A Statistical Field Perspective on Capital Allocation and Accumulation

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Abstract

This paper provides a general method to translate a standard economic model with a large number of agents into a field-formalism model. This formalism preserves the system's interactions and microeconomic features at the individual level but reveals the emergence of collective states.

We apply this method to a simple microeconomic framework of investors and firms. Both macro and micro aspects of the formalism are studied.

At the macro-scale, the field formalism shows that, in each sector, three patterns of capital accumulation may emerge. A distribution of patterns across sectors constitute a collective state. Any change in external parameters or expectations in one sector will affect neighbouring sectors, inducing transitions between collective states and generating permanent fluctuations in patterns and flows of capital. Although changes in expectations can cause abrupt changes in collective states, transitions may be slow to occur. Due to its relative inertia, the real economy is bound to be more affected by these constant variations than the financial markets.

At the micro-scale we compute the transition functions of individual agents and study their probabilistic dynamics in a given collective state, as a function of their initial state. We show that capital accumulation of an individual agent depends on various factors. The probability associated with each firm's trajectories is the result of several contradictory effects: the firm tends to shift towards sectors with the greatest long-term return, but must take into account the impact of its shift on its attractiveness for investors throughout its trajectory. Since this trajectory depends largely on the average capital of transition sectors, a firm's attractiveness during its relocation depends on the relative level of capital in those sectors. Moreover, the firm must also consider the effects of competition in the intermediate sectors that tends to oust under-capitalized firm towards sectors with lower average capital. For investors, capital allocation depends on their short and long-term returns and investors will tend to reallocate their capital to maximize both. The higher their level of capital, the stronger the re-allocation will be.

Key words: Financial Markets, Real Economy, Capital Allocation, Statistical Field Theory, Background fields, Collective states, Multi-Agent Model, Interactions.

JEL Classification: B40, C02, C60, E00, E1, G10

1 Introduction

In large sets of agents, the dynamics of one agent never occurs in a vacuum, but is impacted by the whole set of other agents' trajectories. In such groups, the representative agent is a fiction: the set

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of trajectories gives rise to collective states that will in turn condition individual dynamics. These collective states can be studied analytically at a higher level by using a statistical-field formalism (Gosselin, Lotz, Wambst 2020, 2021).

We have shown in previous papers how such a field-formalism allows to derive the possible collective states of a given economic model while keeping track of distributions and interactions among agents. It acts as a viability test for a large range of classical model by revealing the inherent logic of each system, which a standard economic model with representative agents cannot do. Whereas a standard economic model determines the behavior of optimizing agents, a field model will reveal how such a behavior, once generalized to a large number of heterogeneous agents, would imply for the entire society.

The present paper develops and applies this method to a system composed of a large number of firms and investors spread across numerous sectors and studies the interactions between financial and physical capital as well as the determinants of capital allocation among firms. Because it keeps track of interactions at the macro level, the field formalism reveals what would otherwise remain hidden: capital accumulation is a global mechanism, in which each sector's average capital and number of firms depend on neighbouring sectors. These global characteristics are encoded in the potential collective states of the system, also referred to as background states. Each of these states computes a possible equilibrium distribution of capital and firms across sectors.

This emphasis on collective states does not hinder the extraction of information about agents' individual dynamics. On the contrary, as the field formalism captures agents' interactions at the micro level, it facilitates the study of their probabilistic behavior within a given collective state through the so-called transition functions. The field formalism serves a dual purpose by providing an approach to collective backgrounds arising from agents' interactions and capturing the diverse individual dynamics within such background.

At the macro-level, this work shows that collective states depend on external parameters and expectations, such as short-term and both absolute and relative expected long-term returns. However, depending on external or historical conditions, the interdependency between sectors induces multiple collective states: in each sector, three patterns of accumulation emerge, from low to high. Some are unstable: changes in exogeneous parameters or expectations may induce complete portfolio reallocations, potentially depleting some sectors. At a macro-timescale, any deviation from an equilibrium average capital drives the sector towards the next stable equilibrium, including zero, and if there is none, towards infinity. This notion of instability is sector-relative and context-dependent: variations of parameters may propagate from one sector to another. Sectors may change pattern which induces transitions between collective states. The field formalism thus allows to describe global transitions of the patterns accumulation initiated by one local modification.

To study this systemic instability, we consider a dynamic system involving average capital and endogenized long-term expected returns, so that average capital per sector interacts both with neighbouring ones, and long-term expected returns. This dynamic system differs from those in standard economics: whereas in economics the dynamics are usually studied around static equilibria, we consider the dynamic interactions between potential equilibria and expected long-term returns.

Some solutions are oscillatory: changes in one or several sectors may propagate over the whole space of sectors. We find, for each sector, the conditions of stable or unstable oscillations for the system. Depending on the sector's specific characteristics, oscillations in average capital and expected long-term returns may dampen or increase. Some types of expectations favour overall stability in equilibria, and others deter it.

Eventually, fluctuations in financial expectations impose their pace on the real economy. The combination of expectations both highly sensitive to exogenous conditions and highly reactive to variations in capital implies large fluctuations of capital in the system at the possible expense of the real economy.

At the micro-level, we derive the transition functions for individual agents within a given collective state. These functions describe the probabilistic dynamics of agents in the background field as a function of their initial state. We demonstrate that several factors influence the probability of each firm's relocation path.

First, firms tend to relocate in sectors with highest long-term returns. However, the path followed by the firm to realocate depends on the characteristics of the transition sectors, that are themselves determined by the collective state of the system. The attractiveness of the firm during its relocation process depends on the average capital of the transition sectors it stumbles into. Depending on the sector, investors may over or underinvest in the firm. An under-capitalized firm may fail to attract investors in and either end up being stuck in this sector or be repelled to a less attractive one.

Second, competition along the transition sectors, depends on the background state of the system and impact differently the firm's level or capital and attractiveness. An overcapitalized firm facing many less-endowed competitors, will oust them out of the sector. On the contrary, an under-capitalized firm will be ousted out from its own sector and move towards less-capitalized and denser sectors in average. A capital gain - or loss - may follow. Under-capitalized firms tend to move towards lower than average capitalized sectors, while over-capitalized firms tend to move towards higher than average-capitalized sectors.

Third, investors' capital allocation depends on short and long-term returns. Yet these returns are not independent: short-term returns, dividends and stock prices variations are correlated to the long-term that depend on growth expectations and stock prices expectations. Changes in investors' capital allocation are therefore directly dependent on stock prices' volatility and firms' dividends. Changes in growth expectations impact stock prices and incite investors to reallocate capital to maximize their returns. The higher their level of capital, the stronger the reallocation will be.

The paper is organized as follows. The second section is a literature review. In the first part, we sum up the field theoretic approach to economic models. Section three presents the general method and translation techniques to turn a microeconomic framework with a large number of agents into a field model. In section four we expose the use of the field theoretic formalism to compute both average quantities in the model and the agents' transitions functions. These functions compute the probabilities of the model to evolve from an initial to a final state. Section five details technically the computation of these transition funcions in presence of a given background field. Section six presents and translates a particular microeconomic framework with two types of agents, firms and investors into a field model.

The second part of this paper studies the collective aspect of this model. In sections seven, we describe the resolution of the model and derive the background field for the real economy, the number of firms par sector, the background field for the financial markets and the number of investors per sector. Ultimately, we derive the defining equation for average capital per firm per sector and discuss the main properties of the solution. In section eight, the model is extended to a dynamic system at the macro-time scale by endogenizing the expected long-term revenue. This dynamic system presents some oscillatory solutions whose stability depends on the various patterns of accumulation. Section nine presents, analyses and synthesizes the results.

The third part of this work is devoted to study the individual agents dynamics and interaction within a given background state. In section 10, we derive the transition functions for one and two agents in this particular model. The results are presented and discussed in section 11.

The fourth part of the paper consolidates and presents the main findings. Section twelve discusses our results, their interpretation and their consequences. Section thirteen concludes.

2 Literature review

Several branches of the economic literature seek to replace the representative agent with a collection of heterogeneous ones. Among other things, they differ in the way they model this collection of agents.

The first branch of the literature represents this collection of agents by probability densities. This is the approach followed by mean field theory, heterogeneous agents new Keynesian (HANK) models, and the information-theoretic approach to economics.

Mean field theory studies the evolution of agents' density in the state space of economic variables. It includes the interactions between agents and the population as a whole but does not consider the direct interactions between agents. This approach is thus at an intermediate scale between the macro and micro scale: it does not aggregate agents but replaces them with an overall probability distribution. Mean field theory has been applied to game theory (Bensoussan et al. 2018, Lasry et al. 2010a, b) and economics (Gomes et al. 2015). However, these mean fields are actually probability distributions. In our formalism, the notion of fields refers to some abstract complex functions defined on the state space and is similar to the second-quantized-wave functions of quantum theory. Interactions between agents are included at the individual level. Densities of agents are recovered from these fields and depend directly on interactions.

Heterogeneous agents' new Keynesian (HANK) models use a probabilistic treatment similar to mean fields theory. An equilibrium probability distribution is derived from a set of optimizing heterogeneous agents in a new Keynesian context (see Kaplan and Violante 2018 for an account). Our approach, on the contrary, focuses on the direct interactions between agents at the microeconomic level. We do not look for an equilibrium probability distribution for each agent, but rather directly build a probability density for the system of N agents seen as a whole, that includes interactions, and then translate this probability density in terms of fields. The states' space we consider is thus much larger than those considered in the above approaches. Because it is the space of all paths for a large number of agents, it allows studying the agents' economic structural relations and the emergence of the particular phases or collective states induced by these specific micro-relations, that will in turn impact each agent's stochastic dynamics at the microeconomic level. Other differences are worth mentioning. While HANK models stress the role of an infinite number of heterogeneously-behaved consumers, our formalism dwells on the relations between physical and financial capital[1]. Besides, our formalism does not rely on agents' rationality assumptions, since for a large number of agents, behaviours, be they fully or partly rational, can be modeled as random.

The information theoretic approach to economics (see Yang 2018) considers probabilistic states around the equilibrium. It is close to our methodological stance: it replaces the Walrasian equilibrium with a statistical equilibrium derived from an entropy maximisation program. Our statistical weight is similar to the one they use, but is directly built from microeconomic dynamic equations. The same difference stands for the rational inattention theory (Sims 2006) in which non-gaussian density laws are derived from limited information and constraints: our setting directly includes constraints in the random description of an agent (Gosselin, Lotz, Wambst 2020).

The differences highlighted above between these various approaches and our work also manifest at the micro-scale in the description of agents' dynamics. Actually, in the field framework, once the collective states have been found, we can recover both the types of individual dynamics depending on the initial conditions and the "effective" form of interactions between two or more agents: At the individual level, agents are distributed along some probability law. However, this probability law is directly conditioned by the collective state of the system and the effective interactions. Different

[1] Note that our formalism could also include heterogeneous consumers (see Gosselin, Lotz, Wambst 2020).

collective states, given different parameters, yield different individual dynamics. This approach allows for coming back and forth between collective and individual aspects of the system. Different categories of agents appear in the emerging collective state. Dynamics may present very different patterns, given the collective state's form and the agents' initial conditions.

A second branch of the literature is closest to our approach since it considers the interacting system of agents in itself. It is the multi-agent systems literature, notably agent-based models (see Gaffard Napoletano 2012, Mandel et al. 2010 2012) and economic networks (Jackson 2010).

Agent-based models deal with the macroeconomic level, whereas network models lower-scale phenomena such as contract theory, behaviour diffusion, information sharing, or learning. In both settings, agents are typically defined by and follow various sets of rules, leading to the emergence of equilibria and dynamics otherwise inaccessible to the representative agent setup. Both approaches are however highly numerical and model-dependent and rely on microeconomic relations - such as ad-hoc reaction functions - that may be too simplistic. Statistical fields theory on the contrary accounts for transitions between scales. Macroeconomic patterns do not emerge from the sole dynamics of a large set of agents: they are grounded in behaviours and interaction structures. Describing these structures in terms of field theory allows for the emergence of phases at the macro scale, and the study of their impact at the individual level.

A third branch of the literature, Econophysics, is also related to ours since it often considers the set of agents as a statistical system (for a review, see Abergel et al. 2011a,b and references therein; or Lux 2008, 2016). But it tends to focus on empirical laws, rather than apply the full potential of field theory to economic systems. In the same vein, Kleinert (2009) uses path integrals to model stock prices' dynamics. Our approach, in contrast, keeps track of usual microeconomic concepts, such as utility functions, expectations, and forward-looking behaviours, and includes these behaviours into the analytical treatment of multi-agent systems by translating the main characteristics of optimizing agents in terms of statistical systems. Closer to our approach, Bardoscia et al (2017) study a general equilibrium model for a large economy in the context of statistical mechanics, and show that phase transitions may occur in the system. Our problematic is similar, but our use of field theory deals with a large class of dynamic models.

The literature on interactions between finance and real economy or capital accumulation takes place mainly in the context of DGSE models. (for a review of the literature, see Cochrane 2006; for further developments see Grassetti et al. 2022, Grosshans and Zeisberger 2018, Böhm et al. 2008, Caggese and Orive, Bernanke e al. 1999, Campello et al. 2010, Holmstrom and Tirole 1997, Jermann, and Quadrini 2012, Khan Thomas 2013, Monacelli et al. 2011). Theoretical models include several types of agents at the aggregated level. They describe the interactions between a few representative agents such as producers for possibly several sectors, consumers, financial intermediaries, etc. to determine interest rates, levels of production, and asset pricing, in a context of ad-hoc anticipations.

Our formalism differs from this literature in three ways. First, we consider several groups of a large number of agents to describe the emergence of collective states and study the continuous space of sectors. Second, we consider expected returns and the longer-term horizon as somewhat exogenous or structural. Expected returns are a combination of elements, such as technology, returns, productivity, sectoral capital stock, expectations, and beliefs. These returns are also a function defined over the sectors' space: the system's background fields are functionals of these expected returns. Taken together, the background fields of a field model describe an economic configuration for a given environment of expected returns. As such, expected returns are at first seen as exogenous functions. It is only in the second step, when we consider the dynamics between capital accumulation and expectations, that expectations may themselves be seen as endogenous.

Even then, the form of relations between actual and expected variables specified are general enough to derive some types of possible dynamics.

Last but not least, we do not seek individual or even aggregated dynamics, but rather background fields that describe potential long-term equilibria and may evolve with the structural parameters. For such a background, agents' individual typical dynamics may nevertheless be retrieved through Green functions (see GLW). These functions compute the transition probabilities from one capital-sector point to another. But backgrounds themselves may be considered as dynamical quantities. Structural or long-term variations in the returns' landscape may modify the background and in turn the individual dynamics. Expected returns themselves depend on and interact with, capital accumulation.

Field formalism for economic system with large number of agents and application to a firms-investors model

In the first part of this work, we describe the field formalism for an economic system, its application to derive the potential collective states of the system and the individual dynamics within such collective states. Ultimately we apply this formalism to translate a model with large number of interacting investors and firms.

3 General method of translation

The formalism we propose transforms an economic model of dynamic agents into a statistical field model. In classical models, each agent's dynamics is described by an optimal path for some vector variable, say $A_i(t)$, from an initial to a final point, up to some fluctuations.

But this system of agents could also be seen as probabilistic: each agent could be described by a *probability density* centered around the classical optimal path, up to some idiosyncratic uncertainties[2] [3]. In this probabilistic approach, each possible trajectory of the whole set of N agents has a specific probability. The classical model is therefore described by the set of trajectories of the group of N agents, each one being endowed with its own probability, its statistical weight. The statistical weight is therefore a function that associates a probability with each trajectory of the group.

This probabilistic approach can be translated into a more compact *field formalism*[4] that preserves the essential information encoded in the model but implements a change in perspective. A field model is a structure governed by its own intrinsic rules that encapsulate the economic model chosen. This field model contains all possible realizations that could arise from the initial economic model, i.e. all the possible global outcomes, or collective state, permitted by the economic model. So that, once constructed, the field model provides a unique advantage over the standard economic model: it allows to compute the probabilities of each of the possible outcomes for each collective state of the economic model. These probabilities are computed indirectly through the *action*

[2] Because the number of possible paths is infinite, the probability of each individual path is null. We, therefore, use the word "probability density" rather than "probability".

[3] See Gosselin, Lotz and Wambst (2017, 2020, 2021).

[4] Ibid.

functional of the model, a function that assigns a specific value to each realization of the field. Technically, the random N agents' trajectories $\{\mathbf{A}_i(t)\}$ are replaced by a field, a random variable whose realizations are complex-valued functions Ψ of the variables $\mathbf{A}$, and the statistical weight of the N agents' trajectories $\{\mathbf{A}_i(t)\}$ in the probabilistic approach is translated into a statistical weight for each realization Ψ. They encapsulate the collective states of the system.

Once the probabilities of each collective state computed, the most probable collective state among all other collective states, can be found. In other words, a field model allows to consider the true global outcome induced by any standard economic model. This is what we will call the *expression* of the field model, more usually called the *background field* of the model.

This most probable realization of the field, the expression or background field of the model, should not be seen as a final outcome resulting from a trajectory, but rather as its most recurring realization. Actually, the probability of the realizations of the model is peaked around the expression of the field. This expression, which is characteristic of the system, will determine the nature of individual trajectories within the structure, in the same way as a biased dice would increase the probability of one event. The field in itself is therefore static, insofar as each realization of the system of agents only contributes to the emergence of the proper expression of the field. However, studying variations in the parameters of the system indirectly induce a time parameter at the field or macro level.

3.1 Statistical weight and minimization functions for a classical system

In an economic framework with a large number of agents, each agent is characterized by one or more stochastic dynamic equations. Some of these equations result from the optimization of one or several objective functions. Deriving the statistical weight from these equations is straightforward: it associates, to each trajectory of the group of agents $\{T_i\}$, a probability that is peaked around the set of optimal trajectories of the system, of the form:

$$W\left(s\left(\{T_i\}\right)\right)=\exp\left(-s\left(\{T_i\}\right)\right) \tag{1}$$

where $s\left(\{T_i\}\right)$ measures the distance between the trajectories $\{T_i\}$ and the optimal ones.

This paper considers two types of agents characterized by vector-variables $\{\mathbf{A}_i(t)\}_{i=1,\ldots N}$, and $\left\{\hat{\mathbf{A}}_l(t)\right\}_{i=1,\ldots\hat{N}}$ respectively, where N and $\hat{N}$ are the number of agents of each type, with vectors $\mathbf{A}_i(t)$ and $\hat{\mathbf{A}}_l(t)$ of arbitrary dimension. For such a system, the statistical weight writes:

$$W\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)=\exp\left(-s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)\right) \tag{2}$$

The optimal paths for the system are assumed to be described by the sets of equations:

$$\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l\ldots}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)=\epsilon_i,\ i=1\ldots N \tag{3}$$

$$\frac{d\hat{\mathbf{A}}_l(t)}{dt}-\sum_{i,j,k\ldots}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)=\hat{\epsilon}_l,\ i=1\ldots\hat{N} \tag{4}$$

where the ϵ_i and $\hat{\epsilon}_i$ are idiosynchratic random shocks. These equations describe the general dynamics of the two types agents, including their interactions with other agents. They may encompass the dynamics of optimizing agents where interactions act as externalities so that this set of equations is the full description of a system of interacting agents[5][6].

For equations (3) and (4), the quadratic deviation at time t of any trajectory with respect to the optimal one for each type of agent are:

$$\left(\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l...}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \tag{5}$$

and:

$$\left(\frac{d\hat{\mathbf{A}}_l(t)}{dt}-\sum_{i,j,k...}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \tag{6}$$

Since the function (2) involves the deviations for all agents over all trajectories, the function $s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)$ is obtained by summing (5) and (6) over all agents, and integrate over t. We thus find:

$$\begin{aligned} s\left(\{\mathbf{A}_i(t)\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right) &= \int dt\sum_i\left(\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l...}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \\ &\quad+\int dt\sum_l\left(\frac{d\hat{\mathbf{A}}_l(t)}{dt}-\sum_{i,j,k...}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \end{aligned} \tag{7}$$

There is an alternate, more general, form to (7). We can assume that the dynamical system is originally defined by some equations of type (3) and (4), plus some objective functions for agents i and l, and that these agents aim at minimizing respectively:

$$\sum_{j,k,l...}g\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right) \tag{8}$$

and:

$$\sum_{i,j,k..}\hat{g}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right) \tag{9}$$

In the above equations, the objective functions depend on other agents' actions seen as externalities[7]. The functions (8) and (9) could themselves be considered as a measure of the deviation of a trajectory from the optimum. Actually, the higher the distance, the higher (8) and (9).

Thus, rather than describing the systm by a full system of dynamic equations, we can consider some ad-hoc equations of type (3) and (4) and some objective functions (8) and (9) to write the

[5] Expectations of agents could be included by replacing $\frac{d\mathbf{A}_i(t)}{dt}$ with $E\frac{d\mathbf{A}_i(t)}{dt}$, where E is the expectation operator. This would amount to double some variables by distinguishing "real variables" and expectations. However, for our purpose, in the context of a large number of agents, at least in this work, we discard as much as possible this possibility.

[6] A generalisation of equations (3) and (4), in which agents interact at different times, and its translation in term of field is presented in appendix 1.

[7] We may also assume intertemporal objectives, see (GLW).

alternate form of $s\left(\left\{\mathbf{A}_i(t)\right\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)$ as:

$$
\begin{aligned}
& s\left(\left\{\mathbf{A}_i(t)\right\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right) \qquad (10)\\
= & \int dt \sum_i \left(\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l\ldots} f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)\right)^2 \\
& +\int dt \sum_l \left(\frac{d\hat{\mathbf{A}}_l(t)}{dt}-\sum_{i,j,k\ldots} \hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)\right)^2 \\
& +\int dt \sum_{i,j,k,l\ldots}\left(g\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)+\hat{g}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)\right)
\end{aligned}
$$

In the sequel, we will refer to the various terms arising in equation (10) as the "minimization functions", **i.e**. the functions whose minimization yield the dynamics equations of the system[8].

3.2 Translation techniques

Once the statistical weight $W(s(\{T_i\}))$ defined in (1) iscomputed, it can be translated in terms of field. To do so, and for each type α of agent, the sets of trajectories $\{\mathbf{A}_{\alpha i}(t)\}$ are replaced by a field $\Psi_\alpha(\mathbf{A}_\alpha)$, a random variable whose realizations are complex-valued functions Ψ of the variables $\mathbf{A}_\alpha$[9]. The statistical weight for the whole set of fields $\{\Psi_\alpha\}$ has the form $\exp(-S(\{\Psi_\alpha\}))$. The function $S(\{\Psi_\alpha\})$ is called the *fields action functional*. It represents the interactions among different types of agents. Ultimately, the expression $\exp(-S(\{\Psi_\alpha\}))$ is the statistical weight for the field[10] that computes the probability of any realization $\{\Psi_\alpha\}$ of the field.

The form of $S(\{\Psi_\alpha\})$ is obtained directly from the classical description of our model. For two types of agents, we start with expression (10). The various minimizations functions involved in the definition of $s\left(\left\{\mathbf{A}_i(t)\right\},\left\{\hat{\mathbf{A}}_l(t)\right\}\right)$ will be translated in terms of field and the sum of these translations will produce finally the action functional $S(\{\Psi_\alpha\})$. The translation method can itself be divided into two relatively simple processes, but varies slightly depending on the type of terms that appear in the various minimization functions.

3.2.1 Terms without temporal derivative

In equation (10), the terms that involve indexed variables but no temporal derivative terms are the easiest to translate. They are of the form:

$$\sum_i \sum_{j,k,l,m\ldots} g\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)$$

These terms describe the whole set of interactions both among and between two groups of agents. Here, agents are characterized by their variables $\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t)\ldots$ and $\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots$ respectively, for instance in our model firms and investors.

[8] A generalisation of equation (10), in which agents interact at different times, and its translation in term of field is presented in appendix 1.

[9] In the following, we will use indifferently the term "field" and the notation Ψ for the random variable or any of its realization Ψ.

[10] In general, one must consider the integral of $\exp(-S(\{\Psi_\alpha\}))$ over the configurations $\{\Psi_\alpha\}$. This integral is the partition function of the system.

In the field translation, agents of type $\mathbf{A}_i(t)$ and $\hat{\mathbf{A}}_l(t)$ are described by a field $\Psi(\mathbf{A})$ and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$, respectively.

In a first step, the variables indexed i such as $\mathbf{A}_i(t)$ are replaced by variables $\mathbf{A}$ in the expression of g. The variables indexed j, k, l, m..., such as $\mathbf{A}_j(t)$, $\mathbf{A}_k(t)$, $\hat{\mathbf{A}}_l(t)$, $\hat{\mathbf{A}}_m(t)$... are replaced by $\mathbf{A}'$, $\mathbf{A}''$, $\hat{\mathbf{A}}$, $\hat{\mathbf{A}}'$, and so on for all the indices in the function. This yields the expression:

$$\sum_i \sum_{j,k,l,m...} g\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right)$$

In a second step, each sum is replaced by a weighted integration symbol:

$$\begin{aligned}
\sum_i &\rightarrow \int |\Psi(\mathbf{A})|^2 d\mathbf{A}, \ \sum_j \rightarrow \int |\Psi(\mathbf{A}')|^2 d\mathbf{A}', \ \sum_k \rightarrow \int |\Psi(\mathbf{A}'')|^2 d\mathbf{A}'' \\
\sum_l &\rightarrow \int \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 d\hat{\mathbf{A}}, \ \sum_m \rightarrow \int \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}}'
\end{aligned}$$

which leads to the translation:

$$\begin{aligned}
&\sum_i \sum_j \sum_{j,k...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right) \\
\rightarrow &\int g\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right) |\Psi(\mathbf{A})|^2 |\Psi(\mathbf{A}')|^2 |\Psi(\mathbf{A}'')|^2 \times ...d\mathbf{A}d\mathbf{A}'d\mathbf{A}''... \\
&\times \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 \times ...d\hat{\mathbf{A}}d\hat{\mathbf{A}}'...
\end{aligned} \tag{11}$$

where the dots stand for the products of square fields and integration symbols needed.

3.2.2 Terms with temporal derivative

In equation (10), the terms that involve a variable temporal derivative are of the form:

$$\sum_i \left(\frac{d\mathbf{A}_i^{(\alpha)}(t)}{dt} - \sum_{j,k,l,m...} f^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right)\right)^2 \tag{12}$$

This particular form represents the dynamics of the α-th coordinate of a variable $\mathbf{A}_i(t)$ as a function of the other agents.

The method of translation is similar to the above, but the time derivative adds an additional operation.

In a first step, we translate the terms without derivative inside the parenthesis:

$$\sum_{j,k,l,m...} f^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...\right) \tag{13}$$

This type of term has already been translated in the previous paragraph, but since there is no sum over i in equation (13), there should be no integral over $\mathbf{A}$, nor factor $|\Psi(\mathbf{A})|^2$.

The translation of equation (13) is therefore, as before:

$$\int f^{(\alpha)}\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right) |\Psi(\mathbf{A}')|^2 |\Psi(\mathbf{A}'')|^2 d\mathbf{A}'d\mathbf{A}'' \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}}d\hat{\mathbf{A}}' \tag{14}$$

A free variable $\mathbf{A}$ remains, which will be integrated later, when we account for the external sum $\sum_i$. We will call $\Lambda(\mathbf{A})$ the expression obtained:

$$\Lambda(\mathbf{A}) = \int f^{(\alpha)}\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right) |\Psi(\mathbf{A}')|^2 |\Psi(\mathbf{A}'')|^2 d\mathbf{A}'d\mathbf{A}'' \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}}d\hat{\mathbf{A}}' \tag{15}$$

In a second step, we account for the derivative in time by using field gradients. To do so, and as a rule, we replace :

$$\sum_i \left(\frac{d\mathbf{A}_i^{(\alpha)}(t)}{dt} - \sum_j \sum_{j,k...} f^{(\alpha)} \left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)... \right) \right)^2 \tag{16}$$

by:

$$\int \Psi^\dagger(\mathbf{A}) \left(-\nabla_{\mathbf{A}^{(\alpha)}} \left(\frac{\sigma^2_{\mathbf{A}^{(\alpha)}}}{2} \nabla_{\mathbf{A}^{(\alpha)}} - \Lambda(\mathbf{A}) \right) \right) \Psi(\mathbf{A}) \, d\mathbf{A} \tag{17}$$

The variance $\sigma^2_{\mathbf{A}^{(\alpha)}}$ reflects the probabilistic nature of the model which is hidden behind the field formalism. This variance represents the characteristic level of uncertainty of the system's dynamics. It is a parameter of the model. Note also that in (17), the integral over $\mathbf{A}$ reappears at the end, along with the square of the field $|\Psi(\mathbf{A})|^2$. This square is split into two terms, $\Psi^\dagger(\mathbf{A})$ and $\Psi(\mathbf{A})$, with a gradient operator inserted in between.

3.3 Action functional

The field description is ultimately obtained by summing all the terms translated above and introducing a time dependency. This sum is called the action functional. It is the sum of terms of the form (11) and (17), and is denoted $S(\Psi, \Psi^\dagger)$.

For example, in a system with two types of agents described by two fields $\Psi(\mathbf{A})$and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$, the action functional has the form:

$$\begin{aligned} S(\Psi, \Psi^\dagger) &= \int \Psi^\dagger(\mathbf{A}) \left(-\nabla_{\mathbf{A}^{(\alpha)}} \left(\frac{\sigma^2_{\mathbf{A}^{(\alpha)}}}{2} \nabla_{\mathbf{A}^{(\alpha)}} - \Lambda_1(\mathbf{A}) \right) \right) \Psi(\mathbf{A}) \, d\mathbf{A} \\ &+ \int \hat{\Psi}^\dagger\left(\hat{\mathbf{A}}\right) \left(-\nabla_{\hat{\mathbf{A}}^{(\alpha)}} \left(\frac{\sigma^2_{\hat{\mathbf{A}}^{(\alpha)}}}{2} \nabla_{\hat{\mathbf{A}}^{(\alpha)}} - \Lambda_2(\hat{\mathbf{A}}) \right) \right) \hat{\Psi}\left(\hat{\mathbf{A}}\right) d\hat{\mathbf{A}} \\ &+ \sum_m \int g_m \left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'... \right) |\Psi(\mathbf{A})|^2 |\Psi(\mathbf{A}')|^2 |\Psi(\mathbf{A}'')|^2 \times ...d\mathbf{A}d\mathbf{A}'d\mathbf{A}''... \\ &\times \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 \times ...d\hat{\mathbf{A}}d\hat{\mathbf{A}}'... \end{aligned} \tag{18}$$

where the sequence of functions g_m describes the various types of interactions in the system.

Note that the collective states described by the fields are structural states of the system. The fields have their own dynamics at the macro-scale, which will be discussed later in the paper. This is why the usual microeconomic time variable used in standard models has disappeared in formula (18). However, time dependency may at times be required in fields, so that a time variable, written θ could be introduced by replacing:

$$\begin{aligned} \Psi(\mathbf{A}) &\rightarrow \Psi(\mathbf{A}, \theta) \\ \hat{\Psi}\left(\hat{\mathbf{A}}\right) &\rightarrow \hat{\Psi}\left(\hat{\mathbf{A}}, \theta\right) \end{aligned}$$

More about this point can be found in appendix 1.

4 Use of the field model

Once the field action functional S is found, we can use field theory to study the system of agents. This can be done at two levels: the collective and the individual level. At the collective level, the system is described by the background fields of the system that condition average quantities of economic variables of the system.

At the individual level, the field formalism allows to compute agents' individual dynamics in the state defined by the background fields, through the transition functions of the system.

4.1 Collective level: background fields and averages

At the collective level, the background fields of the system can be computed. These background fields are the particular functions, $\Psi(\mathbf{A})$ and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$, and their adjoints fields $\Psi^{\dagger}(\mathbf{A})$ and $\hat{\Psi}^{\dagger}\left(\hat{\mathbf{A}}\right)$, that minimize the action functional S. Once the background field(s) obtained, the associated density of agents defined by a given A and a given $\hat{A}$ are:

$$|\Psi(\mathbf{A})|^{2}=\Psi^{\dagger}(\mathbf{A})\Psi(\mathbf{A}) \tag{19}$$

and:

$$\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^{2}=\hat{\Psi}^{\dagger}\left(\hat{\mathbf{A}}\right)\hat{\Psi}\left(\hat{\mathbf{A}}\right) \tag{20}$$

respectively. With these density functions at hand, we can compute various average quantities in the collective state. Actually, the averages for the system in the state defined by $\Psi(\mathbf{A})$ and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$ of components $(\mathbf{A})_{k}$ or $\left(\hat{\mathbf{A}}\right)_{l}$ are:

$$\langle(\mathbf{A})_{k}\rangle=\frac{\int(\mathbf{A})_{k}|\Psi(\mathbf{A})|^{2}d\mathbf{A}}{\int|\Psi(\mathbf{A})|^{2}d\mathbf{A}}$$

$$\left\langle\left(\hat{\mathbf{A}}\right)_{l}\right\rangle=\frac{\int\left(\hat{\mathbf{A}}\right)\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^{2}d\hat{\mathbf{A}}}{\int\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^{2}d\hat{\mathbf{A}}}$$

respectively. We can also define both partial densities and averages by integrating some components and fixing the values of others, as will be detailled in the particular model considered in the next sections.

4.2 Individual level: agents transition functions and their field expression

4.2.1 Transition functions in a classical framework

In a classical perspective, the statistical weight (57) can be used to compute the transition probabilities of the system, i.e. the probabilities for any number of agents of both types to evolve from an initial state $\{\mathbf{A}_{l}\}_{l=1,\ldots},\left\{\hat{\mathbf{A}}_{l}\right\}_{l=1,\ldots}$ to a final state in a given timespan. These transition functions describe the dynamic of the agents of the system.

To do so, we first compute the integral of equation (57) over all paths between the initial and the final points considered. Defining $\{\mathbf{A}_{l}(s)\}_{l=1,\ldots,N}$ and $\left\{\hat{\mathbf{A}}_{l}(s)\right\}_{l=1,\ldots,\hat{N}}$ the sets of paths for agents of each type, where N and $\hat{N}$ are the numbers of agents of each type, we consider the set of $N+\hat{N}$ independent paths written:

$$\mathbf{Z}(s)=\left(\{\mathbf{A}_{l}(s)\}_{l=1,\ldots,N},\left\{\hat{\mathbf{A}}_{l}(s)\right\}_{l=1,\ldots,\hat{N}}\right)$$

The weight (57) can now be written $\exp(-W(\mathbf{Z}(s)))$.

The transition functions $T_t\left(\underline{(\mathbf{Z})},\overline{(\mathbf{Z})}\right)$ compute the probability for the $(N,\hat{N}\ )$ agents to evolve from the initial points $Z(0)\equiv\underline{\mathbf{Z}}$ to the final points $Z(t)\equiv\overline{(\mathbf{Z})}$ during a time span t. This probability is defined by:

$$T_t\left(\underline{\mathbf{Z}},\overline{(\mathbf{Z})}\right)=\frac{1}{\mathcal{N}}\int_{\substack{\mathbf{Z}(0)\equiv\underline{\mathbf{Z}}\\ \mathbf{Z}(t)\equiv\overline{(\mathbf{Z})}}}\exp\left(-W\left(\mathbf{Z}\left(s\right)\right)\right)\mathcal{D}\left(\mathbf{Z}\left(s\right)\right) \tag{21}$$

The integration symbol $D\mathbf{Z}(s)$ covers all sets of $N\times\hat{N}$ paths constrained by $\mathbf{Z}(0)\equiv\underline{\mathbf{Z}}$ and $\mathbf{Z}(t)\equiv\overline{(\mathbf{Z})}$. The normalisation factor sets the total probability defined by the weight (57) to 1 and is equal to:

$$\mathcal{N}=\int\exp\left(-W\left(\mathbf{Z}\left(s\right)\right)\right)\mathcal{D}\mathbf{Z}\left(s\right)$$

The interpretation of (21) is straightforward. Instead of studying the full trajectory of one or several agents, we compute their probability to evolve from one configuration to another, and in average, the usual trajectory approach remains valid.

Equation (21) can be generalized to define the transition functions for $k\leqslant N$ and $\hat{k}\leqslant\hat{N}$ agents of each type. The initial and final points respectively for this set of $k+\hat{k}$ agents are written:

$$\mathbf{Z}\left(0\right)^{\left[k,\hat{k}\right]}\equiv\underline{\mathbf{Z}}^{\left[k,\hat{k}\right]}$$

and:

$$\mathbf{Z}\left(t\right)^{\left[k,\hat{k}\right]}\equiv\overline{(\mathbf{Z})}^{\left[k,\hat{k}\right]}$$

The transition function for these agents is written:

$$T_t\left(\underline{(\mathbf{Z})}^{\left[k,\hat{k}\right]},\overline{(\mathbf{Z})}^{\left[k,\hat{k}\right]}\right)$$

and the generalization of equation (21) is:

$$T_t\left(\underline{(\mathbf{Z})}^{\left[k,\hat{k}\right]},\overline{(\mathbf{Z})}^{\left[k,\hat{k}\right]}\right)=\frac{1}{\mathcal{N}}\int_{\substack{\mathbf{Z}(0)^{\left[k,\hat{k}\right]}=\underline{(\mathbf{Z})}^{\left[k,\hat{k}\right]}\\ \mathbf{Z}(t)^{\left[k,\hat{k}\right]}=\overline{(\mathbf{Z})}^{\left[k,\hat{k}\right]}}}\exp\left(-W\left(\left(\mathbf{Z}\left(s\right)\right)\right)\right)\mathcal{D}\left(\left(\mathbf{Z}\left(s\right)\right)\right) \tag{22}$$

The difference with (21) is that only k paths are constrained by their initial and final points.

Ultimately, the Laplace transform of $T_t\left(\underline{(Z)}^{\left[k,\hat{k}\right]},\overline{(Z)}^{\left[k,\hat{k}\right]}\right)$ computes the - time averaged - transition function for agents with random lifespan of mean $\frac{1}{\alpha}$, up to a factor $\frac{1}{\alpha}$, and is given by:

$$G_\alpha\left(\underline{(\mathbf{Z})}^{\left[k,\hat{k}\right]},\overline{(\mathbf{Z})}^{\left[k,\hat{k}\right]}\right)=\int_0^\infty\exp\left(-\alpha t\right)T_t\left(\underline{(\mathbf{Z})}^{\left[k,\hat{k}\right]},\overline{(\mathbf{Z})}^{\left[k,\hat{k}\right]}\right)dt \tag{23}$$

This formulation of the transition functions is relatively intractable. Therefore, we will now propose an alternative method based on the field model.

4.2.2 Field-theoretic expression

The transition functions (22) and (23) can be retrieved using the field theory transition functions - or Green functions, which compute the probability for a variable number $\left(k,\hat{k}\right)$ of agents to transition from an initial state $\underline{(\mathbf{Z},\boldsymbol{\theta})}^{\left[k,\hat{k}\right]}$ to a final state $\overline{(\mathbf{Z},\boldsymbol{\theta})}^{\left[k,\hat{k}\right]}$, where $\underline{(\boldsymbol{\theta})}^{\left[k,\hat{k}\right]}$ and $\overline{(\boldsymbol{\theta})}^{\left[k,\hat{k}\right]}$ are vectors of initial and final times for $k+\hat{k}$ agents respectively.

We will write:

$$T_t\left(\underline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]},\overline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]}\right)$$

the transition function between $\underline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]}$ and $\overline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]}$ with $\overline{(\boldsymbol{\theta})}_i < t$, $\forall i$, and:

$$G_\alpha\left(\underline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]},\overline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]}\right)$$

its Laplace transform. Setting $\underline{(\boldsymbol{\theta})}_i = 0$ and $\overline{(\boldsymbol{\theta})}_i = t$ for $i = 1, ..., k + \hat{k}$, these functions reduce to (22) or (23): the probabilistic formalism of the transition functions is thus a particular case of the field formalism definition. In the sequel we therefore will use the term transition function indiscriminately.

The computation of the transition functions relies on the fact that $\exp(-S(\Psi))$ itself represents a statistical weight for the system. Gosselin, Lotz, Wambst (2020) showed that $S(\Psi)$ can be modified in a straightforward manner to include source terms:

$$S(\Psi, J) = S(\Psi) + \int \left(J(Z,\theta)\Psi^\dagger(Z,\theta) + J^\dagger(Z,\theta)\Psi(Z,\theta)\right) d(Z,\theta) \tag{24}$$

where $J(Z,\theta)$ is an arbitrary complex function, or auxiliary field.

Introducing $J(Z,\theta)$ in $S(\Psi, J)$ allows to compute the transition functions by successive derivatives. Actually, we can show that:

$$G_\alpha\left(\underline{(\mathbf{Z},\theta)}^{[k,\hat{k}]},\overline{(\mathbf{Z},\theta)}^{[k,\hat{k}]}\right) = \left[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{(\mathbf{Z},\theta)}_{i_l}\right)}\frac{\delta}{\delta J^\dagger\left(\overline{(\mathbf{Z},\theta)}_{i_l}\right)}\right)\int \exp(-S(\Psi,J))\mathcal{D}\Psi\mathcal{D}\Psi^\dagger\right]_{J=J^\dagger=0} \tag{25}$$

where the notation $\mathcal{D}\Psi\mathcal{D}\Psi^\dagger$ denotes an integration over the space of functions $\Psi(Z,\theta)$ and $\Psi^\dagger(Z,\theta)$, i.e. an integral in an infinite dimensional space. Even though these integrals can only be computed in simple cases, a series expansion of $G_\alpha\left(\underline{(\mathbf{Z},\theta)}^{[k,\hat{k}]},\overline{(\mathbf{Z},\theta)}^{[k,\hat{k}]}\right)$ can be found using Feynman graphs techniques.

Once $G_\alpha\left(\underline{(\mathbf{Z},\theta)}^{[k,\hat{k}]},\overline{(\mathbf{Z},\theta)}^{[k,\hat{k}]}\right)$ is computed, the expression of $T_t\left(\underline{(\mathbf{Z},\theta)}^{[k,\hat{k}]},\overline{(\mathbf{Z},\theta)}^{[k,\hat{k}]}\right)$ can be retrieved in principle by an inverse Laplace transform. In field theory, formula (25) shows that the transition functions (23) are correlation functions of the field theory with action $S(\Psi)$.

5 Field-theoretic computations of transition functions

The formula (25) provides a precise and compact definition of the transition functions for multiple agents in the system. However, in practice, this formula is not directly applicable and does not shed much light on the connection between the collective and microeconomic aspects of the considered system. To calculate the dynamics of the agents, we will proceed in three steps.

Firstly, we will minimize the system's action functional and determine the background field, which represents the collective state of the system. Once the background field is found, we will perform a series expansion of the action functional around this background field, referred to as the effective action of the system. It is with this effective action that we can compute the transition functions for the state defined by the background field. We will discover that each term in this expansion has an interpretation in terms of a transition function.

Instead of directly computing the transition functions, we can consider a series expansion of the action functional around a specific background field of the system.

5.1 Step 1: finding the background field

For a particular type of agent, background fields are defined as the fields $\Psi_0(Z,\theta)$ that maximize the statistical weight $\exp(-S(\Psi))$ or, alternatively, minimize $S(\Psi)$:

$$\frac{\delta S(\Psi)}{\delta \Psi}\mid_{\Psi_0(Z,\theta)}=0,\ \frac{\delta S(\Psi^{\dagger})}{\delta \Psi^{\dagger}}\mid_{\Psi_0^{\dagger}(Z,\theta)}=0$$

The field $\Psi_0(Z,\theta)$ represents the most probable configuration, a specfic state of the entire system that influences the dynamics of agents. It serves as the background state from which probability transitions and average values can be computed. As we will see, the agents' transitions explicitely depend on the chosen background field $\Psi_0(Z,\theta)$, which represents the macroeconomic state in which the agents evolve.

When considering two or more types of agents, the background field satisfies the following condition:

$$\begin{aligned}
\frac{\delta S\left(\Psi,\hat{\Psi}\right)}{\delta \Psi} &\mid_{\Psi_0(Z,\theta)}=0,\ \frac{\delta S\left(\Psi,\hat{\Psi}\right)}{\delta \Psi^{\dagger}}\mid_{\Psi_0^{\dagger}(Z,\theta)}=0\\
\frac{\delta S\left(\Psi,\hat{\Psi}\right)}{\delta \hat{\Psi}} &\mid_{\hat{\Psi}_0(Z,\theta)}=0,\ \frac{\delta S\left(\Psi,\hat{\Psi}\right)}{\delta \hat{\Psi}^{\dagger}}\mid_{\hat{\Psi}_0^{\dagger}(Z,\theta)}=0
\end{aligned}$$

5.2 Step 2: Series expansion around the background field

In a given background state, the *effective action*[11] is the series expansion of the field functional $S(\Psi)$ around $\Psi_0(Z,\theta)$. We will present the expansion for one type of agent, but generalizing it to two or several agents is straightforward.

The series expansion around the background field simplifies the computations **of transition functions** and provides an interpretation of these functions in terms of individual interactions within the collective state. To perform this series expansion, we decompose Ψ as:

$$\Psi=\Psi_0+\Delta\Psi$$

and write the series expansion of the action functional:

$$\begin{aligned}
S(\Psi) &= S(\Psi_0)+\int \Delta\Psi^{\dagger}(Z,\theta)\,O(\Psi_0(Z,\theta))\,\Delta\Psi(Z,\theta) \\
&\quad +\sum_{k>2}\int\prod_{i=1}^{k}\Delta\Psi^{\dagger}(Z_i,\theta)\,O_k(\Psi_0(Z,\theta),(Z_i))\prod_{i=1}^{k}\Delta\Psi(Z_i,\theta)
\end{aligned}\tag{26}$$

The series expansion can be interpreted economically as follows. The first term, $S(\Psi_0)$, describes the system of all agents in a given macroeconomic state, Ψ_0. The other terms potentially describe all the fluctuations or movements of the agents around this macroeconomic state considered as given. Therefore, the expansion around the background field represents the microeconomic reality of a historical macroeconomic state. More precisely, it describes the range of microeconomic possibilities allowed by a macroeconomic state.

The quadratic approximation is the first term of the expansion and can be written as:

$$S(\Psi)=S(\Psi_0)+\int \Delta\Psi^{\dagger}(Z,\theta)\,O(\Psi_0(Z,\theta))\,\Delta\Psi(Z,\theta)\tag{27}$$

[11] Actually, this paper focuses on the *classical effective action*, which is an approximation sufficient for the computations at hand.

It will allow us to find the transition functions of agents in the historical macro state, where all interactions are averaged. The other terms of the expansion allow us to detail the interactions within the nebula, and are written as follows:

$$\sum_{k>2}\int\prod_{i=1}^{k}\Delta\Psi^{\dagger}\left(Z_{i},\theta\right)O_{k}\left(\Psi_{0}\left(Z,\theta\right),\left(Z_{i}\right)\right)\prod_{i=1}^{k}\Delta\Psi\left(Z_{i},\theta\right)$$

They detail, given the historical macroeconomic state, how the interactions of two or more agents can impact the dynamics of these agents. Mathematically, this corresponds to correcting the transition probabilities calculated in the quadratic approximation.

Here, we provide an interpretation of the third and fourth-order terms.

The third order introduces possibilities for an agent, during its trajectory, to split into two, or conversely, for two agents to merge into one. In other words, the third-order terms take into account or reveal, in the historical macroeconomic environment, the possibilities for any agent to undergo modifications along its trajectory. However, this assumption has been excluded from our model.

The fourth order reveals that in the macroeconomic environment, due to the presence of other agents and their tendency to occupy the same space, all points in space will no longer have the same probabilities for an agent. In fact, the fourth-order terms reveal the notion of geographical or sectoral competition and potentially intertemporal competition. These terms describe the interaction between two agents crossing paths, which leads to a deviation of their trajectories due to the interaction.

We do not interpret higher-order terms, but the idea is similar. The even-order terms (2n) describe interactions among n agents that modify their trajectories. The odd-order terms modify the trajectories but also include the possibility of agents reuniting or splitting into multiple agents. We will see in more detail how these terms come into play in the transition functions.

5.3 Step 3: Computation of the transition functions

5.3.1 Quadratic approximation

In the first approximation, transition functions in a given background field $\Psi_0(Z,\theta)$ can be computed by replacing $S(\Psi)$ in (25), with its quadratic approximation (27). In formula (27), $O(\Psi_0(Z,\theta))$ is a differential operator of second order. This operator depends explicitly on $\Psi_0(Z,\theta)$. As a consequence, transition functions and agent dynamics explicitly depend on the collective state of the system. In this approximation, the formula for the transition functions (25) becomes:

$$\begin{aligned}G_{\alpha}\left(\underline{(Z,\theta)}^{[k]},\overline{(Z,\theta)}^{[k]}\right) &= \left[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{(Z,\theta)}_{i_l}\right)}\frac{\delta}{\delta J^{\dagger}\left(\overline{(Z,\theta)}_{i_l}\right)}\right)\right.\\ &\quad\left.\times\int\exp\left(-\int\Delta\Psi^{\dagger}\left(Z,\theta\right)O\left(\Psi_{0}\left(Z,\theta\right)\right)\Delta\Psi\left(Z,\theta\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}\right]_{J=J^{\dagger}=0}\end{aligned}\tag{28}$$

Using this formula, we can show that the one-agent transition function is given by:

$$G_{\alpha}\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right)=O^{-1}\left(\Psi_{0}\left(Z,\theta\right)\right)\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right)\tag{29}$$

where:

$$O^{-1}\left(\Psi_{0}\left(Z,\theta\right)\right)\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right)$$

is the kernel of the inverse operator $O^{-1}\left(\Psi_0\left(Z,\theta\right)\right)$. It can be seen as the $\left(\underline{\left(Z,\theta\right)}^{[1]},\overline{\left(Z,\theta\right)}^{[1]}\right)$ matrix element of $O^{-1}\left(\Psi_0\left(Z,\theta\right)\right)^{12}$.

