(\left(\frac{1}{\hat{G}\left(\left(T\left(Z,Z_1\right)\right)_{Z_1},\left|\Psi\left(Z\right)\right|^2\right)}\right)^2+\left(\left|\Psi\left(Z\right)\right|^2-\int T\left(Z',Z_1\right)\left|\Psi_0\left(Z\right)\right|^2dZ_1\right)\right)\Psi\left(\theta,Z\right)
\end{aligned}$$

with solutions:

$$\Psi\left(\theta,Z\right)=0$$

or $\left|\Psi\left(Z\right)\right|^2$ satisfying:

$$\left(\frac{1}{\hat{G}\left(\left(T\left(Z,Z_1\right)\right)_{Z_1},\left|\Psi\left(Z\right)\right|^2\right)}\right)^2+\left|\Psi\left(Z\right)\right|^2\simeq\int T\left(Z,Z'\right)\left|\Psi_0\left(Z'\right)\right|^2k\left(Z,Z'\right)dZ'$$

This equation can be approximated by:

$$\begin{aligned}
\left(\frac{1}{\hat{G}\left(\left(T\left(Z,Z_1\right)\right)_{Z_1},\left|\Psi\left(Z\right)\right|^2\right)}\right)^2+\left|\Psi\left(Z\right)\right|^2 &\simeq T\left(Z\right)\frac{\int\left|\Psi_0\left(Z'\right)\right|^2k\left(Z,Z'\right)dZ'}{V}\\
&\equiv T\left(Z\right)\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z
\end{aligned}\tag{240}$$

As a consequence the system is reduced to two variables $T\left(Z\right)$ and $\left|\Psi\left(Z\right)\right|^2$ together with (237) and (240). The average connectivity being then retrieved by (239).

Solving for $\left|\Psi\left(Z\right)\right|^2$ and $T\left(Z\right)$

To solve (237) and (240) for $T\left(Z\right)$ and $\left|\Psi\left(Z\right)\right|^2$, we use (237):

$$T\left(Z\right)=\frac{\lambda\tau\nu c\left(\frac{1}{\tau_D\alpha_D}+\hat{G}\left(T\left(Z\right),\left|\Psi\left(Z\right)\right|^2\right)\left|\Psi\left(Z\right)\right|^2\right)}{\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega+\hat{G}\left(T\left(Z\right),\left|\Psi\left(Z\right)\right|^2\right)\left|\Psi\left(Z\right)\right|^2}$$

and express $\hat{G}\left(T\left(Z\right),\left|\Psi\left(Z\right)\right|^2\right)\left|\Psi\left(Z\right)\right|^2$ as a function of $T\left(Z\right)$:

$$\hat{G}\left(T\left(Z\right),\left|\Psi\left(Z\right)\right|^2\right)\left|\Psi\left(Z\right)\right|^2=\frac{\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega\right)T\left(Z\right)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}{\lambda\tau\nu c-T\left(Z\right)}\tag{241}$$

Inserting this result in (240) yields the equation for $\left|\Psi\left(Z\right)\right|^2$:

$$\left(\frac{\lambda\tau\nu c-T\left(Z\right)}{\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega\right)T\left(Z\right)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}\left|\Psi\left(Z\right)\right|^2\right)^2+\left|\Psi\left(Z\right)\right|^2\simeq T\left(Z\right)\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z\tag{242}$$

with solution:

$$\left|\Psi\left(Z\right)\right|^{2}=\frac{2T\left(Z\right)\left\langle\left|\Psi_{0}\left(Z'\right)\right|^{2}\right\rangle_{Z}}{\left(1+\sqrt{1+4\left(\frac{\lambda\tau\nu c-T(Z)}{\left(\frac{1}{\tau_{D}\alpha_{D}}+\frac{1}{\tau_{C}\alpha_{C}}+\Omega\right)T(Z)-\frac{1}{\tau_{D}\alpha_{D}}\lambda\tau\nu c}\right)^{2}T\left(Z\right)\left\langle\left|\Psi_{0}\left(Z'\right)\right|^{2}\right\rangle_{Z}}\right)} \tag{243}$$