More generally, the k-agents transition functions are the product of individual transition functions:

$$G_\alpha\left(\underline{\left(Z,\theta\right)}^{[k]},\overline{\left(Z,\theta\right)}^{[k]}\right)=\prod_{i=1}^{k}G_\alpha\left(\underline{\left(Z,\theta\right)_i}^{[1]},\overline{\left(Z,\theta\right)_i}^{[1]}\right)\tag{30}$$

The above formula shows that in the quadratic approximation, the transition probability from one state to another for a group is the product of individual transition probabilities. In this approximation, the trajectories of these agents are therefore independent. The agents do not interact with each other and only interact with the environment described by the background field.

The quadratic approximation must be corrected to account for individual interactions within the group, by including higher-order terms in the expansion of the action.

5.3.2 Higher-order corrections

To compute the effects of interactions between agents of a given group, we consider terms of order greater than 2 in the effective action. These terms modify the transition functions. Writing the expansion:

$$\exp\left(-S\left(\Psi\right)\right)=\exp\left(-\left(S\left(\Psi_0\right)+\int\Delta\Psi^{\dagger}\left(Z,\theta\right)O\left(\Psi_0\left(Z,\theta\right)\right)\right)\right)\left(1+\sum_{n\geqslant1}\frac{A^n}{n!}\right)$$

where:

$$A=\sum_{k>2}\int\prod_{i=1}^{k}\Delta\Psi^{\dagger}\left(Z_i,\theta\right)O\left(\Psi_0\left(Z,\theta\right),\left(Z_i\right)\right)\prod_{i=1}^{k}\Delta\Psi\left(Z_i,\theta\right)$$

is the sum of all possible interaction terms, leads to the series expansion of (25):

$$\begin{aligned}G_\alpha\left(\underline{\left(Z,\theta\right)}^{[k]},\overline{\left(Z,\theta\right)}^{[k]}\right) &= \left[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{\left(Z,\theta\right)_{i_l}}\right)}\frac{\delta}{\delta J^{\dagger}\left(\overline{\left(Z,\theta\right)_{i_l}}\right)}\right)\right.\\ &\left.\int\exp\left(-\int\Delta\Psi^{\dagger}\left(Z,\theta\right)O\left(\Psi_0\left(Z,\theta\right)\right)\Delta\Psi\left(Z,\theta\right)\right)\left(1+\sum_{n\geqslant1}\frac{A^n}{n!}\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}\right]_{J=J^{\dagger}=0}\end{aligned}\tag{31}$$

These corrections can be computed using graphs' expansion.

More precisely, the first term of the series:

$$\left[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{\left(Z,\theta\right)_{i_l}}\right)}\frac{\delta}{\delta J^{\dagger}\left(\overline{\left(Z,\theta\right)_{i_l}}\right)}\right)\int\exp\left(-\int\Delta\Psi^{\dagger}\left(Z,\theta\right)O\left(\Psi_0\left(Z,\theta\right)\right)\Delta\Psi\left(Z,\theta\right)\right)\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}\right]_{J=J^{\dagger}=0}\tag{32}$$

is a transition function in the quadratic approximation. The other contributions of the series expansion correct the approximated n agents transtns functions (30).

[12]The differential operator $O\left(\Psi_0\left(Z,\theta\right)\right)$ can be seen as an infinite dimensional matrix indexed by the double (infinite) entries $\left(\underline{\left(Z,\theta\right)}^{[1]},\overline{\left(Z,\theta\right)}^{[1]}\right)$. With this description, the kernel $O^{-1}\left(\Psi_0\left(Z,\theta\right)\right)\left(\underline{\left(Z,\theta\right)}^{[1]},\overline{\left(Z,\theta\right)}^{[1]}\right)$ is the $\left(\underline{\left(Z,\theta\right)}^{[1]},\overline{\left(Z,\theta\right)}^{[1]}\right)$ element of the inverse matrix.

Typically a contribution:

$$G_{\alpha}\left(\underline{(Z,\theta)}^{[k]},\overline{(Z,\theta)}^{[k]}\right) = \left[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{(Z,\theta)}_{i_l}\right)}\frac{\delta}{\delta J^{\dagger}\left(\overline{(Z,\theta)}_{i_l}\right)}\right)\right.$$
$$\left.\int\exp\left(-\int\Delta\Psi^{\dagger}(Z,\theta)\,O\left(\Psi_0(Z,\theta)\right)\Delta\Psi(Z,\theta)\right)\frac{A^n}{n!}\mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}\right]_{J=J^{\dagger}=0} \tag{33}$$

can be depicted by a graph. The power $\frac{A^n}{n!}$ translates that agents interact n times along their path. The trajectories of each agent of the group is broken n times between its initial and final points. At each time of interaction the trajectories of agents are deviated. To such a graph is associated a probility that modifies the quadratic approximation transition functions.

In the sequel we will only focus on the first order corrections to the two-agents transition functions.

6 Application to a microeconomic framework

We will now present the microeconomic framework that will be turned into a field model using our general method. We first describe the microeconomic model, then derive the associated minimization function and the statistical weight associated to the N agents' set of trajectories. We will then translate the minimization functions into an action functional and obtain the statitical weight for the field model associated to the initial microeconomic framework.

6.1 Microeconomic setup

To picture the interactions between the real and the financial economy, we will consider two groups of agents, producers, and investors. In the following, we will refer to producers or firms i indistinctively, and use the upper script $\hat{}$ for variables describing investors.

6.1.1 Firms

Producers are a set of firms operating each in a single sector, so that a single firm with subsidiaries in different countries and/or offering differentiated products can be modeled as a set of independent firms. Similarly, a sector refers to a set of firms with similar productions, so that sectors can be decomposed into sectors per country to account for local specificities, or in several sectors.

Firms move across a vector space of sectors, which is of arbitrary dimension. Firms are defined by their respective sector X_i and physical capital K_i, two variables subject to dynamic changes. They may change their capital stocks over time or altogether shift sectors.

Each firm produces a single differentiated good. However, in the following, we will merely consider the return each producer may provide to its investors.

The return of producer i at time t, denoted r_i, depends on K_i, X_i and on the level of competition in the sector. It is written:

$$r_i = r\left(K_i, X_i\right) - \gamma\sum_j \delta\left(X_i - X_j\right)\frac{K_j}{K_i} \tag{34}$$

where $\delta\left(X_i - X_j\right)$ is the Dirac δ function which is equal to 0 for $X_i \neq X_j$. The first term in formula (34) is an arbitrary function that depends on the sector and the level of capital per firm in this sector. It represents the return of capital in a specific sector X_i under no competition. We deliberately keep the form of $r\left(K_i, X_i\right)$ unspecified, since most of the results of the model rely on the general properties of the functions involved. When needed, we will give a standard Cobb-Douglas form to the returns $r(K_i, X_i)$. The second term in (34) is the decreasing return of capital. In any given sector, it is proportional to both the number of competitors and the specific level of capital per firm used.

We also assume that, for all i, firm i has a market valuation defined by both its price, P_i, and the variation of this price on financial markets, $\dot{P}_i$. This variation is itself assumed to be a function of an expected long-term return denoted R_i, or more precisely the relative return $\bar{R}_i$ of firm i against the whole set of firms:

$$\frac{\dot{P}_i}{P_i} = F_1\left(\bar{R}_i, \frac{\dot{K}_i}{K_i}\right) \tag{35}$$

Formula (35) includes the main features of models of price dynamics. In this equation, the time dependency of variables is implicit. Formula (35) reflects the impact of capital and location on the price of the firm through its expected returns. It also reflects how variations in capital impact its growth prospects, through competition and dividends (see (34)). Actually, the higher the capital of the firm, the lower impact of competition and the higher the dividends.

We assume R_i to have the general form:

$$R_i = R(K_i, X_i)$$

Expected long-term returns depend on the capital and sector in which the firm operates, but also on external parameters, such as technology, ... which are encoded in the shape of $R(K_i, X_i)$.

The relative return $\bar{R}_i$ arising in (35) is defined by:

$$\bar{R}_i = \bar{R}(K_i, X_i) = \frac{R_i}{\sum_l R_l} \tag{36}$$

The function F_1 in (35) is arbitrary and reflects the preferences of the market relatively to the firm's relative returns. We assume that firms relocate in the sector space according to returns, in the direction of the gradient of the expected long-term returns $R(K_i, X_i)$, so that they chose the location X_i that minimizes at each continuous time t the objective function:

$$L_i\left(X_i, \dot{X}_i\right) = \left(\dot{X}_i - \nabla_X R_i H(K_i)\right)^2 + \tau \frac{K_{X_i}}{K_i} \sum_j \delta(X_i - X_j) \tag{37}$$

where $\dot{X}_i$ stands for the continuous version of the discrete variation, $X_i(t+1) - X_i(t)$, and $\delta(X_i - X_j)$ is again the Dirac-δ-function. K_{X_i} is the average capital per firm in sector X_i. The constant τ measures the level of competition between firms, and describes the cost incured to settling in a new sector[13]. The inclusion of the factor $\frac{K_{X_i}}{K_i}$ models that the lower a firm's capital is compared to the sector average, the stronger the effect of competition

Actually, when $\tau = 0$, there are no repulsive forces and the move towards the gradient of R is given by the expression:

[13] Formula (37) represents the continuous version of the following objective function, where the firm aims to maximize its expected long-term revenue at each period:

$$L = R(K_i, X_i) - R(K_i, X_i - \delta X_i) - \frac{1}{2}\left(\frac{1}{H(K_i)}\delta X_i\right)^2 + \tau \frac{K_{X_i}}{K_i} \sum_j \delta(X_i - X_j) \tag{38}$$

Here, $X_i - \delta X_i$ denotes the position of the agent in the previous period. The term $\left(\frac{1}{H(K_i)}\delta X_i\right)^2$ represents the cost of changing sector. The function $H(K_i)$ is increasing with K_i, so that the higher the firm's capital, the larger the shift. The term $\tau \sum_j \delta(X_i - X_j)$ represents an externality increasing the cost of shifting proportionally to the number of firms in sector X_i. The accumulation of agents at any point in the space creates a repulsive force that slows down the shift. In continuous time, formula (38) becomes equivalent to formula (37), up to a constant term.

$$\dot{X}_i = \nabla_{X_i} R_i H(K_i)$$

When $\tau \neq 0$, repulsive forces deviate the trajectory. The dynamic equation associated to the minimization of (37) is given by the general formula of the dynamic optimization:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{X}_i} L_i\left(X_i, \dot{X}_i\right) = \frac{\partial}{\partial X_i} L_i\left(X_i, \dot{X}_i\right) \tag{39}$$

This last equation does not need to be developed further, since formula (37) is sufficient to switch to the field description of the system.

6.1.2 Investors

Each investor j is defined by his level of capital $\hat{K}_j$ and his position $\hat{X}_j$ in the sector space. Investors can invest in the entire sector space, but tend to invest in sectors close to their position.

Besides, investors tend to diversify their capital: each investor j chose to allocate parts of his entire capital $\hat{K}_j$ between various firms i. The capital allocated by investor j to firm i is denoted $\hat{K}_j^{(i)}$, and given by:

$$\hat{K}_j^{(i)}(t) = \left(\hat{F}_2\left(R_i, \hat{X}_j\right)\hat{K}_j\right)(t) \tag{40}$$

where:

$$\hat{F}_2\left(R_i, \hat{X}_j\right) = \frac{F_2(R_i)\, G\left(X_i - \hat{X}_j\right)}{\sum_l F_2(R_l)\, G\left(X_l - \hat{X}_j\right)} \tag{41}$$

The function F_2 is arbitrary. It depends on the expected return of firm i and on the distance between sectors X_i and $\hat{X}_j$. The function $\hat{F}_2\left(R(K_i, X_i), \hat{X}_j\right)$ is the relative version of F_2 and translates the dependency of investments on firms' relative attractivity. Equation (40) is a general form of risk averse portfolio allocation[14].

We now define ε the time scale for capital accumulation. The variation of capital of investor j between t and $t+\varepsilon$ is the sum of two terms: the short-term returns r_i of the firms in which j invested, and the stock price variations of these same firms:

$$\hat{K}_j(t+\varepsilon) - \hat{K}_j(t) = \sum_i \left(r_i + \frac{\dot{P}_i}{P_i}\right)\hat{K}_j^{(i)} = \sum_i \left(r_i + F_1\left(\bar{R}_i, \frac{\dot{K}_i(t)}{K_i(t)}\right)\right)\hat{K}_j^{(i)} \tag{43}$$

Incidentally, note that in equation (37), the time scale of motions within the sectors space was normalized to one. Here, on the contrary, we define this motion time scale as ε, and assume $\varepsilon << 1$: the mobility in the sector space is lower than capital dynamics. To rewrite (43) on the same time-span as $\frac{dX_i}{dt}$, we write:

[14] Actually, an investor allocating capital exclusively in a sector X_i and optimizing the function:

$$\frac{R_i}{\sum_l R_l} s_j - s_j^2 Var\left(\frac{R_i}{\sum_l R_l}\right) \tag{42}$$

where the share s_j satisfies $\sum s_j = 1$, will set $s_j = \frac{R_i}{\sum_l R_l}$. If we were to introduce the possibility of investing in multiple sectors and consider more general preferences than this simple quadratic function, we should introduce the functions $G\left(X_i - \hat{X}_j\right)$ and $F_2(R_i)$ in the solutions of (42), leading to (40).

$$
\begin{aligned}
\hat{K}_j(t+1) - \hat{K}_j(t) &= \sum_{k=1}^{\frac{1}{\varepsilon}} \hat{K}_j(t+k\varepsilon) - \hat{K}_j(t) \\
&= \sum_{k=1}^{\frac{1}{\varepsilon}} \sum_i \left(r_i + \frac{\dot{P}_i}{P_i} \right) \hat{K}_j^{(i)}(t+k\varepsilon) \\
&\simeq \frac{1}{\varepsilon} \sum_i \left(r_i + F_1\left(\bar{R}_i, \frac{\dot{K}_i(t)}{K_i(t)} \right) \right) \hat{K}_j^{(i)}
\end{aligned}
$$

where the quantities in the sum have to be understood as averages over the time span $[t, t+1]$. Using equation (35), equation (43) becomes in the continuous approximation:

$$
\frac{d}{dt}\hat{K}_j = \frac{1}{\varepsilon} \sum_i \left(r_i + F_1\left(\frac{R_i}{\sum_l \delta(X_l - X_i) R_l}, \frac{\dot{K}_i}{K_i} \right) \right) \hat{F}_2\left(R_i, \hat{X}_j \right) \hat{K}_j \tag{44}
$$

where $\frac{d}{dt}\hat{K}_j(t) = \hat{K}_j(t+1) - \hat{K}_j(t)$ is now normalized to the time scale of $\frac{dX_i}{dt}$, i.e. 1.

6.1.3 Link between financial and physical capital

The entire financial capital is, at any time, completely allocated by investors between firms. For producers, there is no alternative source of financing. Self-financing is discarded, since it would amount to considering a producer and an investor as a single agent. The physical capital of a any given firm is thus the sum of all capital allocated to this specific firm by all its investors. Physical capital entirely depends on the financial arbitrage of the financial sector. Firms do not own their capital: they return it fully at the end of each period with a dividend, though possibly negative. Investors then entirely reallocate their capital between firms at the beginning of the next period. This generalisation of the dividend irrelevance theory may not be fully accurate in the short-run but holds in the long-run, since physical capital cannot last long without investment. When investors choose not to finance a firm, this firm is bound to disappear in the long run. Under these assumptions, the following identity holds:

$$
K_i(t+\varepsilon) = \sum_j \hat{K}_j^{(i)} = \sum_j \hat{F}_2\left(R_i, \hat{X}_j(t) \right) \hat{K}_j(t) \tag{45}
$$

where K_i stands for the physical capital of firm i at time t, and $\sum_j \hat{K}_j^{(i)}$ for the sum of capital invested in firm i by investors j. Recall that the parameter ε accounts for the specific time scale of capital accumulation. It differs from that of mobility within the sector space (37), which is normalized to one.

The dynamic equation (45) rewrites:

$$
\frac{K_i(t+\varepsilon) - K_i(t)}{\varepsilon} = \frac{1}{\varepsilon} \left(\sum_j \hat{F}_2\left(R_i, \hat{X}_j(t) \right) \hat{K}_j(t) - K_i(t) \right) \tag{46}
$$

Using the same token as in the derivation of (44), its continuous approximation becomes:

$$
\frac{d}{dt}K_i(t) + \frac{1}{\varepsilon} \left(K_i(t) - \sum_j \hat{F}_2\left(R_i, \hat{X}_j(t) \right) \hat{K}_j(t) \right) = 0 \tag{47}
$$

where $\frac{d}{dt}K_i(t)$ stands for $K_i(t+1) - K_i(t)$.

6.1.4 Capital allocation dynamics

Investors allocate their capital among sectors, and may adjust their portfolio to the returns of the sector or firms they are invested in. This is modelled by a move along the sectors' space in the direction of the gradient of $R(K_i, X_i)$.

Investor j capital reallocation is described by a dynamic equation for $\hat{X}_j$:

$$\frac{d}{dt}\hat{X}_j - \frac{1}{\sum_i \delta\left(X_i - \hat{X}_j\right)} \sum_i \left(\nabla_{\hat{X}} F_0\left(R\left(K_i, \hat{X}_j\right)\right) + \nu \nabla_{\hat{X}} F_1\left(\bar{R}\left(K_i, \hat{X}_j\right), \frac{\dot{K}_i}{K_i}\right)\right) = 0 \tag{48}$$

The factor $\sum_i \delta\left(X_i - \hat{X}_j\right)$ is the firms' density in sector $\hat{X}_j$. The more competitors in a sector, the slower the reallocation. The term $\nabla_{\hat{X}} F_0\left(R\left(K_i, \hat{X}_j\right)\right)$ models the investors' propensity to reallocate in sectors with highest long-term returns. The term $\nu \nabla_{\hat{X}} F_1\left(\bar{R}\left(K_i, \hat{X}_j\right)\right)$ describes the investors' preference for stocks displaying the highest price's increase[15].

Remark on the model Note that consumers do not appear in this framework. This choice of a partial model is deliberate and aims to focus solely on the allocation of capital among a large number of investors and firms.

In the framework of the representative agent, this choice of a partial, consumer-free model could be problematic. A model with a single firm and one investor would be useless: capital would be invested in the single firm and would grow through dividends. In this context, introducing a consumer that can arbitrage between consumption and long-term returns becomes relevant.

Here we rather study how investment varies and how capital is allocated among multiple firms under varying exogeneous conditions, for which our partial model will provide ample results[16].

6.2 Minimization functions

To find the statistical weight associated to the trajectories of the system, we must write the minimization functions of the system's equations. Recall that they are functions whose minimization yields the dynamic equations of the system.

We have seen that the dynamics of the variable X_i comes from the minimization of the function:

$$\left(\frac{dX_i}{dt} - \nabla_X R(K_i, X_i) H(K_i)\right)^2 + \tau \frac{K_{X_i}}{K_i} \sum_j \delta(X_i - X_j)$$

We simply re-use this function and sum over the whole set of agents and yield the minimization function for the capital allocation dynamics:

$$s_1 = \sum_i \left(\frac{dX_i}{dt} - \nabla_X R(K_i, X_i) H(K_i)\right)^2 + \sum_i \tau \frac{K_{X_i}}{K_i} \sum_j \delta(X_i - X_j) \tag{49}$$

The dynamics of K_i, $\hat{K}_i$ et $\hat{X}_i$ are not the result of a minimization, but their associated quadratic functions (10) can easily be found and yield the following minimization functions:

[15] As for (37), equation (48) can be justified by an objective function that would depend on returns and that includes a cost for any sector shift, to translate the loss of information and connections induced by the shift.

[16] Adding a consumer would modify the model by introducing in (40) a factor lower than one and depending on the sector long-term returns. This factor directly arises from the arbitrage between investment and consumption, but appears irrelevant in the present context. Additionally, the firm's dividend would be influenced by the consumption of the good produced by the firm. Once again, a general model would deviate from our intended purpose.

for the physical capital K_i:

$$s_2 = \sum_i \left(\frac{d}{dt}K_i + \frac{1}{\varepsilon}\left(K_i - \sum_j \hat{F}_2\left(R\left(K_i\left(t\right), X_i\left(t\right)\right), \hat{X}_j\left(t\right)\right)\hat{K}_j\left(t\right)\right)\right)^2 \tag{50}$$

for the financial capital $\hat{K}_i$:

$$s_3 = \sum_j \left(\frac{d}{dt}\hat{K}_j - \frac{1}{\varepsilon}\left(\sum_i \left(r_i + F_1\left(\frac{R\left(K_i, X_i\right)}{\sum_l \delta\left(X_l - X_i\right) R\left(K_l, X_l\right)}, \frac{\dot{K}_i\left(t\right)}{K_i\left(t\right)}\right)\right)\hat{F}_2\left(R\left(K_i\left(t\right), X_i\left(t\right)\right), \hat{X}_j\left(t\right)\right)\hat{K}_j\right)\right)^2 \tag{51}$$

and ultimately, for financial capital allocation $\hat{X}_i$:

$$s_4 = \sum_i \left(\frac{d}{dt}\hat{X}_j - \frac{1}{\sum_i \delta\left(X_i - \hat{X}_j\right)}\sum_i \left(\nabla_{\hat{X}} F_0\left(R\left(K_i, \hat{X}_j\right)\right) + \nu \nabla_{\hat{X}} F_1\left(\bar{R}\left(K_i, \hat{X}_j\right)\right)\right)\right)^2 \tag{52}$$

As a consequence, the statistical weight associated to the trajectories is simply:

$$W\left(\left\{K_i\left(t\right), X_i\left(t\right)\right\}, \left\{\hat{K}_i\left(t\right), \hat{X}_j\left(t\right)\right\}\right) = \exp\left(-\int dt\left(s_1 + s_2 + s_3 + s_4\right)\right) \tag{53}$$

6.3 Translation in terms of fields

To translate the previous microeconomic framework into a field model, we must translate the minimization functions (49), (50) for the firms, and (51) and (52) for investors in terms of four functionals of the fields [17]. The sum of these four functionals is the "field action functional"[18] that determine the statistical weight for the field.

We will apply the general method developed in section 3 and start with firms, by translating first (49) and (50).

6.3.1 Real Economy

In both capital allocation dynamics (49) and capital accumulation dynamics (50), time derivatives appear. However, one of them, equation (49), includes time-independent terms and is thus of the form (10), the other, equation (49) is of the type (**??**). Based on the translation rules, appendix 1.3 computes the translation of the various minimization functions.

Using the translation (15) of (13)-type term, the minimization function of physical capital allocation (49) translates into:

$$\begin{aligned} S_1 &= -\int \Psi^{\dagger}\left(K, X\right)\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X - \nabla_X R\left(K, X\right) H\left(K\right)\right)\Psi\left(K, X\right) dK dX \\ &+\tau \frac{K_X}{K}\int \left|\Psi\left(K', X\right)\right|^2 \left|\Psi\left(K, X\right)\right|^2 dK' dK dX \end{aligned} \tag{54}$$

where K_X is the average capital per firm in sector X. We show below that K_X has the field expression:

$$K_X = \frac{\int K \left|\Psi\left(K, X\right)\right|^2 dK}{\left\|\Psi\left(X\right)\right\|^2}$$

[17] The term functional refers to a function of a function, i.e. a function whose argument is itself a function.
[18] Details about the probabilistic step will be given as a reminder along the text and in appendix 1.

Similarly, the minimization function of the physical capital (50), translates into:

$$S_2 = -\int \Psi^{\dagger}(K,X)\nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K + \frac{1}{\varepsilon}\left(K - \int \hat{F}_2\left(R(K,X),\hat{X}\right)\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\hat{K}d\hat{X}\right)\right)\Psi(K,X) \tag{55}$$

with:

$$\hat{F}_2\left(R(K,X),\hat{X}\right) = \frac{F_2(R(K,X))\,G\left(X-\hat{X}\right)}{\int F_2(R(K',X'))\,G\left(X'-\hat{X}\right)|\Psi(K',X')|^2 d(K',X')} \tag{56}$$

6.3.2 Financial markets

The financial capital dynamics (51) and the financial capital allocation (52) both include a time derivative and are thus of type (12). The application of the translation rules is straightforward.

Using the general translation formula of expression (16) in (17), the minimization function (51) for the financial capital dynamics translates into:

$$\begin{aligned} S_3 &= -\int \hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} - \frac{\hat{K}}{\varepsilon}\int\left(r(K,X) - \gamma\frac{\int K'\left\|\Psi(K',X)\right\|^2}{K}\right.\right. \\ &\quad \left.\left. + F_1\left(\bar{R}(K,X),\Gamma(K,X)\right)\right)\hat{F}_2\left(R(K,X),\hat{X}\right)\left\|\Psi(K,X)\right\|^2 d(K,X)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right) \end{aligned} \tag{57}$$

where:

$$\bar{R}(K,X) = \frac{R(K,X)}{\int R(K',X')\left\|\Psi(K',X')\right\|^2 d(K',X')} \tag{58}$$

$$\Gamma(K,X) = \frac{\int \hat{F}_2\left(R(K,X),\hat{X}\right)\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\hat{K}d\hat{X}}{K} - 1 \tag{59}$$

and the function for financial capital allocation (52) translates into:

$$\begin{aligned} S_4 &= -\int \hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right) \\ &\times\left(\nabla_{\hat{X}}\sigma_{\hat{X}}^2\nabla_{\hat{X}} - \int\left(\frac{\nabla_{\hat{X}}F_0\left(R\left(K,\hat{X}\right)\right) + \nu\nabla_{\hat{X}}F_1\left(\bar{R}(K,X),\Gamma(K,X)\right)}{\int\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\right)\left\|\Psi\left(K,\hat{X}\right)\right\|^2 dK\right)\hat{\Psi}\left(\hat{K},\hat{X}\right) \end{aligned} \tag{60}$$

6.3.3 Fields action functional and statistical weight

We can now find the statistical weight of any realization of the fields. Actually, the action functional of the system is the sum of the contributions (54),(55),(57),(60):

$$S\left(\Psi,\hat{\Psi}\right) = S_1 + S_2 + S_3 + S_4$$

and the field statistical weight for the realization $\left(\Psi,\hat{\Psi}\right)$ is:

$$\exp\left(-S\left(\Psi,\hat{\Psi}\right)\right)$$

With no loss of generality, we can find a more compact form for the action S by assuming that investors invest in only one sector, so that:

$$G\left(X-\hat{X}\right) = \delta\left(X-\hat{X}\right) \tag{61}$$

We can thus write the action functional S:

$$\begin{aligned} S &= -\int \Psi^{\dagger}(K,X)\left(\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X - \nabla_X R(K,X)H(K)\right) - \tau\frac{K_X}{K}\left(\int|\Psi(K',X)|^2 dK'\right)\right. & (62)\\ &\left.+\nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K + u\left(K,X,\Psi,\hat{\Psi}\right)\right)\right)\Psi(K,X)\,dKdX \\ &-\int \hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} - \hat{K}f\left(\hat{X},\Psi,\hat{\Psi}\right)\right) + \nabla_{\hat{X}}\left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}} - g\left(K,X,\Psi,\hat{\Psi}\right)\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right) \end{aligned}$$

where each line corresponds to one S_i and where, to simplify, we have defined:

$$u\left(K,X,\Psi,\hat{\Psi}\right) = \frac{1}{\varepsilon}\left(K - \int \hat{F}_2(R(K,X))\hat{K}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 d\hat{K}\right) \quad (63)$$

$$f\left(\hat{X},\Psi,\hat{\Psi}\right) = \frac{1}{\varepsilon}\int\left(r(K,X) - \frac{\gamma\int K'|\Psi(K,X)|^2}{K} + F_1\left(\bar{R}(K,X),\Gamma(K,X)\right)\right) \quad (64)$$

$$\times\hat{F}_2(R(K,X))\left|\Psi\left(K,\hat{X}\right)\right|^2 dK \quad (65)$$

$$g\left(K,\hat{X},\Psi,\hat{\Psi}\right) = \int \frac{\nabla_{\hat{X}}F_0\left(R\left(K,\hat{X}\right)\right) + \nu\nabla_{\hat{X}}F_1\left(\bar{R}\left(K,\hat{X}\right),\Gamma(K,X)\right)}{\int\left|\Psi\left(K',\hat{X}\right)\right|^2 dK'}\left|\Psi\left(K,\hat{X}\right)\right|^2 dK \quad (66)$$

The expression for $\bar{R}(K,X)$ is still given by (58). Under our assumption, the functions $\hat{F}_2$ and Γ become:

$$\hat{F}_2(R(K,X)) = \frac{F_2(R(K,X))}{\int F_2(R(K',X))|\Psi(K',X)|^2 dK'} \quad (67)$$

$$\Gamma(K,X) = \frac{\int \hat{F}_2(R(K,X))\hat{K}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 d\hat{K}}{K} - 1 \quad (68)$$

Despite its compact and abstract form, equation (62) encompasses the main elements of our microeconomic framework. Recall that function $H(K_X)$ encompasses the determinants of the firms' mobility across the sector space. We will specify this function below as a function of expected long term-returns and capital.

Function u describes the evolution of capital of a firm, located at X. This dynamics depends on the relative value of a function F_2 that is itself a function of the firms' expected returns $R(K,X)$. Investors allocate their capital based on their expectations of the firms' long-term returns.

Function f describes the returns of investors located at $\hat{X}$, and investing in sector X a capital K. These returns depend on short-term dividends $r(K,X)$, the field-equivalent cost of capital $\frac{\gamma\int K'\|\Psi(K,X)\|^2}{K}$, and a function F_1 that depends on firms' expected long-term stock valuations. These valuations themselves depend on the relative attractivity of a firm expected long-term returns vis-a-vis its competitors.

Function g describes investors' shifts across the sectors' space. They are driven by the gradient of expected long-term returns and stocks valuations, who themselves depend on the firms' relative expected long-term returns.

Note that we do not introduce a time variable at this stage. Our purpose is to find the collective states of the system, which can be considered as static in a first step. It is only when we will study the impact of exogeneous parameters on the collective states that we will introduce a macro time scale.

6.4 Field model and averages

As detailed above, once the field action functional S is found, we can use field theory to study the system of agents, both at the collective and individual levels. At the collective level we can compute the averages of the system in a given background field. The individual level, described by the transition functions will be studied in the third part.

Recall that the background fields emerging at the collective level are particular functions, $\Psi(K,X)$ and $\hat{\Psi}\left(\hat{K},\hat{X}\right)$, and their adjoints fields $\Psi^{\dagger}(K,X)$ and $\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)$, that minimize the functional S.

Once the background fields are obtained, the associated number of firms and investors per sector for a given average capital K can be computed. They are given by:

$$|\Psi(K,X)|^2 = \Psi^{\dagger}(K,X)\Psi(K,X) \tag{69}$$

and:

$$\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 = \hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\hat{\Psi}\left(\hat{K},\hat{X}\right) \tag{70}$$

With these two density functions at hand, various average quantities in the collective state can be computed.

The number of producers $\|\Psi(X)\|^2$ and investors $\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2$ in sectors are computed using the formula:

$$\|\Psi(X)\|^2 \equiv \int |\Psi(K,X)|^2 dK \tag{71}$$

$$\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2 \equiv \int \left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\hat{K} \tag{72}$$

The total invested capital $\hat{K}_X$ in sector X is defined by a partial average:

$$\hat{K}_{\hat{X}} = \int \hat{K}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 d\hat{K} = \int \hat{K}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2 d\hat{K} \tag{73}$$

and the average invested capital per firm in sector X defined by:

$$K_X = \frac{\int \hat{K}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 d\hat{K}}{\|\Psi(X)\|^2} \tag{74}$$

Note that, given our assumptions, the total physical capital is equal to the total capital invested:

$$\int K|\Psi(K,X)|^2 dK = \int \hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\hat{K}$$

so that K_X is also equal to the average physical capital per firm for sector X, i.e. :

$$K_X = \frac{\int K|\Psi(K,X)|^2 dK}{\|\Psi(X)\|^2} \tag{75}$$

In the following, we will use both expressions (74) or (75) alternately for K_X.

Ultimately, the distributions of invested capital per investor and of capital per firm, given a collective state and a sector X, are $\frac{\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2}{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}$ and $\frac{|\Psi(K,X)|^2}{\|\Psi(X)\|^2}$, respectively.

Gathering equations (71), (72) and (74), each collective state is singularly determined by the collection of data that characterizes each sector: the number of firms, investors, the average capital, and the distribution of capital. All these quantities allow the study of capital allocation among sectors and its dependency in the parameters of the system, such as expected long-term and short-term returns, and any other parameter. This "static" point of view, will be extended by introducing some fluctuations in the expectations, leading to a dynamic of the average capital at the macro-level. In the following, we solve the system for the background fields and compute the average associated quantities.

System at the macro level: background fields and equilibria

In this part, we consider the study of our economic framework at the macro-scale. Starting with the action functional (62), we derive the background fields $\Psi(K,X)$ and $\hat{\Psi}\left(\hat{K},\hat{X}\right)$, and their adjoints fields $\Psi^{\dagger}(K,X)$ and $\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)$, that minimize the functional S. This allow to derive the potential equilibria of the system. We show that at each point of the sector space, several patterns of accumulation may appear, leading to an infinite number of potential background states. Thes pattern are not independent, and describe the system as a whole that may experience global transitions in patterns of accumulation

7 Resolution

Now that the initial framework has been translated into a proper field formalism, we can solve the field model. Average capital per sector (defined in (74) and (73)) depends on the background fields $\Psi(K,X)$ and $\hat{\Psi}\left(\hat{K},X\right)$ and their conjugate $\Psi^{\dagger}(K,X)$ and $\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)$ that minimize the field action S.

To study the influence of investment and financial allocation on the dynamics of the real economy, we must express the quantities relevant to firms as functions of financial quantities.

The order of resolution will thus be the following: we will first minimize the (K,X) part of the fields action (62), i.e. S_1+S_2, to find the real economy background fields $\Psi(K,X)$ and $\Psi^{\dagger}(K,X)$ and the number of firms $|\Psi(K,X)|^2$ as functions of the financial sectors' background fields $\Psi^{\dagger}(K,X)$ and $\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)$ and investors' variables. We will then minimize S_3+S_4, and find the minimal configuration of the investors' field $\hat{\Psi}\left(\hat{K},\hat{X}\right)$ and $\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)$.

7.1 Background field for the real economy

To compute[19] the field of the real economy $\Psi(K,X)$ as a function of the field of the financial sector $\hat{\Psi}\left(\hat{K},\hat{X}\right)$. We first minimize S_1+S_2, i.e. the real economy part of equation (62):

$$\begin{aligned} S_1+S_2 = & -\int \Psi^{\dagger}(K,X)\left(\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X-\nabla_X R(K,X)H(K)\right)-\tau\frac{K_X}{K}\left(\int\left|\Psi(K',X)\right|^2 dK'\right)\right. \\ & \left.+\nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K+u\left(K,X,\Psi,\hat{\Psi}\right)\right)\right)\Psi(K,X)\,dK\,dX \end{aligned} \tag{76}$$

[19] For detailed computations of this subsection, see appendix 2.

Since, in this second part of the paper, we are interested in studying the collective states, we will consider that in first approximation $\frac{K_X}{K}\simeq 1$. This corresponds to consider that in average the fluctuations of capital in one sector X is relatively low with respect to the average K_X. This assumptions will be removed in the third part, when we consider individual dynamics. As a consequence, we consider the replacement:

$$\tau\frac{K_X}{K}\rightarrow\tau$$

For relatively slow fluctuations in X, and up to an exponential change of variable in the fields, we show in appendix 2.1 that the background fields $\Psi(K,X)$ and $\Psi^{\dagger}(K,X)$ decompose as a product:

$$\Psi(K,X)=\Psi^{\dagger}(K,X)=\Psi(X)\Psi_1(K-K_X)\tag{77}$$

where K_X, the average invested capital per firm in sector X, is given by (75) and the functions $\Psi(X)$ and $\Psi_1(K-K_X)$ satisfy the following differential equations:

$$\begin{aligned}0=&\left(-\frac{\sigma_X^2}{2}\nabla_X^2+\frac{(\nabla_X R(X)H(K_X))^2}{2\sigma_X^2}+\frac{\nabla_X^2 R(K,X)}{2}H(K)+2\tau|\Psi(X)|^2\right)\Psi(X)\\&+D\left(\|\Psi\|^2\right)\left(\int\|\Psi(X)\|^2-N\right)+\int\mu(X)\|\Psi(X)\|^2\end{aligned}\tag{78}$$

for $\Psi(X)$, and:

$$0=-\nabla_K^2\Psi_1(K-K_X)+\left(K-\frac{F_2(R(K,X))K_X}{F_2(R(K_X,X))}\right)^2\Psi_1(K-K_X)+\gamma(X)\Psi_1(K-K_X)\tag{79}$$

for $\Psi_1(K-K_X)$.

The constants $D\left(\|\Psi\|^2\right)$, $\mu(X)$ and $\gamma(X)$ arising in (78) and (79) are Lagrange multipliers[20] that implement the constraints:

$$\int\|\Psi(X)\|^2=N,\ \|\Psi(X)\|^2\geqslant 0,\ \|\Psi_1(K-K_X)\|^2=1$$

where N is the total number of firms of the system.

The intuition that the background field can be decomposed as a product, presented in equation (77), is straightforward: in the space of sectors, firms relocate more slowly than capital accumulates, so that firms are first described the position X of their sector, and second, their capital in this sector, distributed around the average capital of the sector, K_X. This is translated in the decomposition (77) of $\Psi(K,X)$ by the two factors $\Psi(X)$ and $\Psi_1(K-K_X)$.

The function $\Psi_1(K-K_X)$, involved in the definitions (77) of the background fields $\Psi(K,X)$ describes the fluctuations of capital in a given sector X around an average value K_X[21]:

$$\Psi_1(K-K_X)=\mathcal{N}\exp\left(-\left(K-\frac{F_2(R(K,X))K_X}{F_2(R(K_X,X))}\right)^2\right)\tag{80}$$

where $\mathcal{N}$ is a normalization factor. The capital accumulated by a firm in a sector X is centered around the average capital K_X in this sector, weighted by a factor $\frac{F_2(R(K,X))}{F_2(R(K_X,X))}$. This factor depends

[20] Incidentally, note that, to keep track of the dependency of the Lagrange multiplier in $\|\Psi\|^2$ in the above, we have chosen the notation $D\left(\|\Psi\|^2\right)$.

[21] It is computed in appendix 2.1.2.

on the firm's expected long-term return. It is relative to the average expected long-term return of the whole sector X described by the function $F_2(R(K_X, X))$[22].

Equation (78) can be solved for the X-dependent part of the background field $\Psi(X)$[23]. From this solution, we can deduce the number of firms $\|\Psi(X)\|^2$ in sector X. However, when fluctuations in capital allocation σ_X^2 are small, we can express directly $\|\Psi(X)\|^2$ as a function of the financial variables.

This number of firms is given by:

$$\|\Psi(X)\|^2 = \frac{D\left(\|\Psi\|^2\right)}{2\tau} - \frac{1}{4\tau}\left((\nabla_X R(X))^2 + \frac{\sigma_X^2 \nabla_X^2 R(K_X, X)}{H(K_X)}\right)\left(1 - \frac{H'\left(\hat{K}_X\right) K_X}{H\left(\hat{K}_X\right)}\right) H^2(K_X) \tag{81}$$

provided that the rhs of equation (81) is positive. It is 0 otherwise.

The Lagrange multiplier $D\left(\|\Psi\|^2\right)$ is obtained by integration of (78) and yields:

$$ND\left(\|\Psi\|^2\right) = 2\tau \int |\Psi(X)|^4 + \frac{1}{2}\int (\nabla_X R(X) H(K_X))^2 \|\Psi(X)\|^2 \tag{82}$$

Formula (81) will be used extensively in the sequel to compute K_X, the average physical capital per firm in sector X.

7.2 Background field for the financial markets

We have computed the background fields for firms, $\Psi(K, X)$ and $\Psi^\dagger(X, K)$, and the number of firms by minimizing $S_1 + S_2$. We can now compute the background fields $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$ and $\hat{\Psi}^\dagger\left(\hat{K}, \hat{X}\right)$ for investors along with the number of investors $\left|\hat{\Psi}\left(\hat{X}, \hat{K}\right)\right|^2$ by minimizing $S_3 + S_4$.

We first rewrite the field action for investors, $S_3 + S_4$. Inserting the number of firms $\|\Psi(X)\|^2$, formula (81), reduces $S_3 + S_4$ to:

$$S_3 + S_4 = -\int \hat{\Psi}^\dagger\left(\hat{K}, \hat{X}\right)\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} - \hat{K} f\left(\hat{X}\right)\right) + \nabla_{\hat{X}}\left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}} - g\left(\hat{X}\right)\right)\right)\hat{\Psi}\left(\hat{K}, \hat{X}\right) \tag{83}$$

where $f\left(K_{\hat{X}}, \hat{X}\right)$ is the short-term return[24]:

$$f\left(K_{\hat{X}}, \hat{X}\right) = \frac{1}{\varepsilon}\left(r\left(K_{\hat{X}}, \hat{X}\right) - \gamma\left\|\Psi\left(\hat{X}\right)\right\|^2 + F_1\left(\bar{R}\left(K_{\hat{X}}, \hat{X}\right)\right)\right) \tag{84}$$

and $g\left(K_{\hat{X}}, \hat{X}\right)$ describes investors capital reallocation:

$$g\left(K_{\hat{X}}, \hat{X}\right) = \left(\nabla_{\hat{X}} F_0\left(R\left(K_{\hat{X}}, \hat{X}\right)\right) + \nu \nabla_{\hat{X}} F_1\left(\bar{R}\left(K_{\hat{X}}, \hat{X}\right)\right)\right) \tag{85}$$

which depends on long-term returns $R\left(K_{\hat{X}}, \hat{X}\right)$ and their relative value $\bar{R}\left(K_{\hat{X}}, \hat{X}\right)$ given by:

$$\bar{R}\left(K_{\hat{X}}, \hat{X}\right) = \frac{R\left(K_{\hat{X}}, \hat{X}\right)}{\int R\left(K'_{X'}, X'\right)\|\Psi(X')\|^2 dX'} \tag{86}$$

[22] See discussion below equation (40).

[23] A method of resolution of (78) and two examples for particular forms of the function $H(K)$ are presented in appendix 2.2.

[24] See appendix 3.1.1.

Recall that F_1 measures the share of capital reallocation that depends on stock prices variation:

$$F_1\left(\bar{R}\left(K_{\hat{X}},\hat{X}\right)\right)=F_1\left(\bar{R}\left(K_{\hat{X}},\hat{X}\right),\Gamma=0\right) \tag{87}$$

In the sequel, any function $h\left(K_{\hat{X}},\hat{X}\right)$ and its partial derivatives $h\left(K_{\hat{X}},\hat{X}\right)$ will be written $h\left(\hat{X}\right)$, $\nabla_{K_{\hat{X}}}h\left(\hat{X}\right)$ and $\nabla_{\hat{X}}h\left(\hat{X}\right)$, respectively.

Using a change of variable:

$$\hat{\Psi}\rightarrow\exp\left(\frac{1}{\sigma_{\hat{X}}^2}\int g\left(\hat{X}\right)d\hat{X}+\frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)\hat{\Psi}$$

the minimization of S_3+S_4, equation (83) yields[25] the equation for $\hat{\Psi}$:

$$0=\left(\frac{\sigma_{\hat{X}}^2\nabla_{\hat{X}}^2}{2}-\frac{\left(g\left(\hat{X}\right)\right)^2}{2\sigma_{\hat{X}}^2}-\frac{\nabla_{\hat{X}}g\left(\hat{X}\right)}{2}\right)\hat{\Psi}+\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2\nabla_{\hat{K}}}{2}-\hat{K}f\left(\hat{X}\right)\right)-F\left(\hat{X}\right)\hat{K}-\hat{\lambda}\right)\hat{\Psi} \tag{88}$$

and the equation for its conjugate $\hat{\Psi}^{\dagger}$:

$$0=\left(\frac{\sigma_{\hat{X}}^2\nabla_{\hat{X}}^2}{2}-\frac{\left(g\left(\hat{X}\right)\right)^2}{2\sigma_{\hat{X}}^2}-\frac{\nabla_{\hat{X}}g\left(\hat{X}\right)}{2}\right)\hat{\Psi}^{\dagger}+\left(\left(\frac{\sigma_{\hat{K}}^2\nabla_{\hat{K}}}{2}+\hat{K}f\left(\hat{X}\right)\right)\nabla_{\hat{K}}-F\left(\hat{X}\right)\hat{K}-\hat{\lambda}\right)\hat{\Psi}^{\dagger} \tag{89}$$

with:

$$F\left(\hat{X}\right)=\nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X}\right)\right)^2}{2\sigma_{\hat{X}}^2}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X}\right)+f\left(\hat{X}\right)\right)\frac{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}+\frac{\left\langle\hat{K}^2\right\rangle_{\hat{X}}\nabla_{K_{\hat{X}}}f^2\left(\hat{X}\right)}{\sigma_{\hat{K}}^2\left\|\Psi\left(\hat{X}\right)\right\|^2} \tag{90}$$

where $\left\langle\hat{K}^2\right\rangle_{\hat{X}}$ denotes the average of $\hat{K}^2$ in sector $\hat{X}$ (see appendix 3.1.2) and $\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2=\int\left|\hat{\Psi}\left(\hat{X},\hat{K}\right)\right|^2d\hat{K}$.

A Lagrange multiplier $\hat{\lambda}$ has been included in the system of equations (88) and (89) to implement the constraint for $\hat{\Psi}$ and $\hat{\Psi}^{\dagger}$:

$$\int\left|\hat{\Psi}\left(\hat{X},\hat{K}\right)\right|^2d\hat{X}d\hat{K}=\hat{N} \tag{91}$$

Incidentally, note that the function $F\left(\hat{X},K_{\hat{X}}\right)$ arising in the minimization equations (88) and (89) describes the impact of individual variations on the collective state (the field $\hat{\Psi}$). It can be neglected in first approximation.

Appendix 3.1.3 computes the solutions for the investors' background fields (equations (88) and (89)). We find an infinite number of solutions for $\hat{\Psi}_{\hat{\lambda}}$ and $\hat{\Psi}_{\hat{\lambda}}^{\dagger}$ parametrized by $\hat{\lambda}\in\mathbb{R}$, which translates the fact that S_3+S_4 has an infinite number of local minima.

However, the eigenvalue $\left|\hat{\lambda}\right|$ computed in appendix 3.1.4.2 has a lower bound M[26] defined by:

$$M=\max_{\hat{X}}\left(A\left(\hat{X}\right)\right) \tag{92}$$

[25] See appendix 3.1.2.

[26] This lower bound is reminiscent of the fact that the Lagrange multiplier λ is the eigenvalue of the second order operator arising in equation (89), and that this operator is bounded from below.

where:

$$A\left(\hat{X}\right)=\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+f\left(\hat{X}\right)+\frac{1}{2}\sqrt{f^{2}\left(\hat{X}\right)}+\nabla_{\hat{X}}g\left(\hat{X}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X}\right)}{2f^{2}\left(\hat{X}\right)} \tag{93}$$

and that $\hat{\Psi}_{-M}$ is the global minimum of S_3+S_4. The background fields are thus $\hat{\Psi}_{-M}$ and its adjoint $\hat{\Psi}^{\dagger}_{-M}$.

For these background fields, the number of agents with capital $\hat{K}$ invested in sector $\hat{X}$ is:

$$\left|\hat{\Psi}_{-M}\left(\hat{K},\hat{X}\right)\right|^{2}=\hat{\Psi}^{\dagger}_{-M}\left(\hat{X},\hat{K}\right)\hat{\Psi}_{-M}\left(\hat{K},\hat{X}\right)$$

We find:

$$\left|\hat{\Psi}_{-M}\left(\hat{K},\hat{X}\right)\right|^{2}\simeq C\left(\bar{p}\right)\exp\left(-\frac{\sigma_{X}^{2}\hat{K}^{4}\left(f'\left(X\right)\right)^{2}}{96\sigma_{\hat{K}}^{2}\left|f\left(\hat{X}\right)\right|}\right)D_{p\left(\hat{X}\right)}^{2}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)^{\frac{1}{2}}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X}\right)}{f^{2}\left(\hat{X}\right)}\right)\right) \tag{94}$$

where D_p is the parabolic cylinder function with parameter $p\left(\hat{X}\right)$ and:

$$p\left(\hat{X}\right)=\frac{M-A\left(\hat{X}\right)}{\sqrt{f^{2}\left(\hat{X}\right)}} \tag{95}$$

The constant $C\left(\bar{p}\right)$ ensures that the constraint (91) is satisfied[27].

Section 9.2 will show that $p\left(\hat{X}\right)$ encompasses the relative expected returns of sector X vis-à-vis its neighbouring sectors.

7.3 Average capital per firm per sector

Now that the number of firms and investors per sector are computed, we can determine the average capital invested per firm in sector $\hat{X}$, i.e. $K_{\hat{X}}$.

We first rewrite the defining equation of $K_{\hat{X}}$ (74) as:

$$K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^{2}=\int\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}d\hat{K} \tag{96}$$

and evaluate this equation for the background field (94):

$$K_{X}\left\|\Psi\left(X\right)\right\|^{2}=\int\hat{K}\left|\hat{\Psi}_{-M}\left(\hat{K},\hat{X}\right)\right|^{2}d\hat{K} \tag{97}$$

Equation (97) allows to find the average capital $K_{\hat{X}}$. Actually, both the densities of agents $\left\|\Psi\left(\hat{X}\right)\right\|^{2}$ and $\left|\hat{\Psi}_{-M}\left(\hat{K},\hat{X}\right)\right|^{2}$, equations (81) and (94), are functions of $K_{\hat{X}}$, so that equation (97) is itself an equation for $K_{\hat{X}}$.

From this general equation, we can find the average capital at point $\hat{X}$. Appendix 3.1.4.2 computes the integral (97) using the financial background field (94).

In the sequel, we will write $p\left(\hat{X}\right)$ defined in (95) as:

$$p\equiv p\left(\hat{X}\right) \tag{98}$$

[27] Its expression is given in appendix 3.1.3.

and equation (97) becomes ultimately:

$$K_{\hat{X}} \left\| \Psi\left(\hat{X}\right) \right\|^2 \left| f\left(\hat{X}\right) \right| = C\left(\bar{p}\right) \sigma_{\hat{K}}^2 \hat{\Gamma}\left(p + \frac{1}{2}\right) \tag{99}$$

with:

$$\begin{aligned} \hat{\Gamma}\left(p+\frac{1}{2}\right) &= \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(p+\frac{1}{2}\right)^2 \left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right) \\ &\times \left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right) - \left(\Gamma\left(-\frac{p}{2}\right)\right)^2}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)} + p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right) - \left(\Gamma\left(-\frac{p-1}{2}\right)\right)^2}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right) \end{aligned} \tag{100}$$

and where Γ is the Gamma function.

This final form of the capital equation, (99), will be central to our following computations. However, it involves some functions, such as f, that have a general form, and functions of the unknown variable $K_{\hat{X}}$ (see for instance equation (84)). Thus, it cannot, in general, be solved analytically.

7.4 Solving for average capital

Except for some particular cases, the final form of the capital equation (99) cannot be solved analytically. Several approaches can nonetheless be used to approximate its solutions and study their behaviours. The detailed results and their derivation are given in appendix 4.

A first and most general approach studies, for each sector, the variation of average capital per firm $K_{\hat{X}}$ with respect to any parameter of the system. This is done by studying the differential form of the capital equation (99) while keeping very general forms for the parameter-functions f and g. This approach allows to study, on a sector, the influence of its neighbours. It is depicted by the variation of $K_{\hat{X}}$ with respect to the sector's relative expected returns. It reveals stable and unstable equilibria in the system but does not yield the sectors' precise levels of capital.

A second approach expands the capital equation (99) around particular solutions. These particular solutions are the average capital in sectors where accumulation is the strongest. This approach confirms the existence of both stable and unstable equilibria, which correspond to several possible average capital in a given sector: depending on initial configurations, an infinite number of collective states may arise[28].

A third approach provides approximate solutions to the capital equation (99) for standard forms of the parameter-functions. The existence of multiple solutions is confirmed, along with the associated stability analysis. Combined, these three approaches confirm and complete each other.

Equation (99) has in general several solutions per sector that can be ranked by their level of average capital; low average, high or very high. For each of these levels of capital, the solutions of equation (99) may be stable or unstable. A stable average capital is one that, when slightly modified, comes back to its initial values, wheras an unstable solution does not. An unstable average capital may be interpreted as a threshold of capital accumulation for firms in the sector. The solutions of (99) depend on short-term returns, long-term returns, absolute and relative, and the dependency of the solutions of (99) depends on the stability of this solution.This results will be interpreted in section 9.

[28] This point will be developed in section 8.

8 Dynamic average capital

So far, we have determined and studied how average capital per firm and per sector react to changes in parameters. However these same parameters may vary over time, and so should average capital values. We thus introduce a macro timescale and design a dynamic model in which average capital and expectations in long-term returns interact and vary over time.

8.1 Average capital and long-run expected returns

We consider how modifications in parameters generate the dynamics for $K_{\hat{X}}$. Assuming that some time-dependent parameters modify expected long-term returns $R(X)$, average capital $K_{\hat{X}}$ becomes a function of the time variable θ. To find how the average physical capital per firm in sector $\hat{X}$, $K_{\hat{X}}$, evolves over time, we must define the equation for $K_{\hat{X}}$, (99), and compute its variation with respect to θ, using the fact that the functions $\left\|\Psi\left(\hat{X}\right)\right\|^2$ and $\hat{\Gamma}\left(p+\frac{1}{2}\right)$ both depend on time θ through $K_{\hat{X}}$ and $R(X)$. The variations of these two functions with respect to the dynamic variables $K_{\hat{X}}$ and $R(X)$ are computed in appendix 5.1. We show that, when $C(\bar{p})$ constant, the variation of (99) writes:

$$k\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}+l\frac{\nabla_\theta R\left(\hat{X}\right)}{R\left(\hat{X}\right)}-2m\frac{\nabla_{\hat{X}}\nabla_\theta R\left(\hat{X}\right)}{\nabla_{\hat{X}}R\left(\hat{X}\right)}+n\frac{\nabla^2_{\hat{X}}\nabla_\theta R\left(\hat{X}\right)}{\nabla^2_{\hat{X}}R\left(\hat{X}\right)}=-C_3\left(p,\hat{X}\right)\frac{\nabla_\theta r\left(\hat{X}\right)}{f\left(\hat{X}\right)} \tag{101}$$

where coefficients k, l, m and n are computed in appendix 5.1.

To make the system self-consistent, and since $K_{\hat{X}}$ already depends on R, we merely need to introduce an endogenous dynamics for R.

To do so, we assume that R depends on $K_{\hat{X}}, \hat{X}$ and $\nabla_\theta K_{\hat{X}}$, and that this dependency has the form of a diffusion process[29]. This leads to write R as a function $R\left(K_{\hat{X}}, \hat{X}, \nabla_\theta K_{\hat{X}}\right)$. The variation of R is of the form:

$$\begin{aligned}\nabla_\theta R\left(\theta,\hat{X}\right) &= a_0\left(\hat{X}\right)\nabla_\theta K_{\hat{X}}+b\left(\hat{X}\right)\nabla^2_{\hat{X}}\nabla_\theta K_{\hat{X}}+c\left(\hat{X}\right)\nabla_\theta\left(\nabla_\theta K_{\hat{X}}\right)+d\left(\hat{X}\right)\nabla^2_\theta\left(\nabla_\theta K_{\hat{X}}\right) \\ &+f\left(\hat{X}\right)\nabla^2_{\hat{X}}\left(\nabla_\theta R\left(\theta,\hat{X}\right)\right)+h\left(\hat{X}\right)\nabla^2_\theta\left(\nabla_\theta R\left(\theta,\hat{X}\right)\right) \\ &+u\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta\left(\nabla_\theta K_{\hat{X}}\right)+v\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta\left(\nabla_\theta R\left(\theta,\hat{X}\right)\right)\end{aligned} \tag{102}$$

We can also assume that the coefficients in the expansion are slowly varying, since they are obtained by computing averages.

The dynamics (102) corresponds to a diffusion process: expected returns in one sector depend on the variations of capital and returns in neighbouring sectors.

To find the intrinsic dynamics for $K_{\hat{X}}$, we assume that the exogenous variation $\frac{\nabla_\theta r(\hat{X})}{r(K_{\hat{X}},\hat{X})}$ is null, and that the system of equations (23) and (102) yields the dynamics for $\nabla_\theta K_{\hat{X}}$ and $\nabla_\theta R\left(\theta,\hat{X}\right)$. Approximating these dynamics to the first order in derivatives, we find in appendix 5.2 the following matricial equation:

$$0=M_1\begin{pmatrix}\nabla_\theta K_{\hat{X}}\\ \nabla_\theta R\end{pmatrix}-M_2\begin{pmatrix}\nabla_\theta K_{\hat{X}}\\ \nabla_\theta R\end{pmatrix}-M_3\begin{pmatrix}\nabla_\theta K_{\hat{X}}\\ \nabla_\theta R\end{pmatrix} \tag{103}$$

[29] See appendix 5.2.

8.2 Oscillatory solutions

We look for oscillating solutions of (103) of the type:

$$\begin{pmatrix} \nabla_\theta K_{\hat{X}} \\ \nabla_\theta R\left(\hat{X}\right) \end{pmatrix} = \exp\left(i\Omega\left(\hat{X}\right)\theta + iG\left(\hat{X}\right)\hat{X}\right)\begin{pmatrix} \nabla_\theta K_0 \\ \nabla_\theta R_0 \end{pmatrix} \tag{104}$$

with slowly varying $G\left(\hat{X}\right)$ and $\Omega\left(\hat{X}\right)$. We are then led to the relation between $\Omega\left(\hat{X}\right)$ and $G\left(\hat{X}\right)$:

$$\begin{aligned} 0 &= \frac{k}{K_{\hat{X}}}\left(1 - ieG - ig\Omega\right) + \left(\frac{l}{R\left(\hat{X}\right)} - i\frac{2m}{\nabla_{\hat{X}} R\left(\hat{X}\right)}G\right)\left(a_0 + iaG + ic\Omega\right) \\ &\quad - \frac{l}{R\left(\hat{X}\right)}\left(d\Omega^2 + bG^2 + u\Omega G\right) + \frac{k}{K_{\hat{X}}}\left(e\Omega^2 + fG^2 + v\Omega G\right) \end{aligned} \tag{105}$$

We will limit our study to the first order terms which yields the expression for Ω as a function of the parameters involved in (101) and (102). Appendix 5.3 computes the expression of Ω.

It also derives the condition of stability for the oscillations. When:

$$\frac{lc}{R\left(\hat{X}\right)}\left(\frac{k}{K_{\hat{X}}} + \frac{a_0 l}{R\left(\hat{X}\right)}\right) + \frac{4m^2 c a_0}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2}G^2 > 0 \tag{106}$$

oscillations are dampened and return to the steady state. Otherwise, oscillations are diverging: the system settles on another steady state, i.e. another background state. Appendix 5.4 studies the condition (106) as a function of the parameter functions $f\left(\hat{X}\right)$ and $R\left(\hat{X}\right)$, the level of average capital $K_{\hat{X}}$, and the coefficients arising in the expectations formations. The results are presented in the next section.

9 Results and nterpretations

For each sector, the equation defining its average capital, equation (99), accepts several solutions, so that each sector could present several average capital. We will discuss the stability of these solutions, before detailing the determinants of average capital, and number of firms and investors per sector. We will then describe the three patterns of capital accumulation that emerge and their possible transitions. Ultimately we will study how average capital per sector interacts with endogeneized long-term returns expectations. We will end up this section by providing a synthesis of the main results.

9.1 Stability of average capital

Each average value of capital solving equation (99) can be either stable or unstable. A stable average capital, when modified, will naturally return to its initial value. An unstable one will not: it will merely act as a potentially varying capital accumulation threshold. Once modified, an unstable average capital will settle at another equilibrium level.

The average capital per firm per sector defined in equation (99) acts as a fixed point of a dynamic equation[30][31] whose (in)stability depends the sector's parameters. An unstable average

[30] The definitions of the parameters are given in appendix 4.1.1.1.

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[31] See section 7.2.1.2 and appendix 4.1.1.1

capital will act as a potential threshold for capital accumulation in a sector: any deviation in this threshold will set firms above or below the threshold, initiate a path towards a new equilibrium and ultimately shift average capital. There is therefore an intrinsic transition dynamics of average capital per sector that is driven by instability and exogenous variations in the system's parameters.

Average capital is potentially unstable in a sector when the following condition is met:

$$\left|B\left(\hat{X}\right)\right| \equiv \left|k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}-\left(\frac{\partial \ln f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}+\frac{\partial \ln\left|\Psi\left(\hat{X},K_{\hat{X}}\right)\right|^{2}}{\partial K_{\hat{X}}}+l\left(\hat{X},K_{\hat{X}}\right)\right)\right|>1 \tag{107}$$

This instability depends on the four parameters of $\left|B\left(\hat{X}\right)\right|$: directly through short-term returns, dividends and price fluctuations, $\frac{\partial \ln f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}$, and the net flow of firms entering the sector, $\frac{\partial \ln\left|\Psi\left(\hat{X},K_{\hat{X}}\right)\right|^{2}}{\partial K_{\hat{X}}}$; indirectly through a change in the background field induced either by a variation in short-term returns $f\left(\hat{X}\right)$ via $l\left(\hat{X},K_{\hat{X}}\right)$, or by the modification of the relative return of sector $\hat{X}$ which depends on the shape of the returns around $\hat{X}$ via $k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}$. Any variation in these parameters will affect the system as a whole and may reshape the collective state through a change in the background field. Altogether, these modifications may magnify or dampen changes in a sector's average capital and impact the stability of the system.

9.2 Determinants of capital accumulation

We will describe the determinants of capital accumulation, before studying the number of firms and investors per sector.

9.2.1 Average capital per sector

The average capital in a sector $\hat{X}$ is determined by short-term returns, $f\left(\hat{X}\right)$ - dividends and price fluctuations - and by the growth prospects of the firm, its expected long-term returns, $R\left(\hat{X}\right)$. These returns are not fully independent since the price fluctuations in short-term returns are driven by expected long-term returns.

Besides, average capital in a sector depends on expected long-term returns of neighbouring sectors. This dependency is measured by $p\left(\hat{X}\right)$ defined in (95)[32]. For positive short-term returns, which is the case here[33], it writes:

$$p(\hat{X})=\frac{M-\left(\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+\nabla_{\hat{X}}g\left(\hat{X}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X}\right)}{2f^{2}\left(\hat{X}\right)}\right)}{f\left(\hat{X}\right)}-\frac{3}{2} \tag{108}$$

Except for the normalization by the short-term return $f\left(\hat{X}\right)$ of sector $\hat{X}$, the function $p(\hat{X})$ is composed of three terms.

The two first terms, $\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}$ [34] and $\nabla_{\hat{X}}g\left(\hat{X}\right)$[35], measure the variations of expected returns across sectors, i.e. the value of expected returns in sector $\hat{X}$ relative to its neighbours. The last term,

[32] See explanation and derivation in appendix 4.1.1.

[33] See explanation in appendix 4.1.3.2.

[34] This term is directly proportional to the gradient of expected long-term returns $\nabla R\left(\hat{X}\right)$. See the definition of the parameter function g, equation (85), and (86).

[35] This term is proportional to the second derivative $\nabla^{2}R\left(\hat{X}\right)$ of $R\left(\hat{X}\right)$.

$\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}$, is a smoothing factor between neighbours' sectors. It can be neglected in the first approximation[36]. The parameter $p(\hat{X})$ is a local maximum when $R\left(\hat{X}\right)$ is itself a local maximum[37], so that it describes the expected long-term returns of a sector relative to its neighbours. The higher $p\left(\hat{X}\right)$, the more attractive is sector $\hat{X}$ relative to its neighbours[38].

Taken altogether, the three parameters $R\left(\hat{X}\right)$, $f\left(\hat{X}\right)$, $p\left(\hat{X}\right)$ are the main determinants of average capital in a sector. However, their influence on capital will depend on the stability of the sector. In stable sectors, average capital values can be understood as equilibria. In unstable ones, they are potential thresholds for the capital accumulation of individual firms. In stable sectors, average capital is increasing in short-term returns $f\left(\hat{X}\right)$, expected long-term returns $R\left(\hat{X}\right)$, and in the sector's relative attractivity $p\left(\hat{X}\right)$, respectively. The higher the short and long-term returns, the higher the capital accumulation. Besides, any increase in relative returns will attract capital from neighbouring sectors and increases the sector average capital. In unstable sectors, average capital is decreasing in these same variables, and any increase in short- or expected long-term, be they absolute or relative, returns reduces the amount of capital required to initiate the capital accumulation process for the individual firms.

9.2.2 Firms per sector

Various parameters determine how firms and investors shift across the sectors' space.

The number of firms per sector defined in equation (81) depends on expected long-term returns:

$$V(X) = \left(\nabla_X R(X)\right)^2 + \frac{\sigma_X^2 \nabla_X^2 R(X)}{H(K_X)}$$

where $\nabla_X R(X)$ is the gradient of expected long-term returns along the sectors space, and $\nabla_X^2 R(K_X, X)$ is the Laplacian, i.e. the generalisation of the second derivative of $R(K_X, X)$ with respect to the sectors' space. The number of firms is a decreasing function of $V(X)$[39].

When expected returns are minimal, $\nabla_X R(X) = 0$ and $\nabla_X^2 R(K_X, X) > 0$, average capital is low, and a large number of small firms provide short-term returns through dividends.

When returns in sector X, $R(X)$, are at a local maximum, $\nabla_X R(X) = 0$ and $\nabla_X^2 R(K_X, X) < 0$, the sector exhibit both a large number of firms and a high level of capital K_X per firm, but this equilibrium is unstable.

Incidentally, competition ensures that sectors with low or minimal expected returns are not completely depleted.

When $\nabla_X R(X) \neq 0$, the sector is "transitory". It is surrounded by neighbouring sectors, with both lower and higher expected returns. Firms head towards sectors with higher returns. The greater the discrepancy between neighbouring returns $\nabla_X R(X)$, the faster firms leave the sector.

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[36] See the discussion following equation (90). This term will also be discussed in section 7.2.2.

[37] Actually, $p(\hat{X})$ is maximal for sectors such that $\nabla R\left(\hat{X}\right) = 0$ and $\nabla^2 R\left(\hat{X}\right) < 0$. It is thus

[38] Note that the parameter $p\left(\hat{X}\right)$ is normalized by short-term returns. It computes the ratio of relative attractivity to short-term returns. This allows to consider these two variables separately.

[39] See equation (81).

9.2.3 Investors per sector

The average number of investors in a sector[40] is an increasing function of the sector short-term returns and relative long-term attractivity p[41]. All else equal, an increase in short-term returns or an improvement of the sector's relative long-term attractivity increases the number of investors and, in turn, firms' disposable capital.

The number of investors in a given sector increases with its relative attractivity $p\left(\hat{X}\right)$ defined in equation (95). The first term in (95) is the sector's relative attractivity towards its neighbours, normalized by its short-term returns $f\left(\hat{X}\right)$. The second term is a factor that smoothes differences between sectors. It is negatively correlated to the variations of the sectors' relative attractivity. Investors and capital will increase in sectors surrounded by significantly more attractive sectors, i.e. sectors with higher average capital and investors[42]: the whole system tends to reach stable configurations, and capital discrepancies are reduced between close neighbours[43].

9.3 Capital accumulation

In each sector, several average capital may exist, and three patterns of capital accumulation arise, defined by their average capital, number of firms, and long- and short-term returns. These parameters will determine the stability of the pattern. Shocks will shift unstable patterns to another one. Any deviation of average capital above or below an unstable equilibrium value will drive firms away from this equilibrium and ultimately shift average capital towards another equilibrium. These transitions provide bridges between patterns of capital. Due to a change in external conditions, sectors may move from one pattern to another.

9.3.1 First pattern: low capital, high short-term returns driven by dividends only

These are sectors where growth prospects are subdued, with a relatively large number of low-capitalized firms. Because firms are small, marginal productivity is high and firms attract capital with short-term returns through dividends, but lack the capital to move towards growth sectors.

These sectors are stable to small fluctuations in growth prospects: any increase in expected long-term returns will only shift moderately investment and average capital. They are unstable to short-term returns: any increase in $f\left(\hat{X}\right)$ will drive dividends higher and attract investors. Average capital will accumulate and reach a stable pattern-2 equilibrium, with more firms and a higher average capital.

However, an adverse shock lowering short-term returns will increase the threshold of capital accumulation and drive the equilibrium towards 0. Producers remain in the sector, but their very lack of capital will prevent them to shifting towards more attractive sectors in the long run (see appendix 3.3).

[40] See formula (94).

[41] See equation (98).

[42] A close inspection of equation (90) reveals that this term contains the -squared- contributions of short-term returns, $f\left(\hat{X}\right)$, and the sector's relative attractivity: $\frac{\left(g\left(\hat{X}\right)\right)^2}{2\sigma_{\hat{X}}^2}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X}\right)$. Both contributions are proportional to the gradient of R with respect to $K_{\hat{X}}$. When this gradient is non-zero, indicating that an increase in capital may enhance either the sector's relative attractiveness or short-term returns, the correction $\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}\right)}{2\sigma_{\hat{X}}^2\left(\sqrt{f^2\left(\hat{X}\right)}\right)^3}$ amplifies $\frac{A\left(\hat{X}\right)}{f\left(\hat{X}\right)}$, and consequently $K_{\hat{X}}$, in most cases.

[43] Derivation of the minimization equation in appendix 3.1.2 shows that the term $F\left(\hat{X}\right)$ arises as a backreaction of the whole system with respect to modifications at one point of the thread.

9.3.2 Second pattern: intermediate-to-high level of capital, short-term returns, long-term expectations

These sectors have moderate growth prospects, so that any increase in short-term, i.e. dividends and stock prices or long-term returns, increases their relative attractivity $p\left(\hat{X}\right)$ and attracts investors and capital. Locally, the higher the relative attractivity of the sector, the higher the capital accumulation. The relatively high number of firms in the sector is a decreasing function of average capital: competition favours higher average capital, and concentration of firms. This is the most standard pattern of capital allocation. It is stable to variations in average capital, except when average capital is high and the firms' density is low.

In this case, any deviation of average capital above its equilibrium increases the threshold and drives the sector backward to a stable pattern 2 equilibrium, i.e. a sector with a large number of average capitalized firms. The lower capital per firm reduces competition and attracts new firms into the sector.

On the contrary, any deviation of average capital below its equilibrium reduces the threshold and favours capital accumulation. The sector is driven towards a stable pattern 3 equilibrium, with a small number of very capitalized firms (see description of this pattern below).

9.3.3 Third pattern: high capital, long-term returns, and relative attractivity

These are sectors where growth prospects are extremely high. Capital accumulation is driven by expectations of long-term returns sustained by ever-higher levels of investment. These are the most attractive sectors. Two cases arise.

When expected long-term returns are not maximal, the sector stabilizes with very few firms with very high capital arises. This extension of pattern 2 corresponds to a few large oligopolistic groups.

When expected long-term returns are maximal, the sector's attractivity allows a large number of firms with high capital to coexist. All else equal, these firms could grow indefinitely, so that such equilibria are bound to be unstable[44].This describes bubble-like, unstable sectors.

An adverse shock drives these unstable sectors towards a stable pattern 3: average capital is approximatively maintained, but an increase in competition evicts the less capitalized firms and the total number of firms is reduced to a small set.

On the contrary, a positive shock reduces the threshold of capital accumulation. Most firms can accumulate without bound, which attracts even higher capital. Capital accumulation is modified in all sectors, which may transform the whole economic landscape. Total available capital is reduced, which modifies the stability conditions for all sectors. Low-capitalized sectors may become unstable and disappear, whereas others may accumulate capital. All in all, the system may end with a reduced sectors space[45].

Global instability

Another source of instability stems from the constraint imposed by the model on the total number of investors.

In our model, we have assumed a fixed number of agents, that are spread across sectors. This hypothesis binds the dynamics of the whole set of sectors. If this constraint were to be lifted, the

[44] See appendix 4.1.

[45] See appendix 4.3 for technical details.

sectors would be independent and could each reach a stable average capital, given their own short and expected long-term returns.

However, the number of agents in a sector is dependent on the whole system's characteristics. Thus there can only be global equilibria for the system. Any change in the parameters induces a perturbation $\delta\Psi\left(\hat{X},K_{\hat{X}}\right)$ that destabilizes the whole system as a whole: the equilibrium is globally unstable[46]. Relaxing the condition on the number of agents amounts to replacing the average capital equation (99) by[47]:

$$K_{\hat{X}}\left\Vert \Psi\left(\hat{X}\right)\right\Vert ^{2}\left|f\left(\hat{X}\right)\right|=C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\hat{\Gamma}\left(\frac{1}{2}\right)=C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\exp\left(-\frac{\sigma_{\hat{X}}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(X\right)\right)^{2}}{384\left|f\left(\hat{X}\right)\right|^{3}}\right) \tag{109}$$

This equation has at least one locally stable solution. The solutions of the modified average capital equation (109) do no longer directly depend on a sector's relative characteristics, but rather on the returns $f\left(\hat{X}\right)$ and on the number of firms in the sector, $\left\Vert \Psi\left(\hat{X}\right)\right\Vert ^{2}$[48][49].

9.4 Dynamic capital accumulation

The dynamic system (103) propagates shocks in capital and expectations across the system[50]: assuming a shock on average capital or long-term returns in a given sector, the interactions between average capital and expected long-term returns induce some volatility around the equilibrium values $K_{\hat{X}}$ and $R\left(\hat{X}\right)$. The fluctuations of long-term returns directly impact average capital and expected return in neighbouring sectors through the induced variation of relative expected returns, which initiates the propagation of the initial perturbation to the whole system.

This propagation is described by the oscillating solutions (104). For a given sector $\hat{X}$, the velocity of oscillations in average capital and expected returns are measured by the frequency $\Omega\left(\hat{X}\right)$, that depends on the sector's characteristics, These oscillations may be dampening (stable oscillations) or widening (unstable oscillations). Three main parameters determine which type of oscillations a sector may experience[51].

1. The elasticity of expected long-term returns with respect to variations of capital, c, that arises in equation (102), determines two relevant forms of expectations. When expectations are highly reactive to variations of capital, $c>0$, and when expected long-term returns increase with any acceleration in capital accumulation, expected long-term returns depend positively on the variations of average capital $K_{\hat{X}}$. When expectations are moderately reactive to variations in the capital, $c<0$, expected long-term returns depend negatively on the variations of average capital $K_{\hat{X}}$.

2. The neighbouring sectors' discrepancy in capital fluctuations at a given time, G. It arises in the oscillatory solutions (104) and measures the inhomogeneity between sectors.

[46] The mechanism of this instability is detailed in appendix 3.3.1.

[47] The derivation is given in appendix 4.1.2.

[48] An intermediate situation between (99) and (109) could also be considered: it would be to assume a constant number of agents in some regions of the sector space.

[49] Alternatively, limiting the number of investors per sector can be achieved through some public regulation to maintain a constant flow of investment in the sector.

[50] See appendix 5.3.

[51] We have already given the condition for dampening oscillations in (106). See appendix 5.4.

3. Last but not least, the sector average level of capital $K_{\hat{X}}$ impacts the type of fluctuations experienced by the sector.

Our results are the following.

9.4.1 Low average capital sectors

When average capital is very low in a sector, the sole relevant parameter to the fluctuations is the reactivity c of the expected return $R\left(\hat{X}\right)$ to an increase in capital[52].

Two cases arise.

When long-term returns strongly react to capital fluctuations, $c > 0$, oscillations are unstable. When they react mildly, $c < 0$, oscillations are stable.

In the first case, expected long-term returns and average capital variations are positively correlated, and any increase in capital will amplify expected returns that will in turn increase capital. In the second case, expected long-term returns and average capital variations are negatively correlated, which will induce dampening oscillations and stabilize the system.

These results show that for expectations mildly reactive to variations of capital, some equilibria with relatively low capital are possible and resilient to oscillations in expectations, a niche effect may exist for some sectors.

9.4.2 High average capital sectors

In high average capital sectors, be they stable or unstable, here again only expectations reactivity to capital increase, i.e. c, matters. Oscillations are dampening for $c > 0$ and explosive for $c < 0$. Highly reactive expectations, $c > 0$, will amplify fluctuations of capital and expected returns:

In the stable case, fluctuations that would otherwise be destabilizing for sectors with low capital may stabilize or maintain sectors with both stable and high levels of capital. A large reactivity between expectations and capital will allow for an intrinsic high level of capital to consolidate. Fluctuations will moderately impact these high-capitalized sectors: for instance, considering an initial increase in returns only, i.e. $\delta R\left(\hat{X}\right) > 0$, will induce a net outflow of capital towards less capitalized sectors with an higher increase in relative returns, while a decreasing return, i.e. $\delta R\left(\hat{X}\right) < 0$, will induce a net inflow of capital dampening the sector's fluctuations.

In the unstable case, an initial increase in capital increases expected long-term returns, while at the same time, the negative correlation between variations in investment and expected return lowers the average capital. An increase in capital will improve the sector profitability, lowering the capital threshold in capital. To put it differently, an initial increase in the average capital amplifies the expected return, which reduces $K_{\hat{X}}$ and offsets the initial increase in capital.

When expectations are mildly reactive, i.e. $c < 0$, the mechanism of dampening oscillations that arises for $c > 0$ is impaired. In the unstable case, for instance, an initial increase in the threshold $K_{\hat{X}}$, impacts only moderately the sector expected returns, and does not offset the initial increase in capital.

9.4.3 Intermediate average capital sectors

In intermediate capital sectors, oscillations depend both on the reactivity of expectations to an increase in capital, c, and discrepancy between sectors, G[53].

[52] See appendix 5.4.
[53] See appendix 5.4.2.

Midly reactive expectations, $c < 0$, and a moderate discrepancy between neighbouring sectors, $G << 1$, oscillations are dampening for sectors with relatively low average capital. The analysis of the first case applies to the extent that indeed some homogeneity in capital between the neighbouring sectors exists.

Strongly reactive expectations, $c > 0$, and a large discrepancy between neighbouring sectors, $G >> 1$, oscillations are dampening for relatively high average capital sectors. The analysis of the second case applies to a locally dominating sector.

9.4.4 The role of expectations in average capital fluctuations

For each sector, the threshold between dampening and explosive oscillations depends on the parameters of the system. Mildly reactive expectations only favour low- to high-capital sectors (patterns 1 and 2). Very reactive expectations will only favour very high-capital sectors (pattern 3) where oscillations will be felt as relatively weak, in absolute value, attracting further capital.

Recall that in extreme cases of pattern 3, both maximal capital and returns act as thresholds that repel low-capital firms and propel high-capital firms to ever higher accumulation. Oscillations in these thresholds generate a high global instability: a constantly oscillating threshold crowds firms out of the sector.

To conclude, the dynamics for average capital and expected returns merely reflect fluctuations in the background fields i.e. the collective states. These fluctuations may destabilize the patterns in some sectors and ultimately switch the collective state and modify the patterns' landscape.

9.5 Synthesis of the results

Let us briefly synthesize our results before discussing them. They can be regrouped along three main axes.

9.5.1 Capital allocation

We have shown that capital allocation by producers and investors differ and interact: these interactions impact the form of the collective state and average capital per sector. The main determinants of capital allocation are short-term returns, expected long-term returns and the sector's relative attractivity. Short-term returns are composed of dividends, and are driven by marginal productivity and variations in stock prices, themselves driven by expectations of long-term returns. Expected long-term returns describe growth prospects, and.the sector's relative attractivity measures the growth prospects of a sector relative to its neighbouring sectors.

Firms tend to relocate in sectors with relatively higher long-term returns at a speed that depends on their capital endowment. However, they can be crowded out by competitors. The higher the firm's capital, the higher the power to overcome competitors. Eventually, firms with the highest capital concentrate in sectors that have the highest expected long-term returns, while the rest relocates in neighbouring and likely lowest expected-return sectors.

Capital allocation depends on short-term returns, dividends and price fluctuations, and expected long-term returns. But since price fluctuations are driven by expected long-term returns, short and long-term returns are not independent. The financial capital allocation also depends on the sector's relative attractivity, which measures the expected returns of a sector relative to its neighbours. However financial capital is volatile. High short-term returns are an incentive, but the relative attractivity of sectors lures investors. Financial capital allocation thus depends on the ratio of sectors' relative attractivity to short-term returns. Since this ratio depends on expectations, it is subject to fluctuations, which in turn impact the collective state.

9.5.2 Capital accumulation

Three stationary[54] patterns of capital accumulation may emerge in each sector. A pattern is characterized by the combination of the firms' average capitalization, the number of firms in the sector, and the type of returns these firms may provide to their investors. The emergence of a given pattern depends on the parameters of this sector:

The first pattern associates a large number of low-capitalised firms. Dividends are determinant in this pattern; the lack of capital, combined with the prospects of competition with better-capitalized firms prevent firms to shift to neighbouring and more profitable sectors.

The second pattern associates a relatively high number of average-to-high capitalised firms and a combination of short and long-term returns. This combination lures intermediate-to-high capital investors in the sector.

In the third pattern, high expected long-term returns generate massive inflows of capital toward highly-capitalized firms. In this pattern, firms with the highest expected returns could theoretically accumulate endlessly. Actually, this accumulation is limited by the amount of available capital.

In each pattern, some sectors are stable, others are unstable. Transitions between patterns occur through exogenous shocks. In pattern 1, some sectors may disappear, whereas in pattern 3, some may grow endlessly and the large amounts of capital they drive may modify the whole system's landscape.

9.5.3 Collective states

We have shown how statistical field theory can describe a microeconomic framework in terms of collective states of sectors composed of a large number of firms.

Each collective state encodes the data characterizing each sector: number of firms, number of investors, average capital, and distribution of capital. These data are theoretical averages over long-term periods, not instantaneous empirical averages.

The collective states are not arbitrary: they directly result from the agents' interactions, and are the most likely stable states of the system. Other states do exist, but they are unstable. A particular collective state can be described by its distribution into patterns of capital accumulation - type 1, 2, or 3 - across sectors. Besides, sectors are connected and benefit from the relative attractiveness of their neighbours: this smoothing effect between sectors materialises in mergers and acquisitions.

The multiple combinations of accumulation patterns in each sector may yield an infinite number of possible collective states. It does not follow that all combinations are possible: sector patterns depend on the relative attractivity of both the sector and its neighbours'. There are also constraints: for instance, massive inflows of capital are needed for the emergence of the pattern 3, which is only driven by high expected long-term returns, while niche effects merely occur for relatively highly productive firms. However, a potentially infinite range of collective states may exist.

The selection of a particular collective state and its sectoral patterns is ultimately determined by exogenous conditions. Structural changes, such as an extra-loose monetary policy or the choice of a pension system are external conditions that modify collective states.

The existence of multiple collective states has a dynamic implication. When parameters vary, a given collective state may switch to another: a change in expectations may, for instance, induce variations in average capital and in turn, induce changes in sectors' patterns of capital accumulation. To study these possible switches, we introduced a dynamic interaction between average capital and

[54] The values of average capital are stationary results: agents accumulate and shift from sectors to other ones, but, in average, the density of firms and average capital per firm per sector are constant.

expected long-term returns, now endogenized. This dynamic interaction depends both on the patterns of accumulation and the way expectations are formed.

In this dynamics system, average capital and expectations present some oscillatory patterns that may dampen equilibria or drive them towards other equilibria. Expectations highly reactive to capital variations stabilize high-capital configurations. They drive low-to-moderate capital sectors towards zero or higher capital, depending on their initial conditions. Inversely, expectations moderately reactive to capital variations stabilize low-to-moderate capital configurations, and drive high-capital sectors towards lower capital equilibria. Amplifying oscillations may modify some sectors' pattern: the ensuing reallocation of capital across the whole sectors' space may initiate a transition in collective states. The mechanism of transition and its implications are discussed below.

System at the individual level: effective action and transition functions

This third part focuses on the micro-scale of the system. In a given background state, we can derive the individual dynamics for agents, firms and investors. This approach stresses the dependence of individual dynamics in the collective states described by the background fields.

10 Computation of transition functions

We use the results of section 5 to compute the agents' transition functions. To do so, we compute the effective action of the system which is given by the series expansion of the action around the background fields. These background fields where computed in the second part of this work (see sections 7.1 and 7.2).

10.1 Effective action expansion

10.1.1 Second-order expansion of effective action

Consider the field action:

$$S = S_1 + S_2 + S_3 + S_4$$

where the S_i are defined by equations (54),(55),(57) and (60)[55]. Expanding the action S to the second-order around the background field will allow us to compute the transition functions of individual agents in the background, without taking into account individual interactions. We can rewrite the fields as follows:

$$\begin{aligned}
\Psi(K,X) &= \Psi_0(K,X) + \Delta\Psi(Z,\theta) \\
\hat{\Psi}\left(\hat{K},\hat{X}\right) &= \hat{\Psi}_0\left(\hat{K},\hat{X}\right) + \Delta\hat{\Psi}(Z,\theta)
\end{aligned}$$

where $\Psi_0(K,X), \hat{\Psi}_0\left(\hat{K},\hat{X}\right)$ are the background fields. This yields the quadratic approximation:

$$S\left(\Psi,\hat{\Psi}\right) = S\left(\Psi_0,\hat{\Psi}_0\right) + \int \left(\Delta\Psi^{\dagger}(Z,\theta), \Delta\hat{\Psi}^{\dagger}(Z,\theta)\right)(Z,\theta)\, O\left(\Psi_0(Z,\theta)\right) \begin{pmatrix} \Delta\Psi(Z,\theta) \\ \Delta\hat{\Psi}(Z,\theta) \end{pmatrix} \tag{110}$$

[55] Recall that at the individual level, we use again the full interaction term $\tau\frac{K_X}{K}$.

with:

$$O\left(\Psi_0\left(Z,\theta\right)\right)=\left(\begin{array}{cc}\frac{\delta^2 S(\Psi)}{\delta\Psi^{\dagger}\delta\Psi} & \frac{\delta^2 S(\Psi)}{\delta\Psi^{\dagger}(Z,\theta)\delta\hat{\Psi}}\\ \frac{\delta^2 S(\Psi)}{\delta\hat{\Psi}^{\dagger}\delta\Psi} & \frac{\delta^2 S(\Psi)}{\delta\hat{\Psi}^{\dagger}\delta\hat{\Psi}}\end{array}\right)_{\substack{\Psi(Z,\theta)=\Psi_0(Z,\theta)\\ \hat{\Psi}(Z,\theta)=\hat{\Psi}_0(Z,\theta)}} \tag{111}$$

The anti-diagonal terms in equation (111) involve crossed derivatives with respect to both the fields of the real economy and the financial economy. These terms represent the interactions between the two economies. However, as explained in (Gosselin Lotz Wambst 2022), the cross-dependency between $\Psi\left(Z,\theta\right)$ and $\hat{\Psi}\left(Z,\theta\right)$ is relatively weak, since these interactions are taken into account by the background fields. In first approximation, the minimization of $S\left(\Psi\right)$ can be separated between S_1+S_2 and S_3+S_4. Therefore, we can write:

$$O\left(\Psi_0\left(Z,\theta\right)\right)\simeq\left(\begin{array}{cc}\frac{\delta^2(S_1+S_2)}{\delta\Psi^{\dagger}\delta\Psi} & 0\\ 0 & \frac{\delta^2(S_3(\Psi)+S_4(\Psi))}{\delta\hat{\Psi}^{\dagger}\delta\hat{\Psi}}\end{array}\right)_{\substack{\Psi(Z,\theta)=\Psi_0(Z,\theta)\\ \hat{\Psi}(Z,\theta)=\hat{\Psi}_0(Z,\theta)}} \tag{112}$$

The second-order expansion then becomes:

$$\begin{aligned}S\left(\Psi,\hat{\Psi}\right) &= S\left(\Psi_0,\hat{\Psi}_0\right)+\Delta S_2\left(\Psi,\hat{\Psi}\right)\\ &= S\left(\Psi_0,\hat{\Psi}_0\right)+\int\Delta\Psi^{\dagger}\left(K,X\right)\frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^{\dagger}\left(Z,\theta\right)\delta\Psi\left(Z,\theta\right)}\Delta\Psi\left(K,\theta\right)\\ &\quad+\int\Delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\frac{\delta^2\left(S_3\left(\Psi\right)+S_4\left(\Psi\right)\right)}{\delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\delta\hat{\Psi}\left(Z,\theta\right)}\Delta\hat{\Psi}\left(Z,\theta\right)\end{aligned} \tag{113}$$

Computing the second order derivatives involved in (113), and using the definition of the background fields (see appendix 6) leads to the formulas:

$$\begin{aligned}\frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^{\dagger}\left(Z,\theta\right)\delta\Psi\left(Z,\theta\right)} &= -\frac{\sigma_X^2}{2}\nabla_X^2-\frac{\sigma_K^2}{2}\nabla_K^2+\left(D\left(\left\|\Psi\right\|^2\right)+2\tau\frac{\left|\Psi\left(X\right)\right|^2\left(K_X-K\right)}{K}\right)\\ &\quad+\frac{1}{2\sigma_K^2}\left(K-\hat{F}_2\left(R\left(K,X\right)\right)K_X\right)^2+\frac{1-\nabla_K\hat{F}_2\left(R\left(K,X\right)\right)K_X}{2}\end{aligned}$$

$$\begin{aligned}&\frac{\delta^2\left(S_3\left(\Psi\right)+S_4\left(\Psi\right)\right)}{\delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\delta\hat{\Psi}\left(Z,\theta\right)}\\ &= \left(-\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2+\frac{\left(g\left(\hat{X}\right)\right)^2+\sigma_{\hat{X}}^2\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}}\right.\\ &\quad\left.-\frac{\sigma_{\hat{K}}^2}{2\sqrt{f^2\left(\hat{X}\right)}}\nabla_{\hat{K}}^2+\left(\frac{\sqrt{f^2\left(\hat{X}\right)}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2}{4\sigma_{\hat{K}}^2}\right)\right)\end{aligned}$$

10.1.2 Fourth-order corrections

Calculating the fourth-order corrections to the effective action is sufficient for deriving the main aspects of the interactions in a given background field. We show in appendix 7 that the third-order terms can be neglected, and that the series expansion of the action to the fourth-order writes:

$$\begin{aligned} S\left(\Psi,\hat{\Psi}\right) &= S\left(\Psi_0,\hat{\Psi}_0\right) \\ &+\int \Delta\Psi^{\dagger}(K,X)\frac{\delta^2(S_1+S_2)}{\delta\Psi^{\dagger}(Z,\theta)\,\delta\Psi(Z,\theta)}\Delta\Psi(K,\theta) \\ &+\int \Delta\hat{\Psi}^{\dagger}(Z,\theta)\frac{\delta^2(S_3(\Psi)+S_4(\Psi))}{\delta\hat{\Psi}^{\dagger}(Z,\theta)\,\delta\hat{\Psi}(Z,\theta)}\Delta\hat{\Psi}(Z,\theta)+\Delta S_4\left(\Psi,\hat{\Psi}\right) \end{aligned} \tag{114}$$

with:

$$\begin{aligned} &\Delta S_4\left(\Psi,\hat{\Psi}\right) \\ \simeq\ & 2\tau\int\left|\Delta\Psi(K',X)\right|^2 dK'\left|\Delta\Psi(K,X)\right|^2 dKdX \\ &-\Delta\Psi^{\dagger}(K,X)\,\Delta\Psi^{\dagger}(K',X')\,\nabla_K\frac{\delta^2 u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\,\delta\Psi^{\dagger}(K',X)}\Delta\Psi(K',X')\,\Delta\Psi(K,X) \\ &-\Delta\Psi^{\dagger}(K,\theta)\,\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\theta\right)\nabla_K\frac{\delta^2 u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}\Delta\hat{\Psi}\left(\hat{K},\theta\right)\Delta\Psi(K,\theta) \\ &-\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\Delta\Psi^{\dagger}(K',\theta)\left\{\nabla_{\hat{K}}\frac{\hat{K}\delta^2 f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\,\delta\Psi^{\dagger}(K',X)}+\nabla_{\hat{X}}\frac{\delta^2 g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\,\delta\Psi^{\dagger}(K',X)}\right\}\Delta\Psi(K',X')\,\Delta\hat{\Psi}\left(\hat{K},\hat{X}\right) \end{aligned} \tag{115}$$

Computing the terms involved in (115) (see appendix 7) allows us to interpret the various terms arising in the correction to the action.

The first term in the right-hand side of (115) describes the direct repulsive interaction between firms due to competition in a given sector.

The second term describes the indirect competition between firms through capital allocation by investors, since:

$$\frac{\delta^2 u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\,\delta\Psi^{\dagger}(K',X)}=-\frac{1}{\varepsilon}\hat{F}_2(s,R(K,X))\,\hat{F}_2(s,R(K',X'))\,\hat{K}_X \tag{116}$$

and this term involves the relative attractiveness of two firms with capital K and K' respectively in sector X.

The third term represents the firms-investors direct interactions through investment, since:

$$\frac{\delta^2 u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}=\frac{1}{\varepsilon}\hat{F}_2(s,R(K,X))\,\hat{K} \tag{117}$$

is the relative attractiveness of a firm with capital K' at sector X.

The last term describes the variation of investement due to the relative short-term and long-term return of a given firm. Specifically, we have:

$$\frac{\delta^2 f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\,\delta\Psi^{\dagger}(K',X)}\simeq\frac{1}{\varepsilon}\left(\Delta f\left(K',\hat{X},\Psi,\hat{\Psi}\right)-\gamma\frac{K'}{K_X}\right) \tag{118}$$

and:

$$\frac{\delta^2 g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\,\delta\Psi^{\dagger}(K',X)}=\frac{1}{\int\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\Delta\left(g\left(K',\hat{X},\Psi,\hat{\Psi}\right)\right) \tag{119}$$

with:

$$\Delta f\left(K',\hat{X},\Psi,\hat{\Psi}\right)=f\left(K',\hat{X},\Psi,\hat{\Psi}\right)-f\left(\hat{X},\Psi,\hat{\Psi}\right)$$

and:

$$\Delta g\left(K',\hat{X},\Psi,\hat{\Psi}\right)=g\left(K',\hat{X},\Psi,\hat{\Psi}\right)-g\left(\hat{X},\Psi,\hat{\Psi}\right)$$

are the relative short-term return and long-term return for firm with capital K' at sector $\hat{X}$ respectively.

10.2 One agent transition functions

Following section 5.3.1, we consider first the "free" transition functions that are given by the inverse operator of:

$$\left(O\left(\Psi_0\left(Z,\theta\right)\right)+\alpha\right)^{-1} \tag{120}$$

Given (112), the inverse (120) reduces to:

$$\begin{pmatrix} \left(\frac{\delta^2(S_1+S_2)}{\delta\Psi^\dagger\delta\Psi}+\alpha\right)^{-1} & 0 \\ 0 & \left(\frac{\delta^2(S_3(\Psi)+S_4(\Psi))}{\delta\hat{\Psi}^\dagger\delta\hat{\Psi}}+\alpha\right)^{-1} \end{pmatrix}_{\substack{\Psi(Z,\theta)=\Psi_0(Z,\theta)\\ \hat{\Psi}(Z,\theta)=\hat{\Psi}_0(Z,\theta)}}$$

This implies that the transition functions can be computed independently for the individual firms and investors. We will write:

$$G_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),\alpha\right)$$

the transition probability for a firm between an initial state (X_i,K_i) and a final state (K_f,X_f) during an average timespan α^{-1} and:

$$G_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right),\alpha\right)$$

the transition probability for a firm between an initial state $\left(\hat{X}_i,\hat{K}_i\right)$ and a final state $\left(\hat{K}_f,\hat{X}_f\right)$ average timespan α^{-1}. Appendix 8 computes these transition functions. We find the following results.