Ultimately, inserting this result in (241) yields the following equation for $T\left(Z\right)$:

$$\begin{aligned}\frac{\hat{\Omega}T\left(Z\right)-\frac{1}{\tau_{D}\alpha_{D}}\lambda\tau\nu c}{\lambda\tau\nu c-T\left(Z\right)} &= \hat{G}\left(T\left(Z\right),\frac{2T\left(Z\right)\left\langle\left|\Psi_{0}\left(Z'\right)\right|^{2}\right\rangle_{Z}}{\left(1+\sqrt{1+4\left(\frac{\lambda\tau\nu c-T(Z)}{\hat{\Omega}T(Z)-\frac{1}{\tau_{D}\alpha_{D}}\lambda\tau\nu c}\right)^{2}T\left(Z\right)\left\langle\left|\Psi_{0}\left(Z'\right)\right|^{2}\right\rangle_{Z}}\right)}\right) \\ &\times\frac{2T\left(Z\right)\left\langle\left|\Psi_{0}\left(Z'\right)\right|^{2}\right\rangle_{Z}}{\left(1+\sqrt{1+4\left(\frac{\lambda\tau\nu c-T(Z)}{\hat{\Omega}T(Z)-\frac{1}{\tau_{D}\alpha_{D}}\lambda\tau\nu c}\right)^{2}T\left(Z\right)\left\langle\left|\Psi_{0}\left(Z'\right)\right|^{2}\right\rangle_{Z}}\right)}\end{aligned} \tag{244}$$

with:

$$\hat{\Omega}=\left(\frac{1}{\tau_{D}\alpha_{D}}+\frac{1}{\tau_{C}\alpha_{C}}+\Omega\right)$$

This equation has in general several solutions (see below) corresponding to several regime of activity, depending on the point. Once $T\left(Z\right)$ is found, one can obtain $\left|\Psi\left(Z\right)\right|^{2}$ using (243). To obtain more precise formula for these solutions, we will detail a particular case below. The system is ultimately determined by finding Ω.

Identification of Ω

The resolution is finalized by using (238) to identify the constant Ω. Writing:

$$\begin{aligned}\Omega &= \frac{1}{V}\int\hat{G}\left(T\left(Z\right),\left|\Psi\left(Z\right)\right|^{2}\right)\left|\Psi\left(Z\right)\right|^{2} \\ &= \frac{1}{V}\int\frac{\left(\frac{1}{\tau_{D}\alpha_{D}}+\frac{1}{\tau_{C}\alpha_{C}}+\Omega\right)T\left(Z\right)-\frac{1}{\tau_{D}\alpha_{D}}\lambda\tau\nu c}{\lambda\tau\nu c-T\left(Z\right)} \\ &\simeq \frac{\left(\frac{1}{\tau_{D}\alpha_{D}}+\frac{1}{\tau_{C}\alpha_{C}}+\Omega\right)\bar{T}-\frac{1}{\tau_{D}\alpha_{D}}\lambda\tau\nu c}{\lambda\tau\nu c-\bar{T}}\end{aligned}$$

where

$$\bar{T}=\frac{1}{V}\int T\left(Z\right)dZ$$

is the average activity of the system, we can find Ω as a function of $\bar{T}$:

$$\Omega=\frac{\left(\frac{1}{\tau_{D}\alpha_{D}}+\frac{1}{\tau_{C}\alpha_{C}}\right)\bar{T}-\frac{1}{\tau_{D}\alpha_{D}}\lambda\tau\nu c}{\lambda\tau\nu c-2\bar{T}} \tag{245}$$

or $\bar{T}$ as a function of Ω:

$$\bar{T}=\frac{\lambda\tau\nu c\left(\Omega+\frac{1}{\tau_{D}\alpha_{D}}\right)}{\left(\frac{1}{\tau_{D}\alpha_{D}}+\frac{1}{\tau_{C}\alpha_{C}}\right)+2\Omega} \tag{246}$$

Inserting this result inside (244), integrating over Z, replacing $T(Z)$ by its average $\bar{T}$ inside the expression and using (245), yields:

$$\Omega=\hat{G}\left(\bar{T},\frac{2\bar{T}\left\langle\left|\Psi_0\right|^2\right\rangle}{\left(1+\sqrt{1+\frac{4\bar{T}}{\Omega^2}\left\langle\left|\Psi_0\right|^2\right\rangle}\right)}\right)\frac{2\bar{T}\left\langle\left|\Psi_0\right|^2\right\rangle}{\left(1+\sqrt{1+\frac{4\bar{T}}{\Omega^2}\left\langle\left|\Psi_0\right|^2\right\rangle}\right)} \tag{247}$$

with:

$$\left\langle\left|\Psi_0\right|^2\right\rangle\equiv\left\langle\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z\right\rangle$$

The system (247) and (246) yields the possible values for Ω and $\bar{T}$. Once these constants derived, they can be replaced in (244) to find $T(Z)$ and finally $|\Psi(Z)|^2$ byusing (133).

Particular case

For G an increasing function of the form $G(x)\simeq b_0x$ for $x<1$, we can solve the system. We start with the derivation of Ω.

Derivation of Ω

In this particular case, we rewrite equation (236) as:

$$\begin{aligned}\omega(Z) &= G\left(\frac{\kappa}{N}V\left((\lambda\tau\nu c-T(Z))\left(\left(\frac{1}{\alpha_D\tau_D}-\frac{T(Z)}{V\alpha_C\tau_C}\right)\omega^{-1}+|\Psi(Z)|^2\right)\right)\right)\\ &\simeq b_0\left(\frac{\kappa}{N}V\left((\lambda\tau\nu c-T(Z))\left(\left(\frac{1}{\alpha_D\tau_D}-\frac{T(Z)}{V\alpha_C\tau_C}\right)\omega^{-1}+|\Psi(Z)|^2\right)\right)\right)\end{aligned}$$

and for $\frac{1}{\alpha_D\tau_D}<<1$ and $\frac{1}{V\alpha_C\tau_C}<<1$, this yields in first approximation:

$$\omega(Z)\simeq b_0\left(\frac{\kappa}{N}\left((\lambda\tau\nu c-T(Z))|\Psi(Z)|^2\right)\right) \tag{248}$$

and equation (247) writes:

$$\Omega=b\left(\left(\lambda\tau\nu c-\bar{T}\right)\frac{2\bar{T}\left\langle\left|\Psi_0\right|^2\right\rangle}{\left(1+\sqrt{1+\frac{4\bar{T}}{\Omega^2}\left\langle\left|\Psi_0\right|^2\right\rangle}\right)}\right)\frac{2\bar{T}\left\langle\left|\Psi_0\right|^2\right\rangle}{\left(1+\sqrt{1+\frac{4\bar{T}}{\Omega^2}\left\langle\left|\Psi_0\right|^2\right\rangle}\right)}$$

We use (246) under approximation $\frac{1}{\tau_D\alpha_D}\simeq\frac{1}{\tau_C\alpha_C}<<1$, so that:

$$\bar{T}\simeq\frac{\lambda\tau\nu c}{2} \tag{249}$$

and (250) reduces to the formula quoted in the paper:

$$\Omega=b\bar{T}\left(\frac{2\bar{T}\left\langle\left|\Psi_0\right|^2\right\rangle}{\left(1+\sqrt{1+\frac{4\bar{T}}{\Omega^2}\left\langle\left|\Psi_0\right|^2\right\rangle}\right)}\right)^2 \tag{250}$$

with $b=b_0\frac{\kappa}{N}V$. Equation (250) can be transformed as:

$$\left(1+\sqrt{1+\frac{4\bar{T}}{\Omega^2}\left\langle|\Psi_0|^2\right\rangle}\right)^2=4\frac{b\bar{T}}{\Omega}\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^2$$

and developping the square yields:

$$1+\frac{4\bar{T}}{\Omega^2}\left\langle|\Psi_0|^2\right\rangle=\left(2\left(\frac{b\bar{T}}{\Omega}\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^2-\frac{\bar{T}\left\langle|\Psi_0|^2\right\rangle}{\Omega^2}\right)-1\right)^2$$

for a final equation:

$$\frac{4\bar{T}}{\Omega^2}\left\langle|\Psi_0|^2\right\rangle=4\left(\frac{b\bar{T}}{\Omega}\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^2-\frac{\bar{T}\left\langle|\Psi_0|^2\right\rangle}{\Omega^2}\right)^2-4\left(\frac{b\bar{T}}{\Omega}\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^2-\frac{\bar{T}\left\langle|\Psi_0|^2\right\rangle}{\Omega^2}\right)$$

This leads to the equation for Ω:

$$1=\frac{b\bar{T}\Omega^3}{\left(b\bar{T}^2\left\langle|\Psi_0|^2\right\rangle\Omega-1\right)^2} \tag{251}$$

Defining $b\bar{T}^2\left\langle|\Psi_0|^2\right\rangle\Omega=X$, this equation becomes:

$$\left(b\bar{T}\right)^2\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^3=\frac{X^3}{\left(X-1\right)^2} \tag{252}$$