One firm transition function

$$\begin{aligned} &G_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right) \\ =\;&\exp\left(D\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)-\alpha_{eff}\left(\Psi,\left(K_f,X_f\right),\left(X_i,K_i\right)\right)\sqrt{\frac{\left(X_f-X_i\right)^2}{2\sigma_X^2}+\frac{\left(K'_f-K'_i\right)^2}{2\sigma_K^2}}\right) \end{aligned} \tag{121}$$

where:

$$D\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)=D_1+D_2+D_3 \tag{122}$$

with:

$$D_1=\int_{X_i}^{X_f}\frac{\nabla_X R\left(K_X,X\right)}{\sigma_X^2}H\left(K_X\right) \tag{123}$$

$$D_2=-\int_{K_i}^{K_f}\left(K-\hat{F}_2\left(s,R\left(K,\bar{X}\right)\right)K_{\bar{X}}\right)dK \tag{124}$$

$$D_3 = \int_{K_i}^{K_f} \left(\left(\frac{X_f - X_i}{2} \right) \nabla_X \hat{F}_2 \left(s, R \left(K, \bar{X} \right) \right) K_{\bar{X}} \right) dK \tag{125}$$

$$\begin{aligned} & \alpha_{eff} \left(\Psi, \left(K_f, X_f \right), \left(X_i, K_i \right) \right) \qquad (126) \\ = \quad & \alpha + D \left(\| \Psi \|^2 \right) + \tau \left(\frac{\left| \Psi \left(X_f \right) \right|^2 \left(K_{X_f} - K_f \right)}{K_f} - \frac{\left| \Psi \left(X_i \right) \right|^2 \left(K_{X_i} - K_i \right)}{K_i} \right) + \frac{\sigma_K^2}{2} K_f' K_i' \end{aligned}$$

and:

$$\begin{aligned} K_i' &= K_i - \hat{F}_2 \left(s, R \left(K_{X_i}, X_i \right) \right) K_{X_i} \\ K_f' &= K_f - \hat{F}_2 \left(s, R \left(K_{X_f}, X_f \right) \right) K_{X_f} \end{aligned}$$

One investor transition function

$$\begin{aligned} & G_2 \left(\left(\hat{K}_f, \hat{X}_f \right), \left(\hat{X}_i, \hat{K}_i \right) \right) \qquad (127) \\ = \quad & \exp \left(D' \left(\left(\hat{K}_f, \hat{X}_f \right), \left(\hat{X}_i, \hat{K}_i \right) \right) \right) \\ & \times \exp \left(-\alpha_{eff}' \left(\left(\hat{K}_f, \hat{X}_f \right), \left(\hat{X}_i, \hat{K}_i \right) \right) \left| \left(\hat{K}_f + \frac{\sigma_{\hat{K}}^2 F \left(\hat{X}_f, K_{\hat{X}_f} \right)}{f^2 \left(\hat{X}_f \right)} \right) - \left(\hat{K}_i + \frac{\sigma_{\hat{K}}^2 F \left(\hat{X}_i, K_{\hat{X}_i} \right)}{f^2 \left(\hat{X}_i \right)} \right) \right| \right) \end{aligned}$$

with:

$$D' \left(\left(\hat{K}_f, \hat{X}_f \right), \left(\hat{X}_i, \hat{K}_i \right) \right) = \frac{1}{\sigma_{\hat{X}}^2} \int_{\hat{X}_i}^{\hat{X}_f} g \left(\hat{X} \right) d\hat{X} + \frac{\hat{K}_f^2}{\sigma_{\hat{K}}^2} f \left(\hat{X}_f \right) - \frac{\hat{K}_i^2}{\sigma_{\hat{K}}^2} f \left(\hat{X}_i \right) \tag{128}$$

and:

$$\begin{aligned} & \alpha_{eff}' \left(\left(\hat{K}_f, \hat{X}_f \right), \left(\hat{X}_i, \hat{K}_i \right) \right) \qquad (129) \\ = \quad & \left(\alpha + \frac{\sigma_{\hat{X}}^2}{2} \left(\hat{K}_f + \frac{\sigma_{\hat{K}}^2 F \left(\hat{X}_f, K_{\hat{X}_f} \right)}{f^2 \left(\hat{X}_f \right)} \right) \left(\hat{K}_i + \frac{\sigma_{\hat{K}}^2 F \left(\hat{X}_i, K_{\hat{X}_i} \right)}{f^2 \left(\hat{X}_i \right)} \right) \right) \sqrt{\frac{\left| f \left(\frac{\hat{X}_f + \hat{X}_i}{2} \right) \right|}{2\sigma_{\hat{X}}^2}} + g^{(R)} \left(\hat{X} \right) \end{aligned}$$

with:

$$g^{(R)} \left(\hat{X} \right) = \int_{\hat{X}_i}^{\hat{X}_f} \frac{\left(g \left(\hat{X} \right) \right)^2 + \sigma_{\hat{X}}^2 \left(f \left(\hat{X} \right) + \nabla_{\hat{X}} g \left(\hat{X}, K_{\hat{X}} \right) - \frac{\sigma_{\hat{K}}^2 F^2 \left(\hat{X}, K_{\hat{X}} \right)}{2 f^2 \left(\hat{X} \right)} \right)}{\left\| \hat{X}_f - \hat{X}_i \right\| \sigma_{\hat{X}}^2 \sqrt{f^2 \left(\hat{X} \right)}}$$

10.3 Two agents transition functions and Interactions between agents

To study the agents interactions within the background field we consider the two-agent transition functions. There are three of them. One for two firms:

$$G_{11} \left(\left[\left(K_f, X_f \right), \left(K_f, X_f \right)' \right], \left[\left(X_i, K_i \right), \left(X_i, K_i \right)' \right] \right)$$

one for one firm and one investor:

$$G_{12} \left(\left[\left(K_f, X_f \right), \left(\hat{K}_f, \hat{X}_f \right) \right], \left[\left(X_i, K_i \right), \left(\hat{X}_i, \hat{K}_i \right) \right] \right)$$

and one for for two investors:

$$G_{22}\left(\left[\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{K}_f,\hat{X}_f\right)'\right],\left[\left(\hat{X}_i,\hat{K}_i\right),\left(\hat{X}_i,\hat{K}_i\right)'\right]\right)$$

If we neglect the terms of order greater than 2 in the effective action, the transition functions reduce to simple products:

$$\begin{aligned}&G_{11}\left(\left[(K_f,X_f),(K_f,X_f)'\right],\left[(X_i,K_i),(X_i,K_i)'\right]\right)\\=\;&G_1\left((K_f,X_f),(X_i,K_i)\right)G_1\left((K_f,X_f)',(X_i,K_i)'\right)\end{aligned}$$

$$\begin{aligned}&G_{12}\left(\left[(K_f,X_f),\left(\hat{K}_f,\hat{X}_f\right)\right],\left[(X_i,K_i),\left(\hat{X}_i,\hat{K}_i\right)\right]\right)\\=\;&G_1\left((K_f,X_f),(X_i,K_i)\right)G_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)\end{aligned}$$

$$\begin{aligned}&G_{22}\left(\left[\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{K}_f,\hat{X}_f\right)'\right],\left[\left(\hat{X}_i,\hat{K}_i\right),\left(\hat{X}_i,\hat{K}_i\right)'\right]\right)\\=\;&G_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)G_2\left(\left(\hat{K}_f,\hat{X}_f\right)',\left(\hat{X}_i,\hat{K}_i\right)'\right)\end{aligned}$$

In first approximation, agents behave independently, solely influenced by the given background state.

To take into account agents interactions we write the expansion:

$$\exp\left(-S\left(\Psi\right)\right)=\exp\left(-\left(S\left(\Psi_0,\hat{\Psi}_0\right)+\Delta S_2\left(\Psi,\hat{\Psi}\right)\right)\right)\left(1+\sum_{n\geqslant 1}\frac{\left(-\Delta S_4\left(\Psi,\hat{\Psi}\right)\right)^n}{n!}\right)$$

as explained in section 5.3.2, the series produces corrective terms to the transition functions. Appendix 9 presents the computations and compute the transitions in the approximations of paths that cross each other one time at some X. In this approximation, we find the following formula:

10.3.1 Firm-firm transition function:

$$\begin{aligned}&G_{11}\left(\left[(K_f,X_f),(K_f,X_f)'\right],\left[(X_i,K_i),(X_i,K_i)'\right]\right)\\\simeq\;&G_1\left((K_f,X_f),(X_i,K_i)\right)G_1\left((K_f,X_f)',(X_i,K_i)'\right)\\&-\left(2\tau-\nabla_K\frac{\delta^2u\left(\bar{K},\bar{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(\bar{K}',\bar{X}\right)\delta\Psi^{\dagger}\left(K',\bar{X}\right)}\right)\hat{G}_1\left((K_f,X_f),(X_i,K_i)\right)\hat{G}_1\left((K_f,X_f)',(X_i,K_i)'\right)\end{aligned}\tag{130}$$

10.3.2 Firm-investor transition function:

$$\begin{aligned}&G_{12}\left(\left[(K_f,X_f),\left(\hat{K}_f,\hat{X}_f\right)'\right],\left[(X_i,K_i),\left(\hat{X},\hat{K}_i\right)'\right]\right)\\\simeq\;&G_1\left((K_f,X_f),(X_i,K_i)\right)G_2\left(\left(\hat{K}_f,\hat{X}_f\right)',\left(\hat{X},\hat{K}_i\right)'\right)\\&+\left(\nabla_K\frac{\delta^2u\left(\bar{K},\bar{X},\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{\bar{K}},\hat{\bar{X}}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{\bar{K}},\hat{\bar{X}}\right)}+\nabla_{\hat{K}}\frac{\hat{K}\delta^2f\left(\overline{\hat{X}},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(\overline{K'},\overline{X}\right)\delta\Psi^{\dagger}\left(\overline{K'},\overline{X}\right)}+\nabla_{\hat{X}}\frac{\delta^2g\left(\overline{\hat{X}},\Psi,\hat{\Psi}\right)}{\delta\Psi\overline{\left(K',X\right)}\delta\Psi^{\dagger}\left(\overline{K'},\overline{X}\right)}\right)\\&\times\hat{G}_1\left((K_f,X_f),(X_i,K_i)\right)\hat{G}_2\left(\left(\hat{K}_f,\hat{X}_f\right)',\left(\hat{X},\hat{K}_i\right)'\right)\end{aligned}\tag{131}$$

10.3.3 Investor-investor transition function:

$$\begin{aligned} & G_{22}\left(\left[\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{K}_f,\hat{X}_f\right)'\right],\left[\left(\hat{X},\hat{K}_i\right),\left(\hat{X},\hat{K}_i\right)'\right]\right) \\ \simeq\ & G_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X},\hat{K}_i\right)\right)G_2\left(\left(\hat{K}_f,\hat{X}_f\right)',\left(\hat{X},\hat{K}_i\right)'\right) \end{aligned} \tag{132}$$

with:

$$\begin{aligned} \left(\bar{X},\bar{K}\right) &= \frac{\left(K_f,X_f\right)+\left(X_i,K_i\right)}{2} \\ \left(\bar{X},\bar{K}\right)' &= \frac{\left(K_f,X_f\right)'+\left(X_i,K_i\right)'}{2} \end{aligned}$$

The derivatives are given in (116), (117), (118), (119) and:

$$\hat{G}_i\left(\left(K_f,X_f\right),\left(X,K\right)\right)\hat{G}_j\left(\left(K_f,X_f\right)',\left(X,K\right)'\right)$$

is the transition function computed on paths that cross once.

11 Results and interpretations

11.1 One-agent transition functions

We present a synthesis of the results for firms and investors transition functions. Some technical details are given in appendix 10.

11.1.1 Firms transition function

For a given background state, the probability of transition for a firm between two states K_i, X_i and K_f, X_f, over an average time of $1/\alpha$, is given by G_1 (see 121). This formula computes the probability that a firm initialy endowed with a capital K_i in sector X_i will relocate to sector X_f with capital K_f. The transition probability is the result of competing effects, as it is composed of several interdependent terms of similar magnitude. Firm transitions occur over the medium to long term but at a slower time scale than transitions for investors. Firms remain in each transitory sector long enough to resettle, and for investors to adjust the capital allocated between firms. Thus, in each transitory sector, firm capital evolves depending on the characteristics of the firm, the sector, and investors expectations.

Attractiveness and sectors shifts The drift term D in formula (122) is the average transition of a firm between its initial and final points (X_i, K_i) and (K_f, X_f), respectively. This term is usually different from zero because firms tend to shift sectors, and their capital evolves. This tendency for a firm to evolve depends both on the transitory sectors and the background field, i.e., the entire landscape in which the transition occurs. In addition, fluctuations around the drift term can alter a firm's trajectory, contributing to the probabilistic nature of the transition.

The drift term of equation (122) is composed of three interacting contributions, D_1, D_2 and D_3.

The first component D_1 shows that firms tend to relocate to sectors with higher long-term returns, shifts which in turn modify their present and future attractiveness to investors.

The second component D_2 shows that the shift alters the capital of the firm. Specifically, the amount of investment that investors are willing to make in the firm, $\hat{F}_2\left(R\left(K,\bar{X}\right)\right)K_{\bar{X}}$ depends on three key parameters: the average capital of the new sector, $K_{\bar{X}}$, the absolute average return on capital in the sector, $R\left(K,\bar{X}\right)$, and the propensity of investors, $\hat{F}_2$ to invest in the firm based on its given capital compared to the average capital of firms in the sector.

When a firm begins the process of relocating to a nearby sector, its capitalization may differ from that of firms already present in that sector, which in turn affects its attractiveness to investors, represented by $\hat{F}_2$. The shape of $\hat{F}_2$ reflects the propensity of investors to invest in the firm. When $\hat{F}_2$ is concave, this propensity marginally decreases, while a convex shape results in a marginal increase.

The equilibrium capital of the firm in the new sector is $\hat{F}_2\left(s,R\left(K,\bar{X}\right)\right)K_{\bar{X}}$. When a firm relocates, its capital may turn out to be below or above this equilibrium level. For each of these cases, two possibilities arise depending on the shape s of $\hat{F}_2$.

If $\hat{F}_2$ is concave, the marginal propensity of investors to invest is decreasing: once the firm has entered the sector, its capital will converge towards the sector's average capital. It will either increase or decrease, depending on whether its initial level of capital is above or below the equilibrium capital, respectively. If $\hat{F}_2\left(s,R\left(K,\bar{X}\right)\right)$ is convex, the marginal propensity of investors to invest is increasing: the dynamics of its capital accumulation is unstable. Investors will tend to over or underinvest in the firm.

The third contribution D_3 reflects the firm's relative attractiveness in different transition sectors. If the the firm's relative attractiveness is reduced during the shift, such that it attracts less capital than the average capital of the transition sector, it may become stuck in an intermediate sector.

Impact of competition The coefficient α_{eff} defined in equation (126) represents the inverse of the average mobility of a firm. This mobility depends on the competition in transitional sectors which is captured by the two first terms on the right-hand side of (126).

The first term, $D\left(\|\Psi\|^2\right)$ is a constant that characterizes the background state of the firms and is correlated with the total number of firms in the space of sectors. As competition increases, α_{eff} rises and firms' mobility decreases.

The second term measures the local competition that firms face as they move through the sector space. It is determined by the density of agents in the sector, multiplied by the variation, along the path, of the firm's excess capital with respect to the average capital of the sector. A well-capitalized firm facing many less-capitalized competitors will repel them and create its own market share. Relocation will occur towards sectors that are denser and less capitalized. Under-capitalized firms will be forced out of their sectors and into denser, less capitalized sectors. The relocation process may result in a capital gain or loss. However, holding capital constant, initially under-capitalized firms will tend to move towards sectors with lower average capital, whereas over-capitalized firms tend to move towards sectors with higher average capital.

Stabilization terms: The square-rooted term multiplying α_{eff} is written:

$$\sqrt{\frac{\left(X_f-X_i\right)^2}{2\sigma_X^2}+\frac{\left(\tilde{K}_f-\tilde{K}_i\right)^2}{2\sigma_K^2}} \tag{133}$$

and the last term in the right-hand side of equation (122) both describe random oscillations around a path of zero marginal capital demand. Changes in equity, investments, for instance, may modify (133). and the oscillations are of magnitude $\frac{\sigma_K^2}{2}$. However, these oscillations do not necessarily imply a return to the initial point. The larger the deviation from the average, the more likely firms are to deviate from the average, and possibly shift to a new trajectory. Therefore, a capital increase above the average may induce a shift in sector, which in turn may modify the firm's accumulation and prospects. Thus, oscillations do not prevent trends and may even initiate them. However, such "random shifts" may prove disadvantageous as they could harm the firm's position and reduce its capital compared to the sector.

Possible paths Overall, what are the possible dynamics for a firm in terms of capital and sector? If a firm experiences capital growth in a sector where the investor propensity, F_2, is concave, the accumulation of its capital could cause the firm to shift to a higher-return sector, but this may result in the firm being perceived as less attractive by investors in this new sector.

This shift can lead to a change in the firm's attractiveness to investors, F_2. The growth or decline of the firm in the new sector will depend on both its capital level and the shape of F_2. These factors will also determine the speed of this change. If the firm's capital level gradually declines, it may have time to react and reposition itself. However, if the decline in capital is sudden, the firm may not have enough resources to reposition itself. The new sector may turn out to be a capital trap.

The patterns of possible trajectories are various and may be irregular, with some transitions occurring at a constant rate, while others may involve discontinuities and sudden increases or reductions in capital, depending on the characteristics of the landscape such as expected returns in sectors, density of firms, and other background factors.

11.1.2 Investors transition functions

Drift term Short-term and long-term returns are the two parameters that determine investors' capital allocation. Short-term returns include the firm's dividends and increase with the value of its shares, while long-term returns reflect the market's expectations for the firm's future growth potential, which in turn affect expectations for higher dividends and share price appreciation. Both types of returns are captured in the drift term D', which is defined in equation (128). The most likely paths are those that maximize both short-term and long-term returns.

However, these returns are not independent, since faltering share prices in the short-term impact long-term returns expectations, and vice versa.

Ideally, to maximize their capital, investors seek both higher short-term and long-term returns. Therefore, capital allocation within and across sectors will depend on firms share prices volatility and dividends.

A sector in which share prices increase tends to attract capital, since investors can maximize both short-term and long-term returns: an increase in share prices sustains the firm's growth expectations. Investors tend to move towards the next local maximum of long-term returns while also maximizing their short-term return. In this case, there is no trade-off between the two objectives.

In a sector where stock prices fall or remain stagnant, investors are faced with a trade-off between short- and long-term returns. When stock prices no longer support long-term earnings expectations, capital allocation is determined by short-term dividends. Capital reallocation will depend on the level of capital held by investors. While investors may consider long-term expectationsthey must also generate short-term returns to maintain their capital. An investor who ignores dividends in a context of falling share prices would eventually see his capital depleted, which could hinder or impair his ability to reallocate capital in the long term.

Stabilization terms: Similarly to firms, investors have an effective inverse mobility α'_{eff}, defined in equation (129). This formula shows that mobility $\frac{1}{\alpha'_{eff}}$ decreases with the average short-term return along the path : the higher the returns, the lower the incentive to switch from one sector to another. Similarly, mobility increases with $g^{(R)}\left(\hat{X}\right)$, which measures the relative long-term return of the sectors along the path. The higher this relative return, the lower the incentive to switch to another sector.

Moreover, $\frac{1}{\alpha'_{eff}}$ decreases with the final level of capital $\hat{K}_f$ increases, impairing the firm's capacity to reach high levels of capital. Conversely, $\frac{1}{\alpha'_{eff}}$ decreases with the initial capital $\hat{K}_i$ decreases, indicating that investors with high capitalization are less likely to experience significant capital losses. This is supported by the factor multiplying α'_{eff}:

$$\left|\left(\hat{K}_f+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X}_f,K_{\hat{X}_f}\right)}{f^2\left(\hat{X}_f\right)}\right)-\left(\hat{K}_i+\frac{\sigma_{\hat{K}}^2 F\left(\hat{X}_i,K_{\hat{X}_i}\right)}{f^2\left(\hat{X}_i\right)}\right)\right|$$

As a result, the probability for an investor to deviate significantly from its initial capital value, apart from the smoothing term which can be neglected, is relatively low.

11.2 Two-agent transition functions

First, it should be noted that the transition function G_{22}, as defined in equation (132), does not include any interaction corrections. Specifically, the transition probability for two investors is simply the product of their individual transition probabilities. In our model, investors do not directly interact with each other, but only through their investments in various firms. Only two transition functions are affected by these indirect interactions.

11.2.1 Firm-firm interactions

First the transition G_{11} is modified by the term:

$$I=2\tau-\nabla_K\frac{\delta^2 u\left(\bar{K},\bar{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(\bar{K}',\bar{X}\right)\delta\Psi^{\dagger}\left(K',\bar{X}\right)}$$

The interaction I measures the interactions between two firms in the same sector. The first contribution to I describes a direct competition between firms in a given sector, whereas the second term describes the competition of the firms to attract investors that share their investments between the two firms. Given that the 2-agents transition functions are modified by (see (130)):

$$\left(2\tau-\nabla_K\frac{\delta^2 u\left(\bar{K},\bar{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(\bar{K}',\bar{X}\right)\delta\Psi^{\dagger}\left(K',\bar{X}\right)}\right)\hat{G}_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)\hat{G}_1\left(\left(K_f,X_f\right)',\left(X_i,K_i\right)'\right)$$

and since $I>0$, the contribution to the green function of paths crossing at some point are underweighted. The competition between the two firms repell them from the sector where they interact. If we consider that the competion factor τ is capital-dependent (see (**??**)), the less capitalized firm is relatively more repelled than the more capitalized one.

11.2.2 Firm-investor interactions

Second, the firm-investor transition function G_{12} is modified by the term:

$$\left(\nabla_K\frac{\delta^2 u\left(\bar{K},\bar{X},\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}+\left\{\nabla_{\hat{K}}\frac{\hat{K}\delta^2 f\left(\overline{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(\overline{K'},\overline{X}\right)\delta\Psi^{\dagger}\left(\overline{K'},\overline{X}\right)}+\nabla_{\hat{X}}\frac{\delta^2 g\left(\overline{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\overline{(K',X)}\delta\Psi^{\dagger}\left(\overline{K',X}\right)}\right\}\right)$$

Given (117), (**??**), (119), this term depends mainly on three contributions:

$$\nabla_K \frac{F_2(s, R(K', X)) \hat{K}}{\int F_2(s, R(K', X)) \|\Psi(K', X)\|^2 dK'}$$

$$\nabla_{\hat{K}} \Delta f\left(K', \hat{X}, \Psi, \hat{\Psi}\right)$$

$$\nabla_{\hat{X}} \Delta\left(g\left(K', \hat{X}, \Psi, \hat{\Psi}\right)\right)$$

each of this contribution describes the relative perspectives of the firm in his path through the sectors.

The first one represents the gradient of firm"s attractiveness with respect to capital. The investor willl decide to invest or not depending on the marginal gain of long term returns of the firm.

The second term represents the marginal short-term return of an investment of the firm, and the third one measures the reltive attractiveness of the firm with respect to his neighbours (see Gosselin Lotz Wambst 2022).

The interaction between the firm and the investor is a combination of these three quantities.

When the combination of these term is positive, the firm has positive perspective either in terms of short or long term returns, or relatively to his neighbors.

In this case the associated corrections to the path crossing at some points is positive and this paths will be overweighted: in probability, this translates by the fact that paths in which a firm presents above average perspectives in his capital accumaltion and shift in sectors, will be favoured by an increase in investment. The firm will take advantage from its interaction with the investor, except if this one experiences, for any reason, an decrease of capital. On the contrary, a firm perceived as moving toward lower perspective will experience in average a decrease in investment. This decrease will be dampened if its investor has itself low capital to invest. Some mixed situation may arise: good short term perspective, but uncertain long term expectations may cancel or compensate each other.

Discussion of the main outcomes

Field formalism presents an alternate point of view about the economic reality that surround us. In such a formalism, representative agents do not exist, only collective states do. They emerge from the interactions of a large number of agents, and condition the behaviours and the economic activity. In this context, agents only randomly carry out the possible trajectories authorized by the system.

Collective states can be multiple and present transitions. The economic dynamic is not limited to fluctuations around an average trajectory which would be a dynamic equilibrium, but rather by dynamic transitions between collective states, which completely condition the fluctuations. apparent individual dynamics. The collective states dynamics depend on the form of short-term and long-term return functions, that are exogenous, and more broadly on a whole landscape of technological and economic conditions. But as a system, they have their own internal dynamics: the system is not inert. We have considered these two types of variations in the paper.

First, collective states are sensitive to structural changes. Any such change in expectations, economic and/or monetary conditions may alter expected returns and in turn impact the collective state. Unstable type-3 sectors are particularly sensitive to these changes in long-term growth, inflation, and interest rates. Higher expectations in these sectors attract investment which in turn increase expectations. This seemingly endless expected growth spirals until the outlook flattens or deteriorates. This would be the growth model of a company whose ever-broadening range of products fuels higher expected long-term returns and stock price increases. Type-1 and -2 sectors attract capital through dividends and, although only partially and for high capital type-2 sectors, expected returns. Under higher expectations, these sectors are relatively less attractive than the nearby type-3 sector. They may nonetheless survive in the long-term provided their short-term returns and dividends are high enough. This may be done by cutting costs or investment, at the expense of future growth. Moreover, advert signaling may emerge: an increase in dividends can be interpreted as faltering growth prospects. Conversely, any increase in long-term uncertainty impact expected returns and drive sector-3 capital towards other patterns. External shocks, inflation, and monetary policy impact expectations, reduce long-term investment and either drive capital out of sectors 3 to sector 1 or 2 or favour other pattern-3 sectors.

Second, any deviation of capital from its equilibrium value may initiate oscillations in the collective state of the entire system. A temporary deviation will induce an unstable redistribution of capital, growth expectations and returns, and generate intersectoral capital reallocation and global oscillations that can either dampen or drive the system toward a new collective state. There are thus potential transitions between collective states, which occur at a slower, larger timescale than that of market fluctuations. In the long run, once the transition has occured, both sectors' averages and patterns may have changed: pattern 2 may morph in, say, pattern 3 stable or unstable, or sectors may simply disappear. Concretely, any significant modification in average capital in a sector could induce oscillations and initiate a transition.

Moreover, once endogenous expectations are introduced, they react to variations in the capital: collective states of mixed 1-2-3 patterns are difficult to maintain. Highly reactive expectations favour pattern 3: expected returns magnify capital accumulation at the expense of other patterns. Mildly reactive expectations favour patterns 1 and 2: their oscillations, which are actually induced by uncertainties, dampen. Type-3 sectors on the contrary experience strong fluctuations in capital : attracting capital is less effective with fading expectations. The threshold in capital accumulation shifts upwards and the least-profitable firms are ousted from the sector. The recent evolution in performances between value and growth investment strategies exemplifies these shifts in investors' sentiment between expected growth and real returns. In periods of uncertainties, fluctuations affect capital accumulation in growth sectors and today's tech companies, and strengthen more dividend-driven investments. Note however that the most profitable and best-capitalized firms, that remain above the threshold, maintain relatively high levels of capital. Here our versatile notion of firms[56] proves convenient: any firm that accumulates enough capital to be able to buy back, in periods of volatility, its own stocks is actually acting as an autonomous investor. When volatility is high, the most likely investors for the best-capitalized firms are, first and foremost, the best-capitalized companies themselves. They react, so to speak, as pools of closely held investors. In other words, provided firms have high enough capital, they can always cushion the impact of price fluctuations and adverse shocks through buybacks. Similarly, they also could choose to acquire companies in their sector or in neighbouring sectors.

Fluctuations in financial expectations impose their pace on the real economy. Expected returns are both exogenous and endogenous. Being exogenous, they may change quickly. Expected returns theoretically reflect long-term perspectives, but actually rely on short-term sentiments: any incoming information, change in the global economic outlook or adverse shock will modify long-term

[56] We modeled a single company as a set of independent firms. Similarly, the notion of sector merely refers to a group of entities with similar activities.

expectations and shift capital from sector to sector. But expected returns are also endogenous. Being expectations, they react to changes within the system. When high levels of capital seek to maximize returns, expectations react strongly to capital changes. Expectations both highly sensitive to exogenous conditions and highly reactive to variations in capital induce large fluctuations of capital in the system. Creating or inflating expectations may attract capital, at times unduly. When expectations can no longer be worked on, the sole remaining tool to reduce capital outflows is a high dividend policy, which may be done at the expense of the labour force, capital expenditures and future growth.

At the level of individual dynamics, macro fluctuations condition agents transitions. In fact, the field formalism encompasses both macro and microeconomic elements: the macro scale keeps track of the entire set of agents and, in turn, influences the microeconomic scale, allowing for two-level interpretations. We showed that individual dynamics heavily rely on underlying macroeconomic parameters such as average capital and the number of firms per sector. Consequently, our model also describes the microeconomic impact of the present macroeconomic states.

In the face of macroeconomic fluctuations, investors may experience capital losses. However they can always shield their capital by reallocating it to more profitable or stable sectors. In doing so, they may amplify global capital fluctuations for firms, which are unable to react at the same pace. Financial risk is therefore limited in our model. Investors can always reposition themselves and, as a result, do not bear the same risk as firms that move to attract investors. The primary burden of risk falls on the firms themselves, not the financial sector. Our model demonstrates that investors do not experience the eviction phenomenon that firms do. However, investors may face eviction from certain investment sectors if their capital no longer allows them to invest in sectors perceived as the most promising, based on returns and share prices.

We posit that firms have a natural inclination to switch sectors. Indeed, firms tend to change due to the continuous evolution and transformation of sectors and the changing economic environment. In our model, the historical development of a sector is not depicted by a specific variable, but rather by firms shifts between closely related sectors. In the shift, the initial sector is the past state of the sector, and the final sector its present state. Thus, firms transitions captures both firm reorientations and their adaptation to an evolving environment.

Attracting investors is crucial to firms and can be achieved through continuous expansion. However, firms face higher uncertainty and risk than investors. Specifically, firms face two distinct risks:

First, the individual risk associated with seeking higher returns. Switching to more attractive sectors may expose firms to higher competition and faltering investors sentiment. For example, a firm shifting to a high-capitalized sector will experience a stronger competition and weaker prospects, potentially deterring any present or additional investment. When these two phenomena combine, they may induce a substantial loss of capital, and trap the firm in the sector, evict it towards less-capitalized and less-attractive ones, and impact its ability to position itself for future sectoral changes and transformations.

Second, the global risk, caused by exogenous and macro fluctuations. This risk can alter sectoral growth prospects and, consequently, affect individual dynamics. Our model captures these potential instabilities at the individual level. Within a sector, sub-sectors may emerge, some presenting more promising opportunities than others. However, the entire sector can be impacted. Even though, on average, the collective state may exhibit some stability, fluctuations among a set of similar firms can be substantial at the individual dynamics level. Consequently, fluctuations in this context magnify the uncertainty at the individual level, making it difficult to identify and capitalize on profitable shifts while also increasing the risk of making detrimental moves. To sum up, both collective and individual results suggest that firms with high initial capitalization are generally less exposed to market fluctuations. Note incidentally that these risks may be amplified by swift financial reallocation in the face of global uncertainties.

Therefore, firms can undergo sharp changes in dynamics due to variations in the landscape of expected returns, reactivity of expectations, relative attractiveness compared to neighboring sectors, or the number of competing firms.

The present paper also advocates that field formalism, in addition to mixing macro and micro analysis, provides some precise insights about the structures of interactions inside the macroeconomic state. The technique of series expansion of the effective action induces emerging interactions that are not detected in the classical formalism, such as indirect emerging competition among agents. More precisely, interactions between firms within a sector reveal phenomena of specialization and eviction. Competition is at first determined by the firms relative levels of capital. This is the direct form of competition. The firm with the highest capital is more likely to evict its competitors. However, field formalism reveals that competition also revolves around attracting investor capital. This is the indirect form of competition. A firm that successfully differentiates itself within a sector, through specialization, has the potential to attract capital and mitigate or reverse a possible eviction. However, specialization makes the firm dependent on its investors. If investors suffer capital losses, the firm is directly impacted.

Interactions between firms and their investors detail the impact of investment at the individual level. A firm that attracts more investors will be better positioned in the sector, as it enjoys a stronger position, whereas its competitors will be compelled to reorient themselves. To attract investors, a firm needs to demonstrate a high growth potential, which may favor new entrants in a sector, provided they have the necessary capital to position themselves, or better growth prospects. Note that from this perspective the concept of comparative advantage is not relevant in our model. Indeed, given that changes are inevitable within sectors, any comparative advantage is bound to be swept away, potentially even by relatively distant and unexpected causes. Actually, exogenous fluctuations, such as the perception of the sector and the firm within it (by investors), as well as competition among firms to retain their position and attract investors, create inherent instability within a specific sector. Specializing in a single sector exposes firms to the risk of eventual eviction, forming a trap.

12 Conclusion

This paper has shown how a statistical field model could be constructed from a simple microeconomic model. Using a simple economic framework involving two types of agents, firms and investors, we have studied the impact financial capital could have on physical capital allocation. and shown the complexity of the collective states reached in this very simple case. We have examined how variations in external parameters could induce transitions in these collective states.

At the individual level, we derived the probabilistic dynamics of agents in this environment. We identified several types of dynamics for producers, depending on the firms' landscape, returns, and the firms' and sectors' relative attractiveness. A firm's dynamics also depends on its initial sector and level of capital, and may exhibit turning points Modifications of the macroeconomic state may lead to significant fluctuations in a firm's growth trajectory.

However, in this work, to examine the impact of financial allocation, we concentrated on interactions between firms and between firms and investors. Investors only interact indirectly, through firms. For a more comprehensive study, we will include interactions between investors in a subsequent work, where they invest in each other.

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Appendix 1 From large number of agents to field formalism

This appendix summarizes the most useful steps of the method developed in Gosselin, Lotz and Wambst (2017, 2020, 2021), to switch from the probabilistic description of the model to the field theoretic formalism and summarizes the translation of a generalization of (10) involving different time variables. By convention and unless otherwise mentioned, the symbol of integration $\int$ refers to all the variables involved.

A1.1 Probabilistic formalism

The probabilistic formalism for a system with N identical economic agents in interaction is based on the minimization functions described in the text. Classically, the dynamics derives through the optimization problem of these functions. The probabilistic formalism relies on the contrary on the fact, that, due to uncertainties, shocks... agents do not optimize fully these functions. Moreover, given the large number of agents, there may be some discrepancy between agents minimization functions, and this fact may be translated in an uncertainty of behavior around some average minimization, or objective function.

We thus assume that each agent chooses for his action a path randomly distributed around the optimal path. The agent's behavior can be described as a weight that is an exponential of the intertemporal utility, that concentrates the probability around the optimal path. This feature models some internal uncertainty as well as non-measurable shocks. Gathering all agents, it yields a probabilistic description of the system in terms of a probabilistic weight.

In general, this weight includes utility functions and internalizes forward-looking behaviors, such as intertemporal budget constraints and interactions among agents. These interactions may for instance arise through constraints, since income flows depend on other agents demand. The probabilistic description then allows to compute the transition functions of the system, and in turn compute the probability for a system to evolve from an initial state to a final state within a given time span. They have the form of Euclidean path integrals.

In the context of the present paper, we have seen that the minimization functions for the system considered in this work have the form:

$$\int dt \left(\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right)\right)^2 + \sum_i \left(\sum_{j,k,l...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right)\right)\right) \tag{134}$$

The minimization of this function will yield a dynamic equation for N agents in interaction described by a set of dynamic variables $\mathbf{A}_i(t)$ during a given timespan T.

The probabilistic description is straightforwardly obtained from (134). The probability associated to a configuration $\left(\mathbf{A}_i(t)\right)_{\substack{i=1,...,N\\ 0\leqslant t\leqslant T}}$ is directly given by:

$$\mathcal{N} \exp\left(-\frac{1}{\sigma^2} \int dt \left(\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right)\right)^2 + \sum_i \left(\sum_{j,k,l...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...\right)\right)\right)\right) \tag{135}$$

where $\mathcal{N}$ is a normalization factor and σ^2 is a variance whose magnitude describes the amplitude of deviations around the optimal path.

As in the paper, the system is in general modelled by several equations, and thus, several minimization function. The overall system is thus described by several functions, and the minimization function of the whole system is described by the set of functions:

$$\int dt \left(\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f^{(\alpha)}(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...) \right)^2 + \sum_i \left(\sum_{j,k,l...} g^{(\alpha)}(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...) \right) \right) \tag{136}$$

where α runs over the set equations describing the system's dynamics. The associated weight is then:

$$\mathcal{N} \exp \left(- \left(\sum_{i,\alpha} \frac{1}{\sigma_\alpha^2} \int dt \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f^{(\alpha)}(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...) \right)^2 + \sum_{i,\alpha} \left(\sum_{j,k,l...} g^{(\alpha)}(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...) \right) \right) \right) \tag{137}$$

The appearance of the sum of minimization functions in the probabilistic weight (137) translates the hypothesis that the deviations with respect to the optimization of the functions (136) are assumed to be independent.

For a large number of agents, the system described by (137) involves a large number of variables $K_i(t)$, $P_i(t)$ and $X_i(t)$ that are difficult to handle. To overcome this difficulty, we consider the space H of complex functions defined on the space of a single agent's actions. The space H describes the collective behavior of the system. Each function Ψ of H encodes a particular state of the system. We then associate to each function Ψ of H a statistical weight, i.e. a probability describing the state encoded in Ψ. This probability is written $\exp(-S(\Psi))$, where $S(\Psi)$ is a functional, i.e. the function of the function Ψ. The form of $S(\Psi)$ is derived directly from the form of (137) as detailed in the text. As seen from (137), this translation can in fact be directly obtained from the sum of "classical" minimization functions weighted by the factors $\frac{1}{\sigma_\alpha^2}$:

$$\sum_{i,\alpha} \frac{1}{\sigma_\alpha^2} \int dt \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f^{(\alpha)}(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...) \right)^2 + \sum_{i,\alpha} \left(\sum_{j,k,l...} g^{(\alpha)}(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \mathbf{A}_l(t)...) \right)$$

This is this shortcut we used in the text.

A1.2 Interactions between agents at different times

A straightforward generalization of (10) involve agents interactions at different times. The terms considered have the form:

$$\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} \int f\left(\mathbf{A}_i(t_i), \mathbf{A}_j(t_j), \mathbf{A}_k(t_k), \mathbf{A}_l(t_l)..., \mathbf{t}\right) \mathbf{dt} \right)^2 \tag{138}$$
$$+\sum_i \sum_{j,k,l...} \int g\left(\mathbf{A}_i(t_i), \mathbf{A}_j(t_j), \mathbf{A}_k(t_k), \mathbf{A}_l(t_l)..., \mathbf{t}\right) \mathbf{dt}$$

where $\mathbf{t}$ stands for $(t_i, t_j, t_k, t_l...)$ and $\mathbf{dt}$ stands for $dt_i dt_j dt_k dt_l...$

The translation is straightforward. We introduce a time variable θ on the field side and the fields write $|\Psi(\mathbf{A}, \theta)|^2$ and $\left|\hat{\Psi}\left(\hat{\mathbf{A}}, \hat{\theta}\right)\right|^2$. The second term in (138) becomes:

$$\sum_i \sum_j \sum_{j,k...} \int g\left(\mathbf{A}_i(t_i), \mathbf{A}_j(t_j), \mathbf{A}_k(t_k), \mathbf{A}_l(t_l)..., \mathbf{t}\right) \mathbf{dt}$$
$$\rightarrow \int g\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'..., \boldsymbol{\theta}, \hat{\boldsymbol{\theta}}\right) |\Psi(\mathbf{A}, \theta)|^2 \left|\Psi(\mathbf{A}', \theta')\right|^2 \left|\Psi(\mathbf{A}'', \theta'')\right|^2 d\mathbf{A}d\mathbf{A}'d\mathbf{A}'' \tag{139}$$
$$\times \left|\hat{\Psi}\left(\hat{\mathbf{A}}, \hat{\theta}\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}', \hat{\theta}'\right)\right|^2 d\hat{\mathbf{A}}d\hat{\mathbf{A}}'\mathbf{d}\boldsymbol{\theta}\mathbf{d}\hat{\boldsymbol{\theta}}$$

where $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}$ are the multivariables $(\theta, \theta', \theta''...)$ and $\left(\hat{\theta}, \hat{\theta}'...\right)$ respectively and $\mathbf{d}\boldsymbol{\theta}\mathbf{d}\hat{\boldsymbol{\theta}}$ stands for $d\theta d\theta' d\theta''...$ and $d\hat{\theta}d\hat{\theta}'...$

Similarly, the first term in (138) translates as:

$$\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} \int f\left(\mathbf{A}_i(t_i), \mathbf{A}_j(t_j), \mathbf{A}_k(t_k), \mathbf{A}_l(t_l)..., \mathbf{t}\right) \mathbf{dt} \right)^2 \tag{140}$$
$$\rightarrow \int \Psi^\dagger(\mathbf{A}, \theta) \left(-\nabla_{\mathbf{A}^{(\alpha)}} \left(\frac{\sigma^2_{\mathbf{A}^{(\alpha)}}}{2} \nabla_{\mathbf{A}^{(\alpha)}} - \Lambda(\mathbf{A}, \theta)\right)\right) \Psi(\mathbf{A}, \theta) d\mathbf{A}d\theta \tag{141}$$

by:

$$\Lambda(\mathbf{A}, \theta) = \int f^{(\alpha)}\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'..., \boldsymbol{\theta}, \hat{\boldsymbol{\theta}}\right) \left|\Psi(\mathbf{A}', \theta')\right|^2 \left|\Psi(\mathbf{A}'', \theta'')\right|^2 d\mathbf{A}'d\mathbf{A}'' \tag{142}$$
$$\times \left|\hat{\Psi}\left(\hat{\mathbf{A}}, \theta\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}', \theta''\right)\right|^2 d\hat{\mathbf{A}}d\hat{\mathbf{A}}'\mathbf{d}\bar{\boldsymbol{\theta}}\mathbf{d}\hat{\boldsymbol{\theta}}$$

with $\mathbf{d}\bar{\boldsymbol{\theta}} = d\theta' d\theta''$.

Ultimately, as in the text, additional terms (445):

$$\Psi^\dagger(\mathbf{A}, \theta) \left(-\nabla_\theta \left(\frac{\sigma^2_\theta}{2} \nabla_\theta - 1\right)\right) \Psi(\mathbf{A}, \theta) \tag{143}$$
$$+\hat{\Psi}^\dagger\left(\hat{\mathbf{A}}, \theta\right) \left(-\nabla_\theta \left(\frac{\sigma^2_\theta}{2} \nabla_\theta - 1\right)\right) \hat{\Psi}\left(\hat{\mathbf{A}}, \theta\right) + \alpha |\Psi(\mathbf{A})|^2 + \alpha \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2$$

are included to the action functional to take into account for the time variable.

A1.3 Translation of the minimization functions

Real economy

Translation of the minimization function: Physical capital allocation Let us start by translating in terms of fields the expression (49):

$$\sum_i \left(\left(\frac{dX_i}{dt} - \nabla_X R(K_i, X_i) H(K_i) \right)^2 + \tau \frac{K_{X_i}}{K_i} \sum_j \delta(X_i - X_j) \right) \tag{144}$$

To do so, we first consider the last term $\tau \frac{K_{X_i}}{K_i} \sum_i \sum_j \delta(X_i - X_j)$. This term contains no derivative. The form of the translation is given by formula (11). Since the expression contains two indices, both of them are summed.

The first step of the translation is to replace X_i and X_j by two variables X et X', and substitute:

$$\tau \frac{K_{X_i}}{K_i} \delta(X_i - X_j) \rightarrow \tau \frac{K_X}{K} \delta(X - X')$$

where K_X is the average capital per firm in sector X. The sum over i and the sum over j are then replaced directly by the integrals $\int |\Psi(K, X)|^2 d(K, X)$, $\int |\Psi(K', X')|^2 d(K', X')$, which leads to the following translation:

$$\begin{aligned} \tau \frac{K_{X_i}}{K_i} \sum_i \sum_j \delta(X_i - X_j) &\rightarrow \int |\Psi(K, X)|^2 d(K, X) \int |\Psi(K', X')|^2 d(K', X') \tau \frac{K_X}{K} \delta(X - X') \\ &= \int \tau \frac{K_X}{K} |\Psi(K, X)|^2 |\Psi(K', X)|^2 d(K, X) dK' \end{aligned} \tag{145}$$

To translate the first term in formula (144):

$$\sum_i \left(\frac{dX_i}{dt} - \nabla_X R(K_i, X_i) H(K_i) \right)^2 \tag{146}$$

We use the translation (17) of a type-(16) expression. The gradient term appearing in equation (17) is ∇_X. We thus obtain the translation:

$$\begin{aligned} &\sum_i \left(\frac{dX_i}{dt} - \nabla_X R(K_i, X_i) H(K_i) \right)^2 \\ \rightarrow &\int \Psi^\dagger(K, X) \left(-\nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X + \Lambda(X, K) \right) \right) \Psi(K, X) dK dX \end{aligned} \tag{147}$$

Note that the variance σ_X^2 reflects the probabilistic nature of the model hidden behind the field formalism. This σ_X^2 represents the characteristic level of uncertainty of the sectors space dynamics. It is a parameter of the model. The term $\Lambda(X, K)$ is the translation of the term $-\nabla_X R(K_i, X_i) H(K_i)$ in the parenthesis of (146). This term is a function of one sole index "i". In that case, the term Λ is simply obtained by replacing (K_i, X_i) by (K, X). We use the translation (15) of (13)-type term, so that Λ writes:

$$\Lambda(X, K) = -\nabla_X R(K, X) H(K)$$

and the translation of expression (146) is:

$$\begin{aligned} &\sum_i \left(\frac{dX_i}{dt} - \nabla_X R(K_i, X_i) H(K_i) \right)^2 \\ \rightarrow &\int \Psi^\dagger(K, X) \left(-\nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - \nabla_X R(K, X) H(K) \right) \right) \Psi(K, X) dK dX \end{aligned} \tag{148}$$

Using equations (145) and (148), the translation of (144) is thus:

$$\begin{aligned} S_1 &= -\int \Psi^{\dagger}(K,X)\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X-\nabla_X R(K,X)H(K)\right)\Psi(K,X)\,dKdX \\ &\quad +\tau\frac{K_X}{K}\int\left|\Psi(K',X)\right|^2\left|\Psi(K,X)\right|^2 dK'dKdX \end{aligned} \tag{149}$$

Translation of the minimization function: Physical capital We can now turn to the translation of the second equation (50):

$$\sum_i\left(\frac{d}{dt}K_i+\frac{1}{\varepsilon}\left(K_i(t)-\frac{F_2(R(K_i(t),X_i(t)))\,G\left(X_i(t)-\hat{X}_j(t)\right)}{\sum_l F_2(R(K_l(t),X_l(t)))\,G\left(X_l(t)-\hat{X}_j(t)\right)}\hat{K}_j(t)\right)\right)^2 \tag{150}$$

To detail the computations, we have kept the expanded formula (41) for $F_2\left(R(K_i(t),X_i(t)),\hat{X}_j(t)\right)$ Once again, we use the translation (15) of (13)-type term, and start by building the field functional associated to the term inside the square:

$$K_i(t)-\sum_j\frac{F_2(R(K_i(t),X_i(t)))\,G\left(X_i(t)-\hat{X}_j(t)\right)}{\sum_l F_2(R(K_l(t),X_l(t)))\,G\left(X_l(t)-\hat{X}_j(t)\right)}\hat{K}_j(t)$$

We replace:

$$\begin{aligned} (K_i(t),X_i(t)) &\rightarrow (K,X) \\ (K_l(t),X_l(t)) &\rightarrow (K',X') \\ \left(\hat{K}_j(t),\hat{X}_j(t)\right) &\rightarrow \left(\hat{K},\hat{X}\right) \end{aligned}$$

and:

$$K_i(t)-\sum_j\frac{F_2(R(K_i(t),X_i(t)))\,G\left(X_i(t)-\hat{X}_j\right)}{\sum_l F_2(R(K_l(t),X_l(t)))\,G\left(X_l(t)-\hat{X}_j\right)}\hat{K}_j(t)\rightarrow K-\sum_j\frac{F_2(R(K,X))\,G\left(X-\hat{X}\right)}{\sum_l F_2(R(K',X'))\,G\left(X'-\hat{X}\right)}\hat{K} \tag{151}$$

The sum over l is then replaced by an integral $\int\left|\Psi(K',X')\right|^2 d(K',X')$:

$$\begin{aligned} &K_i(t)-\sum_j\frac{F_2(R(K_i(t),X_i(t)))\,G\left(X_i(t)-\hat{X}_j\right)}{\sum_l F_2(R(K_l(t),X_l(t)))\,G\left(X_l(t)-\hat{X}_j\right)}\hat{K}_j(t) \\ \rightarrow\;& K-\sum_j\frac{F_2(R(K,X))\,G\left(X-\hat{X}\right)}{\int\left|\Psi(K',X')\right|^2 d(K',X')\,F_2(R(K',X'))\,G\left(X'-\hat{X}_j\right)}\hat{K} \end{aligned} \tag{152}$$

Recall that investors' variables are denoted with an upper script $\hat{}$.

Finally, the sum over j and the second field are replaced by $\int\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\left(\hat{K},\hat{X}\right)$. After introducing the characteristic factor $\frac{1}{\varepsilon}$ of the capital accumulation time scale (see (46)), the translation becomes:

$$\begin{aligned}
& \frac{1}{\varepsilon}\left(K_i(t)-\sum_j \frac{F_2\left(R\left(K_i(t), X_i(t)\right)\right) G\left(X_i(t)-\hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l(t), X_l(t)\right)\right) G\left(X_l(t)-\hat{X}_j\right)} \hat{K}_j(t)\right) \\
\rightarrow \quad & \frac{1}{\varepsilon}\left(K-\int\left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2 d\left(\hat{K}, \hat{X}\right) \frac{F_2(R(K, X)) G\left(X-\hat{X}\right) \hat{K}}{\int\left|\Psi\left(K^{\prime}, X^{\prime}\right)\right|^2 d\left(K^{\prime}, X^{\prime}\right) F_2\left(R\left(K^{\prime}, X^{\prime}\right)\right) G\left(X^{\prime}-\hat{X}\right)}\right) \\
\equiv \quad & \Lambda(K, X) \qquad (153)
\end{aligned}$$

Using the translation (17) of (16)-type term, we are led to the translation of (150). Since the square (150) includes a derivative $\frac{d}{dt}K_i$, the expression starts with a gradient with respect to K, and we have:

$$\begin{aligned}
& \sum_i\left(\frac{d}{d t} K_i+\frac{1}{\varepsilon}\left(K_i-\sum_j \frac{F_2\left(R\left(K_i(t), X_i(t)\right)\right) G\left(X_i(t)-\hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l(t), X_l(t)\right)\right) G\left(X_l(t)-\hat{X}_j\right)} \hat{K}_j(t)\right)\right)^2 \qquad (154) \\
\rightarrow \quad & \int \Psi^{\dagger}(K, X)\left(-\nabla_K\left(\frac{\sigma_K^2}{2} \nabla_K+\Lambda(K, X)\right)\right) \Psi(K, X)\, d K\, d X
\end{aligned}$$

where, here again, the variance σ_K^2 reflects the probabilistic nature of the model that is hidden behind the field formalism. Recall that it represents the characteristic level of uncertainty in the dynamics of capital.

Inserting result (153) in equation (154), the translation of (**??**) becomes:

$$S_2=-\int \Psi^{\dagger}(K, X) \nabla_K\left(\frac{\sigma_K^2}{2} \nabla_K+\frac{1}{\varepsilon}\left(K-\int \hat{F}_2\left(R(K, X), \hat{X}\right) \hat{K}\left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2 d \hat{K} d \hat{X}\right)\right) \Psi(K, X) \qquad (155)$$

with:

$$\hat{F}_2\left(R(K, X), \hat{X}\right)=\frac{F_2(R(K, X)) G\left(X-\hat{X}\right)}{\int F_2(R(K, X)) G\left(X-\hat{X}\right)|\Psi(K, X)|^2}$$

as quoted in the text.