For $d=\left(b\bar{T}\right)^2\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^3<\frac{27}{4}$ there is one solution:

$$X\simeq\sqrt[3]{\left(b\bar{T}\right)^2}\bar{T}\left\langle|\Psi_0|^2\right\rangle$$

with:

$$\Omega\simeq\left(b\bar{T}\right)^{-\frac{1}{3}}<<1$$

For $d=\left(b\bar{T}\right)^2\left\langle|\Psi_0|^2\right\rangle^3>\frac{27}{4}$ there are three solutions. The first one is:

$$X\simeq\left(b\bar{T}\right)^2\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^3$$

with:

$$\Omega\simeq b\bar{T}\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^2$$

The two other solutions are centered around 1. We set $X=1\pm\delta$ and (**??**) becomes:

$$\left(b\bar{T}\right)^2\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^3\simeq\frac{1}{\delta^2} \tag{253}$$

so that:

$$X=1\pm\sqrt{\frac{1}{\left(b\bar{T}\right)^2\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^3}}$$

with:

$$\Omega=\frac{1\pm\sqrt{\frac{1}{\left(b\bar{T}\right)^2\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^3}}}{b\bar{T}^2\left\langle|\Psi_0|^2\right\rangle}$$

Computation of $T(Z)$

In this paragraph we compute also Y defined by:

$$Y=\frac{\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega\right)T(Z)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}{\lambda\tau\nu c-T(Z)} \tag{254}$$

that will be used to derive the connectivity function $T(Z,Z')$. Equation (244) rewrites in this particular case:

$$Y=b\left(2\bar{T}-T(Z)\right)\left(\frac{2T(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z}{\left(1+\sqrt{1+\frac{4T(Z)}{Y^2}\left\langle|\Psi_0(Z')|^2\right\rangle_Z}\right)}\right)^2 \tag{255}$$

Similar computations as for Ω reduce equation (255) to:

$$1=\frac{b\left(2\bar{T}-T(Z)\right)Y^3}{\left(b\left(2\bar{T}-T(Z)\right)T(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z Y-1\right)^2} \tag{256}$$

Equations (254) and (256) form a system allowing to find $T(Z)$. To do so, we solve (256) for $T(Z)$:

$$T(Z)=\frac{\lambda\tau\nu c\left(\frac{1}{\tau_D\alpha_D}+Y\right)}{\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega+Y}\simeq\frac{2\bar{T}Y}{\Omega+Y} \tag{257}$$

Along with (256), equation (257) leads to the equation defining Y:

$$1=\frac{b\frac{2\bar{T}\Omega}{\Omega+Y}Y^3}{\left(b\frac{2\bar{T}\Omega}{\Omega+Y}\frac{2\bar{T}Y}{\Omega+Y}\left\langle|\Psi_0(Z')|^2\right\rangle_Z Y-1\right)^2} \tag{258}$$

Lowest order approximation If we assume that the fluctations of $\left\langle|\Psi_0(Z')|^2\right\rangle_Z$ around $|\Psi_0|^2$ are relatively low, we can assume in first approximation that:

$$\Omega\simeq Y$$

and thus;

$$T(Z)\simeq\frac{\lambda\tau\nu c\left(\frac{1}{\tau_D\alpha_D}+\Omega\right)}{\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+2\Omega}\simeq\frac{\lambda\tau\nu c}{2}$$

Then the resolution of (258) is similar to the previous section.

For $d=\left(b\bar{T}\right)^2\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^3<\frac{27}{4}$ there is one solution:

$$Y\simeq\left(b\bar{T}\right)^{-\frac{1}{3}}<<1 \tag{259}$$

For $d=\left(b\bar{T}\right)^2\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^3>\frac{27}{4}$ there are three solutions. The first one is:

$$Y\simeq\left(b\bar{T}\right)^2\left(\bar{T}\left\langle|\Psi_0(Z')|^2\right\rangle_Z\right)^3 \tag{260}$$

The two other solutions are centered around $\frac{1}{b\bar{T}^2\left\langle|\Psi_0|^2\right\rangle}$. We obtain:

$$Y=\frac{1\pm\sqrt{\frac{1}{\left(b\bar{T}\right)^2\left(\bar{T}\left\langle|\Psi_0(Z')|^2\right\rangle_Z\right)^3}}}{b\bar{T}^2\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z} \tag{261}$$

We write Y_+ the solution (260) and $Y_{-(\pm)}$ the solutions (261). In the sequel, we will consider only these three solutions, written $Y_{+,-(\pm)}$, and neglect (259) which corresponds to $\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z^3<<1$, a point with low activity.

First order corrections Note also that the average connectivity at Z, i.e. $T\left(Z\right)$ can be computed including the first order correction with respect to $\frac{\lambda\tau\nu c}{2}$, by using:

$$T\left(Z\right)=\frac{\lambda\tau\nu cY+\lambda\tau\nu c\frac{1}{\tau_D\alpha_D}}{Y+\hat{\Omega}} \tag{262}$$

with:

$$\hat{\Omega}=\frac{1}{\tau_C\alpha_C}+\frac{1}{\tau_D\alpha_D}+\Omega$$

We compute these corrections, in the approximation $\frac{1}{\tau_C\alpha_C}<<1$ and $\frac{1}{\tau_D\alpha_D}<<1$. We write equation (258) as:

$$1=\frac{b\frac{2\bar{T}(\delta+Y)}{\delta+x+2Y}\left(Y+x\right)^3}{\left(b\frac{2\bar{T}(Y+\delta)}{\delta+x+2Y}\frac{2\bar{T}(Y+x)}{\delta+x+2Y}\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z\left(Y+x\right)-1\right)^2} \tag{263}$$

wth x is the deviation of Y from its zeroth rdr value and where δ is the difference between Ω and Y at the first order:

$$\delta=\Omega-Y\simeq3\left(b\bar{T}\right)^2\left(\bar{T}\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z\right)^2\bar{T}\left(\left\langle|\Psi_0|^2\right\rangle^2-\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z^2\right)$$

if (260) is used, or:

$$\delta=\Omega-Y\simeq\mp\frac{1}{T^3b\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z^2}\left(\frac{5}{2}\sqrt{\frac{1}{T^5b^2\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z^3}}\pm1\right)\bar{T}\left(\left\langle|\Psi_0|^2\right\rangle^2-\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z^2\right)$$

if (261) is considered. Expanding equation (263) to the first order yields the correction ΔY to the lowest order solution, whatever its form:

$$\Delta Y=-\frac{\delta}{\left(5-4\frac{bT^2Y\left\langle|\Psi_0(Z')|^2\right\rangle_Z}{\left(bT^2Y\left\langle|\Psi_0(Z')|^2\right\rangle_Z-1\right)}\right)}$$

which corrects (260):

$$Y_+\simeq\left(b\bar{T}\right)^2\left(\bar{T}\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z\right)^3-\frac{3\left(b\bar{T}\right)^2\left(\bar{T}\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z\right)^2}{5-4\frac{bT^2Y\left\langle|\Psi_0(Z')|^2\right\rangle_Z}{\left(bT^2Y\left\langle|\Psi_0(Z')|^2\right\rangle_Z-1\right)}}\bar{T}\left(\left\langle|\Psi_0|^2\right\rangle^2-\left\langle\left|\Psi_0\left(Z'\right)\right|^2\right\rangle_Z^2\right) \tag{264}$$

with:

$$\begin{aligned} T_{+}(Z) &\simeq \frac{2\bar{T}Y}{\Omega+Y}=\frac{2\bar{T}Y}{2Y+\delta}\simeq\bar{T}-\frac{\bar{T}}{2Y}\delta \\ &\simeq \bar{T}-\frac{3\bar{T}\left(\left\langle|\Psi_0|^2\right\rangle^2-\left\langle|\Psi_0(Z')|^2\right\rangle_Z^2\right)}{2\left\langle|\Psi_0(Z')|^2\right\rangle_Z} \end{aligned}$$

and (261):

$$\begin{aligned} Y_{-\pm} &\simeq Y=\frac{1\pm\sqrt{\frac{1}{(b\bar{T})^2\left(\bar{T}\left\langle|\Psi_0(Z')|^2\right\rangle_Z\right)^3}}}{b\bar{T}^2\left\langle|\Psi_0(Z')|^2\right\rangle_Z} \\ &\pm\frac{\frac{1}{T^3b\left\langle|\Psi_0(Z')|^2\right\rangle_Z^2}\left(\frac{5}{2}\sqrt{\frac{1}{T^5b^2\left\langle|\Psi_0(Z')|^2\right\rangle_Z^3}}\pm1\right)}{2\left(5-4\frac{bT^2Y\left\langle|\Psi_0(Z')|^2\right\rangle_Z}{\left(bT^2Y\left\langle|\Psi_0(Z')|^2\right\rangle_Z-1\right)}\right)}\bar{T}\left(\left\langle|\Psi_0|^2\right\rangle^2-\left\langle|\Psi_0(Z')|^2\right\rangle_Z^2\right) \end{aligned}$$

with:

$$\begin{aligned} T_{-\pm}(Z) &\simeq \bar{T}-\frac{\bar{T}}{2Y}\delta \\ &\simeq \bar{T}\pm\frac{b\bar{T}\left(\frac{5}{2}\sqrt{\frac{1}{T^5b^2\left\langle|\Psi_0(Z')|^2\right\rangle_Z^3}}\pm1\right)}{\left(b\bar{T}\left(\bar{T}\left\langle|\Psi_0(Z')|^2\right\rangle_Z\right)\pm\sqrt{\frac{1}{\bar{T}\left\langle|\Psi_0(Z')|^2\right\rangle_Z}}\right)}\bar{T}\left(\left\langle|\Psi_0|^2\right\rangle^2-\left\langle|\Psi_0(Z')|^2\right\rangle_Z^2\right) \end{aligned}$$

we define also the average of $T_{-+}(Z)$ and $T_{--}(Z)$:

$$T_{-}(Z)\simeq\bar{T}+\frac{5}{2}\sqrt{\frac{1}{\bar{T}^2b^2\left\langle\bar{T}|\Psi_0(Z')|^2\right\rangle_Z^5}}\bar{T}\left(\left\langle|\Psi_0|^2\right\rangle^2-\left\langle|\Psi_0(Z')|^2\right\rangle_Z^2\right)$$

Connectivity functions

We ultimately write the connectivity functions:

$$\begin{aligned} T(Z,Z') &= \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{1+\frac{\alpha_D}{\alpha_C}\frac{\frac{1}{\tau_C}+\alpha_C\omega'|\Psi(Z')|^2}{\frac{1}{\tau_D}+\alpha_D\omega|\Psi(Z)|^2}} \\ &= \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)}{1+\frac{\frac{1}{\alpha_C\tau_C}+\hat{G}\left(T(Z'),|\Psi(Z')|^2\right)|\Psi(Z')|^2}{\frac{1}{\tau_D\alpha_D}+\hat{G}\left(T(Z),|\Psi(Z)|^2\right)|\Psi(Z)|^2}} \end{aligned}$$

Since:

$$Y=\hat{G}\left(T(Z),|\Psi(Z)|^2\right)|\Psi(Z)|^2$$

we write $T(Z,Z')$:

$$T(Z,Z') = \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left(\frac{1}{\tau_D\alpha_D}+Y(Z)\right)}{\frac{1}{\tau_D\alpha_D}+Y(Z)+\frac{1}{\alpha_C\tau_C}+Y(Z')}$$

with $Y(Z)$ given by (260) and (261). There are nine possibilities for the connectivity function:

$$T\left(Z_{+,-(\pm)},Z'_{+,-(\pm)}\right) = \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left(\frac{1}{\tau_D\alpha_D}+Y_{+,-(\pm)}(Z)\right)}{\frac{1}{\tau_D\alpha_D}+Y_{+,-(\pm)}(Z)+\frac{1}{\alpha_C\tau_C}+Y_{+,-(\pm)}(Z')}$$

Given that the solution (261) are both centered around $\frac{1}{b\left\langle|\Psi_0(Z')|^2\right\rangle_Z}$, and thus relatively close from each other, we can gather them in one approximative solution $\frac{1}{b\left\langle|\Psi_0(Z')|^2\right\rangle_Z}$ and replace $Y_{+,-(\pm)}$ by $Y_\pm(Z)$ given by:

$$T\left(Z_\pm,Z'_\pm\right) = \frac{\lambda\tau\exp\left(-\frac{|Z-Z'|}{\nu c}\right)\left(\frac{1}{\tau_D\alpha_D}+Y_\pm(Z)\right)}{\frac{1}{\tau_D\alpha_D}+Y_\pm(Z)+\frac{1}{\alpha_C\tau_C}+Y_\pm(Z')}$$

and this yields four possibilities as detailed in the text.

Note that (262) can also be written as:

$$T(Z_\pm) = \frac{\lambda\tau\nu c Y_\pm(Z)+\lambda\tau\nu c\frac{1}{\tau_D\alpha_D}}{Y_\pm(Z)+\frac{1}{\tau_D\alpha_D}+\frac{1}{\alpha_C\tau_C}+\Omega_\pm}$$

where:

$$\Omega_\pm = (\Omega_+,\Omega_-)$$

with:

$$\Omega_+ = b\bar{T}\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^2, \Omega_- = \frac{1}{b\bar{T}^2\left\langle|\Psi_0|^2\right\rangle}$$