Financial markets

The functions to be translated are those of the financial capital dynamics (51) and of the financial capital allocation (52). Both expressions include a time derivative and are thus of type (12). As for the real economy, the application of the translation rules is straightforward.

Translation of the minimization function: Financial capital dynamics We consider the function (51):

$$\sum_j\left(\frac{d}{d t} \hat{K}_j-\frac{1}{\varepsilon}\left(\sum_i\left(r_i+F_1\left(\frac{R\left(K_i, X_i\right)}{\sum_l \delta\left(X_l-X_i\right) R\left(K_l, X_l\right)}, \frac{\dot{K}_i(t)}{K_i(t)}\right)\right) \frac{F_2\left(R\left(K_i, X_i\right)\right) G\left(X_i-\hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l, X_l\right)\right) G\left(X_l-\hat{X}_j\right)} \hat{K}_j\right)\right)^2 \qquad (156)$$

which translates, using the general translation formula of expression (16) in (17), into:

$$\int \hat{\Psi}^{\dagger}\left(\hat{K}, \hat{X}\right)\left(-\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}}+\Lambda\left(\hat{K}, \hat{X}\right)\right)\right) \hat{\Psi}\left(\hat{K}, \hat{X}\right) d \hat{K} d \hat{X}$$

The function $\Lambda\left(\hat{K},\hat{X}\right)$ is obtained, as before, by translating the term following the derivative in the function (156):

$$\frac{1}{\varepsilon}\sum_{i}\left(r_{i}+F_{1}\left(\frac{R\left(K_{i},X_{i}\right)}{\sum_{l}\delta\left(X_{l}-X_{i}\right)R\left(K_{l},X_{l}\right)},\frac{\dot{K}_{i}\left(t\right)}{K_{i}\left(t\right)}\right)\right)\frac{F_{2}\left(R\left(K_{i},X_{i}\right)\right)G\left(X_{i}-\hat{X}_{j}\right)}{\sum_{l}F_{2}\left(R\left(K_{l},X_{l}\right)\right)G\left(X_{l}-\hat{X}_{j}\right)}\hat{K}_{j}\rightarrow\Lambda\left(\hat{K},\hat{X}\right) \tag{157}$$

First, we use the price dynamics equation (35) at the zero-th order in fluctuations to translate the capital dynamics $\frac{\dot{K}_{i}(t)}{K_{i}(t)}$:

$$\begin{aligned}\frac{\dot{K}_{i}\left(t\right)}{K_{i}\left(t\right)} &= \sum_{j}\frac{F_{2}\left(R\left(K_{i}\left(t\right),X_{i}\left(t\right)\right)\right)G\left(X_{i}\left(t\right)-\hat{X}_{j}\right)}{K_{i}\sum_{l}F_{2}\left(R\left(K_{l}\left(t\right),X_{l}\left(t\right)\right)\right)G\left(X_{l}\left(t\right)-\hat{X}_{j}\right)}\hat{K}_{j}\left(t\right)-K_{i}\left(t\right)\\ &\rightarrow \Gamma\left(K,X\right)\end{aligned}$$

where:

$$\begin{aligned}\Gamma\left(K,X\right) &= \frac{\int\frac{F_{2}\left(R\left(K,X\right)\right)G\left(X-\hat{X}\right)}{\int F_{2}\left(R\left(K,X\right)\right)G\left(X-\hat{X}\right)\left\|\Psi\left(K,X\right)\right\|^{2}}\hat{K}\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^{2}d\left(\hat{K},\hat{X}\right)-K}{K}\\ &= \int\frac{F_{2}\left(R\left(K,X\right)\right)G\left(X-\hat{X}\right)}{K\int F_{2}\left(R\left(K,X\right)\right)G\left(X-\hat{X}\right)\left\|\Psi\left(K,X\right)\right\|^{2}}\hat{K}\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^{2}d\left(\hat{K},\hat{X}\right)-1\end{aligned} \tag{158}$$

Then, using the translation (15) of (13), we translate expression (157) by replacing:

$$\begin{aligned}\left(K_{i},X_{i}\right) &\rightarrow \left(K,X\right)\\ \left(K_{l},X_{l}\right) &\rightarrow \left(K',X'\right)\\ \left(\hat{K}_{j},\hat{X}_{j}\right) &\rightarrow \left(\hat{K},\hat{X}\right)\end{aligned}$$

We also replace the sums by integrals times the appropriate square of field, which yields:

$$\begin{aligned}\Lambda\left(\hat{K},\hat{X}\right) &= -\frac{\hat{K}}{\varepsilon}\int\left(r\left(K,X\right)-\gamma\frac{\int K'\left\|\Psi\left(K',X\right)\right\|^{2}}{K}+F_{1}\left(\frac{R\left(K,X\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^{2}d\left(K',X'\right)},\Gamma\left(K,X\right)\right)\right)\\ &\times\frac{F_{2}\left(R\left(K,X\right)\right)G\left(X-\hat{X}\right)}{\int F_{2}\left(R\left(K',X'\right)\right)G\left(X'-\hat{X}\right)\left\|\Psi\left(K',X'\right)\right\|^{2}d\left(K',X'\right)}\left\|\Psi\left(K,X\right)\right\|^{2}d\left(K,X\right)\end{aligned}$$

Ultimately, the translation of (51) is:

$$\begin{aligned}S_{3} &= -\int\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^{2}}{2}\nabla_{\hat{K}}-\frac{\hat{K}}{\varepsilon}\int\left(r\left(K,X\right)-\gamma\frac{\int K'\left\|\Psi\left(K',X\right)\right\|^{2}}{K}\right.\right.\\ &\left.+F_{1}\left(\frac{R\left(K,X\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^{2}d\left(K',X'\right)},\Gamma\left(K,X\right)\right)\right)\\ &\left.\times\frac{F_{2}\left(R\left(K,X\right)\right)G\left(X-\hat{X}\right)}{\int F_{2}\left(R\left(K',X'\right)\right)G\left(X'-\hat{X}\right)\left\|\Psi\left(K',X'\right)\right\|^{2}d\left(K',X'\right)}\left\|\Psi\left(K,X\right)\right\|^{2}d\left(K,X\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right)\end{aligned}$$

Using expressions (56) and (58) yields the expression of the text.

Translation of the minimization function: Financial capital allocation The translation of the function for financial capital allocation (52) follows the previous pattern. We obtain:

$$\begin{aligned} S_4 &= -\int \hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\nabla_{\hat{X}}\left(\sigma_{\hat{X}}^2\nabla_{\hat{X}} - \int\left(\nabla_{\hat{X}}F_0\left(R\left(K,\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K,\hat{X}\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)}\right)\right)\right. \\ &\quad\left.\times\frac{\left\|\Psi\left(K,\hat{X}\right)\right\|^2 dK}{\int\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\right)\hat{\Psi}\left(\hat{K},\hat{X}\right) \end{aligned}$$

and (58) yields the formula quoted in the text.

Appendix 2 expression of $\Psi(K,X)$ as function of financial variables

A2.1 Finding $\Psi(K,X)$: principle

In this paragraph, we give the principle of resolution for $\Psi(K,X)$ for an arbitrary function H. The full resolution for some particular cases is given below. Given a particular state $\hat{\Psi}$, we aim at minimizing the action functional $S_1+S_2+S_3+S_4$. However, given our assumptions, the action functional S_3+S_4 depends on $\Psi(K,X)$, through average quantities, and moreover, we have assumed that physical capital dynamics depends on financial accumulation. Consequently, we can neglect, in first approximation, the impact of $\Psi(K,X)$ on S_3+S_4 and consider rather the minimization of S_1+S_2 which is given by:

$$\begin{aligned} S_1+S_2 &= -\int \Psi^{\dagger}(K,X)\left(\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X - \nabla_X R(K,X)H(K)\right) - \tau\frac{K_X}{K}\left(\int\left|\Psi\left(K',X\right)\right|^2 dK'\right)\right. \\ &\quad\left.+\nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K + u\left(K,X,\Psi,\hat{\Psi}\right)\right)\right)\Psi(K,X)\,dKdX \end{aligned} \tag{159}$$

with:

$$u\left(K,X,\Psi,\hat{\Psi}\right)=\frac{1}{\varepsilon}\left(K-\int\frac{F_2\left(R(K,X)\right)G\left(X-\hat{X}\right)}{\int F_2\left(R(K,X)\right)G\left(X-\hat{X}\right)\left\|\Psi(K,X)\right\|^2}\hat{K}\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^2 d\hat{K}d\hat{X}\right) \tag{160}$$

and:

$$\Gamma(K,X)=\int\frac{F_2\left(R(K,X)\right)G\left(X-\hat{X}\right)}{K\int F_2\left(R(K,X)\right)G\left(X-\hat{X}\right)\left\|\Psi(K,X)\right\|^2}\hat{K}\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^2 d\left(\hat{K},\hat{X}\right)-1$$

This is done in two steps. First, we find $\Psi(X)$, the background field for X when K determined by X. We then find the corrections to the particular cases considered and compute $\Psi(K,X)$.

A2.1.1 Particular case: K determined by X

A simplification arises, assuming K adapting to X. We assume that in first approximation K is a function of X, written K_X:

$$K=K_X=\int\frac{F_2\left(R\left(K_X,X\right)\right)G\left(X-\hat{X}\right)}{\int F_2\left(R\left(K'_{X'},X'\right)\right)G\left(X'-\hat{X}\right)\left\|\Psi\left(X'\right)\right\|^2 dX'}\hat{K}\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^2 d\left(\hat{K},\hat{X}\right) \tag{161}$$

This means that for any sector X, the capital of all agents in this sector are equal. At the individual level, this corresponds to set $\frac{d}{dt}K_i(t)=0$. The level of capital adapts faster than the motion in sector space and reaches quickly its equilibrium value. Incindently, (161) implies that $\Gamma(K,X)=0$. Actually, using (161):

$$\begin{aligned}
\Gamma(K,X) &= \int \frac{F_2(R(K,X))\,G\left(X-\hat{X}\right)}{K\int F_2(R(K,X))\,G\left(X-\hat{X}\right)\|\Psi(K,X)\|^2}\hat{K}\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^2 d\left(\hat{K},\hat{X}\right)-1\\
&= \int \frac{F_2(R(K,X))\,G\left(X-\hat{X}\right)\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\int F_2(R(K_X,X))\,G\left(X-\hat{X}\right)\hat{K}\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^2 d\left(\hat{K},\hat{X}\right)}-1\\
&= 0
\end{aligned}$$

A2.1.1.1 Justification of approximation (161) Approximation (161) justifies in the following way. When F_2 is slowly varying with K, we perform the following change of variable in (159):

$$\begin{aligned}
\Psi &\rightarrow \Psi\exp\left(-\frac{\int u\left(K,X,\Psi,\hat{\Psi}\right)dK}{\sigma_K^2}\right)\simeq\Psi\exp\left(-\frac{1}{2\sigma_K^2}\varepsilon u^2\left(K,X,\Psi,\hat{\Psi}\right)\right)\\
\Psi^\dagger &\rightarrow \Psi^\dagger\exp\left(\frac{1}{\sigma_K^2}\int u\left(K,X,\Psi,\hat{\Psi}\right)dK\right)\simeq\Psi^\dagger\exp\left(-\frac{1}{2\sigma_K^2}\varepsilon u^2\left(K,X,\Psi,\hat{\Psi}\right)\right)
\end{aligned}$$

and this replaces S_2 in (159) by:

$$-\int\Psi^\dagger(K,X)\left(\frac{\sigma_K^2}{2}\nabla_K^2-\frac{u^2}{2\sigma_K^2}\left(K,X,\Psi,\hat{\Psi}\right)+\frac{1}{2}\nabla_K u\left(K,X,\Psi,\hat{\Psi}\right)\right)\Psi(K,X)\,dKdX \tag{162}$$

The change of variable modifies S_1 in (159). Actually, the derivative ∇_X acts on $\exp\left(-\frac{1}{2\sigma_K^2}u^2\left(K,X,\Psi,\hat{\Psi}\right)\right)$ and the term:

$$-\int\Psi^\dagger(K,X)\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X-\nabla_X R(K,X)H(K)\right)\Psi(K,X)\,dKdX$$

becomes:

$$\begin{aligned}
&-\int\Psi^\dagger(X)\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X-\nabla_X R(K,X)H(K)\right)\Psi(X)\,dKdX\\
&+\varepsilon\int\Psi^\dagger(K,X)\left(\frac{\sigma_X^2}{2\sigma_K^2}u\nabla_X u\right)\nabla_X\Psi(K,X)\,dKdX+\varepsilon\int\Psi^\dagger(K,X)\left(\frac{\sigma_X^2}{2\sigma_K^2}\left((\nabla_X u)^2+u\nabla_X^2 u\right)\right)\Psi(K,X)\,dKdX\\
&-\int\Psi^\dagger(K,X)\left(\varepsilon\frac{u\nabla_X u}{\sigma_K^2}\nabla_X R(K,X)H(K)+\varepsilon^2\frac{\sigma_X^2}{2\sigma_K^4}(u\nabla_X u)^2\right)\Psi(K,X)\,dKdX
\end{aligned} \tag{163}$$

Using that u is of order $\frac{1}{\varepsilon}$ (see(160)), the minimum of S_1+S_2 is obtained when the potential:

$$\begin{aligned}
&\int\Psi^\dagger(K,X)\left(\frac{u^2}{2\sigma_K^2}-\frac{1}{2}\nabla_K u\right)\Psi(K,X)\,dKdX\\
&+\varepsilon\int\Psi^\dagger(K,X)\left(\frac{\sigma_X^2}{2\sigma_K^2}\left((\nabla_X u)^2+u\nabla_X^2 u\right)\right)\Psi(K,X)\,dKdX\\
&-\int\Psi^\dagger(K,X)\left(\varepsilon\frac{u\nabla_X u}{\sigma_K^2}\nabla_X R(K,X)H(K)+\varepsilon^2\frac{\sigma_X^2}{2\sigma_K^4}(u\nabla_X u)^2\right)\Psi(K,X)\,dKdX
\end{aligned} \tag{164}$$

is nul. The dominant term in (164) for $\varepsilon << 1$ is:

$$\int \Psi^{\dagger}(K,X)\left(\frac{u^2}{2\sigma_K^2}-\varepsilon^2\frac{\sigma_X^2}{2\sigma_K^4}(u\nabla_X u)^2\right)\Psi(K,X)\,dKdX \tag{165}$$

For $\sigma_X^2 << \sigma_K^2$ it implies that the minimum for S_1+S_2 is obtained for:

$$u\left(K,X,\Psi,\hat{\Psi}\right)\simeq 0$$

with solution (161).

A2.1.1.2 Rewriting the action S_1+S_2 With our choice $G\left(X-\hat{X}\right)=\delta\left(X-\hat{X}\right)$ we find:

$$K_X=\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\left\|\Psi(X)\right\|^2} \tag{166}$$

and $\Psi(K,X)$ becomes a function $\Psi(X)$:

$$\Psi(K,X)\rightarrow\Psi(X)$$

To find the action for $\Psi(X)$ we evaluate (164) using $u\left(K_X,X,\Psi,\hat{\Psi}\right)=0$, and compute the first term in (165) for $\Psi(X)=\Psi(K_X,X)\,\delta(u)$ by replacing:

$$\delta(u)\rightarrow\frac{\exp\left(-\varepsilon u^2\right)}{\sqrt{2\pi\varepsilon}}$$

We obtain:

$$\begin{aligned}-\int \Psi^{\dagger}(K,X)\left(\frac{\sigma_K^2}{2}\nabla_K^2\right)\Psi(K,X)\,dKdX &= \frac{\sigma_K^2}{2}\int\left|\Psi(X)\right|^2 dX\int\frac{\exp\left(-\varepsilon u^2\right)}{\sqrt{2\pi\varepsilon}}\nabla_K^2\frac{\exp\left(-\varepsilon u^2\right)}{\sqrt{2\pi\varepsilon}}dK\\ &\simeq \frac{\sigma_K^2}{2\varepsilon}\int\left|\Psi(X)\right|^2 dX\end{aligned}$$

and the action S_1 restricted to the variable X is given by:

$$\begin{aligned}S_1 &= \int\Psi^{\dagger}(X)\left(-\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X-\left(\nabla_X R(X)\,H(K_X)\right)\right)+\tau\frac{K_X}{K}\left|\Psi(X)\right|^2\right)\Psi(X)\\ &+\int\Psi^{\dagger}(K,X)\left(\frac{\sigma_X^2}{4\sigma_K^2}\left(\nabla_X u\left(K_X,X,\Psi,\hat{\Psi}\right)\right)^2\right)\Psi(K,X)\,dKdX\\ &+\int\left(\frac{\sigma_K^2}{2\varepsilon}-\frac{1}{2}\nabla_K u\left(K_X,X,\Psi,\hat{\Psi}\right)\right)\left|\Psi(X)\right|^2 dX\end{aligned}$$

In our order of approximation $\nabla_K u\left(K_X,X,\Psi,\hat{\Psi}\right)\simeq\varepsilon$. Ultimately, for $\sigma_X^2<<\sigma_K^2$, action S_1 reduces to:

$$S_1=\int\Psi^{\dagger}(X)\left(-\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X-\left(\nabla_X R(X)\,H(K_X)\right)\right)+\tau\frac{K_X}{K}\left|\Psi(X)\right|^2+\frac{\sigma_K^2-1}{2\varepsilon}\right)\Psi(X) \tag{167}$$

and we look for $\Psi(X)$ minimizing (167). As explained in the text, we will consider at the collective level that we can replace:

$$\tau\frac{K_X}{K}\rightarrow\tau$$

A2.1.1.3 Minimization of (167) To minimize (167), we assume for the sake of simplicity, that for $i \neq j$:

$$\left|\nabla_{X_i}\nabla_{X_j}R(X)\right| << \left|\nabla^2_{X_i}R(X)\right|$$

which is the case for example if $R(X)$ is a function with separated variables : $R(X) = \sum R_i(X_i)$. This can be also realized if locally, one chooses the variables X_i to diagonalize $\nabla_{X_i}\nabla_{X_j}R(X)$ at some points in the sector space.

We then perform the change of variables:

$$\exp\left(\int^X \frac{\nabla_X R(X)}{\sigma_X^2 \|\nabla_X R(X)\|} H(K_X)\right)\Psi(X) \rightarrow \Psi(X)$$

and:

$$\exp\left(-\int^X \nabla_X R(X) H(K_X)\right)\Psi^\dagger(X) \rightarrow \Psi^\dagger(X)$$

so that (167) becomes:

$$\int \Psi^\dagger(X)\left(-\frac{\sigma_X^2}{2}\nabla_X^2 + \frac{1}{2\sigma_X^2}(\nabla_X R(X) H(K_X))^2 + \frac{\nabla_X^2 R(K_X, X)}{2} H(K_X) + \tau|\Psi(X)|^2 + \frac{\sigma_K^2 - 1}{2\varepsilon}\right)\Psi(X) \tag{168}$$

which is of second order in derivatives with a potential:

$$\tau\|\Psi(X)\|^4 + \frac{1}{2\sigma_X^2}\int(\nabla_X R(X) H(K_X))^2\|\Psi(X)\|^2$$

We assume the number of agents fixed equal to N. We must minimize (168) with the constraint $\|\Psi(X)\|^2 \geqslant 0$ and $\int\|\Psi(X)\|^2 = N$. We thus replace (168) by:

$$\int \Psi^\dagger(X)\left(-\frac{\sigma_X^2\nabla_X^2}{2} + \frac{(\nabla_X R(X) H(K_X))^2}{2\sigma_X^2} + \frac{\nabla_X^2 R(K_X, X)}{2} H(K_X) + \tau|\Psi(X)|^2 + \frac{\sigma_K^2 - 1}{2\varepsilon}\right)\Psi(X)$$
$$+D\left(\|\Psi\|^2\right)\left(\int\|\Psi(X)\|^2 - N\right) + \int\mu(X)\|\Psi(X)\|^2 \tag{169}$$

we have written $D\left(\|\Psi\|^2\right)$ the Lagrange multiplier for $\int\|\Psi(X)\|^2$, to keep track of its dependency multiplier in $\|\Psi\|^2$. By a redefinition $D\left(\|\Psi\|^2\right) - \frac{\sigma_K^2-1}{2\varepsilon} \rightarrow D\left(\|\Psi\|^2\right)$, $\frac{D\left(\|\Psi\|^2\right)}{D\left(\|\Psi\|^2\right) - \frac{\sigma_K^2}{2\varepsilon}}N \rightarrow N$ we can write (169) as:

$$\int \Psi^\dagger(X)\left(-\frac{\sigma_X^2}{2}\nabla_X^2 + \frac{1}{2\sigma_X^2}(\nabla_X R(X) H(K_X))^2 + \frac{H(K_X)\nabla_X^2 R(K_X, X)}{2} + \tau|\Psi(X)|^2\right)\Psi(X) \tag{170}$$
$$+D\left(\|\Psi\|^2\right)\left(\int\|\Psi(X)\|^2 - N\right) + \int\mu(X)\|\Psi(X)\|^2$$

Introducing the change of variable for $\nabla_X R(X)$ for the sake of simplicity:

$$(\nabla_X R(X))^2 + \sigma_X^2\frac{\nabla_X^2 R(K_X, X)}{H(K_X)} \rightarrow (\nabla_X R(X))^2 \tag{171}$$

the minimization of the potential yields, for $\sigma_X^2 << 1$:

$$\begin{aligned}
& iD\left(\|\Psi\|^2\right)+\mu(X) & (172)\\
= \; & 2\tau\|\Psi(X)\|^2-\frac{H'\left(\frac{\int\hat{K}\left\|\hat{\Psi}(\hat{K},X)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)}{2\sigma_X^2 H\left(\frac{\int\hat{K}\left\|\hat{\Psi}(\hat{K},X)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)} \\
& \times\left(\nabla_X R(X)\, H\left(\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)^2\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^4}\|\Psi(X)\|^2 \\
& +\frac{1}{2\sigma_X^2}\left(\nabla_X R(X)\, H\left(\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)^2
\end{aligned}$$

Moreover, multiplying (172) by $\|\Psi(X)\|^2$ and integrating yields:

$$\begin{aligned}
D\left(\|\Psi\|^2\right)N \; = \; & 2\tau\int|\Psi(X)|^4 & (173)\\
& -\int\frac{H'\left(\frac{\int\hat{K}\left\|\hat{\Psi}(\hat{K},X)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)}{2\sigma_X^2 H\left(\frac{\int\hat{K}\left\|\hat{\Psi}(\hat{K},X)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)}\left(\nabla_X R(X)\, H\left(\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)^2\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K} \\
& +\frac{1}{2\sigma_X^2}\int\left(\nabla_X R(X)\, H\left(\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)^2\|\Psi(X)\|^2 \\
\simeq \; & 2\tau\int|\Psi(X)|^4+\frac{1}{2\sigma_X^2}\int\left(\nabla_X R(X)\, H\left(\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)^2\|\Psi(X)\|^2
\end{aligned}$$

Note that in first approximation, for $H' << 1$, (172) and (173) become:

$$D\left(\|\Psi\|^2\right)+\mu(X)=2\tau\|\Psi(X)\|^2+\frac{1}{2\sigma_X^2}\left(\nabla_X R(X)\right)^2 H^2\left(\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right) \tag{174}$$

and:

$$ND\left(\|\Psi\|^2\right)=2\tau\int|\Psi(X)|^4+\frac{1}{2\sigma_X^2}\int\left(\nabla_X R(X)\, H\left(\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)^2\|\Psi(X)\|^2 \tag{175}$$

A2.1.1.4 Resolution of (174) and (175) Two cases arise in the resolution:

Case 1: $\|\Psi(X)\|^2>0$ For $\|\Psi(X)\|^2>0$, (172) writes:

$$D\left(\|\Psi\|^2\right)=2\tau\|\Psi(X)\|^2+\frac{1}{2\sigma_X^2}\left(\nabla_X R(X)\right)^2 H^2\left(\frac{\hat{K}_X}{\|\Psi(X)\|^2}\right)\left(1-\frac{H'\left(\hat{K}_X\right)}{H\left(\hat{K}_X\right)}\frac{\hat{K}_X}{\|\Psi(X)\|^2}\right) \tag{176a}$$

with:

$$\hat{K}_X = \int \hat{K} \left\| \hat{\Psi}\left(\hat{K}, X\right)\right\|^2 d\hat{K} = K_X \left\|\Psi(X)\right\|^2 \tag{177}$$

Note that restoring the initial variable:

$$\left(\nabla_X R(X)\right)^2 \rightarrow \left(\nabla_X R(X)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R(K_X, X)}{H(K_X)} \tag{178}$$

yields (81) in the text.

Given the setup, we can assume that

$$H^2\left(\frac{\hat{K}_X}{\left\|\Psi(X)\right\|^2}\right)\left(1 - \frac{H'\left(\hat{K}_X\right)}{H\left(\hat{K}_X\right)}\frac{\hat{K}_X}{\left\|\Psi(X)\right\|^2}\right)$$

is a decreasing function of $\left\|\Psi(X)\right\|^2$. Assume a minimum $\Psi_0(X)$ for the right hand side of (176a). It leads to a condition for $D\left(\left\|\Psi\right\|^2\right)$:

$$D\left(\left\|\Psi\right\|^2\right) > 2\tau\left\|\Psi_0(X)\right\|^2 + \frac{1}{2\sigma_X^2}\left(\nabla_X R(X)\right)^2 H^2\left(\frac{\hat{K}_X}{\left\|\Psi_0(X)\right\|^2}\right)\left(1 - \frac{H'\left(\hat{K}_X\right)}{H\left(\hat{K}_X\right)}\frac{\hat{K}_X}{\left\|\Psi_0(X)\right\|^2}\right) \tag{179}$$

and the solution of (176a) writes:

$$\left\|\Psi\left(X, \left(\nabla_X R(X)\right)^2, \frac{\hat{K}_X}{\hat{K}_{X,0}}\right)\right\|^2 \tag{180}$$

where $\hat{K}_{X,0}$ is a constant representing some average to normalize $\frac{\hat{K}_X}{\hat{K}_{X,0}}$ as a dimensionless number.

Case 2 $\left\|\Psi(X)\right\|^2 = 0$ On the other hand, if:

$$D\left(\left\|\Psi\right\|^2\right) < 2\tau\left\|\Psi_0(X)\right\|^2 + \frac{1}{2\sigma_X^2}\left(\nabla_X R(X)\right)^2 H^2\left(\frac{\hat{K}_X}{\left\|\Psi_0(X)\right\|^2}\right)\left(1 - \frac{H'\left(\hat{K}_X\right)}{H\left(\hat{K}_X\right)}\frac{\hat{K}_X}{\left\|\Psi_0(X)\right\|^2}\right) \tag{181}$$

the solution of (176a) is $\left\|\Psi(X)\right\|^2 = 0$

Gathering both cases The value of $\left\|\Psi\right\|^2$ thus depends on the conditions (179) and (181). To compute the value of $D\left(\left\|\Psi\right\|^2\right)$ we integrate (176a) over V/V_0 with V_0 locus where $\left\|\Psi(X)\right\|^2 = 0$. V_0 will be then defined by (181) once $D\left(\left\|\Psi\right\|^2\right)$ found. For H slowly varying, we can replace $\frac{\hat{K}_X}{\left\|\Psi(X)\right\|^2}$ by:

$$\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K}, X\right)\right\|^2 d\hat{K}dX}{\int \left\|\Psi(X)\right\|^2 dX} = \frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K}, X\right)\right\|^2 d\hat{K}dX}{N}$$

so that the integration of (181) over X yields:

$$D\left(\|\Psi\|^2\right)(V-V_0) \simeq 2\tau N+\frac{1}{2\sigma_X^2}\int\left(\nabla_X R(X)\right)^2 H^2\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}dX}{N}\right)$$

$$\times\left(1-\frac{H'\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}dX}{N}\right)}{H\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}dX}{N}\right)}\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}dX}{N}\right)$$

$$= 2\tau N+\frac{1}{2\sigma_X^2}\left(\nabla_X R(X)\right)^2 H^2\left(\frac{\left\langle\hat{K}\right\rangle}{N}\right)\left(1-\frac{H'\left(\frac{\langle\hat{K}\rangle}{N}\right)}{H\left(\frac{\langle\hat{K}\rangle}{N}\right)}\frac{\left\langle\hat{K}\right\rangle}{N}\right)$$

Consequently:

$$D\left(\|\Psi\|^2\right)\simeq 2\tau\frac{N}{V-V_0}+\frac{1}{2\sigma_X^2}\left\langle\left(\nabla_X R(X)\right)^2\right\rangle_{V/V_0} H^2\left(\frac{\left\langle\hat{K}\right\rangle}{N}\right)\left(1-\frac{H'\left(\frac{\langle\hat{K}\rangle}{N}\right)}{H\left(\frac{\langle\hat{K}\rangle}{N}\right)}\frac{\left\langle\hat{K}\right\rangle}{N}\right)$$

and V_0 is defined by (181):

$$2\tau\frac{N}{V-V_0}+\frac{1}{2\sigma_X^2}\left\langle\left(\nabla_X R(X)\right)^2\right\rangle_{V/V_0} H^2\left(\frac{\left\langle\hat{K}\right\rangle}{N}\right)\left(1-\frac{H'\left(\frac{\langle\hat{K}\rangle}{N}\right)}{H\left(\frac{\langle\hat{K}\rangle}{N}\right)}\frac{\left\langle\hat{K}\right\rangle}{N}\right) \tag{182}$$
$$< 2\tau\left\|\Psi_0(X)\right\|^2+\frac{1}{2\sigma_X^2}\left(\nabla_X R(X)\right)^2 H^2\left(\frac{\hat{K}_X}{\left\|\Psi_0(X)\right\|^2}\right)\left(1-\frac{H'\left(\hat{K}_X\right)}{H\left(\hat{K}_X\right)}\frac{\hat{K}_X}{\left\|\Psi_0(X)\right\|^2}\right)$$

On V/V_0, $\|\Psi\|^2$ is given by (180) and on V_0, $\|\Psi\|^2=0$.

Below, we give explicitaly the form of $\Psi(X)$ form two different form of the function H.

A2.1.2 Introducing the K dependency

A2.1.2.1 First order condition To go beyond approximation (161) and solve for the field $\Psi(K,X)$ that minimizes (159), we come back to the full system for K and X:

$$\int \Psi^{\dagger}(K,X)\left(\left(-\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X-\left(\frac{\nabla_X R(K,X)}{\left\|\nabla_X R(K,X)\right\|}\right)H(K)+\tau\left|\Psi(K,X)\right|^2\right)\right)\right. \tag{183}$$
$$\left.-\nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K+u\left(K,X,\Psi,\hat{\Psi}\right)\right)-\frac{1}{2}\nabla_K u\left(K,X,\Psi,\hat{\Psi}\right)\right)\Psi(K,X)$$

with $u\left(K,X,\Psi,\hat{\Psi}\right)$ given by (160). We then look for a minimum of (183) of the form:

$$\Psi(K,X)=\Psi(X)\Psi_1(K-K_X) \tag{184}$$

with K_X given in (166):

$$K_X=\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\left\|\Psi(X)\right\|^2} \tag{185}$$

and Ψ_1 peaked around 0 and of norm 1. When $H(K)$ is slowly varying around K_X, the minimization of (183) for $\Psi_1(K-K_X)$ writes:

$$\nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K+u\left(K,X,\Psi,\hat{\Psi}\right)+\frac{1}{2}\nabla_K u\left(K,X,\Psi,\hat{\Psi}\right)\right)\Psi_1(K-K_X)=0 \tag{186}$$

Then, using that, in first approximation:

$$\int F_2\left(R\left(K',X\right)\right)\left\|\Psi\left(K',X\right)\right\|^2 dK'\simeq F_2\left(R\left(K_X,X\right)\right)\left\|\Psi(X)\right\|^2$$

Equation (186) becomes:

$$\nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K+K-\frac{F_2\left(R\left(K,X\right)\right)K_X}{F_2\left(R\left(K_X,X\right)\right)}\right)\Psi_1(K-K_X)=0 \tag{187}$$

A2.1.2.2 Solving (187) To solve the first order condition (187) we perform the change of variable:

$$\Psi_1(K-K_X)\rightarrow\exp\left(\frac{1}{\sigma_K^2}\int\left[K-\frac{F_2\left(R\left(K,X\right)\right)K_X}{F_2\left(R\left(K_X,X\right)\right)}\right]dK\right)\Psi_1(K-K_X)$$

and (187) is transformed into

$$-\frac{\sigma_K^2}{2}\nabla_K^2\Psi_1(K-K_X)+\frac{1}{2\sigma_K^2}\left(K-\frac{F_2\left(R\left(K,X\right)\right)K_X}{F_2\left(R\left(K_X,X\right)\right)}\right)^2\Psi_1(K-K_X)=0 \tag{188}$$

This equation can be solved by implementing the constraint:

$$\int\left\|\Psi_1(K-K_X)\right\|^2=1$$

and we find:

$$\begin{aligned}\Psi_1(K-K_X)&\simeq\mathcal{N}\exp\left(-\frac{1}{\sigma_K^2}\left(K-\frac{F_2\left(R\left(K,X\right)\right)K_X}{F_2\left(R\left(K_X,X\right)\right)}\right)^2\right)\\&\simeq\mathcal{N}\exp\left(-\frac{1}{\sigma_K^2}\left(K-K_X-(K-K_X)\frac{\partial_K R\left(K_X,X\right)F_2'\left(R\left(K_X,X\right)\right)}{F_2\left(R\left(K_X,X\right)\right)}K_X\right)^2\right)\\&=\mathcal{N}\exp\left(-\frac{1}{\sigma_K^2}\left(1-\frac{\partial_K R\left(K_X,X\right)F_2'\left(R\left(K,X\right)\right)}{F_2\left(R\left(K_X,X\right)\right)}K_X\right)^2(K-K_X)^2\right)\end{aligned}$$

with the normalization factor $\mathcal{N}$ given by:

$$\mathcal{N}=\sqrt{\frac{c}{\sigma_K^2\left(1-\frac{\partial_K R(K_X,X)F_2'(R(K,X))}{F_2(R(K_X,X))}K_X\right)^2}}$$

A2.1.2.3 Expression for the density of firms $\left\|\Psi(K,X)\right\|^2$ Having found Ψ_1, and using (180) and (184) we obtain the expression for $\left\|\Psi(K,X)\right\|^2$:

$$\begin{aligned}\left\|\Psi(K,X)\right\|^2&=\mathcal{N}\left\|\Psi\right\|^2\left(X,\left(\nabla_X R(X)\right)^2,\frac{\hat{K}_X}{\hat{K}_{X,0}}\right)\\&\quad\times\exp\left(-\frac{1}{\sigma_K^2}\left(K-\frac{F_2\left(R\left(K,X\right)\right)}{F_2\left(R\left(K_X,X\right)\right)\left\|\Psi(X)\right\|^2}\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}\right)^2\right)\\&=\left\|\Psi\right\|^2\left(X,\left(\nabla_X R(X)\right)^2,\frac{\hat{K}_X}{\hat{K}_{X,0}}\right)\frac{c\exp\left(-\frac{1}{\sigma_K^2}\left(1-\frac{\partial_K R(K_X,X)F_2'(R(K,X))}{F_2(R(K_X,X))}K_X\right)^2(K-K_X)^2\right)}{\frac{1}{\sigma_K^2}\left(1-\frac{\partial_K R(K_X,X)F_2'(R(K,X))}{F_2(R(K_X,X))}K_X\right)^2}\end{aligned} \tag{189}$$

for $X \in V/V_0$ and $\|\Psi(K,X)\|^2 = 0$ otherwise.

As stated in the text, note that the form of the exponential in (189) implies that:

$$\int K \|\Psi(K,X)\|^2 d\hat{K} = \int \hat{K} \left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}$$

A2.2 Examples

We solve the minimization for $\Psi(K,X)$ for two particular forms of the function $H(K)$.

A2.2.1 Example 1

We compute $\Psi(K,X)$ for the specific function:

$$H(y) = \left(\frac{y}{1+y}\right)^{\varsigma},\ H'(y) = \varsigma \frac{\left(\frac{y}{y+1}\right)^{\varsigma}}{y(y+1)}$$

We use the simplified equations (174) and (175) that yield:

$$D\left(\|\Psi\|^2\right) + \mu(X) = \tau \|\Psi(X)\|^2 + \frac{\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 \left(\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)^{\varsigma}\right)^2 \left(1 - \varsigma \frac{1}{\left(\frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2} + \left\langle\hat{K}\right\rangle\right)}\right)}{\left(\left\langle\hat{K}\right\rangle + \frac{\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)^{2\varsigma}}$$

or equivalently:

$$\begin{aligned} D\left(\|\Psi\|^2\right) + \mu(X) &= \tau \|\Psi(X)\|^2 \\ &+ \frac{\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 \left(\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}\right)^{2\varsigma}}{\left(\left\langle\hat{K}\right\rangle \|\Psi(X)\|^2 + \int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}\right)^{2\varsigma+1}} \\ &\times \left(\int \hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K} + (1-\varsigma)\left\langle\hat{K}\right\rangle \|\Psi(X)\|^2\right) \end{aligned}$$

For $\varsigma \simeq \frac{1}{2}$, this reduces to:

$$D\left(\|\Psi\|^2\right) + \mu(X) = \tau \|\Psi(X)\|^2 + \frac{\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 \hat{K}_X \left(\hat{K}_X + \frac{1}{2}\left\langle\hat{K}\right\rangle \|\Psi(X)\|^2\right)}{\left(\left\langle\hat{K}\right\rangle \|\Psi(X)\|^2 + \hat{K}_X\right)^2}$$

and for $\left\langle\hat{K}\right\rangle \|\Psi(X)\|^2 << \hat{K}_X$ this becomes:

$$D\left(\|\Psi\|^2\right) + \mu(X) \simeq \tau \|\Psi(X)\|^2 + \frac{\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 \hat{K}_X}{\left(\left\langle\hat{K}\right\rangle \|\Psi(X)\|^2 + \hat{K}_X\right)} \tag{190}$$

Two cases arise.

When $\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 << \tau$:

$$\begin{aligned}
\|\Psi(X)\|^2 &= \frac{\left(D\left(\|\Psi\|^2\right)-\tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\right)+\sqrt{\left(D\left(\|\Psi\|^2\right)-\tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\right)^2-4\tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\left(\frac{(\nabla_X R(X))^2}{\sigma_X^2}-D\left(\|\Psi\|^2\right)\right)}}{2\tau} \quad (191)\\
&= \frac{4\tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\left(\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2-D\left(\|\Psi\|^2\right)\right)}{2\tau\left(\left(D\left(\|\Psi\|^2\right)-\tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\right)-\sqrt{\left(D\left(\|\Psi\|^2\right)-\tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\right)^2-4\tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\left(\frac{(\nabla_X R(X))^2}{\sigma_X^2}-D\left(\|\Psi\|^2\right)\right)}\right)}
\end{aligned}$$

This is positive on the set:

$$\left\{\left(D\left(\|\Psi\|^2\right)-\tau\frac{\hat{K}_X}{\left\langle\hat{K}\right\rangle}\right)>0\right\}\cup\left\{\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2-D\left(\|\Psi\|^2\right)<0\right\} \quad (192)$$

To detail these two conditions, we write (190) for $\|\Psi(X)\|^2>0$:

$$D\left(\|\Psi\|^2\right)\simeq\tau\|\Psi(X)\|^2+\frac{\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2\frac{\hat{K}_X}{\langle\hat{K}\rangle}}{\left(\|\Psi(X)\|^2+\frac{\hat{K}_X}{\langle\hat{K}\rangle}\right)}$$

which is equivalent to:

$$\frac{\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2-D\left(\|\Psi\|^2\right)}{\left(\|\Psi(X)\|^2+\frac{\hat{K}_X}{\langle\hat{K}\rangle}\right)}\frac{\hat{K}_X}{\left\langle\hat{K}\right\rangle}=\frac{-\tau\|\Psi(X)\|^2+D\left(\|\Psi\|^2\right)-\tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}}{\left(\|\Psi(X)\|^2+\frac{\hat{K}_X}{\langle\hat{K}\rangle}\right)}\|\Psi(X)\|^2$$

Then, we have the implication:

$$\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2-D\left(\|\Psi\|^2\right)>0\Rightarrow D\left(\|\Psi\|^2\right)-\tau\frac{\hat{K}_X}{\left\langle\hat{K}\right\rangle}>0 \quad (193)$$

This implies that (192) is always satisfied, and formula (191) is valid for all X.

The second case arises when $\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 << \tau$. In this case, the solution is:

$$\|\Psi(X)\|^2=\frac{\left(D\left(\|\Psi\|^2\right)-\tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\right)-\sqrt{\left(D\left(\|\Psi\|^2\right)-\tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\right)^2-4\tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\left(\frac{(\nabla_X R(X))^2}{\sigma_X^2}-D\left(\|\Psi\|^2\right)\right)}}{2\tau}$$

This solution is valid, i.e. $\|\Psi(X)\|^2>0$, under the conditions:

$$\left\{D\left(\|\Psi\|^2\right)-\tau\frac{\hat{K}_X}{\left\langle\hat{K}\right\rangle}>0\right\}\cap\left\{\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2-D\left(\|\Psi\|^2\right)>0\right\} \quad (194)$$

and $\|\Psi\|^2=0$ for:

$$\left\{D\left(\|\Psi\|^2\right)-\tau\frac{\hat{K}_X}{\left\langle\hat{K}\right\rangle}<0\right\}\cup\left\{\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2-D\left(\|\Psi\|^2\right)<0\right\}$$

To detail these two conditions, we use the implication (193) that is equivalent to:

$$D\left(\|\Psi\|^2\right)-\tau\frac{\hat{K}_X}{\left\langle\hat{K}\right\rangle}<0\Rightarrow\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2-D\left(\|\Psi\|^2\right)<0$$

Consequently, $\|\Psi\left(X\right)\|^2=0$ only if:

$$\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2-D\left(\|\Psi\|^2\right)<0 \tag{195}$$

We find $D\left(\|\Psi\|^2\right)$ by integration of:

$$D\left(\|\Psi\|^2\right)+\mu\left(X\right)\simeq\tau\left\|\Psi\left(X\right)\right\|^2+\frac{\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2\hat{K}_X}{\left(\left\langle\hat{K}\right\rangle\left\|\Psi\left(X\right)\right\|^2+\hat{K}_X\right)} \tag{196}$$

and this leads to:

$$\begin{aligned}\int_{V/V_0}D\left(\|\Psi\|^2\right)&\simeq\tau N+\int_{V/V_0}\frac{\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2\frac{\hat{K}_X}{\langle\hat{K}\rangle}}{\left(\left\|\Psi\left(X\right)\right\|^2+\frac{\hat{K}_X}{\langle\hat{K}\rangle}\right)}\\&\simeq\tau N+\frac{1}{2}\int_{V/V_0}\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2=\tau N+\frac{1}{2}\left(V-V_0\right)\left\langle\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2\right\rangle_{V/V_0}\end{aligned}$$

we thus have:

$$D\left(\|\Psi\|^2\right)\simeq\frac{\tau N}{V-V_0}+\frac{1}{2}\left\langle\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2\right\rangle_{V/V_0} \tag{197}$$

and V_0 is defined using (195). It is the set of points X such that:

$$\frac{\tau N}{V-V_0}+\frac{1}{2}\left\langle\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2\right\rangle_{V/V_0}-\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2>0 \tag{198}$$

Similarly, the set V/V_0 is defined by:

$$\frac{\tau N}{V-V_0}+\frac{1}{2}\left\langle\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2\right\rangle_{V/V_0}-\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2<0 \tag{199}$$

To each function $R\left(X\right)$ and any $d>0$, we associate two functions that depend on the form of $\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2$ over the whole space. First, $v\left(V-V_0\right)$ is a decreasing function of $V-V_0$, defined by:

$$V\left(\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2>v\left(V-V_0\right)\right)=V-V_0 \tag{200}$$

Second, for every $d\geqslant 0$, the function $h\left(d\right)$ is given by:

$$h\left(d\right)=\frac{1}{\int_{\nabla_X R(X)>d}dX}\int_{\nabla_X R(X)>d}\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2dX \tag{201}$$

This is an increasing function of d.

Thus, we can rewrite (199) as:

$$\frac{\tau N}{V-V_0}+\frac{1}{2}\left\langle\frac{1}{\sigma_X^2}\left(\nabla_X R\left(X\right)\right)^2\right\rangle_{V/V_0}=v\left(V-V_0\right) \tag{202}$$

and moreover, by integration of (199) over V/V_0:

$$\left\langle \frac{1}{\sigma_X^2} \left(\nabla_X R(X)\right)^2 \right\rangle_{V/V_0} = h\left(\frac{\tau N}{V-V_0} + \frac{1}{2}\left\langle \frac{1}{\sigma_X^2} \left(\nabla_X R(X)\right)^2 \right\rangle_{V/V_0}\right) \tag{203}$$

Equations (202) and (203) combine as:

$$2\left(v\left(V-V_0\right) - \frac{\tau N}{V-V_0}\right) = h\left(v\left(V-V_0\right)\right) \tag{204}$$

which is an equation depending on the form of $R(X)$. If it has a solution, the set on which $\|\Psi(X)\|^2 = 0$ is defined by:

$$\frac{1}{\sigma_X^2} \left(\nabla_X R(X)\right)^2 < v\left(V-V_0\right)$$

and $D\left(\|\Psi\|^2\right)$ is given by

$$D\left(\|\Psi\|^2\right) \simeq v\left(V-V_0\right)$$

Once the solution of (204) is known, the constant $D\left(\|\Psi\|^2\right)$ is given by (197) and:

$$\|\Psi(X)\|^2 = \frac{2\frac{\hat{K}_X}{\langle\hat{K}\rangle}\left(\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 - D\left(\|\Psi\|^2\right)\right)}{D\left(\|\Psi\|^2\right) - \tau\frac{\hat{K}_X}{\langle\hat{K}\rangle} + \sqrt{\left(D\left(\|\Psi\|^2\right) - \tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\right)^2 - 4\tau\frac{\hat{K}_X}{\langle\hat{K}\rangle}\left(\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 - D\left(\|\Psi\|^2\right)\right)}} \tag{205}$$

for $X \in V/V_0$.

A2.2.2 Example 2

We choose $H(y) = y$ and equations (174) and (175) yield:

$$D\left(\|\Psi\|^2\right) \simeq \tau\|\Psi(X)\|^2 + \frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 \frac{\hat{K}_X}{\|\Psi(X)\|^2}$$

If:

$$D\left(\|\Psi\|^2\right) > 2\sqrt{\tau\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 \hat{K}_X} \tag{206}$$

then:

$$\|\Psi(X)\|^2 = \frac{1}{2\tau}\left(D\left(\|\Psi\|^2\right) - \sqrt{\left(D\left(\|\Psi\|^2\right)\right)^2 - 4\hat{K}_X\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 \tau}\right) > 0$$

To solve (197) and to find V_0, we compute $D\left(\|\Psi\|^2\right)$ by integrating (196) and (197) is still valid:

$$D\left(\|\Psi\|^2\right) \simeq \frac{\tau N}{V-V_0} + \frac{1}{2}\left\langle \frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 \right\rangle_{V/V_0} \tag{207}$$

We proceed as in the previous paragraph to find $D\left(\|\Psi\|^2\right)$ and V_0. Using (207), (206) becomes:

$$\frac{1}{4\tau\hat{K}_X}\left(\frac{\tau N}{V-V_0} + \frac{1}{2}\left\langle \frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 \right\rangle_{V/V_0}\right)^2 > \frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2 \tag{208}$$

Definitions (200) and (201) allow to rewrite (207) and (208):

$$\frac{1}{4\tau\hat{K}_X}\left(\frac{\tau N}{V-V_0}+\frac{1}{2}\left\langle\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2\right\rangle_{V/V_0}\right)^2=v\left(V-V_0\right)$$

$$\left\langle\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2\right\rangle_{V/V_0}=h\left(v\left(V-V_0\right)\right)$$

that reduce to an equation for $V-V_0$:

$$2\left(2\sqrt{\tau v\left(V-V_0\right)\hat{K}_X}-\frac{\tau N}{V-V_0}\right)=h\left(v\left(V-V_0\right)\right)$$

If it has a solution, the set on which $\|\Psi(X)\|^2=0$ is defined by:

$$\frac{1}{\sigma_X^2}\left(\nabla_X R(X)\right)^2<v\left(V-V_0\right)$$

and $D\left(\|\Psi\|^2\right)$ is given by

$$D\left(\|\Psi\|^2\right)\simeq 2\sqrt{\tau v\left(V-V_0\right)\hat{K}_X}$$

Appendix 3. Computation of the background field $\hat{\Psi}\left(\hat{K},\hat{X}\right)$ and average capital $\hat{K}_X$

A3.1 System for $\hat{\Psi}\left(\hat{K},\hat{X}\right)$

A3.1.1 Replacing quantities depending on (K,X)

Having found $\Psi(K,X)$, we can rewrite an action functional for $\hat{\Psi}\left(\hat{K},\hat{X}\right)$. To do so, we first replace the quantities depending on $\Psi(K,X)$ in the action (62). Given the form of this function we can use the approximation $K\simeq K_X$: at the collective level, the relevant quantity, from the point of view of investors are the sectors.