Activities

We can rewrite the activity (248) as:

$$\begin{aligned}\omega_+(Z) &\simeq b_0\left(\frac{\kappa}{N}(\lambda\tau\nu c - T_+(Z))|\Psi(Z)|^2\right)\\ &= b_0\frac{\kappa}{N}\bar{T}\left(1-\frac{3\left(\left\langle|\Psi_0(Z')|^2\right\rangle_Z^2-\left\langle|\Psi_0|^2\right\rangle^2\right)}{2\left\langle|\Psi_0(Z')|^2\right\rangle_Z^2}\right)|\Psi(Z)|^2\end{aligned}$$

and:

$$\begin{aligned}\omega_-(Z)b_0 &\simeq \left(\frac{\kappa}{N}(\lambda\tau\nu c - T_+(Z))|\Psi(Z)|^2\right)\\ &\simeq b_0\frac{\kappa}{N}\bar{T}\left(\left(1+\frac{5}{2}\sqrt{\frac{1}{\bar{T}^2b^2\left\langle\bar{T}|\Psi_0(Z')|^2\right\rangle_Z^5}}\left(\left\langle|\Psi_0(Z')|^2\right\rangle_Z^2-\left\langle|\Psi_0|^2\right\rangle^2\right)\right)|\Psi(Z)|^2\right)\end{aligned}$$

where $\omega_-(Z)$ gathers the formula for $\omega_-(Z)$

The square $|\Psi(Z)|^2$ depends on $\Omega_\pm$. Thus we write:

$$|\Psi_\pm(Z)|^2 = \frac{2T_\pm(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z^2}{\left(1+\sqrt{1+4\left(\frac{\lambda\tau\nu c-T_\pm(Z)}{\left(\frac{1}{\tau_D\alpha_D}+\frac{1}{\tau_C\alpha_C}+\Omega_\pm\right)T_\pm(Z)-\frac{1}{\tau_D\alpha_D}\lambda\tau\nu c}\right)^2 T_\pm(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z}\right)}$$

Given that:

$$\sqrt{1+4\left(\frac{\lambda\tau\nu c-T_\pm(Z)}{(\Omega_\pm)T_\pm(Z)}\right)^2 T_\pm(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z} \simeq \sqrt{1+\left(\frac{1}{\Omega_\pm}\right)^2 T_\pm(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z}$$

we find:

$$|\Psi_+(Z)|^2 = \frac{2T_+(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z^2}{\left(1+\sqrt{1+\frac{T_+(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z}{\left(b\bar{T}\left(\bar{T}\left\langle|\Psi_0|^2\right\rangle\right)^2\right)^2}}\right)} \simeq 2\bar{T}\left\langle|\Psi_0(Z')|^2\right\rangle_Z^2$$

and:

$$|\Psi_-(Z)|^2 = \frac{2T_-(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z^2}{\left(1+\sqrt{1+4\left(b\bar{T}^2\left\langle|\Psi_0|^2\right\rangle\right)^2 T_-(Z)\left\langle|\Psi_0(Z')|^2\right\rangle_Z}\right)} << 1$$

for $\bar{T} >> 1$.

Thus, given our assumptions:

$$\omega_-(Z) << \omega_+(Z)$$

Appendix 4 Solution of classical action's first order condition

To solve equations (176) and (175), the dependency of $\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)$ in $|\Psi|^2$ has to be explicited. Note that in first approximation, the solution of (175) is:

$$\begin{aligned}\delta\Psi(\theta,Z) &\simeq -\frac{\nabla_\theta\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)}{U''(X_0)}\Psi_0(\theta,Z)\\ &= \frac{\nabla_\theta\omega\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi_0|^2\right)}{U''(X_0)\,\omega^2\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi_0|^2\right)}\Psi_0(\theta,Z)\end{aligned}$$

and this approximation is sufficient as a first approximation.

However, to find a more precise expression for $\delta\Psi(\theta,Z)$, we use (**??**) that defines $\omega^{-1}\left(J(\theta),\theta,Z,\mathcal{G}_0+|\Psi|^2\right)$ at the classical order:

$$\begin{aligned}&\omega^{-1}\left(J,\theta,Z,|\Psi|^2\right) \qquad (265)\\ &= G\left(J(\theta,Z)+\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\right.\\ &\quad\times\left.\left(\mathcal{G}_0(Z_1)+\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1\right)\end{aligned}$$

Using (265), the defining equation (176) for $\delta\Psi(\theta, Z)$ becomes:

$$\begin{aligned}
&G^{-1}\left(-\frac{U''(X_0)}{X_0}\int^{\theta}\delta\Psi(\theta, Z)\right)\\
=&\int\frac{\kappa}{N}\frac{\omega\left(J,\theta-\frac{|Z-Z_1|}{c},Z_1,\Psi\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)\left(\mathcal{G}_0(Z_1)+\left|\Psi\left(\theta-\frac{|Z-Z_1|}{c},Z_1\right)\right|^2\right)dZ_1
\end{aligned}$$

This equation can be rewritten in the local approximation:

$$G^{-1}\left(-\frac{U''(X_0)}{X_0}\int^{\theta}\delta\Psi(\theta, Z)\right)\simeq J(\theta,Z)+\frac{\left(-\Gamma\nabla_\theta+\Gamma'\nabla_Z^2\right)\left(\omega(J,\theta,Z)\left(\mathcal{G}_0(Z)+|\Psi(\theta,Z)|^2\right)\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\tag{266}$$

where Γ and Γ' are defined by:

$$\begin{aligned}
\Gamma &= \int\frac{\kappa}{N}\frac{|Z-Z_1|}{c}T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)dZ_1\\
\Gamma' &= \int\frac{\kappa}{N}|Z-Z_1|^2\,T\left(Z,\theta,Z_1,\theta-\frac{|Z-Z_1|}{c}\right)dZ_1
\end{aligned}$$

At the lowest order in derivatives, equation (266) becomes:

$$\begin{aligned}
G^{-1}\left(-\frac{U''(X_0)}{X_0}\int^{\theta}\delta\Psi(\theta, Z)\right) &\simeq J(\theta,Z)-\frac{\Gamma\nabla_\theta\omega(J,\theta,Z)\left(\mathcal{G}_0(Z_1)+|\Psi(\theta,Z)|^2\right)}{\omega\left(J,\theta,Z,|\Psi|^2\right)}\\
&= J(\theta,Z)-\Gamma\nabla_\theta|\Psi(\theta,Z)|^2+\Gamma\frac{\delta\Psi(\theta,Z)}{\int^{\theta}\delta\Psi(\theta,Z)}\left(\mathcal{G}_0(Z)+|\Psi(\theta,Z)|^2\right)\\
&\simeq J(\theta,Z)-\Gamma\sqrt{X_0}\nabla_\theta\delta\Psi(\theta,Z)+\Gamma\frac{\mathcal{G}_0(Z)+X_0+\sqrt{X_0}\delta\Psi(\theta,Z)}{\int^{\theta}\delta\Psi(\theta,Z)}\delta\Psi(\theta,Z)\\
&\simeq J(\theta,Z)+\Gamma\frac{\mathcal{G}_0(Z_1)+X_0}{\int^{\theta}\delta\Psi(\theta,Z)}\delta\Psi(\theta,Z)
\end{aligned}\tag{267}$$

We set:

$$Y=\ln\left(\int\delta\Psi(\theta,Z)\right)$$

and (267) writes:

$$\begin{aligned}
G^{-1}\left(-\frac{U''(X_0)}{X_0}\exp Y\right) &= J(\theta,Z)+\Gamma\left(\mathcal{G}_0(Z_1)+X_0\right)\nabla_\theta Y\\
&\simeq \langle J\rangle(Z)+\Gamma\left(\mathcal{G}_0(Z_1)+X_0\right)\nabla_\theta Y
\end{aligned}\tag{268}$$

where $\langle J\rangle(Z)$ is the current averaged over time. The solution of (268) is:

$$\int\delta\Psi(\theta,Z)=\exp(Y)=\exp\left(H^{-1}\left(\frac{\theta}{\Gamma\left(\mathcal{G}_0(Z_1)+X_0\right)}+d\right)\right)$$

with:

$$H(Y)=\int\frac{dY}{G^{-1}\left(-\frac{U''(X_0)}{X_0}\exp Y\right)-\langle J\rangle(Z)}$$

and:

$$\delta\Psi\left(\theta,Z\right) = \left(G^{-1}\left(-\frac{U''\left(X_0\right)}{X_0}\exp\left(H^{-1}\left(\frac{\theta}{\Gamma\left(\mathcal{G}_0\left(Z_1\right)+\sqrt{X_0}\right)}+d\right)\right)\right)-\left\langle J\right\rangle\left(Z\right)\right) \times\exp\left(H^{-1}\left(\frac{\theta}{\Gamma\left(\mathcal{G}_0\left(Z_1\right)+\sqrt{X_0}\right)}+d\right)\right) \quad (269)$$

The constant d is chosen so that $\lim_{\theta\to\infty}\delta\Psi\left(\theta,Z\right)=0$. For slowly varyng currents, $\left\langle J\right\rangle\left(Z\right)$ can replaced by $J\left(\theta,Z\right)$ in the formula.