Using that:

$$\frac{R(K,X)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)}\simeq\frac{R(K,X)}{\int R\left(K'_{X'},X'\right)\left\|\Psi\left(X'\right)\right\|^2 dX'}$$

we first start by rewriting F_1 and we have:

$$F_1\left(\frac{R(K,X)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)},\Gamma(K,X)\right)\simeq F_1\left(\frac{R\left(K_X,X\right)}{\int R\left(K'_{X'},X'\right)\left\|\Psi\left(X'\right)\right\|^2 dX'},\Gamma(K,X)\right)$$

As explained in appendix 1, when $K\simeq K_X$, we also have:

$$\Gamma(K,X)=\int\frac{F_2\left(R(K,X)\right)}{K_X F_2\left(R\left(K_X,X\right)\right)\left\|\Psi\left(K_X,X\right)\right\|}\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}-1=0$$

Then, we rewrite the expression involving F_2 in (62):

$$\begin{aligned}\frac{F_2\left(R\left(K,\hat{X}\right)\right)}{\int F_2\left(R\left(K',\hat{X}\right)\right)\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\left\|\Psi\left(K,\hat{X}\right)\right\|^2 &\simeq \frac{F_2\left(R\left(K,\hat{X}\right)\right)}{F_2\left(R\left(K_{\hat{X}},\hat{X}\right)\right)\left\|\Psi\left(\hat{X}\right)\right\|^2}\left\|\Psi\left(K,\hat{X}\right)\right\|^2\\ &= \frac{F_2\left(R\left(K,\hat{X}\right)\right)\left\|\Psi_0\left(K-K_{\hat{X}}\right)\right\|^2}{F_2\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}\end{aligned}$$

and the $\hat{\Psi}\left(\hat{K},\hat{X}\right)$ part of the action functional (62) writes:

$$\begin{aligned}S_3+S_4=&-\int\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}-\hat{K}f\left(K,X,\Psi,\hat{\Psi}\right)\right)\right.\\ &\left.+\nabla_{\hat{X}}\left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}-g\left(K,X,\Psi,\hat{\Psi}\right)\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right)\end{aligned}\tag{209}$$

where:

$$\begin{aligned}f\left(\hat{X},\Psi,\hat{\Psi}\right) &= \frac{1}{\varepsilon}\int\left(\nabla_K R(K,X)-\gamma\frac{\int K'\left\|\Psi\left(K',X\right)\right\|^2}{K}+F_1\left(\frac{R(K,X)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)}\right)\right)\\ &\times\frac{F_2\left(R\left(K,\hat{X}\right)\right)\left\|\Psi_0\left(K-K_{\hat{X}}\right)\right\|^2}{F_2\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}dK\end{aligned}\tag{210}$$

$$\begin{aligned}g\left(\hat{X},\Psi,\hat{\Psi}\right) &= \int\left(\frac{\nabla_{\hat{X}}F_0\left(R\left(K,\hat{X}\right)\right)}{\left\|\nabla_{\hat{X}}R\left(K,\hat{X}\right)\right\|}+\nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K,\hat{X}\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)},\Gamma(K,X)\right)\right)\\ &\times\frac{\left\|\Psi\left(K,\hat{X}\right)\right\|^2 dK}{\int\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\end{aligned}\tag{211}$$

Another simplification arises for the function $F_2\left(R\left(K,\hat{X}\right)\right)$. Actually:

$$\begin{aligned}&\frac{F_2\left(R\left(K,\hat{X}\right)\right)}{\int F_2\left(R\left(K',\hat{X}\right)\right)\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\left\|\Psi\left(K,\hat{X}\right)\right\|^2\\ \simeq\ &\frac{F_2\left(R\left(K,\hat{X}\right)\right)}{\int F_2\left(R\left(K_{\hat{X}},\hat{X}\right)\right)\left\|\Psi\left(\hat{X}\right)\right\|^2}\left\|\Psi\left(K,\hat{X}\right)\right\|^2\\ \simeq\ &\frac{F_2\left(R\left(K,\hat{X}\right)\right)}{F_2\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}\left\|\Psi\left(K-K_{\hat{X}}\right)\right\|^2\end{aligned}$$

and by integration in (210) and (211), we have:

$$\begin{aligned}f\left(\hat{X},\Psi,\hat{\Psi}\right) &= \frac{1}{\varepsilon}\left(r\left(K_{\hat{X}},\hat{X}\right)-\gamma\left\|\Psi\left(\hat{X}\right)\right\|^2+F_1\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\int R\left(K'_{X'},X'\right)\left\|\Psi\left(X'\right)\right\|^2 dX'}\right)\right)\end{aligned}\tag{212}$$

$$g\left(\hat{X},\Psi,\hat{\Psi}\right) = \frac{\nabla_{\hat{X}}F_0\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}{\left\|\nabla_{\hat{X}}R\left(K_{\hat{X}},\hat{X}\right)\right\|}+\nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\int R\left(K'_{X'},X'\right)\left\|\Psi\left(X'\right)\right\|^2 dX'}\right)$$

In the sequel, for the sake of simplicity, we will write $f\left(\hat{X}\right)$ and $g\left(\hat{X}\right)$ for $f\left(\hat{X},K_{\hat{X}}\right)$ and $g\left(\hat{X},K_{\hat{X}}\right)$ respectively. We then perform the following change of variable in (209):

$$\begin{aligned}\hat{\Psi} &\rightarrow \exp\left(\frac{1}{\sigma_{\hat{X}}^{2}}\int g\left(\hat{X}\right)d\hat{X}\right)\hat{\Psi}\\ \hat{\Psi}^{\dagger} &\rightarrow \exp\left(\frac{1}{\sigma_{\hat{X}}^{2}}\int g\left(\hat{X}\right)d\hat{X}\right)\hat{\Psi}^{\dagger}\end{aligned}$$

so that (209) becomes:

$$\begin{aligned}S_{3}+S_{4} &= -\int\hat{\Psi}^{\dagger}\left(\frac{\sigma_{\hat{X}}^{2}}{2}\nabla_{\hat{X}}^{2}-\frac{1}{2\sigma_{\hat{X}}^{2}}\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}-\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\hat{\Psi} \qquad (213)\\ &-\int\hat{\Psi}^{\dagger}\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^{2}}{2}\nabla_{\hat{K}}-\hat{K}f\left(\hat{X},K_{\hat{X}}\right)\right)\right)\hat{\Psi}\end{aligned}$$

This action functional for $\hat{\Psi}$ will be minimized in the next paragraph. Note that we should also include to (213), the action functional S_1+S_2 evaluated at the background field Ψ, since this one depends on $\hat{\Psi}$. However, we have seen that at the background field Ψ, for $K\simeq K_X$, $u\left(K,X,\Psi,\hat{\Psi}\right)\simeq 0$ and the action functional S_1+S_2 defined in (159) reduces to:

$$S_{1}+S_{2}\simeq\int\Psi^{\dagger}\left(X\right)\left(-\nabla_{X}\left(\frac{\sigma_{X}^{2}}{2}\nabla_{X}-\left(\nabla_{X}R\left(X\right)H\left(K_{X}\right)\right)\right)+\tau\left|\Psi\left(X\right)\right|^{2}+\frac{\sigma_{K}^{2}-1}{2\varepsilon}\right)\Psi\left(X\right) \qquad (214)$$

and this depends on through K_X. Then, due to the first order condition for $\Psi(X)$, one has:

$$\frac{\delta}{\delta\hat{\Psi}}\left(S_{1}+S_{2}\right)=\frac{\delta K_{X}}{\delta\hat{\Psi}}\frac{\partial}{\partial K_{X}}\left(S_{1}+S_{2}\right)$$

We have assumed previously that $H(K_X)$ is slowly varying. Moreover, due to is definition:

$$\frac{\delta K_{X}}{\delta\hat{\Psi}\left(\hat{K},X\right)}=\frac{\hat{K}}{\left\Vert\Psi\left(X\right)\right\Vert^{2}}$$

In most of the cases, this reduces to:

$$\frac{\delta K_{X}}{\delta\hat{\Psi}\left(\hat{K},X\right)}\simeq\frac{\hat{K}}{D\left(\left\Vert\Psi\right\Vert^{2}\right)}<<\hat{K}$$

Consequently, we can assume that $\frac{\delta}{\delta\hat{\Psi}}\left(S_1+S_2\right)$ will be negligible with respect to the other quantities in the minimization with respect to $\hat{\Psi}\left(\hat{K},X\right)$. The rationale for this approximation is the following. The field action S_1+S_2 for $\Psi(X)$ depends on the global quantity $\int\hat{K}\left\Vert\hat{\Psi}\left(\hat{K},X\right)\right\Vert^{2}d\hat{K}$ that represents the total investment in sector X. While minimizing the field action S_1+S_2 with respect to $\hat{\Psi}\left(\hat{K},X\right)$, we compute the change in this action with respect to an individual variation $\hat{\Psi}\left(\hat{K},X\right)$, and the impact of this variation is, consequently, negligible.

A3.1.2 Minimization for $\hat{\Psi}\left(\hat{K},\hat{X}\right)$

Adding the Lagrange multiplier $\hat{\lambda}$ implementing the constraint $\int\left\Vert \hat{\Psi}\left(\hat{K},\hat{X}\right)\right\Vert ^{2}=\hat{N}$, the minimization of (417) with the functions given by (212) leads to the first order conditions:

$$0 = \left(\frac{\sigma_{\hat{X}}^{2}\nabla_{\hat{X}}^{2}}{2}-\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}}{2\sigma_{\hat{X}}^{2}}-\frac{\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)}{2}\right)\hat{\Psi}+\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^{2}\nabla_{\hat{K}}}{2}-\hat{K}f\left(\hat{X},K_{\hat{X}}\right)-\hat{\lambda}\right)\hat{\Psi} \quad (215)$$
$$-\left(\int\hat{\Psi}^{\dagger}\frac{\delta}{\delta\hat{\Psi}^{\dagger}}\left(\frac{1}{2\sigma_{\hat{X}}^{2}}\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\hat{\Psi}\right)-\left(\int\hat{\Psi}^{\dagger}\nabla_{\hat{K}}\frac{\delta}{\delta\hat{\Psi}^{\dagger}}\left(\hat{K}f\left(\hat{X},K_{\hat{X}}\right)\right)\hat{\Psi}\right)$$

Using that:

$$\frac{\delta}{\delta\hat{\Psi}^{\dagger}}K_{\hat{X}}=\frac{\hat{K}}{\left\Vert \Psi\left(\hat{X}\right)\right\Vert ^{2}}\hat{\Psi}$$

equation (215) becomes:

$$0 = \left(\frac{\sigma_{\hat{X}}^{2}}{2}\nabla_{\hat{X}}^{2}-\frac{1}{2\sigma_{\hat{X}}^{2}}\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}-\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\hat{\Psi} \quad (216)$$
$$+\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^{2}}{2}\nabla_{\hat{K}}-\hat{K}f\left(\hat{X},K_{\hat{X}}\right)\right)-\hat{\lambda}\right)\hat{\Psi}-F\left(\hat{X},K_{\hat{X}}\right)\hat{K}\hat{\Psi}$$

with:

$$F\left(\hat{X},K_{\hat{X}}\right)=\frac{\left\langle \nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}}{2\sigma_{\hat{X}}^{2}}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\right\rangle }{\left\Vert \Psi\left(\hat{X}\right)\right\Vert ^{2}}+\frac{\left\langle \nabla_{\hat{K}}\left(\hat{K}\nabla_{K_{\hat{X}}}f\left(\hat{X},K_{\hat{X}}\right)\right)\right\rangle }{\left\Vert \Psi\left(\hat{X}\right)\right\Vert ^{2}} \quad (217)$$

The brackets in (217) are given by:

$$\left\langle \nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}}{2\sigma_{\hat{X}}^{2}}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\right\rangle$$
$$= \int\hat{\Psi}^{\dagger}\left(\hat{X},\hat{K}\right)\nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}}{2\sigma_{\hat{X}}^{2}}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\hat{\Psi}\left(\hat{X},\hat{K}\right)d\hat{K}$$
$$\equiv \nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}}{2\sigma_{\hat{X}}^{2}}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\left\Vert \hat{\Psi}\left(\hat{X}\right)\right\Vert ^{2}$$

$$\left\langle \nabla_{\hat{K}}\left(\hat{K}\nabla_{K_{\hat{X}}}f\left(\hat{X},K_{\hat{X}}\right)\right)\right\rangle$$
$$= \int\hat{\Psi}^{\dagger}\left(\hat{X},K_{\hat{X}}\right)\nabla_{\hat{K}}\left(\hat{K}\nabla_{K_{\hat{X}}}f\left(\hat{X},K_{\hat{X}}\right)\right)\hat{\Psi}\left(\hat{X},K_{\hat{X}}\right)d\hat{K}$$
$$= -\nabla_{K_{\hat{X}}}f\left(\hat{X},K_{\hat{X}}\right)\int\left(\hat{K}\nabla_{\hat{K}}\left\Vert \hat{\Psi}\left(\hat{X},K_{\hat{X}}\right)\right\Vert ^{2}-\frac{2\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\left\Vert \hat{\Psi}\left(\hat{X},K_{\hat{X}}\right)\right\Vert ^{2}\right)d\hat{K}$$
$$= \nabla_{K_{\hat{X}}}f\left(\hat{X},K_{\hat{X}}\right)\left\Vert \hat{\Psi}\left(\hat{X}\right)\right\Vert ^{2}+\frac{\nabla_{K_{\hat{X}}}f^{2}\left(\hat{X},K_{\hat{X}}\right)}{\sigma_{\hat{K}}^{2}}\left\langle \hat{K}^{2}\right\rangle _{\hat{X}} \quad (218)$$

Where the average $\left\langle \hat{K}^2 \right\rangle_{\hat{X}}$ is defined by:

$$\left\langle \hat{K}^2 \right\rangle_{\hat{X}} = \int \left\| \hat{\Psi}\left(\hat{X}, \hat{K}\right) \right\|^2 d\hat{K}$$

The previous expression (218) for $F\left(\hat{X}, K_{\hat{X}}\right)$ can also be rewritten as:

$$\begin{aligned} F\left(\hat{X}, K_{\hat{X}}\right) &= \frac{\left\langle \nabla_{K_{\hat{X}}} \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right) \right\rangle}{\left\| \Psi\left(\hat{X}\right) \right\|^2} + \frac{\left\langle \nabla_{\hat{K}} \left(\hat{K} \nabla_{K_{\hat{X}}} f\left(\hat{X}, K_{\hat{X}}\right) \right) \right\rangle}{\left\| \Psi\left(\hat{X}\right) \right\|^2} \qquad (219) \\ &= \nabla_{K_{\hat{X}}} \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) + f\left(\hat{X}, K_{\hat{X}}\right) \right) \frac{\left\| \hat{\Psi}\left(\hat{X}\right) \right\|^2}{\left\| \Psi\left(\hat{X}\right) \right\|^2} \\ &\quad + \frac{\nabla_{K_{\hat{X}}} f^2\left(\hat{X}, K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2 \left\| \Psi\left(\hat{X}\right) \right\|^2} \left\langle \hat{K}^2 \right\rangle_{\hat{X}} \end{aligned}$$

It will be useful to rewrite the last term as:

$$\frac{\nabla_{K_{\hat{X}}} f^2\left(\hat{X}, K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2 \left\| \Psi\left(\hat{X}\right) \right\|^2} \left\langle \hat{K}^2 \right\rangle_{\hat{X}} \simeq \frac{\nabla_{K_{\hat{X}}} f^2\left(\hat{X}, K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2} \left\langle \hat{K} \right\rangle_{\hat{X}}^2 = \frac{\nabla_{K_{\hat{X}}} f^2\left(\hat{X}, K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2} \frac{\left\| \Psi\left(\hat{X}\right) \right\|^2}{\left\| \hat{\Psi}\left(\hat{X}\right) \right\|^2} \qquad (220)$$

Consequently:

$$\begin{aligned} F\left(\hat{X}, K_{\hat{X}}\right) &= \nabla_{K_{\hat{X}}} \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) + f\left(\hat{X}, K_{\hat{X}}\right) \right) \frac{\left\| \hat{\Psi}\left(\hat{X}\right) \right\|^2}{\left\| \Psi\left(\hat{X}\right) \right\|^2} \qquad (221) \\ &\quad + \frac{\nabla_{K_{\hat{X}}} f^2\left(\hat{X}, K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2} \frac{\left\| \Psi\left(\hat{X}\right) \right\|^2}{\left\| \hat{\Psi}\left(\hat{X}\right) \right\|^2} \end{aligned}$$

We also have an equation for $\hat{\Psi}^{\dagger}$ similar to (216):

$$\begin{aligned} 0 &= \left(\frac{\sigma_{\hat{X}}^2}{2} \nabla_{\hat{X}}^2 - \frac{1}{2\sigma_{\hat{X}}^2} \left(g\left(\hat{X}, K_{\hat{X}}\right) \right)^2 - \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) \right) \hat{\Psi}^{\dagger} \qquad (222) \\ &\quad + \left(\left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} + \hat{K} f\left(\hat{X}, K_{\hat{X}}\right) \right) \nabla_{\hat{K}} - \hat{\lambda} \right) \hat{\Psi} - F\left(\hat{X}, K_{\hat{X}}\right) \hat{K} \hat{\Psi}^{\dagger} \end{aligned}$$

A3.1.3 Resolution of (216)

A3.1.3.1 zeroth order in σ_X^2 We consider $\sigma_X^2 << 1$ (which means that fluctuation in $X <<$ fluctuation in K). Thus (216) writes at the lowest order:

$$\left(\nabla_{\hat{K}} \left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} - \hat{K} f\left(\hat{X}, K_{\hat{X}}\right) \right) - \frac{\left(g\left(\hat{X}\right)\right)^2}{2\sigma_{\hat{X}}^2} - \frac{\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)}{2} - F\left(\hat{X}, K_{\hat{X}}\right) \hat{K} - \hat{\lambda} \right) \hat{\Psi} = 0 \qquad (223)$$

Performing the change of variable:

$$\hat{\Psi} \rightarrow \exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}\right)\right)\hat{\Psi}$$

leads to the equation for $\hat{K}$:

$$\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}^2\hat{\Psi} - \left(\frac{\hat{K}^2}{2\sigma_{\hat{K}}^2} f^2\left(\hat{X}\right) + F\left(\hat{X}, K_{\hat{X}}\right)\hat{K} + \frac{1}{2} f\left(\hat{X}, K_{\hat{X}}\right) + \frac{\left(g\left(\hat{X}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) + \hat{\lambda}\right)\hat{\Psi} \simeq 0 \tag{224}$$

This equation can be normalized by dividing by $f^2\left(\hat{X}\right)$:

$$\frac{\sigma_{\hat{K}}^2\nabla_{\hat{K}}^2\hat{\Psi}}{2f^2\left(\hat{X}\right)} - \left(\frac{\hat{K}^2}{2\sigma_{\hat{K}}^2} + \frac{F\left(\hat{X}, K_{\hat{X}}\right)\hat{K}}{f^2\left(\hat{X}\right)} + \frac{\frac{f\left(\hat{X}, K_{\hat{X}}\right)}{2} + \frac{\left(g\left(\hat{X}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) + \hat{\lambda}}{f^2\left(\hat{X}\right)}\right)\hat{\Psi} \simeq 0$$

We then define:

$$y = \frac{\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}}{\sqrt{\sigma_{\hat{K}}^2}}\left(f^2\left(\hat{X}\right)\right)^{\frac{1}{4}}$$

and (216) is transformed into:

$$\nabla_y^2\hat{\Psi} - \left(\frac{y^2}{4} + \frac{\left(g\left(\hat{X}\right)\right)^2 + \sigma_{\hat{X}}^2\left(f\left(\hat{X}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} + \hat{\lambda}\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}}\right)\Psi \simeq 0 \tag{225}$$

Solutions of (225) are obtained by rewriting (225):

$$\hat{\Psi}'' + \left(p\left(\hat{X}, \hat{\lambda}\right) + \frac{1}{2} - \frac{1}{4}y^2\right)\hat{\Psi}$$

where:

$$p\left(\hat{X}, \hat{\lambda}\right) = -\frac{\left(g\left(\hat{X}\right)\right)^2 + \sigma_{\hat{X}}^2\left(f\left(\hat{X}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} + \hat{\lambda}\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}} - \frac{1}{2} \tag{226}$$

The solution of (225) is thus:

$$\hat{\Psi}_{\hat{\lambda}, C}^{(0)}\left(\hat{X}, \hat{K}\right) = \sqrt{C} D_{p\left(\hat{X}, \hat{\lambda}\right)}\left(\left(\left|f\left(\hat{X}\right)\right|\right)^{\frac{1}{2}}\frac{\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)}{\sigma_{\hat{K}}}\right) \tag{227}$$

where D_p denotes the parabolic cylinder function with parameter p and C is a normalization constant that will be computed as a function of λ using the constraint $\int\left\|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right\|^2 = \hat{N}$.

A similar equation to (223) can be obtained for $\hat{\Psi}^{\dagger}$. The equivalent of (215) is (222):

$$0 = \left(\frac{\sigma_{\hat{X}}^{2}}{2}\nabla_{\hat{X}}^{2}-\frac{1}{2\sigma_{\hat{X}}^{2}}\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}-\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\hat{\Psi} + \left(\left(\frac{\sigma_{\hat{K}}^{2}}{2}\nabla_{\hat{K}}+\hat{K}f\left(\hat{X},K_{\hat{X}}\right)\right)\nabla_{\hat{K}}-\hat{\lambda}\right)\hat{\Psi}-F\left(\hat{X},K_{\hat{X}}\right)\hat{K}\hat{\Psi} \tag{228}$$

The change of variable:

$$\hat{\Psi}^{\dagger}\rightarrow\exp\left(-\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)\hat{\Psi}^{\dagger}$$

and the approximation $\sigma_{\hat{X}}^{2}<<1$ lead ultimately to:

$$\frac{\sigma_{\hat{K}}^{2}}{2}\nabla_{\hat{K}}^{2}\hat{\Psi}^{\dagger}-\left(\frac{\hat{K}^{2}}{2\sigma_{\hat{K}}^{2}}f^{2}\left(\hat{X}\right)+\frac{1}{2}\nabla_{\hat{X}}f\left(\hat{X},K_{\hat{X}}\right)+\frac{1}{2\sigma_{\hat{X}}^{2}}\left(g\left(\hat{X}\right)\right)^{2}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)+F\left(\hat{X},K_{\hat{X}}\right)+\hat{\lambda}\right)\hat{\Psi}^{\dagger}\simeq 0 \tag{229}$$

which is the same equation as (224). Consequently, the solutions of (229) write:

$$\hat{\Psi}_{\lambda,C}^{(0)\dagger}\left(\hat{X},\hat{K}\right)=\hat{\Psi}_{\lambda,C}^{(0)}\left(\hat{X},\hat{K}\right)=\sqrt{C}D_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\left|f\left(\hat{X}\right)\right|\right)^{\frac{1}{2}}\frac{\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)}{\sigma_{\hat{K}}}\right) \tag{230}$$

To conclude this section, we detail the expressions for $\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}$ and $\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}$. Given the expression for $F\left(\hat{X},K_{\hat{X}}\right)$ in (221), the term $\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}$ arising in (227) and (230)

$$\begin{aligned}\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X},K_{\hat{X}}\right)} &= \frac{\sigma_{\hat{K}}^{2}}{f^{2}\left(\hat{X}\right)}\nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}}{2\sigma_{\hat{X}}^{2}}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)+f\left(\hat{X},K_{\hat{X}}\right)\right)\frac{\left\Vert\hat{\Psi}\left(\hat{X}\right)\right\Vert^{2}}{\left\Vert\Psi\left(\hat{X}\right)\right\Vert^{2}} \\ &\quad +\frac{\nabla_{K_{\hat{X}}}f^{2}\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X},K_{\hat{X}}\right)}\frac{\left\Vert\Psi\left(\hat{X}\right)\right\Vert^{2}}{\left\Vert\hat{\Psi}\left(\hat{X}\right)\right\Vert^{2}} \\ &\simeq \frac{\nabla_{K_{\hat{X}}}f\left(\hat{X},K_{\hat{X}}\right)}{f\left(\hat{X},K_{\hat{X}}\right)}\frac{\left\Vert\Psi\left(\hat{X}\right)\right\Vert^{2}}{\left\Vert\hat{\Psi}\left(\hat{X}\right)\right\Vert^{2}}\end{aligned} \tag{231}$$

$\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}$ arising in the definition (226) of $p\left(\hat{X},\hat{\lambda}\right)$ is equal to:

$$\begin{aligned}\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)} &= \frac{\sigma_{\hat{K}}^{2}}{2}\left(\left(\frac{\nabla_{K_{\hat{X}}}\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}+\sigma_{\hat{X}}^{2}\left(\nabla_{\hat{X}}^{2}g\left(\hat{X},K_{\hat{X}}\right)+\nabla_{K_{\hat{X}}}f\left(\hat{X},K_{\hat{X}}\right)\right)}{2\sigma_{\hat{X}}^{2}f\left(\hat{X},K_{\hat{X}}\right)}\right)\frac{\left\Vert\hat{\Psi}\left(\hat{X}\right)\right\Vert^{2}}{\left\Vert\Psi\left(\hat{X}\right)\right\Vert^{2}}\right. \\ &\quad \left.+2\nabla_{K_{\hat{X}}}f\left(\hat{X},K_{\hat{X}}\right)\frac{\left\Vert\Psi\left(\hat{X}\right)\right\Vert^{2}}{\left\Vert\hat{\Psi}\left(\hat{X}\right)\right\Vert^{2}}\right)^{2}\end{aligned} \tag{232}$$

and this simplifies as:

$$\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} \simeq 2\sigma_{\hat{K}}^2 \left(\nabla_{K_{\hat{X}}} f\left(\hat{X}, K_{\hat{X}}\right) \frac{\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}\right)^2 \tag{233}$$

since:

$$\begin{aligned}
&\frac{\nabla_{K_{\hat{X}}}\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2 + \sigma_{\hat{X}}^2\left(\nabla_{\hat{X}}^2 g\left(\hat{X}, K_{\hat{X}}\right) + \nabla_{K_{\hat{X}}} f\left(\hat{X}, K_{\hat{X}}\right)\right)}{2\sigma_{\hat{X}}^2 f\left(\hat{X}, K_{\hat{X}}\right)} \frac{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} \\
\sim\ & \frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2 + \sigma_{\hat{X}}^2\left(\nabla_{\hat{X}}^2 g\left(\hat{X}, K_{\hat{X}}\right) + \nabla_{K_{\hat{X}}} f\left(\hat{X}, K_{\hat{X}}\right)\right)}{2\sigma_{\hat{X}}^2 f\left(\hat{X}, K_{\hat{X}}\right)} \left(\frac{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2}\right) << 1
\end{aligned}$$

A3.1.3.2 Corrections in σ_X^2: To introduce the corrections in σ_X^2 in (216) we factor the solution as:

$$\begin{aligned}
\hat{\Psi}_{\lambda,C}\left(\hat{K}, \hat{X}\right) &= \sqrt{C}\exp\left(\frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}\right)\right) D_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right) \hat{\Psi}^{(1)}\left(\hat{K}, \hat{X}\right) \\
&\equiv \hat{\Psi}_{\lambda,C}^{(0)}\left(\hat{K}, \hat{X}\right) \hat{\Psi}^{(1)}\left(\hat{K}, \hat{X}\right)
\end{aligned}$$

and we look for $\hat{\Psi}^{(1)}$ of the form:

$$\hat{\Psi}^{(1)} = \exp\left(\sigma_X^2 h\left(K, X\right)\right) \tag{234}$$

Introducing the postulated form in (216) we are led to:

$$\frac{\sigma_X^2}{2}\nabla_{\hat{X}}^2\left(\hat{\Psi}^{(1)}\hat{\Psi}_{\lambda,C}^{(0)}\right) + \left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}^2\hat{\Psi}^{(1)}\right)\hat{\Psi}_{\lambda,C}^{(0)} + \left(\nabla_{\hat{K}}\hat{\Psi}^{(1)}\right)\left(\sigma_{\hat{K}}^2\nabla_{\hat{K}}\hat{\Psi}_{\lambda,C}^{(0)} - \hat{K}f\left(\hat{X}\right)\hat{\Psi}_{\lambda,C}^{(0)}\right) = 0$$

Written in terms of $h\left(\hat{K}, \hat{X}\right)$, this equation becomes at the first order in σ_X^2:

$$\frac{\nabla_{\hat{X}}^2\hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}} + \sigma_{\hat{K}}^2\nabla_{\hat{K}}^2 h\left(\hat{K}, \hat{X}\right) + 2\left(\nabla_{\hat{K}} h\left(\hat{K}, \hat{X}\right)\right)\left(\sigma_{\hat{K}}^2\frac{\nabla_{\hat{K}}\hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}} - \hat{K}f\left(\hat{X}\right)\right) = 0 \tag{235}$$

The solution of (235) is of the type:

$$\nabla_{\hat{K}}\left(h\left(K, X\right)\right) = C\left(\hat{K}, X\right)\exp\left(-2\int\left(\frac{\nabla_{\hat{K}}\hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}} - \frac{\hat{K}f\left(\hat{X}\right)}{\sigma_{\hat{K}}^2}\right)d\hat{K}\right) = C\left(\hat{K}, X\right)\exp\left(-\left(2\ln\hat{\Psi}_{\lambda,C}^{(0)} - \frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)\right)$$

where $C\left(X\right)$ satisfies:

$$C'\left(\hat{K}, X\right) = -\frac{\nabla_{\hat{X}}^2\hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}\sigma_{\hat{K}}^2}\exp\left(2\ln\hat{\Psi}_{\lambda,C}^{(0)} - \frac{\hat{K}^2 f\left(\hat{X}\right)}{\sigma_{\hat{K}}^2}\right) = -\frac{\nabla_{\hat{X}}^2\hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}}\left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^2\exp\left(-\frac{\hat{K}^2 f\left(\hat{X}\right)}{\sigma_{\hat{K}}^2}\right)$$

and the solution of (235) is:

$$\nabla_{\hat{K}}\left(h\left(K,X\right)\right)=\exp\left(-\left(2\ln\hat{\Psi}_{\lambda,C}^{(0)}-\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)\right)\left(C-\int\frac{\nabla_{\hat{X}}^{2}\hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}\sigma_{\hat{K}}^{2}}\left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^{2}\exp\left(-\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)d\hat{K}\right)$$

letting $C=0$, we obtain:

$$\nabla_{\hat{K}}\left(h\left(K,X\right)\right)=-\frac{1}{\sigma_{\hat{K}}^{2}\left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^{2}}\exp\left(\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)\left(\int\frac{\nabla_{\hat{X}}^{2}\hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}}\left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^{2}\exp\left(-\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)d\hat{K}\right)\quad(236)$$

To compute $h\left(K,X\right)$, we must estimate $\frac{\nabla_{\hat{X}}^{2}\hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}}$ in (236). To do so, we write, for $\varepsilon<<1$, i.e. $\left|f\left(\hat{X}\right)\right|>>1$:

$$\begin{aligned}
&\exp\left(\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)D_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)^{\frac{1}{2}}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)\right)\\
\simeq\;&\exp\left(\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)-\frac{\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\left|f\left(\hat{X}\right)\right|}{4\sigma_{\hat{K}}^{2}}\right)\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)^{\frac{1}{2}}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)\right)^{p\left(\hat{X},\hat{\lambda}\right)}\\
=\;&\exp\left(\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)-\frac{\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\left|f\left(\hat{X}\right)\right|}{4\sigma_{\hat{K}}^{2}}\right)\\
&\times\exp\left(\left(p\left(\hat{X},\hat{\lambda}\right)\right)\ln\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)^{\frac{1}{2}}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)\right)\right)
\end{aligned}$$

which allows to compute the successives derivatives of $\hat{\Psi}$. We find, for $f>0$:

$$\begin{aligned}\frac{\nabla_{\hat{X}}^{2}\hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}} &\simeq \left(\frac{-f'\sigma_{\hat{X}}^{2}\hat{\lambda}-g^{2}f'+2fgg'}{\sigma_{\hat{X}}^{2}f^{2}}\ln\left(\left(\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)\left(\frac{f\left(\hat{X}\right)}{\sigma_{\hat{K}}^{2}}\right)\right)^{\frac{1}{2}}\right)\right. \\ &+\frac{1}{2}\left(\frac{\left(g\left(\hat{X}\right)\right)^{2}+\sigma_{\hat{X}}^{2}\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}+\hat{\lambda}\right)}{\sigma_{\hat{X}}^{2}\sqrt{f^{2}\left(\hat{X}\right)}}+\frac{1}{2}\right)\frac{f'}{f} \\ &\left.+\frac{\hat{K}^{2}-\left(\frac{\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}}{2}\right)^{2}}{\sigma_{\hat{K}}^{2}}f'\right)^{2} \\ &\simeq \left(\frac{\left(4\hat{K}^{2}-\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\right)f'\left(X\right)}{4\sigma_{\hat{K}}^{2}}\right)^{2}\end{aligned} \tag{237}$$

The same approximation is valid for $f<0$ and we find for this case:

$$\frac{\nabla_{\hat{X}}^{2}\hat{\Psi}_{\lambda,C}^{(0)}}{\hat{\Psi}_{\lambda,C}^{(0)}}\simeq\left(\frac{\left(4\hat{K}^{2}+\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\right)f'\left(X\right)}{4\sigma_{\hat{K}}^{2}}\right)^{2}$$

Then, introducing $\mp$ to account for the sign of $-f$, (236) becomes:

$$\begin{aligned}
\nabla_{\hat{K}}\left(h\left(K,X\right)\right) = & -\frac{1}{\sigma_{\hat{K}}^{2}\left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^{2}}\exp\left(\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)\int\left(\frac{\left(4\hat{K}^{2}\mp\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\right)f'\left(X\right)}{4\sigma_{\hat{K}}^{2}}\right)^{2} \\
& \times\left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^{2}\exp\left(-\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)d\hat{K}
\end{aligned} \tag{238}$$

$$\begin{aligned}
\simeq & -\frac{1}{\sigma_{\hat{K}}^{2}\left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^{2}}\exp\left(\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)\int\left(\frac{\left(4\hat{K}^{2}\mp\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\right)f'\left(X\right)}{4\sigma_{\hat{K}}^{2}}\right)^{2} \\
& \times\exp\left(\frac{\hat{K}^{2}f\left(\hat{X}\right)-\frac{1}{2}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)d\hat{K}
\end{aligned} \tag{239}$$

$$\begin{aligned}
\simeq & -\frac{1}{\sigma_{\hat{K}}^{2}\left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^{2}}\exp\left(\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)\int\left(\sigma_{\hat{K}}^{2}\frac{\left(\hat{K}^{2}\mp\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\right)f'\left(X\right)}{\left(2\hat{K}f\left(\hat{X}\right)-\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)^{2}}\right)^{2} \\
& \times\partial_{\hat{K}}^{4}\exp\left(\frac{\hat{K}^{2}f\left(\hat{X}\right)-\frac{1}{2}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)d\hat{K}
\end{aligned} \tag{240}$$

Assuming $\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}<<1$, we have ultimately:

$$\begin{aligned}
\nabla_{\hat{K}}\left(h\left(K,X\right)\right) &\simeq -\frac{1}{\sigma_{\hat{K}}^{2}\left(\hat{\Psi}_{\lambda,C}^{(0)}\right)^{2}}\exp\left(\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)\left(\sigma_{\hat{K}}^{2}\frac{\left(\hat{K}^{2}\mp\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\right)f'\left(X\right)}{2\hat{K}f\left(\hat{X}\right)-\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|}\right)^{2} \\
&\quad\times\partial_{\hat{K}}^{3}\exp\left(\frac{\hat{K}^{2}f\left(\hat{X}\right)-\frac{1}{2}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right) \\
&= -\frac{\left(\frac{\left(\hat{K}^{2}-\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\right)f'(X)}{\sigma_{\hat{K}}^{2}}\right)^{2}}{2\hat{K}f\left(\hat{X}\right)-\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|} \\
&= -\frac{\left(\left(\hat{K}^{2}\mp\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\right)f'\left(X\right)\right)^{2}}{\left(\sigma_{\hat{K}}^{2}\right)^{2}\left(2\hat{K}f\left(\hat{X}\right)-\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}
\end{aligned}$$

Replacing in first approximation $\hat{K}$ by $\frac{\left\|\Psi\left(\hat{X}\right)\right\|^{2}\hat{K}_{\hat{X}}}{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^{2}}$ in (237), and using (236) and (234) leads to:

$$\hat{\Psi}^{(1)}\left(\hat{X}\right)=\sqrt{C}\exp\left(-\int\frac{\left(\left(\hat{K}^{2}\mp\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\right)f'\left(X\right)\right)^{2}}{\left(\sigma_{\hat{K}}^{2}\right)^{2}\left(2\hat{K}f\left(\hat{X}\right)-\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}d\hat{K}\right)$$

with C a constant to be computed using the normalization condition.

To find $\Psi^{\dagger}$, we need also $\hat{\Psi}^{(1)\dagger}$. Writing:

$$\hat{\Psi}^{(1)\dagger}=\exp\left(\sigma_{X}^{2}g\left(K,X\right)\right)$$

with a function $g\left(K,X\right)$ that satisfies:

$$\frac{\nabla_{\hat{X}}^{2}\hat{\Psi}_{\lambda,C}^{(0)\dagger}}{\hat{\Psi}_{\lambda,C}^{(0)}}+\sigma_{\hat{K}}^{2}\nabla_{\hat{K}}^{2}g\left(\hat{K},\hat{X}\right)+2\left(\nabla_{\hat{K}}g\left(\hat{K},\hat{X}\right)\right)\left(\sigma_{\hat{K}}^{2}\frac{\nabla_{\hat{K}}\hat{\Psi}_{\lambda,C}^{(0)\dagger}}{\hat{\Psi}_{\lambda,C}^{(0)}}+\hat{K}f\left(\hat{X}\right)\right)=0$$

with:

$$\hat{\Psi}_{\lambda,C}^{(0)\dagger}=\exp\left(-\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)D_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)^{\frac{1}{2}}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)\right)$$

we find:

$$\nabla_{\hat{K}}\left(g\left(K,X\right)\right)=-\frac{\nabla_{\hat{X}}^{2}\hat{\Psi}_{\lambda,C}^{(0)\dagger}}{\hat{\Psi}_{\lambda,C}^{(0)}}\exp\left(-\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)\left(\int\frac{\nabla_{\hat{X}}^{2}\hat{\Psi}_{\lambda,C}^{(0)\dagger}}{\hat{\Psi}_{\lambda,C}^{(0)\dagger}}\left(\hat{\Psi}_{\lambda,C}^{(0)\dagger}\right)^{2}\exp\left(\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)d\hat{K}\right)$$

and:

$$\hat{\Psi}^{(1)\dagger}\left(\hat{X}\right)=\sqrt{C}\exp\left(\int\frac{\left(\left(\hat{K}^{2}\pm\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\right)f'\left(X\right)\right)^{2}}{\sigma_{\hat{K}}^{2}\left(2\hat{K}f\left(\hat{X}\right)+\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}d\hat{K}\right)$$

where $\pm$ accounts for the sign of f.

Ultimately, coming back to the initial definition of the fields we obtain for $\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right)$ and $\hat{\Psi}_{\lambda,C}^{\dagger}\left(\hat{K},\hat{X}\right)$:

$$\begin{aligned}
\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right) &= \sqrt{C}\exp\left(-\sigma_{X}^{2}\int\frac{\left(\left(\hat{K}^{2}\mp\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\right)f'\left(X\right)\right)^{2}}{\left(\sigma_{\hat{K}}^{2}\right)^{2}\left(2\hat{K}f\left(\hat{X}\right)-\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}d\hat{K}\right) \\
&\quad\times\exp\left(\frac{1}{\sigma_{\hat{X}}^{2}}\int g\left(\hat{X}\right)d\hat{X}+\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)D_{p\left(\hat{X},\hat{\lambda}\right)}\left(\hat{K}\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)^{\frac{1}{2}}\right) \\
\hat{\Psi}_{\lambda,C}^{\dagger}\left(\hat{K},\hat{X}\right) &= \sqrt{C}\exp\left(\sigma_{X}^{2}\int\frac{\left(\left(\hat{K}^{2}\pm\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}\right)f'\left(X\right)\right)^{2}}{\left(\sigma_{\hat{K}}^{2}\right)^{2}\left(2\hat{K}f\left(\hat{X}\right)+\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}d\hat{K}\right) \\
&\quad\times\exp\left(-\left(\frac{1}{\sigma_{\hat{X}}^{2}}\int g\left(\hat{X}\right)d\hat{X}+\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)\right)D_{p\left(\hat{X},\hat{\lambda}\right)}\left(\hat{K}\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)^{\frac{1}{2}}\right)
\end{aligned}$$

A3.1.3.3 Computation of $\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^2$ As a consequence of the previsous result, we can compute $\left\|\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right)\right\|^2$. We start with $\hat{\Psi}^{(1)\dagger}\hat{\Psi}^{(1)}$. We have:

$$\begin{aligned}
\hat{\Psi}^{(1)\dagger}\hat{\Psi}^{(1)} &= C\exp\left(-\sigma_X^2\int\left(\frac{\left(\left(\hat{K}^2\mp\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'(X)\right)^2}{\left(\sigma_{\hat{K}}^2\right)^2\left(2\hat{K}f\left(\hat{X}\right)-\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}\right.\right.\\
&\left.\left.-\frac{\left(\left(\hat{K}^2\pm\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'(X)\right)^2}{\left(\sigma_{\hat{K}}^2\right)^2\left(2\hat{K}f\left(\hat{X}\right)+\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}d\hat{K}\right)\right)\\
&= C\exp\left(-\sigma_X^2\int\left(\frac{\left(\left(\hat{K}^2-\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'(X)\right)^2}{\sigma_{\hat{K}}^2\left(2\hat{K}\left|f\left(\hat{X}\right)\right|-\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}\right.\right.\\
&\left.\left.--\frac{\left(\left(\hat{K}^2+\frac{1}{4}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\right)f'(X)\right)^2}{\left(\sigma_{\hat{K}}^2\right)^2\left(2\hat{K}\left|f\left(\hat{X}\right)\right|+\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\left|f\left(\hat{X}\right)\right|\right)}d\hat{K}\right)\right)
\end{aligned}$$

And for $\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}<<1$:

$$\begin{aligned}
\hat{\Psi}^{(1)\dagger}\hat{\Psi}^{(1)} &\simeq C\exp\left(-\sigma_X^2\int\left(\frac{\left(\frac{3}{4}\hat{K}^2f'(X)\right)^2}{\left(\sigma_{\hat{K}}^2\right)^2\hat{K}\left|f\left(\hat{X}\right)\right|}-\frac{\left(\frac{5}{4}\hat{K}^2f'(X)\right)^2}{3\left(\sigma_{\hat{K}}^2\right)^2\hat{K}f\left(\hat{X}\right)}\right)d\hat{K}\right)\\
&= C\exp\left(-\frac{\sigma_X^2\hat{K}^4\left(f'(X)\right)^2}{96\left(\sigma_{\hat{K}}^2\right)^2\left|f\left(\hat{X}\right)\right|}\right)
\end{aligned}$$

Gathering the previous results, we obtain the norm of $\left\|\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right)\right\|^2$:

$$\left\|\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right)\right\|^2\simeq C\exp\left(-\frac{\sigma_X^2\hat{K}^4\left(f'(X)\right)^2}{96\left(\sigma_{\hat{K}}^2\right)^2\left|f\left(\hat{X}\right)\right|}\right)D^2_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right)\tag{241}$$

with:

$$f\left(\hat{X},K_{\hat{X}}\right) = \left(r\left(K_{\hat{X}},\hat{X}\right)-\gamma\left\|\Psi\left(\hat{X}\right)\right\|^2+F_1\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\int R\left(K'_{X'},X'\right)\left\|\Psi\left(X'\right)\right\|^2dX'}\right)\right)\tag{242}$$

$$g\left(\hat{X},K_{\hat{X}}\right) = \left(\frac{\nabla_{\hat{X}}F_0\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}{\left\|\nabla_{\hat{X}}R\left(K_{\hat{X}},\hat{X}\right)\right\|}+\nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\int R\left(K'_{X'},X'\right)\left\|\Psi\left(X'\right)\right\|^2dX'}\right)\right)\tag{243}$$

The solutions are parametrized by C and $\hat{\lambda}$ and $\hat{K}_{\hat{X}}$. Using the constraint $\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^2=\hat{N}$ will reduce the solutions to a one-parameter set of solutions. The computation of the average capital over this set will lead to the defining equation for $\hat{K}_{\hat{X}}$.

Replacing in first approximation $\hat{K}$ by its average $\frac{\|\Psi(\hat{X})\|^2\hat{K}_{\hat{X}}}{\|\hat{\Psi}(\hat{X})\|^2}$ in the first term yields:

$$\left\|\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right)\right\|^2\simeq C\exp\left(-\frac{\sigma_{\hat{X}}^2\left(\frac{\|\Psi(\hat{X})\|^2\hat{K}_{\hat{X}}}{\|\hat{\Psi}(\hat{X})\|^2}\right)^4\left(f'\left(X\right)\right)^2}{96\left(\sigma_{\hat{K}}^2\right)^2\left|f\left(\hat{X}\right)\right|}\right)D^2_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right) \tag{244}$$

A3.1.4 Estimation of $S_3\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right)+S_4\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right)$

For later purposes, we compute an estimation of $S_3\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right)+S_4\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right)$ for any background field $\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)$. We multiply (88)by $\hat{\Psi}^{\dagger}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)$ on the left and integrate the equation over $\hat{K}$ and $\hat{X}$. It yields:

$$0=S_3\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right)+S_4\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right)-\hat{\lambda}\int\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right\|^2d\hat{K}d\hat{X}-\int F\left(\hat{X},K_{\hat{X}}\right)\hat{K}\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right\|^2d\hat{K}d\hat{X}$$

Using the constraint about the number of investors:

$$\int\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right\|^2d\hat{K}=\hat{N}$$

we find:

$$S_3\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right)+S_4\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right)=\hat{\lambda}\hat{N}+\int F\left(\hat{X},K_{\hat{X}}\right)\hat{K}\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right\|^2d\hat{K}d\hat{X}$$

Moreover, equation (90) implies[57]:

$$\begin{aligned}&\int F\left(\hat{X},K_{\hat{X}}\right)\hat{K}\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right\|^2d\hat{K}d\hat{X} \qquad (245)\\=&\int K_{\hat{X}}\nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)+f\left(\hat{X},K_{\hat{X}}\right)\right)\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2d\hat{X}\\&+\int K_{\hat{X}}\frac{\nabla_{K_{\hat{X}}}f^2\left(\hat{X},K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\left\langle\hat{K}^2\right\rangle_{\hat{X}}d\hat{X}\end{aligned}$$

In our applications the involved functions are roughly power functions in $K_{\hat{X}}$, and consequently, the integral $\int F\left(\hat{X},K_{\hat{X}}\right)\hat{K}\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right\|^2d\hat{K}d\hat{X}$ is of order:

$$\int\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)+f\left(\hat{X},K_{\hat{X}}\right)\right)\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2d\hat{X}+\int\frac{f^2\left(\hat{X},K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\left\langle\hat{K}^2\right\rangle_{\hat{X}}d\hat{X} \tag{246}$$

[57] All averages in the next formula are computed in state $\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)$.

Since $\left\langle \hat{K}^2 \right\rangle_{\hat{X}} \simeq K_{\hat{X}}^2 \frac{\|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2}$, the second term in (??) is negligible if we assume $\frac{\|\Psi(\hat{X})\|^2}{\|\hat{\Psi}(\hat{X})\|^2} << 1$, i.e. the number of firms is smaller than the number of investors. Consequently, (??) reduces to:

$$\begin{aligned}\int \left(\frac{\left(g\left(\hat{X}, K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) + f\left(\hat{X}, K_{\hat{X}}\right) \right) \left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2 d\hat{X} &\lesssim \int M \left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2 d\hat{X} \\ &= M\hat{N}\end{aligned}$$

where M is the lowest bound for $\left|\hat{\lambda}\right|$, computed below in (267) and (268). Our previous estimation relies on $\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} << 1$,which is true for $f^2\left(\hat{X}\right) >> 1$. As a consequence:

$$S_3\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right) + S_4\left(\hat{\Psi}_{\hat{\lambda}}\left(\hat{K}, \hat{X}\right)\right) = \left(\hat{\lambda} + M\right)\hat{N} = -\left(\left|\hat{\lambda}\right| - M\right)\hat{N} \tag{247}$$

A3.1.4 Identification of $K_{\hat{X}}$ and $\left\|\Psi\left(\hat{X}\right)\right\|^2$:

A3.1.4.1 Formula depending on $\hat{\lambda}$ and C In this paragraph, we compute the average capital $K_{\hat{X}}$ and the density of investors $\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2$ at $\hat{X}$ that are defined by using (177):

$$\begin{aligned}K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2 &= \int_0^\infty \hat{K} C \exp\left(-\frac{\sigma_{\hat{X}}^2 u\left(\hat{X}, \hat{K}_{\hat{X}}\right)}{\left(\sigma_{\hat{K}}^2\right)^2}\right) \\ &\times D_{p\left(\hat{X},\hat{\lambda}\right)}^2\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right) d\hat{K}\end{aligned} \tag{248}$$

and:

$$\begin{aligned}\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2 &= C\int_0^\infty \exp\left(-\frac{\sigma_{\hat{X}}^2 u\left(\hat{X}, \hat{K}_{\hat{X}}\right)}{\left(\sigma_{\hat{K}}^2\right)^2}\right) \\ &\times D_{p\left(\hat{X},\hat{\lambda}\right)}^2\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right) d\hat{K}\end{aligned}$$

with:

$$u\left(\hat{X}, \hat{K}_{\hat{X}}\right) = \frac{\left(\frac{\|\Psi(\hat{X})\|^2 \hat{K}_{\hat{X}}}{\|\hat{\Psi}(\hat{X})\|^2}\right)^4 \left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|} \tag{249}$$

Note that in these formulas, $K_{\hat{X}}$ and $\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2$ depend implicitely of $\hat{\lambda}$ since they have been computed in the state defined by the background field $\hat{\Psi}_{\lambda,C}\left(\hat{K}, \hat{X}\right)$. In the sequel, for the sake of simplicity, $\hat{\Psi}_{\lambda,C}\left(\hat{K}, \hat{X}\right)$, the indices λ and C may be omitted.

We will also need $\frac{K_{\hat{X}}\left\|\Psi(\hat{X})\right\|^2}{\left\|\hat{\Psi}(\hat{X})\right\|^2}$ that arises in (249):

$$\frac{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}=\frac{\int_0^{\infty}\hat{K}D^2_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma^2_{\hat{K}}}\right)^{\frac{1}{2}}\left(\hat{K}+\frac{\sigma^2_{\hat{K}}F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\right)d\hat{K}}{\int^{\infty}_{\frac{\sigma^2_{\hat{K}}F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}}\hat{K}D^2_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma^2_{\hat{K}}}\right)^{\frac{1}{2}}\hat{K}\right)d\hat{K}}$$

By a change of variable $\hat{K}+\frac{\sigma^2_{\hat{K}}F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\rightarrow\hat{K}$ we can also write:

$$K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\simeq C\exp\left(-\frac{\sigma^2_{\hat{X}}u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma^2_{\hat{K}}}\right)\int^{\infty}_{\frac{\sigma^2_{\hat{K}}F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}}\hat{K}D^2_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma^2_{\hat{K}}}\right)^{\frac{1}{2}}\hat{K}\right)d\hat{K}$$

$$\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2\simeq C\exp\left(-\frac{\sigma^2_{\hat{X}}u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{16\sigma^2_{\hat{K}}}\right)\int^{\infty}_{\frac{\sigma^2_{\hat{K}}F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}}D^2_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma^2_{\hat{K}}}\right)^{\frac{1}{2}}\hat{K}\right)d\hat{K}$$

and by a zeroth order expansion around 0 of $\hat{K}D^2_{p\left(\hat{X},\hat{\lambda}\right)}$ and $D^2_{p\left(\hat{X},\hat{\lambda}\right)}$ we have:

$$K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\simeq C\exp\left(-\frac{\sigma^2_{\hat{X}}u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{16\sigma^2_{\hat{K}}}\right)\int_0^{\infty}\hat{K}D^2_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma^2_{\hat{K}}}\right)^{\frac{1}{2}}\hat{K}\right)d\hat{K}\tag{250}$$

$$\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2\simeq C\exp\left(-\frac{\sigma^2_{\hat{X}}u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{16\sigma^2_{\hat{K}}}\right)\left(\int_0^{\infty}D^2_{p\left(\hat{X},\hat{\lambda}\right)}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma^2_{\hat{K}}}\right)^{\frac{1}{2}}\hat{K}\right)d\hat{K}-\frac{\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma^2_{\hat{K}}}\right)^{-\frac{1}{2}}2^{\frac{p\left(\hat{X},\hat{\lambda}\right)}{2}}\sqrt{\pi}}{\Gamma\left(\frac{1-p\left(\hat{X},\hat{\lambda}\right)}{2}\right)}\frac{\sigma^2_{\hat{K}}F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\tag{251}$$

To compute $\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2$ we use that the function D satisfies:

$$\int D^2_p=\frac{\sqrt{\pi}}{2^{\frac{3}{2}}}\frac{\text{Psi}\left(\frac{1}{2}-\frac{p}{2}\right)-\text{Psi}\left(-\frac{p}{2}\right)}{\Gamma\left(-p\right)}$$

The computation of the norm implies a second change of variable $\hat{K}\rightarrow\hat{K}\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma^2_{\hat{K}}}\right)^{\frac{1}{2}}$ and we obtain for (251):

$$\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2=\int\left\|\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right)\right\|^2 d\hat{K} \tag{252}$$

$$= C\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{16\sigma_{\hat{K}}^2}\right)\left(\int D^2_{p\left(\hat{X},\hat{\lambda}\right)}\left(\hat{K}\left(f^2\left(\hat{X}\right)\right)^{\frac{1}{4}}\right)dK-\frac{\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}}2^{\frac{p\left(\hat{X},\hat{\lambda}\right)}{2}}\sqrt{\pi}}{\Gamma\left(\frac{1-p\left(\hat{X},\hat{\lambda}\right)}{2}\right)}\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)$$

$$= C\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{16\sigma_{\hat{K}}^2}\right)\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}}$$

$$\times\left(\frac{\sqrt{\pi}}{2^{\frac{3}{2}}}\frac{\text{Psi}\left(\frac{1-p\left(\hat{X},\hat{\lambda}\right)}{2}\right)-\text{Psi}\left(-\frac{p\left(\hat{X},\hat{\lambda}\right)}{2}\right)}{\Gamma\left(-p\left(\hat{X},\hat{\lambda}\right)\right)}-\frac{2^{\frac{p\left(\hat{X},\hat{\lambda}\right)}{2}}\sqrt{\pi}}{\Gamma\left(\frac{1-p\left(\hat{X},\hat{\lambda}\right)}{2}\right)}\frac{\sigma_{\hat{K}}^2 F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)$$

Expression (250) is computed using that:

$$\int_0^\infty zD_p^2(z)\,dz=\int_0^\infty D_{p+1}(z)\,D_p(z)\,dz+p\int_0^\infty D_{p-1}(z)\,D_p(z)\,dz$$

$$\int_0^\infty zD_p^2(z)\,dz=\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(-\frac{p}{2}\right)}{2^{p+2}\Gamma(-p-1)\,\Gamma(-p)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma(-p)\,\Gamma(-p+1)}$$

and:

$$\int\hat{K}D^2_{p\left(\hat{X},\hat{\lambda}\right)}\left(\hat{K}\left(f^2\left(\hat{X}\right)\right)^{\frac{1}{4}}\right)=\left(f\left(\hat{X}\right)\right)^{-1}\int uD^2_{p\left(\hat{X},\hat{\lambda}\right)}(u)$$

We obtain:

$$K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2 \simeq \exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{16\left(\sigma_{\hat{K}}^2\right)^2}\right)\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{-1}C \tag{253}$$

$$\times\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma(-p-1)\,\Gamma(-p)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma(-p)\,\Gamma(-p+1)}\right)$$

where:

$$p=p\left(\hat{X},\hat{\lambda}\right) \tag{254}$$

Ultimately we can compute $\frac{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}$:

$$\frac{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2} \simeq \left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}}\frac{\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma(-p-1)\Gamma(-p)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma(-p)\Gamma(-p+1)}}{\frac{\sqrt{\pi}}{2^{\frac{3}{2}}}\frac{\text{Psi}\left(\frac{1-p}{2}\right)-\text{Psi}\left(-\frac{p}{2}\right)}{\Gamma(-p)}}$$

$$\equiv \left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}}h(p)$$

$$\simeq \left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}}\sqrt{p+\frac{1}{2}}$$

so that:

$$\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\left(\sigma_{\hat{K}}^2\right)^2}\right)\simeq\exp\left(-\frac{\sigma_X^2\left(p+\frac{1}{2}\right)^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)\tag{255}$$

We end this section by finding asymptotic form for $\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2$ and $K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2$

For $\varepsilon<<1$ an asymptotic form yields that:

$$D_{p\left(\hat{X},\hat{\lambda}\right)}\left(\hat{K}\left(f^2\left(\hat{X}\right)\right)^{\frac{1}{4}}\right)\simeq\exp\left(-\frac{\hat{K}^2\left|f\left(\hat{X}\right)\right|}{4\sigma_{\hat{K}}^2}\right)\left(\hat{K}\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\right)^{p\left(\hat{X},\hat{\lambda}\right)}\tag{256}$$

and we obtain:

$$\begin{aligned}\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2 &= C\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\right)\\ &\quad\times\int_0^\infty\exp\left(-\frac{\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\left|f\left(\hat{X}\right)\right|}{2\sigma_{\hat{K}}^2}\right)\left(\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\right)^{2p\left(\hat{X},\hat{\lambda}\right)}d\hat{K}\end{aligned}$$

A change of variable $w=\frac{\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2\left|f\left(\hat{X}\right)\right|}{2\sigma_{\hat{K}}^2}$ leads to:

$$\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2\simeq C\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\right)\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{-\frac{1}{2}}\left(2^{p\left(\hat{X},\hat{\lambda}\right)-\frac{1}{2}}\Gamma\left(p\left(\hat{X},\hat{\lambda}\right)+\frac{1}{2}\right)-\frac{2^{\frac{p\left(\hat{X},\hat{\lambda}\right)}{2}}\sqrt{\pi}}{\Gamma\left(\frac{1-p\left(\hat{X},\hat{\lambda}\right)}{2}\right)}\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)\tag{257}$$

By the same token we can use the asymptotic form (256) to find $K_{\hat{X}}$:

$$\begin{aligned}K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2 &\simeq C\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\right)\int\hat{K}\exp\left(-\frac{\hat{K}^2\left|f\left(\hat{X}\right)\right|}{2\sigma_{\hat{K}}^2}\right)\left(\hat{K}\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\right)^{2p\left(\hat{X},\hat{\lambda}\right)}d\hat{K}\\ &= \frac{\sigma_{\hat{K}}^2C\exp\left(-\frac{\sigma_X^2u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\right)}{\left|f\left(\hat{X}\right)\right|}\int y\exp\left(-\frac{y^2}{2}\right)y^{2p\left(\hat{X},\hat{\lambda}\right)}\end{aligned}$$

We set $y=\sqrt{2w}$ and we obtain:

$$\begin{aligned}K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2 &\simeq C\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\right)2^{p\left(\hat{X},\hat{\lambda}\right)}\frac{\sigma_{\hat{K}}^2}{\left|f\left(\hat{X}\right)\right|}\int\exp\left(-w\right)w^{p\left(\hat{X},\hat{\lambda}\right)}dw\\ &= C\exp\left(-\frac{\sigma_X^2 u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^2}\right)2^{p\left(\hat{X},\hat{\lambda}\right)}\frac{\sigma_{\hat{K}}^2}{\left|f\left(\hat{X}\right)\right|}\Gamma\left(p\left(\hat{X},\hat{\lambda}\right)+1\right)\end{aligned}$$

A3.1.4.2 Computation of C as a function of $\hat{\lambda}$: Ultimately, we need to determine the value of the Lagrange multiplier $\hat{\lambda}$ and of the associated value of C. We do so by integrating (241) and the result is constrained to be $\hat{N}$, the total number of agents:

$$\hat{N}=\int\left\|\hat{\Psi}_{\lambda,C}\left(\hat{K},\hat{X}\right)\right\|^{2}d\hat{K}d\hat{X}=\int\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^{2}d\hat{X}$$

Using (252) and (255), we have:

$$\begin{aligned}\hat{N} &= \int\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^{2}\simeq\int C\exp\left(-\frac{\sigma_{X}^{2}u\left(\hat{X},\hat{K}_{\hat{X}}\right)}{\sigma_{\hat{K}}^{2}}\right)\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)^{-\frac{1}{2}} \\ &\quad\times\left(\frac{\sqrt{\pi}}{2^{\frac{3}{2}}}\frac{\mathrm{Psi}\left(\frac{1-p\left(\hat{X},\hat{\lambda}\right)}{2}\right)-\mathrm{Psi}\left(-\frac{p\left(\hat{X},\hat{\lambda}\right)}{2}\right)}{\Gamma\left(-p\left(\hat{X},\hat{\lambda}\right)\right)}-\frac{2^{\frac{p\left(\hat{X},\hat{\lambda}\right)}{2}}\sqrt{\pi}}{\Gamma\left(\frac{1-p\left(\hat{X},\hat{\lambda}\right)}{2}\right)}\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)d\hat{X} \\ &\simeq \int C\exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}\right)\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)^{-\frac{1}{2}}\frac{\sqrt{\pi}}{2^{\frac{3}{2}}}\frac{\mathrm{Psi}\left(\frac{1-p\left(\hat{X},\hat{\lambda}\right)}{2}\right)-\mathrm{Psi}\left(-\frac{p\left(\hat{X},\hat{\lambda}\right)}{2}\right)}{\Gamma\left(-p\left(\hat{X},\hat{\lambda}\right)\right)}d\hat{X}\end{aligned}\tag{258}$$

with f and g given by (242) and (243). We thus obtain C as a function of $\hat{\lambda}$. For $f\left(\hat{X}\right)$ slowly varying around its average we can replace $\left|f\left(\hat{X}\right)\right|$ and $f'\left(X\right)$ by $\left\langle\left|f\left(\hat{X}\right)\right|\right\rangle$ and $\left\langle f'\left(X\right)\right\rangle$, where the bracket $\left\langle A\left(\hat{X}\right)\right\rangle$ represents the average of the quantity $A\left(\hat{X}\right)$ over the sectors space. Given that the integrated function is of order $\Gamma\left(p\right)$, we can replace the integral by the maximal values of the integrand. Consequently, we have:

$$C\left(\bar{p}\left(\hat{\lambda}\right)\right)\simeq\frac{\exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(\frac{\left(\bar{p}\left(\hat{\lambda}\right)+\frac{1}{2}\right)f'\left(X_{0}\right)}{f\left(\hat{X}_{0}\right)}\right)^{2}}{96\left|f\left(\hat{X}_{0}\right)\right|}\right)\hat{N}\Gamma\left(-\bar{p}\left(\hat{\lambda}\right)\right)}{\left(\frac{\left\langle\left|f\left(\hat{X}\right)\right|\right\rangle}{\sigma_{\hat{K}}^{2}}\right)^{-\frac{1}{2}}V_{r}\left(\mathrm{Psi}\left(-\frac{\bar{p}\left(\hat{\lambda}\right)-1}{2}\right)-\mathrm{Psi}\left(-\frac{\bar{p}\left(\hat{\lambda}\right)}{2}\right)\right)}\tag{259}$$

where:

$$\bar{p}\left(\hat{\lambda}\right)=\left(-\frac{\frac{\left(g\left(\hat{X}_{0}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+\left(f\left(\hat{X}_{0}\right)+\frac{1}{2}\left|f\left(\hat{X}_{0}\right)\right|+\nabla_{\hat{X}}g\left(\hat{X}_{0},K_{\hat{X}_{0}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X}_{0},K_{\hat{X}_{0}}\right)}{2f^{2}\left(\hat{X}_{0}\right)}+\hat{\lambda}\right)}{\left|f\left(\hat{X}_{0}\right)\right|}\right)\tag{260}$$

and:

$$\hat{X}_{0}=\arg\min_{\hat{X}}\left(\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(\frac{\left(p\left(\hat{\lambda}\right)+\frac{1}{2}\right)f'\left(X\right)}{f\left(\hat{X}\right)}\right)^{2}}{96\left|f\left(\hat{X}\right)\right|}\right)\tag{261}$$

and V_r is the volume of the reduced space where the maximum is reached defined by:

$$V_{r}=\sum_{\hat{X}/p\left(\hat{X},\hat{\lambda}\right)=\bar{p}\left(\hat{\lambda}\right)}\frac{1}{\left|\frac{\left(\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^{2}\right)''}{C}\right|}$$

We thus can replace C by $C\left(\hat{\lambda}\right)$ and we are left with an infinite number of solutions of (223) parametrized by $\hat{\lambda}$ and given by (241). We write $\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right\|^{2}$ the solution for $\hat{\lambda}$.

A3.1.4.2 Identification equation for $K_{\hat{X}}$ To each state $\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{K},\hat{X}\right)\right\|^{2}$, we can associate an average level of $K_{\hat{X},\hat{\lambda}}$ satisfying (253) rewritten as a function of $\hat{\lambda}$. Using (255) we find:

$$\begin{aligned}
K_{\hat{X},\hat{\lambda}}\left\|\hat{\Psi}_{\hat{\lambda}}\left(\hat{X}\right)\right\|^{2} &= \hat{K}_{\hat{X},\hat{\lambda}} \\
&= \exp\left(-\frac{\sigma_{\hat{X}}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}\right)\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)^{-1} \\
&\quad\times C\left(\bar{p}\left(\hat{\lambda}\right)\right)\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right)
\end{aligned} \tag{262}$$

where:

$$p\left(\hat{X},\hat{\lambda}\right)=-\frac{\left(g\left(\hat{X}\right)\right)^{2}+\sigma_{\hat{X}}^{2}\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}+\hat{\lambda}\right)}{\sigma_{\hat{X}}^{2}\sqrt{f^{2}\left(\hat{X}\right)}}-\frac{1}{2} \tag{263}$$

As explained in the core of the paper, to compute $K_{\hat{X}}$ we must average (262) over $\hat{\lambda}$ with the weight $\exp\left(-\left(S_3+S_4\right)\right)$. Given equation (216), a solution (241) for a given $\hat{\lambda}$ and taking into account the constraint $\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^{2}=\hat{N}$, has the associated normalized weight (see (247)):

$$w\left(\left|\hat{\lambda}\right|\right)=\frac{\exp\left(-\left(\left|\hat{\lambda}\right|-M\right)\hat{N}\right)}{\int_{\left|\hat{\lambda}\right|>M}\exp\left(-\left(\left|\hat{\lambda}\right|-M\right)\hat{N}\right)d\left|\hat{\lambda}\right|}$$

with M is the lower bound for $\left|\hat{\lambda}\right|$.

This lower bound is found by considering (224) and adding the term proportional to $\frac{\sigma_{\hat{X}}^{2}}{2}$:

$$\frac{\sigma_{\hat{X}}^{2}}{2}\nabla_{\hat{X}}^{2}\hat{\Psi}+\nabla_{y}^{2}\hat{\Psi}-\left(\sqrt{f^{2}\left(\hat{X}\right)}\frac{y^{2}}{4}+\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}+\hat{\lambda}\right)\right)\Psi \tag{264}$$

multiplying (264) by $\hat{\Psi}^{\dagger}$ and integrating. It yields:

$$\begin{aligned}
0 &= -\frac{\sigma_{\hat{X}}^{2}}{2}\int\left(\nabla_{\hat{X}}\hat{\Psi}^{\dagger}\right)\left(\nabla_{\hat{X}}\hat{\Psi}\right) \\
&\quad-\frac{1}{2}\int\sqrt{f^{2}\left(\hat{X}\right)}\left(\left(\nabla_{y}\hat{\Psi}^{\dagger}\right)\left(\nabla_{y}\hat{\Psi}\right)+\hat{\Psi}^{\dagger}\frac{y^{2}}{4}\hat{\Psi}\right)+\int\hat{\Psi}_{y=0}^{\dagger}\left(\nabla_{y}\hat{\Psi}\right)_{y=0} \\
&\quad-\int\hat{\Psi}^{\dagger}\left(\sqrt{f^{2}\left(\hat{X}\right)}\frac{y^{2}}{4}+\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}+\hat{\lambda}\right)\right)\Psi
\end{aligned} \tag{265}$$

The first part of the right hand side in (265):

$$-\frac{\sigma_{\hat{X}}^{2}}{2}\int\left(\nabla_{\hat{X}}\hat{\Psi}^{\dagger}\right)\left(\nabla_{\hat{X}}\hat{\Psi}\right)-\int\sqrt{f^{2}\left(\hat{X}\right)}\left(\frac{1}{2}\left(\nabla_{y}\hat{\Psi}^{\dagger}\right)\left(\nabla_{y}\hat{\Psi}\right)+\hat{\Psi}^{\dagger}\frac{y^{2}}{4}\hat{\Psi}\right) \tag{266}$$

includes the hamiltonian of a sum of harmonic oscillators, and thus (266) is lower than $-\frac{\int \hat{\Psi}^{\dagger}\sqrt{f^2(\hat{X})}\hat{\Psi}}{2}$. Consequently, we have the inequality for all $\hat{X}$:

$$\hat{\Psi}^{\dagger}_{y=0}\left(\nabla_y\hat{\Psi}\right)_{y=0}+\int\hat{\Psi}^{\dagger}\left(\left|\hat{\lambda}\right|-\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma^2_{\hat{X}}}-\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma^2_{\hat{K}}F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)\right)\Psi d\hat{K}$$
$$>\quad\frac{\int\hat{\Psi}^{\dagger}\sqrt{f^2\left(\hat{X}\right)}\hat{\Psi}d\hat{K}}{2}$$

Since:

$$\left|\hat{\lambda}\right|\int\left|\Psi\right|^2 d\hat{K}=\left|\hat{\lambda}\right|\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2$$

and $\hat{\Psi}^{\dagger}_{y=0}\left(\nabla_y\hat{\Psi}\right)_{y=0}$ is of order $1<<\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2$ since it is integrated over $\hat{X}$ only. Consequently, the condition reduces to:

$$\left|\hat{\lambda}\right|\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2>\int\hat{\Psi}^{\dagger}\left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma^2_{\hat{X}}}+f\left(\hat{X}\right)+\frac{1}{2}\sqrt{f^2\left(\hat{X}\right)}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma^2_{\hat{K}}F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)\Psi d\hat{K}$$

that is:

$$\left|\hat{\lambda}\right|>\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma^2_{\hat{X}}}+f\left(\hat{X}\right)+\frac{1}{2}\sqrt{f^2\left(\hat{X}\right)}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma^2_{\hat{K}}F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}$$

for each $\hat{X}$, and we have:

$$M=\max_{\hat{X}}\left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma^2_{\hat{X}}}+f\left(\hat{X}\right)+\frac{1}{2}\sqrt{f^2\left(\hat{X}\right)}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma^2_{\hat{K}}F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)\tag{267}$$

Note that in general, for $\varepsilon<<1$, $f\left(\hat{X}\right)>>1$ and:

$$\frac{\sigma^2_{\hat{K}}F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}<<\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma^2_{\hat{X}}}+f\left(\hat{X}\right)+\frac{1}{2}\sqrt{f^2\left(\hat{X}\right)}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)$$

so that:

$$M\simeq\max_{\hat{X}}\left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma^2_{\hat{X}}}+f\left(\hat{X}\right)+\frac{1}{2}\sqrt{f^2\left(\hat{X}\right)}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\tag{268}$$

Having found M, this yields:

$$w\left(\left|\hat{\lambda}\right|\right)=\hat{N}\exp\left(-\left(\left|\hat{\lambda}\right|-M\right)\hat{N}\right)\tag{269}$$

Consequently, averaging equation (262) yields:

$$K_{\hat{X}}=\int K_{\hat{X},\hat{\lambda}}\hat{N}\exp\left(-\left(\left|\hat{\lambda}\right|-M\right)\hat{N}\right)d\hat{\lambda}$$

$$K_{\hat{X}}\left\Vert \Psi\left(\hat{X}\right)\right\Vert ^{2} = \int C\left(\hat{\lambda}\right)w\left(\left|\hat{\lambda}\right|\right)\exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}\right)\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)^{-1} \tag{270}$$

$$\times\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right)d\hat{\lambda}$$

with $C\left(\bar{p}\left(\hat{\lambda}\right)\right)$ given by (259). Given (269), the average value of $\left|\hat{\lambda}\right|$ is $M+\frac{1}{\hat{N}}$ and have:

$$K_{\hat{X}}\left\Vert \Psi\left(\hat{X}\right)\right\Vert ^{2}\left|f\left(\hat{X}\right)\right| = C\left(\bar{p}\left(-\left(M-\frac{1}{\hat{N}}\right)\right)\right)\sigma_{\hat{K}}^{2} \tag{271}$$

$$\times\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right)$$

with:

$$p=-\frac{\left(g\left(\hat{X}\right)\right)^{2}+\sigma_{\hat{X}}^{2}\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}-\left(M-\frac{1}{\hat{N}}\right)\right)}{\sigma_{\hat{X}}^{2}\sqrt{f^{2}\left(\hat{X}\right)}}-\frac{1}{2}$$

We can consider that $\frac{1}{\hat{N}}<<1$ so that $C\left(\bar{p}\left(-\left(M-\frac{1}{\hat{N}}\right)\right)\right)\simeq C\left(\bar{p}\left(-M\right)\right)$. It amounts to consider $\left|\hat{\lambda}\right|=M$. We will also write $\bar{p}\left(-M\right)=\bar{p}$ and given (**??**) we have:

$$\bar{p}=\left(\frac{M-\frac{\left(g\left(\hat{X}_{0}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+\left(f\left(\hat{X}_{0}\right)+\frac{1}{2}\left|f\left(\hat{X}_{0}\right)\right|+\nabla_{\hat{X}}g\left(\hat{X}_{0},K_{\hat{X}_{0}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X}_{0},K_{\hat{X}_{0}}\right)}{2f^{2}\left(\hat{X}_{0}\right)}\right)}{\left|f\left(\hat{X}_{0}\right)\right|}\right) \tag{272}$$

and:

$$p=\frac{M-\left(\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+\left(f\left(\hat{X}\right)+\frac{\sqrt{f^{2}\left(\hat{X}\right)}}{2}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}\right)\right)}{\sqrt{f^{2}\left(\hat{X}\right)}} \tag{273}$$

Equation (259) rewrites:

$$C\left(\bar{p}\right)\simeq\frac{\exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(\frac{\left(\bar{p}\left(\hat{\lambda}\right)+\frac{1}{2}\right)f'\left(X_{0}\right)}{f\left(\hat{X}_{0}\right)}\right)^{2}}{96\left|f\left(\hat{X}_{0}\right)\right|}\right)\hat{N}\Gamma\left(-\bar{p}\right)}{\left(\frac{\left\langle\left|f\left(\hat{X}\right)\right|\right\rangle}{\sigma_{\hat{K}}^{2}}\right)^{-\frac{1}{2}}V_{r}\left(\text{Psi}\left(-\frac{\bar{p}-1}{2}\right)-\text{Psi}\left(-\frac{\bar{p}}{2}\right)\right)} \tag{274}$$

and (271) reduces to:

$$K_{\hat{X}}\left\Vert \Psi\left(\hat{X}\right)\right\Vert ^{2}\left|f\left(\hat{X}\right)\right|=C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\hat{\Gamma}\left(p+\frac{1}{2}\right) \tag{275}$$

with:

$$\hat{\Gamma}\left(p+\frac{1}{2}\right) = \exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}\right) \tag{276}$$

$$\times\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right)$$

We note that, asymptotically:

$$\hat{\Gamma}\left(p+\frac{1}{2}\right) \sim_{\infty} \exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)\Gamma\left(p+\frac{3}{2}\right) \tag{277}$$

A3.1.4.3 Replacing $\|\Psi(X)\|^2$ in the $K_{\hat{X}}$equation We can isolate $K_{\hat{X}}$ in (271) by using (191) and (205) to rewrite $\left\|\Psi\left(\hat{X}\right)\right\|^2$:

Using (176a):

$$\begin{aligned} D\left(\|\Psi\|^2\right) &= 2\tau\|\Psi(X)\|^2+\frac{1}{2\sigma_X^2}\left(\nabla_X R(X)\right)^2 H^2\left(\frac{\hat{K}_X}{\|\Psi(X)\|^2}\right)\left(1-\frac{H'\left(\hat{K}_X\right)}{H\left(\hat{K}_X\right)}\frac{\hat{K}_X}{\|\Psi(X)\|^2}\right) \\ &= 2\tau\|\Psi(X)\|^2+\frac{1}{2\sigma_X^2}\left(\nabla_X R(X)\right)^2 H^2\left(K_X\right)\left(1-\frac{H'\left(K_X\right)}{H\left(K_X\right)}K_X\right) \end{aligned}$$

We rewrite $\|\Psi(X)\|^2$ as a function of K_X:

$$\|\Psi(X)\|^2=\frac{D\left(\|\Psi\|^2\right)-\frac{1}{2\sigma_X^2}\left(\nabla_X R(X)\right)^2 H^2\left(K_X\right)\left(1-\frac{H'(K_X)}{H(K_X)}K_X\right)}{2\tau}\equiv D-\bar{H}\left(X,K_X\right) \tag{278}$$

Ultimately, the equation (275) for $K_{\hat{X}}$ can be rewritten:

$$K_{\hat{X}}\left|f\left(\hat{X}\right)\right|=\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}{\|\Psi(X)\|^2}\hat{\Gamma}\left(p+\frac{1}{2}\right)=\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}{D-\bar{H}\left(X,K_X\right)}\hat{\Gamma}\left(p+\frac{1}{2}\right) \tag{279}$$

with $C\left(\bar{p}\right)$ given by (274), $\hat{\Gamma}\left(p+\frac{1}{2}\right)$ defined in (276) and p given by (273).

13 Appendix 4. Approaches to solutions for $K_{\hat{X}}$

We present the three approaches to the average capital equation (99) and their results in the first section. The details of the computations are given in section two of this appendix.

A4.1 Solutions for average capital across sectors

We stated in the text that the final form of the capital equation, (99), cannot be solved analytically, except for some particular cases[58] but that several approaches can be used to study the behaviour of its solutions or approximate its solutions. Combined, these three approaches confirm and complete with each other.

A.4.1.1 Stability of average capital and dependency on exoenous parameters

One way to better understand equation (99) is to study its differential form.

Assume at point $\hat{X}$ of the system, a variation $\delta Y\left(\hat{X}\right)$ for any parameter, in which the parameter $Y\left(\hat{X}\right)$ can be either $R(X)$, its gradient, or any parameter arising in the definition of $f\left(\hat{X}\right)$and

[58] These particular cases will be studied in the following sections.

$g\left(\hat{X}\right)$. This variation $\delta Y\left(\hat{X}\right)$ induces in turn a variation $\delta K_{\hat{X}}$ in average capital expressed by differentiating (99):

$$\delta K_{\hat{X}} = \left(-\left(\frac{\partial \ln f\left(\hat{X}, K_{\hat{X}}\right)}{\partial K_{\hat{X}}}+\frac{\partial \ln \left|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right|^{2}}{\partial K_{\hat{X}}}+l\left(\hat{X}, K_{\hat{X}}\right)\right)+k(p) \frac{\partial p}{\partial K_{\hat{X}}}\right) K_{\hat{X}} \delta K_{\hat{X}} + \frac{\partial}{\partial Y\left(\hat{X}\right)}\left(\frac{\sigma_{\hat{K}}^{2} C(\bar{p}) 2 \hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left|f\left(\hat{X}, K_{\hat{X}}\right)\right|\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^{2}}\right) \delta Y\left(\hat{X}\right) \quad (280)$$

where the coefficients $l\left(\hat{X}, K_{\hat{X}}\right)$ and $k(p)$ are computed in appendix 4.2.1. The parameter $l\left(\hat{X}, K_{\hat{X}}\right)$ accounts for the variation of the short-term returns across sectors, while $k(p)$ describes the impact of relative returns variations across sectors.

Equation (280) will be used to compute the dependency of average capital per firm in sector $\hat{X}$, i.e. $K_{\hat{X}}$, as a function of any parameter $Y(\hat{X})$, and more fundamentally to investigate the stability of the solutions of (99) with respect to the variations in parameters.

A.4.1.1.1 Local stability

The differential form given by equation (280), computes the effect of a variation $\delta Y\left(\hat{X}\right)$ in the parameters on the average capital $K_{\hat{X}}$. Moreover, equation (280) can be understood as the fixed-point equation of a dynamical system of the following mechanism: each variation $\delta Y\left(\hat{X}\right)$ in the parameters impacts directly the average capital through the second term in the RHS of (280). In a second step, the variation $\delta K_{\hat{X}}$ impacts the various functions implied in (99), and indirectly modifies $K_{\hat{X}}$ through the first term in the rhs of (280)[59].

What matters here is the condition of stability. We show that the fixed point is stable when:

$$\left|k(p) \frac{\partial p}{\partial K_{\hat{X}}}-\left(\frac{\partial \ln f\left(\hat{X}, K_{\hat{X}}\right)}{\partial K_{\hat{X}}}+\frac{\partial \ln \left|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right|^{2}}{\partial K_{\hat{X}}}+l\left(\hat{X}, K_{\hat{X}}\right)\right)\right|<1 \quad (281a)$$

and unstable otherwise.

Thus, two types of solutions emerge for the average capital per firm $K_{\hat{X}}$. The stable solutions $K_{\hat{X}}$ can be considered as the potential equilibrium averages for sector $\hat{X}$. However unstable solutions must rather be considered as thresholds: when $K_{\hat{X}}$ is driven away from this threshold, it may either converge toward a stable solution of (99), or diverge towards 0 or infinity.

Remark The variation in average capital induced by a change in parameter reveals a shift $\delta\hat{\Psi}\left(\hat{K}, \hat{X}\right)$ in the background state $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$. The new configuration $\hat{\Psi}\left(\hat{K}, \hat{X}\right)+\delta\hat{\Psi}\left(\hat{K}, \hat{X}\right)$ may not be a minimum of the action functional. We must therefore determine whether the system will settle on a background state, slightly modified with a different $K_{\hat{X}}$, or be driven towards an altogether different equilibrium. To this end, we will study the dynamics equation for $K_{\hat{X}}$ in appendix 5.

A.4.1.1.2 Dependency of average capital in system's parameters

Once the notion of stability understood, we can use equation (280) to compute the impact of the variation of any parameter $Y(\hat{X})$ on $\delta K_{\hat{X}}$. Two applications are of particular interest to us.

[59] The computations and formula for the dynamics' fixed points are given in appendix 3.2.1.

A.4.1.1.2.1 Dependency in relative expected returns The main application of equation (280) is to consider the dependency of the average capital $K_{\hat{X}}$ in the parameter $p(\hat{X})$ defined in (95). This parameter encompasses sector $\hat{X}$ relative expected returns vis-à-vis its neighbours.

Using (280), we show (see appendix 4.2.1) that the variation of $K_{\hat{X}}$ with respect to $p(\hat{X})$ depends on the notion of equilibrium stability defined in (281a).

For a stable equilibrium where the expected return $f\left(\hat{X}\right)$ is positive[60], we find that:

$$\frac{\delta K_{\hat{X}}}{\delta p\left(\hat{X}\right)} > 0 \tag{282}$$

so that $p(\hat{X})$ writes as:

$$p(\hat{X}) = \frac{M - \left(\frac{\left(g(\hat{X})\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}\right) - \frac{\sigma_{\hat{K}}^2 F^2(\hat{X})}{2f^2(\hat{X})}\right)}{f\left(\hat{X}\right)} - \frac{3}{2} \tag{283}$$

The definitions (85) and (86) show that $g\left(\hat{X}\right)$ and $\nabla_{\hat{X}} g\left(\hat{X}\right)$ are proportional to $\nabla_{\hat{X}} R\left(\hat{X}\right)$ and $\nabla_{\hat{X}}^2 R\left(\hat{X}\right)$ respectively. Thus, when the expected long-term return $R\left(\hat{X}\right)$ is a maximum, $p(\hat{X})$ is maximal too[61]: under a stable equilibrium, capital accumulation is maximal in sectors where the expected long-term return $R\left(\hat{X}\right)$ is maximal.

On the other hand, when the equilibrium is unstable we have:

$$\frac{\delta K_{\hat{X}}}{\delta p\left(\hat{X}\right)} < 0 \tag{284}$$

Actually, the capital $K_{\hat{X}}$ is minimal for $R\left(\hat{X}\right)$ maximal. Actually, as seen above, in the instability range, the average capital $K_{\hat{X}}$ acts as a threshold. When, due to variations in the system's parameters, the average capital per firm is shifted above the threshold $K_{\hat{X}}$, capital will either move to the next stable equilibrium, possibly zero, or tend to infinity. Our results show that when the expected long-term return of a sector increases, the threshold $K_{\hat{X}}$ decreases, which favours capital accumulation.

A.4.1.1.2.2 Dependency in short term returns A second use of equation (280) is to consider $Y(\hat{X})$ as any parameter-function involved in the definition of $f\left(\hat{X}, K_{\hat{X}}\right)$ that may condition either real short-term returns or the price-dividend ratio. We show in appendix 4.2.1 that again the result depends on the stability of the solution.

Around a stable equilibrium, in most cases:

$$\frac{\delta K_{\hat{X}}}{\delta f\left(\hat{X}\right)} > 0$$

A higher short-term return, decomposed as a sum of dividend and price variation, induces a higher average capital. This effect is magnified for larger levels of capital: the third approach will confirm that, in most cases, the return $f\left(\hat{X}\right)$ is asymptotically a constant $c << 1$ when capital is high: $K_{\hat{X}} >> 1$.

[60] which is the case of interest for us (see section 11.3.3)
[61] See also section 14.1 for more details.

Turning now to the case of an unstable equilibrium, we find:

$$\frac{\delta K_{\hat{X}}}{\delta f\left(\hat{X}\right)}<0$$

In the instability range, and due to this very instability, an increase in returns $f\left(\hat{X}\right)$ reduces the threshold of capital accumulation for low levels of capital. When short-term returns $f\left(\hat{X}\right)$ increase, a lower average capital will trigger capital accumulation towards an equilibrium. Otherwise, when average capital $K_{\hat{X}}$ is below this threshold, it will converge toward 0.

A.4.1.2 Accumulation points of capital

The second approach to equation (99) is to find the average capital at some particular points $\hat{X}$, and then by first order expansion, the solutions in the neighbourhood of these particular points. We choose as particular points $\hat{X}$ those such that $A\left(\hat{X}\right)$ defined in (93) is maximal. At these points $\left(\hat{X}_M, K_{\hat{X}_M}\right)$, we have[62]:

$$A\left(\hat{X}_M\right)=M=\max_{\hat{X}} A\left(\hat{X}\right) \tag{285}$$

and $p=0$, given (98). These sectors are characterized by expected returns that exhibit a local maximum when compared to their neighboring sectors. As a result, these sectors serve as points of capital accumulation.

A.4.1.2.1 Particular solutions for capital when $p=0$

For $p=0$ equation (99) at points $\left(\hat{X}_M, K_{\hat{X}_M}\right)$ reduces to:

$$K_{\hat{X},M}\left|f\left(\hat{X},K_{\hat{X},M}\right)\right|\left\|\Psi\left(\hat{X},K_{\hat{X},M}\right)\right\|^2\simeq\sigma_{\hat{K}}^2C\left(\bar{p}\right)\exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(f'\left(X_M,K_{\hat{X},M}\right)\right)^2}{384\left|f\left(\hat{X}_M,K_{\hat{X},M}\right)\right|^3}\right) \tag{286}$$

Given that $\left\|\Psi\left(\hat{X}_M,K_{\hat{X},M}\right)\right\|^2$ is decreasing in $K_{\hat{X}}$ (see (278)), and assuming $f\left(\hat{X}\right)$ is decreasing too, as is usual for marginal decreasing returns, equation (99) has two solutions.

For some particular values of the parameters, an approximate form can be found for these solutions. Here, we will merely consider a power law for $f\left(\hat{X}\right)$:

$$f\left(\hat{X}\right)\simeq B\left(X\right)K_{\hat{X}}^{\alpha} \tag{287}$$

The parameter $B\left(X\right)$ is the productivity in sector X, and equation (287) shows that the return $f\left(\hat{X}\right)$ is increasing in $B\left(X\right)$.

The stable case corresponds to an intermediate level of capital, $K_{\hat{X}}^{\alpha}<<D$. In such a case, given the density of producers (81), we can assume that this density satisfies $\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2\simeq D$. The solution to equation (99) is then:

$$K_{\hat{X}}^{\alpha}=\left(\frac{DB\left(X\right)}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}\right)^{-\frac{\alpha}{\alpha+1}}\exp\left(W_0\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B'\left(X\right)\right)^2\alpha}{384\left(B\left(X\right)\right)^3\left(\alpha+1\right)}\left(\frac{DB\left(X\right)}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}\right)^{\frac{\alpha}{\alpha+1}}\right)\right) \tag{288}$$

[62] See equation (92).

where W_0 is the Lambert W function.

For $B(X) << 1$, we can check that $K_{\hat{X}}^{\alpha}$ is increasing with $B(X)$, i.e. with short-term returns $f\left(\hat{X}\right)$[63], which confirms the results found in the first approach: in the stable case, capital equilibrium increases with short-term returns $f\left(\hat{X}\right)$.

The unstable case corresponds to a higher level of capital. Given the density of producers (81), this case amounts to consider, in first approximation, that capital K_X is concentrated among a small group of agents, that is $\left|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right|^2 << 1$. Considering a power law for $H^2(K_X)$:

$$H^2(K_X) = K_X^{\alpha}$$

the solution (286) can be written as:

$$\begin{aligned}
K_{\hat{X}}^{\alpha} \simeq & \frac{2D}{\left(\nabla_X R(X)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R(X)}{H(K_X)}} \\
& - \left(\frac{\left(\nabla_X R(X)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R(X)}{H(K_X)}}{2D}\right)^{\frac{1}{\alpha}} \frac{\sigma_{\hat{K}}^2 C(\bar{p})}{DB(X)} \\
& \times \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B'(X)\right)^2}{768D\left(B(X)\right)^3}\left(\left(\nabla_X R(X)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R(X)}{H(K_X)}\right)\right)
\end{aligned} \tag{289}$$

at the first order in D, plus corrections of order $\frac{1}{D}$, with:

$$f\left(\hat{X}\right) \simeq B(X) K_{\hat{X}}^{\alpha}$$

The (in)stability analysis of the previous approach applies. In the range $B(X) << 1$, when $f\left(\hat{X}\right)$ increases, or which is equivalent, $B(X)$ increases, average capital must reduce to preserve the possibility of unstable equilibria. Likewise, equilibrium capital is higher when expected returns $R(X)$ are minimal. When expected returns increase, the threshold defined by the unstable equilibrium decreases.

A.4.1.2.2 Expansion around particular solutions for $p = 0$

To better understand the behaviour of the solutions of the average capital equation (99), we expand this equation around the points $\left(\hat{X}, K_{\hat{X},M}\right)$ that solve equation (99). Appendix 4.2.2 computes this expansion at the second order around $\hat{X}_M$ and $K_{\hat{X},M}$. This yields the form of the solutions of (99) in the vincinity of the points $\left(\hat{X}, K_{\hat{X},M}\right)$. We find:

[63] See equation (287).

$$\left(K_{\hat{X}}-K_{\hat{X},M}\right) = \frac{1}{D}\left(\sigma_X^2\sigma_{\hat{K}}^2\frac{3\left(f'\left(\hat{X}\right)\right)^3-2f'\left(X\right)f''\left(\hat{X}\right)\left|f\left(\hat{X}\right)\right|}{120\left|f\left(\hat{X}\right)\right|^4} - \frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial\hat{X}}}{f\left(\hat{X},K_{\hat{X}}\right)} - \frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial\hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}}\left(\hat{X}-\hat{X}_M\right) + \frac{1}{D}\frac{b}{2}\left(\hat{X}-\hat{X}_M\right)\nabla_{\hat{X}}^2\left(\frac{M-A\left(\hat{X}\right)}{f\left(\hat{X}\right)}\right)_{K_{\hat{X},M}}\left(\hat{X}-\hat{X}_M\right) \tag{290}$$

with $\frac{A(\hat{X})}{f(\hat{X})}$ given in formula (**??**), and b and D are coefficients given in the appendix 4.2.2. In fact we recover the analysis of the first approach in terms of stability. the case $D>0$ corresponds to a stable equilibrium, and $D<0$ to an unstable one. The expansion (290) describes the local variations of $K_{\hat{X}}$ in the neighbourhood of the points $K_{\hat{X},M}$. This approximation (290) suffices to understand the role of the parameters of the system. The whole analysis is performed in appendix 4.2.2 and confirms and refines our previous results.

A.4.1.3 Solutions to the capital equation in some particular cases

A third approach computes the approximate solutions of (99) for the average capital per firm per sector X. To do so, we choose some general forms for the three parameter-functions arising in the definition of the action functional: f that defines short-term returns, that include dividend and expected long-term price variations, and is given by equation (84); g that describes investors' mobility in the sector space, given by (85), and the function $H\left(K_X\right)$ involved in the firms' background field, that describes firms' moves in the sectors space and is given by equation (81).

Once these parameter-functions chosen, the approximate solutions of (99) for average capital per firm per sector can be found. We have already seen in the second approach that this equation has in general several solutions. To find them, we must consider several relevant ranges for average capital, namely a very large level of capital, $K_X>>1$, a very low one, $K_X<<1$, and an intermediate range $\infty>K_X>1$. We will derive the solutions for K_X within these various ranges. Details of the computations are given in appendix 4.2.3.

A.4.1.3.1 Choice of parameter functions

Our choices for the parameter functions f, g and $H^2\left(K_X\right)$ are the following.

Firms' intersectoral moves $H^2\left(K_X\right)$ We can choose for $H^2\left(K_X\right)$ a power function of K_X, so that equation (81) rewrites:

$$\left\|\Psi\left(X\right)\right\|^2=D-L\left(X\right)\left(\nabla_X R\left(X\right)\right)^2 K_X^\eta \tag{291}$$

with $L\left(X\right)$ given in the appendix.

Short-term returns f To determine the function f, we must first assume a form for $r(K,X)$, the physical capital marginal returns, and for F_1, the function that measures the impact of expected long-term return on investment choices.

We assume Cobb-Douglas production functions, i.e. $B(X)K^{\alpha}$ with $B(X)$ a productivity factor. We also choose the expected long-term return F_1 to be an increasing function of the arctan type, so that investments increase linearly with expected returns and capital for small-capitalized firms, but is bounded for large values of capital.

Under these assumption, the short-term return can be written in a compact form as:

$$f\left(\hat{X},\Psi,\hat{\Psi}\right)=B_1\left(\hat{X}\right)K_{\hat{X}}^{\alpha-1}+B_2\left(\hat{X}\right)K_{\hat{X}}^{\alpha}-C\left(\hat{X}\right) \tag{292}$$

The coefficients $B_1\left(\hat{X}\right)$, $B_2\left(\hat{X}\right)$ and $C\left(\hat{X}\right)$ are given in the appendix 4.2.3.

investors' mobility in the sector space g To determine the form of the investors' mobility in the sector space g, given by (85), we must first choose a form for F_0, the investors' mobility towards higher long-term returns[64].

Here again, we choose an arctan type function of the expected long-term return, so that the velocity in the sectors' space g increases with capital, and is bounded and maximal when $K_{\hat{X}}^{\alpha}\rightarrow\infty$.

Appendix 3.2.2 shows that $g\left(\hat{X},\Psi,\hat{\Psi}\right)$ can be written:

$$g\left(\hat{X},\Psi,\hat{\Psi}\right)\nabla_{\hat{X}}R\left(\hat{X}\right)A\left(\hat{X}\right)K_{\hat{X}}^{\alpha} \tag{293}$$

where the function $A\left(\hat{X}\right)$ is given in appendix 4.2.2.

A.4.1.3.2 Solutions for the average capital

Now that the particular functions have been chosen, we can find approximate solutions to (99) in several ranges of sector X's average capital: Very large and stable capitalization, very large and unstable, i.e. bubble-like, capitalization, large capitalization stable or unstable, the intermediate case of mid-capitalization and ultimately small capitalization. Besides, we only consider positive short-term returns[65], $f>0$.

We consider the several type of solutions separately.

Case 1 *Very large and stable capitalization,* $K_{\hat{X}}>>>1$

When returns are either slowing or increasing in $\hat{X}$, i.e. $\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2\neq 0$, a solution the capital equation (99) may exist with $K_{\hat{X}}>>>1$. In this case, only a small number of firms are present in the sector. Indeed, in such a case, the competition-deterent factor $L\left(\hat{X}\right)$ in (291) is very large, and we can assume, in first approximation, that:

$$\left\|\Psi\left(\hat{X}\right)\right\|^2<<1 \tag{294}$$

A sector in which average capital is very large implies a very high competition, that act as a barrier to the entry of other firms. In this case, we can show that $f\left(\hat{X}\right)\simeq c$, for some constant c. Appendix

[64] See section 4.4.

[65] Solutions for negative returns, $f<0$, are discussed below.

4.2.3.2 solves equation (99) given these assumptions. The average capital is given by:

$$K_{\hat{X}}^{\alpha} \simeq \frac{D}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2} - \frac{C(\bar{p})\,\sigma_{\hat{K}}^2\sqrt{\frac{M-c}{c}}}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{2\left(1-\frac{1}{\alpha}\right)} D^{\frac{1}{\alpha}} c} - \frac{d}{R\left(\hat{X}\right)} \frac{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}} C(\bar{p})\,\sigma_{\hat{K}}^2 \left(\sqrt{\frac{M-c}{c}} + \frac{\frac{M}{c}+\nabla_{\hat{X}}^2 R\left(\hat{X}\right)\frac{f}{d}}{2\sqrt{\frac{M-c}{c}}}\right)}{c^2 D^{1+\frac{1}{\alpha}}\left(1-\frac{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}} C(\bar{p})\sigma_{\hat{K}}^2}{cD^{1+\frac{1}{\alpha}}}\sqrt{\frac{M-c}{c}}\right)} \tag{295}$$

which shows that $K_{\hat{X}}^{\alpha}$ is increasing in $f\left(\hat{X}\right)$ and $R\left(\hat{X}\right)$ for $K_{\hat{X}}^{\alpha}$ large, $f\left(\hat{X}\right) \simeq c << 1$ and $D >> 1$. Using (281a) shows that this corresponds to a stable local equilibrium.

Case 2 *Very large and unstable, i.e. bubble-like, capitalization,* $K_{\hat{X}} >>> 1$

This case arises when the expected long term returns is a local maximum, i.e. when $\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 \rightarrow 0$ and $\nabla_X^2 R(K_X, X) < 0$[66]. This describes a sector with a large number of firms and very high level of capital. Actually, the number of firms given in (291) shows that:

$$\left\|\Psi\left(\hat{X}\right)\right\|^2 > D >> 1 \tag{296}$$

and appendix 4.2.3.2 shows that the average capital is given by:

$$K_{\hat{X}} = \left(\frac{C(\bar{p})\,\sigma_{\hat{K}}^2}{\left|\nabla_X^2 R(X)\right| c}\Gamma\left(\frac{M-\nabla_{\hat{X}} g\left(\hat{X}\right)}{c}\right)\right)^{\frac{2}{3\alpha}} \tag{297}$$

where $f\left(\hat{X}\right) \simeq c << 1$ for some constant c and $D >> 1$.

The case (297) is unstable. Actually, in this case K_X is decreasing in $f\left(\hat{X}\right)$. When returns increase, an equilibrium arises only for a relatively low average capital. Otherwise, capital tends to accumulate infinitely. When the sector's expected returns are at a local maximum, the pattern of accumulation becomes unstable. Note that an equilibrium with $K_{\hat{X}} >>> 1$ is merely possible for $c << 1$. Otherwise, there is no equilibrium for $R(K_X, X)$ maximum.

Case 3 *Large capitalization,* $K_{\hat{X}} >> 1$

For a very large and stable capitalization, i.e. when average capital $K_{\hat{X}}$ is large but below a given threshold, we can assume in first approximation that the density of firms in sector X (291) becomes:

$$\left\|\Psi(X)\right\|^2 \simeq D \tag{298}$$

Appendix 4.2.3.2 shows that average capital in sector X is :

$$K_X^{\alpha} \simeq \frac{C(\bar{p})\,\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)}{Df(X)} + \frac{d}{f(X)\,R(X)}\left(1+M\,\mathrm{Psi}\left(\frac{M}{c}\right)\left(1+\frac{\nabla_{\hat{X}}^2 R(X)}{M}\right)\right) \tag{299}$$

[66] The case $\nabla_X^2 R(K_X, X) > 0$ i.e. a minimum for the expected long term return is studied in appendix 3.2.3.2 which shows that this equilibrium is unlikely and can be discarded.

where $\mathrm{Psi}(x)=\frac{\Gamma'(x)}{\Gamma(x)}$, and d and c are some constant parameters. This solution only holds when $f(X)>0$ and $\frac{C(\bar{p})\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)}{Df(X)}>1$.

Formula (299) shows that this dependency of $K_{\hat{X}}^{\alpha}$ in $R\left(\hat{X}\right)$ depends in turns on the sign of the second term in the rhs of (299).

When the condition:

$$1+M\,\mathrm{Psi}\left(\frac{M}{c}\right)\left(1+\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)}{M}\right)>0$$

holds, average capital in sector $\hat{X}$, $K_{\hat{X}}^{\alpha}$, is a decreasing function of both returns $R\left(\hat{X}\right)$ and the short-term returns $f\left(\hat{X}\right)$. The stability analysis (281a) thus implies that the solution (299) is unstable.

On the contrary, when:

$$1+M\,\mathrm{Psi}\left(\frac{M}{c}\right)\left(1+\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)}{M}\right)<0$$

a stable equilibrium is possible. In this case, the average capital in sector X, $K_{\hat{X}}^{\alpha}$, is increasing with both returns $R\left(\hat{X}\right)$ and short-term returns $f\left(\hat{X}\right)$. This case arises when, for already maximum returns, $\nabla_{\hat{X}}^2 R\left(\hat{X}\right)<<0$, a further increase in long-term returns $R\left(\hat{X}\right)$ occurs. This increases the number of firms $\left\|\Psi\left(\hat{X}\right)\right\|^2$ in the sector without impairing average capital per firm. Note that stable equilibrium is an extreme case of the next case, intermediate level of capital.

Case 4 *intermediate case, mid-capitalization* $\infty>>K_{\hat{X}}>1$

To solve equation (99) in this general case, we consider that $\sigma_X^2<<1$ and the following simplifying assumptions:

$$f\left(\hat{X}\right)\simeq B_2(X)\,K_{\hat{X}}^{\alpha} \tag{300}$$

and:

$$\left\|\Psi\left(\hat{X}\right)\right\|^2\simeq D$$

Eventually, appendix 4.2.3.2 shows that:

$$\begin{aligned}K_{\hat{X}}^{\alpha} &= \left(\frac{8C(\bar{p})}{D}\sqrt{\frac{3\sigma_{\hat{K}}^2\left|B_2(X)\right|}{\sigma_X^2\left(B_2'(X)\right)^2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)\right)^{\frac{2\alpha}{1+\alpha}} \\ &\quad\times\exp\left(-W_0\left(-\frac{48\alpha}{1+\alpha}\left(\sqrt{\frac{3\sigma_{\hat{K}}^2}{\sigma_X^2}}\frac{8C(\bar{p})}{D}\right)^{\frac{2\alpha}{1+\alpha}}\frac{\left|B_2(X)\right|^{3+\frac{\alpha}{1+\alpha}}}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^{2+\frac{2\alpha}{1+\alpha}}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^{2+\frac{\alpha}{1+\alpha}}\right)\right)\end{aligned} \tag{301}$$

where W_0 is the Lambert W function and $\bar{p}$ a constant.

In first approximation, equation (301) implies that $K_{\hat{X}}^{\alpha}$ is an increasing function of $B_2(X)$. Given our simplifying assumption (300), average capital is higher in high short-term returns sectors.

Moreover, $K_{\hat{X}}^{\alpha}$ is a decreasing function of $\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2$ and $\nabla_{\hat{X}}^2 R\left(\hat{X}\right)$: capital accumulation is locally maximal when expected returns $R\left(\hat{X}\right)$ of sector $\hat{X}$ are at a local maxima, i.e. $\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2=0$ and $\nabla_{\hat{X}}^2 R\left(\hat{X}\right)<0$.

Thus, in the intermediate case, the average values $K_{\hat{X}}$ are stable. In addition, both short-term and long term returns matter in the intermediate range.

Case 5 *Small capitalization* $K_{\hat{X}} << 1$

When average physical capital per firm in sector $\hat{X}$ is very low, we can use our assumptions about $g\left(\hat{X}\right)$ equation (293), and assume that:

$$f\left(\hat{X}\right) \simeq B_1\left(\hat{X}\right) K_{\hat{X}}^{\alpha-1} >> 1,\ g\left(\hat{X}\right) \simeq 0 \tag{302}$$

and:

$$\left\|\Psi\left(\hat{X}\right)\right\|^2 \simeq D$$

For these conditions, the solution of (99) is locally stable. We show in appendix 4.2.3.2 that the solution for average capital is at the first order[67]:

$$K_{\hat{X}} = \left(\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\hat{\Gamma}\left(-1\right)}{DB_1\left(\hat{X}\right)}\right)^{\frac{1}{\alpha}} + \frac{\frac{C(\bar{p})\sigma_{\hat{K}}^2}{D}\hat{\Gamma}'\left(-1\right)\left(M - \left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}g\left(\hat{X}\right)\right)\right)}{B_1^{\frac{1}{\alpha}}\left(\hat{X}\right)\left(\frac{C(\bar{p})\sigma_{\hat{K}}^2\hat{\Gamma}(-1)}{D}\right)^{1-\frac{1}{\alpha}}} \tag{303}$$

Equation (303) shows that average capital $K_{\hat{X}}$ increases with $M - \left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}}g\left(\hat{X}\right)\right)$: when expected long-term returns increase, more capital is allocated to the sector. Equation (302) also shows that average capital $K_{\hat{X}}$ is maximal when returns $R\left(\hat{X}\right)$ are at a local maximum, i.e. when $\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} = 0$ and $\nabla_{\hat{X}}g\left(\hat{X}, K_{\hat{X}}\right) < 0$.

Inversely, the same equations (303) and (302) show that average capital $K_{\hat{X}}$ is decreasing in $f\left(\hat{X}\right)$. The equilibrium is unstable. Recall that in this unstable equilibrium, $K_{\hat{X}}$ must be seen as a threshold. The rise in $f\left(\hat{X}\right)$ reduces the threshold $K_{\hat{X}}$, which favours capital accumulation and increases the average capital $K_{\hat{X}}$. Actually, when average capital is very low, i.e. $K_{\hat{X}} << 1$, which is the case studied here, marginal returns are high. Any increase in capital above the threshold $K_{\hat{X}}$, or any shift reducing the threshold, widely increases returns, which drives capital towards the next stable equilibrium, with higher $K_{\hat{X}}$.

This case is thus an exception: the dependency of $K_{\hat{X}}$ in $R\left(\hat{X}\right)$ is stable, but the dependency in $f\left(\hat{X}\right)$ is unstable. This saddle path type of instability may lead the sector, either towards a higher level of capital (case 4 below) or towards 0. where the sector disappears.

Remark: The case of negative short-term returns $f < 0$

In the four cases described above, we have only considered the case where a sector $\hat{X}$ short-term returns are positive $f\left(\hat{X}\right) > 0$. We can nonetheless extend our analysis to the case $f\left(\hat{X}\right) < 0$.

In such a case, the equilibria, whether stable or unstable, defined in cases 1, 2 with $K_{\hat{X}} >> 1$, and 4 with $K_{\hat{X}} > 1$, are still valid, and capital allocation relies on expectations of high long-term returns. If we consider that $f\left(\hat{X}\right) < 0$ is an extreme case, where expectations of large future profits must offset short-term losses. However, such equilibria become unsustainable when $R\left(\hat{X}\right)$

[67] Given our hypotheses, $D >> 1$, which implies that $K_{\hat{X}} << 1$, as needed.

decreases to such an extent that it does not compensate for the loss $f\left(\hat{X}\right)$. Case 3, $K_{\hat{X}}<1$ is the only case that is no longer possible when $f\left(\hat{X}\right)<0$, since the returns that matter in this case are dividends. If they turn negative, the equilibrium is no longer sustainable.

A 4.2 Details of the computations

A 4.2.1 First approach: Differential form of (99)

To understand the behavior of the solutions of (99), we can write its differential version. Assume a variation $\delta Y\left(\hat{X}\right)$ for any parameter of the system at point $\hat{X}$. This parameter $Y\left(\hat{X}\right)$ can be either $R(X)$, its gradient, or any parameter arising in the definition of f and g. This induces a variation $\delta K_{\hat{X}}$ for the average capital. The equation for $\delta K_{\hat{X}}$ is obtained by differentiation of (99):

$$\begin{aligned}\delta K_{\hat{X}} &= \left(-\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X},K_{\hat{X}}\right)}+\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}+l\left(\hat{X},K_{\hat{X}}\right)\right)+k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}\right)K_{\hat{X}}\delta K_{\hat{X}} \\ &+\frac{\partial}{\partial Y\left(\hat{X}\right)}\left(\frac{\sigma_{\hat{K}}^2 C\left(\bar{p}\right)2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)\delta Y\left(\hat{X}\right)\end{aligned} \tag{304}$$

where we define:

$$\begin{aligned}l\left(\hat{X},K_{\hat{X}}\right) &= \frac{\sigma_{\hat{X}}^2\sigma_{\hat{K}}^2\left(\nabla_{K_{\hat{X}}}\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right|-3\left(\nabla_{K_{\hat{X}}}\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2\right)\left(p+\frac{1}{2}\right)^2}{120\left|f\left(\hat{X}\right)\right|^4} \\ &+\frac{\partial p}{\partial K_{\hat{X}}}\frac{\sigma_{\hat{X}}^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}\end{aligned}$$

$$k\left(p\right)=\frac{\frac{d}{dp}\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\hat{\Gamma}\left(p+\frac{1}{2}\right)}\sim_{\infty}\sqrt{\frac{p-\frac{1}{2}}{2}}-\frac{\sigma_{\hat{X}}^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3} \tag{305}$$

and:

$$\frac{\partial p}{\partial K_{\hat{X}}}=\frac{\partial}{\partial K_{\hat{X}}}\frac{M-A\left(\hat{X},K_{\hat{X}}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}=-\frac{\partial_{K_{\hat{X}}}\left|f\left(\hat{X},K_{\hat{X}}\right)\right|p+\partial_{K_{\hat{X}}}A\left(\hat{X},K_{\hat{X}}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}$$

with:

$$\begin{aligned}A\left(\hat{X},K_{\hat{X}}\right) &= \frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2}+\left(f\left(\hat{X},K_{\hat{X}}\right)+\frac{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}{2}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X},K_{\hat{X}}\right)}\right) \\ &\simeq \frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2}+f\left(\hat{X},K_{\hat{X}}\right)+\frac{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}{2}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\end{aligned}$$

A. 4.2.1.1 Expanded form of (304) In an expanded form (304) writes:

$$\begin{aligned}
\delta K_{\hat{X}} &= \Big(k(p)\,\partial_{K_{\hat{X}}}(p) \\
&\quad -\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}\left(p+\mathcal{H}\left(f\left(\hat{X},K_{\hat{X}}\right)\right)+\frac{1}{2}\right)k(p)}{f\left(\hat{X},K_{\hat{X}}\right)}+\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}+l\left(\hat{X},K_{\hat{X}}\right)\right)\Big)K_{\hat{X}}\delta K_{\hat{X}} \\
&\quad +\frac{\partial}{\partial Y}\left(\frac{\sigma_{\hat{K}}^2 C(\bar{p})\,2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)\delta Y
\end{aligned}$$

with $\mathcal{H}$ the heaviside function. Moreover:

$$\begin{aligned}
&\frac{\partial}{\partial Y}\left(\frac{\sigma_{\hat{K}}^2 C(\bar{p})\,2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)\delta Y \\
&= \Big(k(p)\,\partial_Y p \\
&\quad -\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial Y}\left(p+\mathcal{H}\left(f\left(\hat{X},K_{\hat{X}}\right)\right)+\frac{1}{2}\right)k(p)}{f\left(\hat{X},K_{\hat{X}}\right)}+\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial Y}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}+m_Y\left(\hat{X},K_{\hat{X}}\right)\right)\Big)K_{\hat{X}}\delta Y
\end{aligned}$$

with:

$$\begin{aligned}
m_Y\left(\hat{X},K_{\hat{X}}\right) &= \frac{\sigma_X^2\sigma_{\hat{K}}^2\left(\nabla_Y\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right|-3\left(\nabla_Y\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2\right)\left(p+\frac{1}{2}\right)^2}{120\left|f\left(\hat{X}\right)\right|^4} \\
&\quad +\nabla_Y p\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'(X)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}
\end{aligned}$$

so that:

$$\begin{aligned}
\frac{\delta K_{\hat{X}}}{K_{\hat{X}}} &= \Big(k(p)\,\partial_Y(p) \\
&\quad -\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial Y}\left(p+\mathcal{H}\left(f\left(\hat{X},K_{\hat{X}}\right)\right)+\frac{1}{2}\right)k(p)}{f\left(\hat{X},K_{\hat{X}}\right)}+\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial Y}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}+m_Y\left(\hat{X},K_{\hat{X}}\right)\right)\Big)\frac{\delta Y}{D}
\end{aligned}$$

with:

$$\begin{aligned}
D &= 1+\Bigg(\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}\left(p+\mathcal{H}\left(f\left(\hat{X},K_{\hat{X}}\right)\right)+\frac{1}{2}\right)k(p)}{f\left(\hat{X},K_{\hat{X}}\right)}+\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}+l\left(\hat{X},K_{\hat{X}}\right)\right) \qquad (306) \\
&\quad -k(p)\,\partial_{K_{\hat{X}}}(p)\Bigg)K_{\hat{X}}
\end{aligned}$$

A. 4.2.1.2 Local stability As explained in the text, equation (280) can be understood as the fixed-point equation of a dynamical system through the following mechanism**.**

Each variation $\delta Y\left(\hat{X}\right)$ in the parameters impacts the average capital, which must then be computed with the new parameters. The first change induced is written $\delta K_{\hat{X}}^{(1)}$:

$$\delta K_{\hat{X}}^{(1)}=\frac{\partial}{\partial Y\left(\hat{X}\right)}\left(\frac{\sigma_{\hat{K}}^{2}C\left(\bar{p}\right)2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}\right)\delta Y\left(\hat{X}\right) \tag{307}$$

In a second step, the variation $\delta K_{\hat{X}}$ impacts the various functions implied in (99), and indirectly modifies $K_{\hat{X}}$ through the first term in the rhs of (280):

$$\left(-\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X},K_{\hat{X}}\right)}+\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}\right)+k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}\right)K_{\hat{X}}\delta K_{\hat{X}}^{(1)} \tag{308}$$

These two effects combined, (307) and (308), yield the total variation $\delta K_{\hat{X}}$.

Importantly, note that if we can interpret $\delta K_{\hat{X}}^{(1)}$ as a variation at time t, we can also infer from the indirect effect (308) that $\delta K_{\hat{X}}$ is itself a variation at time $t+1$. Equation (280) can thus be seen as the fixed point equation of a dynamical system written:

$$\begin{aligned}\delta K_{\hat{X}}\left(t+1\right) &= \left(-\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X},K_{\hat{X}}\right)}+\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}+l\left(\hat{X},K_{\hat{X}}\right)\right)+k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}\right)K_{\hat{X}}\delta K_{\hat{X}}\left(t\right) \\ &\quad +\frac{\partial}{\partial Y\left(\hat{X},t\right)}\left(\frac{\sigma_{\hat{K}}^{2}C\left(\bar{p}\right)2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}\right)\delta Y\left(\hat{X},t\right)\end{aligned} \tag{309}$$

whose fixed point is the solution of (280):

$$\delta K_{\hat{X}}=\frac{\frac{\partial}{\partial Y\left(\hat{X}\right)}\left(\frac{\sigma_{\hat{K}}^{2}C\left(\bar{p}\right)2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}\right)}{1+\left(\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X},K_{\hat{X}}\right)}+\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}+l\left(\hat{X},K_{\hat{X}}\right)\right)-k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}\right)K_{\hat{X}}}\delta Y\left(\hat{X}\right) \tag{310}$$

This solution (**??**) is stable when:

$$\left|k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}-\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X},K_{\hat{X}}\right)}+\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}+l\left(\hat{X},K_{\hat{X}}\right)\right)\right|<1 \tag{311a}$$

i.e. when D, defined in (**??**), is positive, and unstable otherwise. So that the stability of this average capital depends, in last analysis, on the sign of D.

A 4.2.1.3 Applications of the differential form: dependency in expected returns The main application of equation (**??**) is to consider a parameter denoted $Y(\hat{X})$, that encompasses the relative expected returns of sector X vis-à-vis its neighbouring sectors, and defined as:

$$Y(\hat{X})=p\left(\hat{X}\right) \tag{312}$$

Interpretations are given in the text. To compute the dependency of averagecapital in this parameter, we use (**??**), and we have:

$$\frac{\delta K_{\hat{X}}}{K_{\hat{X}}}=\frac{\frac{k(p)}{f(\hat{X},K_{\hat{X}})}}{D}\delta p\left(\hat{X}\right) \tag{313}$$

Given equation (**??**), $k\left(p\right)$ is positive at the first order in σ_{X}^{2}. More precisely, using equation (**??**):

$$k\left(p\right)\sim_{\infty}\sqrt{\frac{p-\frac{1}{2}}{2}}-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^{2}}{48\left|f\left(\hat{X}\right)\right|^{3}}$$

along with equation (**??**), we can infer that $\sqrt{\frac{p-\frac{1}{2}}{2}}$ is of order $\frac{1}{\sigma_{X}}$ and $\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)\left(f'(X)\right)^{2}}{48\left|f\left(\hat{X}\right)\right|^{3}}\sim 1$.

Consequently, in a stable equilibrium, i.e. for $D>0$, equation (282) implies that the dependency of $K_{\hat{X}}$ in the parameter $p\left(\hat{X}\right)$ is positive:

$$\frac{\delta K_{\hat{X}}}{\delta p\left(\hat{X}\right)}>0$$

We have seen above that $p\left(\hat{X}\right)$ is maximal for a maximum expected long-term return $R\left(\hat{X},K_{\hat{X}}\right)$: when the equilibrium is stable, capital accumulation is maximal for sectors that are themselves a local maximum for $R\left(\hat{X},K_{\hat{X}}\right)$.

On the other hand, when the equilibrium is unstable, i.e. for $D<0$, the capital $K_{\hat{X}}$ is minimal for $R\left(\hat{X},K_{\hat{X}}\right)$ maximal.

Actually, as seen above, in the instability range $D<0$,the average capital $K_{\hat{X}}$ acts as a threshold. When, due to variations in the system's parameters, the average capital per firm is shifted above the threshold $K_{\hat{X}}$, capital will either move to the next stable equilibrium, possibly zero, or tend to infinity. Our results show that when the expected long-term return of a sector increases, the threshold $K_{\hat{X}}$ decreases, which favours capital accumulation.

A 4.2.1.3 Applications of the differential form: dependency in short term returns A second use of equation (**??**) is to consider $Y(\hat{X})$ as any parameter-function involved in the definition of $f\left(\hat{X},K_{\hat{X}}\right)$ that may condition either real short-term returns or the price-dividend ratio.

We can see that in this case, $Y(\hat{X})$ only impacts $f\left(\hat{X},K_{\hat{X}}\right)$, so that equation (**??**) simplifies and yields:

$$\begin{aligned}\frac{\delta K_{\hat{X}}}{K_{\hat{X}}} &= -\frac{m_{Y}\left(\hat{X},K_{\hat{X}}\right)}{D}\delta Y \\ &-\frac{1}{D}\left(\frac{\frac{\partial f(\hat{X},K_{\hat{X}})}{\partial Y}\left(1+\left(p+H\left(f\left(\hat{X},K_{\hat{X}}\right)\right)+\frac{1}{2}\right)k\left(p\right)\right)}{f\left(\hat{X},K_{\hat{X}}\right)}+\frac{\frac{\partial\left\|\Psi(\hat{X},K_{\hat{X}})\right\|^{2}}{\partial Y}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}\right)\delta Y\end{aligned} \tag{314}$$

Incidentally, note that p being proportional to $f^{-1}\left(\hat{X}\right)$, $m_Y\left(\hat{X},K_{\hat{X}}\right)$ rewrites:

$$-m_Y\left(\hat{X},K_{\hat{X}}\right) = \frac{\sigma_{\hat{X}}^2\sigma_{\hat{K}}^2\left(3\left(\nabla_Y\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2-\nabla_Y\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right|\right)\left(p+\frac{1}{2}\right)^2}{120\left|f\left(\hat{X}\right)\right|^4} + \nabla_Y\left|f\left(\hat{X}\right)\right|\frac{\sigma_{\hat{X}}^2\sigma_{\hat{K}}^2 p\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^4} \tag{315}$$

The first term in the rhs of (314) is the impact of an increase in investors' short-term returns. The second is the variation in capital needed to maintain investors' overall returns.

The sign of $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}}$ given by equation (314) can be studied under two cases: the stable and the unstable equilibrium.

Let us first consider the case of a stable equilibrium, i.e. $D>0$.

The first term in the rhs of (314), the variation induced by an increase in short-term returns, is in general positive for $f'\left(\hat{X}\right)$ proportional to $f\left(\hat{X}\right)$, that is for instance when the function $f\left(\hat{X}\right)$, that describes short-term returns and prices, depends on the variable $K_{\hat{X}}$ raised to some arbitrary power.

Indeed in that case:

$$3\left(\nabla_Y\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2-\nabla_Y\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right| = \left(\nabla_Y\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2$$

The second term in the rhs of (314) is in general negative. When $\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial Y}>0$, i.e. when returns are increasing in Y, a rise in Y increases returns and decreases the capital needed to maintain these returns. Similarly, when $\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial Y}>0$, i.e. when the number of agents in sector $\hat{X}$ is increasing in Y, a rise in Y increases the number of agents that move towards point $\hat{X}$, and the average capital per firm diminishes.

The net variation (314) of $K_{\hat{X}}$ is the sum of these two contributions. Considering an expansion of (314) in powers of $\sigma_{\hat{X}}^2$, the first contribution $-m_Y\left(\hat{X},K_{\hat{X}}\right)$ is of magnitude $\left(\sigma_{\hat{X}}^2\right)^{-1}$, whereas the second is proportional to $k\left(p\right)\sim\left(\sigma_X\right)^{-1}$. The variation $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}}$ is thus positive: $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}}>0$. In most cases, a higher short-term return, decomposed as a sum of dividend and price variation, induces a higher average capital. This effect is magnified for larger levels of capital: the third approach will confirm that, in most cases, the return $f\left(\hat{X}\right)$ is asymptotically a constant $c<<1$ when capital is high: $K_{\hat{X}}>>1$.

Turning now to the case of an unstable equilibrium, i.e. $D<0$, the variation $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}}$ is negative: $\frac{\delta K_{\hat{X}}}{K_{\hat{X}}}<0$. In the instability range, and due to this very instability, an increase in returns $f\left(\hat{X}\right)$ reduces the threshold of capital accumulation for low levels of capital. When short-term returns $f\left(\hat{X}\right)$ increase, a lower average capital will trigger capital accumulation towards an equilibrium. Otherwise, when average capital $K_{\hat{X}}$ is below this threshold, it will converge toward 0.

A 4.2.2 Second approach: Expansion around particular solutions

As explained in the text, we choose to expand (275), or equivalently (279), around solutions with $p=0$.

A 4.2.2.1 Equation (99) for $p=0$ To find the solution with $p=0$, we maximize the function:

$$A\left(\hat{X}\right)=\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+f\left(\hat{X}\right)+\frac{1}{2}\sqrt{f^{2}\left(\hat{X}\right)}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}$$

We write:

$$M=\max_{\hat{X}}A\left(\hat{X}\right) \tag{316}$$

and denote by $\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)$ the solutions $\hat{X}_{M}$ of (316) with $K_{\hat{X}_{M}}$ their associated value of average capital per firm.

Given that

$$\hat{\Gamma}\left(\frac{1}{2}\right)=\exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(X,K_{\hat{X},M}\right)\right)^{2}}{384\left|f\left(\hat{X},K_{\hat{X},M}\right)\right|^{3}}\right) \tag{317}$$

(275) becomes at points $\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)$ and $p=0$:

$$K_{\hat{X},M}\left|f\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)\right|\left\|\Psi\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)\right\|^{2}\simeq\sigma_{\hat{K}}^{2}C\left(\bar{p}\right)\exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)\right)^{2}}{384\left|f\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)\right|^{3}}\right) \tag{318}$$

This equation has in general several solutions, depending on the assumptions on $f\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)$.

Note that once a solution $K_{\hat{X}}$ of (279) is found, the value of $C\left(\bar{p}\right)$ can be obtained by solving (272) and using (274). These solutions are discussed in the text.

The next paragraph computes the expansion of (275) around these solutions with $p=0$. Remark that coming back to (275) and (279) for general values of p defined in (273), the value of $C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}$ can be replaced by $K_{\hat{X}_{M}}\left|f\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)\right|\left\|\Psi\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)\right\|^{2}$ for any solution $\left(\hat{X}_{M},K_{\hat{X}_{M}}\right)$.

A 4.2.2.2 Expansion around particular solutions To better understand the behavior of the solutions of equation (99), we expand this equation around the points $\left(\hat{X},K_{\hat{X},M}\right)$ that solve equation (99). We can find approximate solutions to (275):

$$K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^{2}\left|f\left(\hat{X}\right)\right|=C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\hat{\Gamma}\left(p+\frac{1}{2}\right) \tag{319}$$

with:

$$\begin{aligned}\hat{\Gamma}\left(p+\frac{1}{2}\right) &= \exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}\right)\\ &\times\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right)\end{aligned} \tag{320}$$

for general form of the functions $f\left(\hat{X}\right)$ and $g\left(\hat{X}\right)$ by expanding (319), for each $\hat{X}$,around the closest point $\hat{X}_{M}$ satisfying (319) with $p=0$. We use that:

$$\begin{aligned}&\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right)\\ &= 1-p\left(\gamma_{0}+\ln 2-2\right)+o\left(p\right)\end{aligned} \tag{321}$$

with γ_0 the Euler-Mascheroni constant, as well as the following relations:

$$\nabla_{K_{\hat{X}}}\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2 h(p)\left(f'\left(\hat{X}\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)_{p=0}\simeq-\frac{\nabla_{K_{\hat{X}}}\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right|-3\left(\nabla_{K_{\hat{X}}}\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2}{120\left|f\left(\hat{X}\right)\right|^4}$$

and:

$$\begin{aligned}&\nabla_{\hat{X}}\left(-\frac{h(p)\left(f'\left(\hat{X}\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)_{p=0}\\ \simeq\ &-\frac{2f'(X)f''\left(\hat{X}\right)\left|f\left(\hat{X}\right)\right|-3\left(f'\left(\hat{X}\right)\right)^3}{120\left|f\left(\hat{X}\right)\right|^4}=\frac{f'(X)\left(3\left(f'\left(\hat{X}\right)\right)^2-2f''(X)\left|f\left(\hat{X}\right)\right|\right)}{120\left|f\left(\hat{X}\right)\right|^4}\end{aligned}$$

the expansion of (319) at the lowest order, is:

$$\begin{aligned}&\left(1+\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X},K_{\hat{X}}\right)}+\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}}\left(K_{\hat{X}}-K_{\hat{X},M}\right)+\left(\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial\hat{X}}}{f\left(\hat{X},K_{\hat{X}}\right)}+\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial\hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}}\left(\hat{X}-\hat{X}_M\right)\\ \simeq\ &-\left(\sigma_X^2\sigma_{\hat{K}}^2\frac{\nabla_{K_{\hat{X}}}\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right|-3\left(\nabla_{K_{\hat{X}}}\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2}{120\left|f\left(\hat{X}\right)\right|^4}\right)_{K_{\hat{X},M}}\left(K_{\hat{X}}-K_{\hat{X},M}\right)\\ &-\left(\sigma_X^2\sigma_{\hat{K}}^2\frac{2f'(X)f''\left(\hat{X}\right)\left|f\left(\hat{X}\right)\right|-3\left(f'\left(\hat{X}\right)\right)^3}{120\left|f\left(\hat{X}\right)\right|^4}\right)_{K_{\hat{X},M}}\left(\hat{X}-\hat{X}_M\right)\\ &-b\frac{\partial_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2}+\left(f\left(\hat{X},K_{\hat{X}}\right)+\frac{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}{2}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\right)\left(K_{\hat{X}}-K_{\hat{X},M}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}\\ &-b\frac{\partial_{\hat{X}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2}+\left(f\left(\hat{X},K_{\hat{X}}\right)+\frac{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}{2}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\right)\left(\hat{X}-\hat{X}_M\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}\end{aligned}$$

Given the maximization (316), the two last terms in the right hand side is equal to 0.

$$\begin{aligned}\left(K_{\hat{X}}-K_{\hat{X},M}\right)=\ &\frac{1}{D}\left(\sigma_X^2\sigma_{\hat{K}}^2\frac{3\left(f'\left(\hat{X}\right)\right)^3-2f'(X)f''\left(\hat{X}\right)\left|f\left(\hat{X}\right)\right|}{120\left|f\left(\hat{X}\right)\right|^4}\right.\\ &\left.-\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial\hat{X}}}{f\left(\hat{X},K_{\hat{X}}\right)}-\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial\hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}}\left(\hat{X}-\hat{X}_M\right)\\ &-\frac{1}{D}\frac{b}{2}\left(\hat{X}-\hat{X}_M\right)\nabla_{\hat{X}}^2\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2}+\frac{3}{2}f\left(\hat{X},K_{\hat{X}}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)_{K_{\hat{X},M}}\left(\hat{X}-\hat{X}_M\right)\end{aligned}\tag{322}$$

with:

$$D=\left(1+\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}}{f\left(\hat{X},K_{\hat{X}}\right)}+\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial K_{\hat{X}}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}+\frac{\sigma_{\hat{X}}^2\sigma_{\hat{K}}^2\left(\nabla_{K_{\hat{X}}}\left(f'\left(\hat{X}\right)\right)^2\left|f\left(\hat{X}\right)\right|-3\left(\nabla_{K_{\hat{X}}}\left|f\left(\hat{X}\right)\right|\right)\left(f'\left(\hat{X}\right)\right)^2\right)}{120\left|f\left(\hat{X}\right)\right|^4}\right)_{K_{\hat{X}_M}}$$

and $K_{\hat{X}_M}$ solution of:

$$\begin{aligned}&K_{\hat{X},M}\left|f\left(\hat{X},K_{\hat{X},M}\right)\right|\left\|\Psi\left(\hat{X},K_{\hat{X},M}\right)\right\|^2\\ \simeq\ &\sigma_{\hat{K}}^2\exp\left(-\frac{\sigma_{\hat{X}}^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)C\left(\bar{p}\right)\simeq\sigma_{\hat{K}}^2C\left(\bar{p}\right)\end{aligned}$$

The maximization condition (316) cancels the contribution due to:

$$\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2}+\left(f\left(\hat{X},K_{\hat{X}}\right)+\frac{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|}{2}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)$$

To find a contribution due to this term, we must expand (319) to the second order. The second order contributions proportional to $\left(K_{\hat{X}}-K_{\hat{X},M}\right)^2$ modifies slightly (322) and the term $\left(K_{\hat{X}}-K_{\hat{X},M}\right)\left(\hat{X}-\hat{X}_M\right)$ shifts D at the first order. Both modifications do not alter the interpretation for (322). We can thus consider the sole term:

$$\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|}$$

Due to (279), for $H\left(K_{\hat{X}}\right)$ slowly varying, the contribution due to the derivatives of $\left\|\Psi\left(\hat{X}\right)\right\|^2$ can be neglected. Moreover the contribution due to the derivative of $\left|f\left(\hat{X}\right)\right|$ are negligible with respect to the first order terms. We can thus consider only the second order contributions due to $\hat{\Gamma}\left(p+\frac{1}{2}\right)$. In the rhs of (320), the second term is dominant. Moreover, we can check that in the second order expansion of (321), the term in p^2 can be neglected compared to $-p\left(\gamma_0+\ln 2-2\right)$. Consequently, the relevant second order correction to (322) is :

$$b\left(\hat{X}-\hat{X}_M\right)\nabla_{\hat{X}}^2p\left(\hat{X}-\hat{X}_M\right)=b\left(\hat{X}-\hat{X}_M\right)\nabla_{\hat{X}}^2\left(\frac{M-\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2}+\frac{3}{2}f\left(\hat{X},K_{\hat{X}}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)}{\left|f\left(\hat{X}\right)\right|}\right)\left(\hat{X}-\hat{X}_M\right)$$

and the relevant contributions to (322) are:

$$\left(K_{\hat{X}}-K_{\hat{X},M}\right) = \frac{1}{D}\left(\sigma_X^2\sigma_{\hat{K}}^2\frac{3\left(f'\left(\hat{X}\right)\right)^3-2f'\left(X\right)f''\left(\hat{X}\right)\left|f\left(\hat{X}\right)\right|}{120\left|f\left(\hat{X}\right)\right|^4}\right.$$
$$\left.-\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial\hat{X}}}{f\left(\hat{X},K_{\hat{X}}\right)}-\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial\hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}}\left(\hat{X}-\hat{X}_M\right)$$
$$+\frac{1}{D}\frac{b}{2}\left(\hat{X}-\hat{X}_M\right)\nabla_{\hat{X}}^2\left(\frac{M-\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{\sigma_{\hat{X}}^2}+\frac{3}{2}f\left(\hat{X},K_{\hat{X}}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)}{\left|f\left(\hat{X}\right)\right|}\right)_{K_{\hat{X},M}}\left(\hat{X}-\hat{X}_M\right) \tag{323}$$

A 4.2.2.3 Interpretation of (323) As in the first approach, $D>0$ corresponds to a stable equilibrium, and $D<0$ to an unstable one. The expansion (290) describes the local variations of $K_{\hat{X}}$ in the neighbourhood of the points $K_{\hat{X},M}$. This approximation (290) suffices to understand the role of the parameters of the system.

We consider the case of stable equilibria, i.e. $D>0$. Note that under unstable equilibria, $D<0$, the interpretations are inverted, since $K_{\hat{X}}$ is interpreted as a threshold[68].

The equation (290), that expands average capital at sector $\hat{X}_M$, is composed of a first order and a second order contributions.

The first order part in the expansion (290) writes:

$$\frac{1}{D}\left(\sigma_X^2\sigma_{\hat{K}}^2\frac{3\left(f'\left(\hat{X}\right)\right)^3-2f'\left(X\right)f''\left(\hat{X}\right)\left|f\left(\hat{X}\right)\right|}{120\left|f\left(\hat{X}\right)\right|^4}-\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial\hat{X}}}{f\left(\hat{X},K_{\hat{X}}\right)}-\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial\hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}}\left(\hat{X}-\hat{X}_M\right) \tag{324}$$

It represents the variation of equilibrium capital as a function of its position. It is decomposed in three contributions:

For $f'\left(\hat{X}\right)>0$, the second contribution in (324):

$$-\frac{\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial\hat{X}}\left(\hat{X}-\hat{X}_M\right)}{f\left(\hat{X},K_{\hat{X}}\right)}$$

is positive. It represents the decrease in capital needed to reach equilibrium. Actually, the return is higher at point $\hat{X}$ than at $\hat{X}_M$: a lower capital will yield the same overall return at point $\hat{X}$. On the contrary, the first contribution in (324):

$$\frac{\sigma_X^2\sigma_{\hat{K}}^2\frac{3\left(f'\left(\hat{X}\right)\right)^3-2f'\left(X\right)f''\left(\hat{X}\right)\left|f\left(\hat{X}\right)\right|}{120\left|f\left(\hat{X}\right)\right|^4}\left(\hat{X}-\hat{X}_M\right)}{D}$$

describes the "net" variation of capital due to a variation in $f\left(X\right)$. When returns are decreasing, i.e. when $f'\left(\hat{X}\right)>0$ and $f''\left(\hat{X}\right)<0$, this first contribution has the sign of $f'\left(\hat{X}\right)$. An increase in returns attracts capital.

[68] See the first approach.

The third term in (324):

$$-\frac{1}{D}\left(\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial\hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}\right)_{K_{\hat{X},M}}\left(\hat{X}-\hat{X}_M\right)$$

represents the number effect. Actually, when:

$$\frac{\frac{\partial\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}{\partial\hat{X}}}{\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^2}>0$$

the number of agents is higher at $\hat{X}$ than at $\hat{X}_M$: the average capital per agent is reduced.

The second order contribution in (290) represents the effect of the neighbouring sector space on each sector. Given the first order condition (285):

$$\nabla_{\hat{X}}^2\left(\frac{M-A\left(\hat{X}\right)}{f\left(\hat{X}\right)}\right)_{K_{\hat{X},M}}=\left(\frac{\nabla_{\hat{X}}^2\left(M-A\left(\hat{X}\right)\right)}{f\left(\hat{X}\right)}\right)_{K_{\hat{X},M}}$$

and since $A\left(\hat{X}_M\right)$ is a maximum, we have:

$$\left(\hat{X}-\hat{X}_M\right)\nabla_{\hat{X}}^2\left(\frac{M-A\left(\hat{X}\right)}{f\left(\hat{X}\right)}\right)_{K_{\hat{X},M}}\left(\hat{X}-\hat{X}_M\right)>0$$

When $f\left(\hat{X}\right)$ is constant, $A\left(\hat{X}_M\right)$ is a local maximum, and $K_{\hat{X}_M}$ is a minimum. To put it differently, $K_{\hat{X}}$ is a decreasing function of $A\left(\hat{X}\right)$. This is in line with the definition of $A\left(\hat{X}\right)$[69], which measures the relative attractiveness of sector $\hat{X}$'s neighbours: the higher $A\left(\hat{X}\right)$, the lower the incentive for capital to stay in sector $\hat{X}$.

A 4.2.3 Third approach: Resolution for particular form for the functions

As stated in the text, we can find approximate solutions to (279) by choosing some forms for the parameters functions. The solutions are then studied in some ranges for average capital per firm K_X: $K_X >> 1$, $K_X >>> 1$, $K_X << 1$ and the intermediate range $\infty > K_X > 1$ In the case $K_X >>> 1$, the distinction between stable and unstable cases has to be made.

A 4.2.3.1 Function $H^2\left(K_X\right)$ We can choose for $H^2\left(K_X\right)$ a power function of K_X:

$$H\left(K_X\right)=K_X^{\eta} \tag{325}$$

so that equation (81) rewrites:

$$\left\|\Psi\left(X\right)\right\|^2\simeq\frac{D\left(\left\|\Psi\right\|^2\right)-\frac{F}{2\sigma_X^2}\left(\left(\nabla_X R\left(X\right)\right)^2+\frac{2\sigma_X^2\nabla_X^2 R\left(K_X,X\right)}{H\left(K_X\right)}\right)K_X^{\eta}}{2\tau}\equiv D-L\left(X\right)\left(\nabla_X R\left(X\right)\right)^2 K_X^{\eta} \tag{326}$$

[69] See discussions after equations (??) and (??).

A 4.2.3.2 Function f To determine the function f, we must first assume a form for $r(K,X)$, the physical capital marginal returns, and for F_1, the function that measures the impact of expected long-term return on investment choices.

Assuming the production functions are of Cobb-Douglas type, i.e. $B(X)K^{\alpha}$ with $B(X)$ a productivity factor, we have for $r(K,X)$:

$$r(K,X)=\frac{\partial r(K,X)}{\partial K}=\alpha B(X)K^{\alpha-1} \tag{327}$$

For function F_1, the simplest choice would be a linear form:

$$F_1\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\int R\left(K'_{X'},X'\right)\left\|\Psi(X')\right\|^2 dX'}\right)\simeq F_1\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\left\langle K_{\hat{X}}^{\alpha}\right\rangle\left\langle R\left(\hat{X}\right)\right\rangle}\right)=b\left(\frac{K_{\hat{X}}^{\alpha}R\left(\hat{X}\right)}{\left\langle K_X^{\alpha}\right\rangle\left\langle R(X)\right\rangle}-1\right)$$

where, for any function $u\left(\hat{X}\right)$, $\left\langle u\left(\hat{X}\right)\right\rangle$ denotes its average over the sector space, and b an arbitrary parameter.

However, when capital $K_{\hat{X}}^{\alpha}\rightarrow\infty$ and is concentrated at $\hat{X}$, we have $\left\langle K_X^{\alpha}\right\rangle\simeq\frac{K_{\hat{X}}^{\alpha}}{N^{\alpha}(X)}$, so that $\frac{K_{\hat{X}}^{\alpha}R\left(\hat{X}\right)}{\left\langle K_X^{\alpha}\right\rangle\left\langle R(X)\right\rangle}\rightarrow\frac{N^{\alpha}(X)R\left(\hat{X}\right)}{\left\langle R(X)\right\rangle}>>1$. To impose some bound on moves in the sector space we rather choose:

$$F_1\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\left\langle K_{\hat{X}}^{\alpha}\right\rangle\left\langle R\left(\hat{X}\right)\right\rangle}\right)\simeq b\arctan\left(\frac{K_{\hat{X}}^{\alpha}R\left(\hat{X}\right)}{\left\langle K_X^{\alpha}\right\rangle\left\langle R(X)\right\rangle}-1\right) \tag{328}$$

so that $F_1\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\left\langle K_{\hat{X}}^{\alpha}\right\rangle\left\langle R\left(\hat{X}\right)\right\rangle}\right)>0$ when $\frac{K_{\hat{X}}^{\alpha}R\left(\hat{X}\right)}{\left\langle K_X^{\alpha}\right\rangle\left\langle R(X)\right\rangle}>1$.

Given the above assumptions, the general formula for f given in equation (84) rewrites:

$$f\left(\hat{X},\Psi,\hat{\Psi}\right)=\frac{1}{\varepsilon}\left(r\left(\hat{X}\right)K_{\hat{X}}^{\alpha-1}-\gamma\left\|\Psi\left(\hat{X}\right)\right\|^2+b\arctan\left(\frac{K_{\hat{X}}^{\alpha}R\left(\hat{X}\right)}{\left\langle K_X^{\alpha}\right\rangle\left\langle R(X)\right\rangle}-1\right)\right) \tag{329}$$

This general formula can be approximated for $\frac{K_{\hat{X}}^{\alpha}R\left(\hat{X}\right)}{\left\langle K_X^{\alpha}\right\rangle\left\langle R(X)\right\rangle}\simeq 1$, when average capital in sector $\hat{X}$ is close to the average capital of the whole space, which is usually the case.

Using our choices (291), (327) and (328) for $\left\|\Psi(X)\right\|^2$ $r\left(\hat{X}\right)$ and F_1 respectively, the equation (84) for $f\left(\hat{X},\Psi,\hat{\Psi}\right)$ becomes:

$$f\left(\hat{X},\Psi,\hat{\Psi}\right)=\frac{1}{\varepsilon}\left(\left(r\left(\hat{X}\right)+\frac{bR\left(\hat{X}\right)K_{\hat{X}}^{\alpha}}{\left\langle K_{\hat{X}}^{\alpha}\right\rangle\left\langle R\left(\hat{X}\right)\right\rangle}\right)+\gamma L\left(\hat{X}\right)K_X^{\eta}-\gamma D-b\right)$$

We may assume without impairing the results that $\eta=\alpha$. We thus have:

$$\begin{aligned}f\left(\hat{X},\Psi,\hat{\Psi}\right) &= \frac{1}{\varepsilon}\left(\left(\frac{r\left(\hat{X}\right)}{K_{\hat{X}}^{\alpha}}+\frac{bR\left(\hat{X}\right)}{\left\langle K_{\hat{X}}^{\alpha}\right\rangle\left\langle R\left(\hat{X}\right)\right\rangle}+\gamma L\left(\hat{X}\right)\right)K_{\hat{X}}^{\alpha}-\gamma D-b\right) \\ &\equiv B_1\left(\hat{X}\right)K_{\hat{X}}^{\alpha-1}+B_2\left(\hat{X}\right)K_{\hat{X}}^{\alpha}-C\left(\hat{X}\right)\end{aligned} \tag{330}$$

where:

$$\begin{aligned}
B_1\left(\hat{X}\right) &= \frac{\alpha B\left(\hat{X}\right)}{\varepsilon} \\
B_2\left(\hat{X}\right) &= \frac{bR\left(\hat{X}\right)}{\varepsilon\left\langle K_{\hat{X}}^{\alpha}\right\rangle\left\langle R\left(\hat{X}\right)\right\rangle} + \frac{\gamma}{\varepsilon} \\
C\left(\hat{X}\right) &= \gamma D + b
\end{aligned}$$

A 4.2.3.3 Function g To determine the form of function g, equation (85), we must first choose a form for the function F_0.

We assume that:

$$F_0\left(R\left(\hat{X}, K_{\hat{X}}\right)\right) = a \arctan\left(K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)\right) \tag{331}$$

where is a an arbitrary constant.

Combined to our assumption for F_1, (328), the formula (85) for g can be written:

$$g\left(\hat{X}, \Psi, \hat{\Psi}\right) = a\nabla_{\hat{X}} \arctan\left(K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)\right) + b\nabla_{\hat{X}} \arctan\left(\frac{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)}{\left\langle K_{X}^{\alpha}\right\rangle\left\langle R\left(X\right)\right\rangle} - 1\right) \tag{332}$$

where the arctan function ensures that the velocity in the sector space g increases with capital and is maximal when average capital per firm in sector $\hat{X}$ tends to infinity, i.e. $K_{\hat{X}}^{\alpha} \to \infty$.

This general formula, equation (**??**), can be approximated for $\frac{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)}{\left\langle K_{X}^{\alpha}\right\rangle\left\langle R(X)\right\rangle} \simeq 1$, when average capital in sector $\hat{X}$ is close to the average capital of the whole space. It then reduces to:

$$g\left(\hat{X}, \Psi, \hat{\Psi}\right) \simeq \frac{K_{\hat{X}}^{\alpha}}{\left\langle K_{\hat{X}}^{\alpha}\right\rangle} \nabla_{\hat{X}} R\left(\hat{X}\right)\left(1 + \frac{b}{\left\langle R\left(\hat{X}\right)\right\rangle}\right) \equiv \nabla_{\hat{X}} R\left(\hat{X}\right) A\left(\hat{X}\right) K_{\hat{X}}^{\alpha} \tag{333}$$

which in turn allows to approximate the gradient of g, $\nabla_{\hat{X}} g\left(\hat{X}, \Psi, \hat{\Psi}\right)$, by:

$$\nabla_{\hat{X}} g\left(\hat{X}, \Psi, \hat{\Psi}\right) \simeq \frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)}{\left\langle K_{\hat{X}}^{\alpha}\right\rangle}\left(1 + \frac{b}{\left\langle K_{\hat{X}}^{\alpha}\right\rangle\left\langle R\left(\hat{X}\right)\right\rangle}\right) K_{\hat{X}}^{\alpha} \equiv \nabla_{\hat{X}}^2 R\left(\hat{X}\right) A\left(\hat{X}\right) K_{\hat{X}}^{\alpha} \tag{334}$$

A 4.2.3.4 Solving (279) Equation (279) can be studied by considering five cases presented in the text:

Case 1. Very high capital, $K_{\hat{X}} >>> 1$**, stable case** In that case, $K_{\hat{X}} >>> 1$, and we assume in first approximation that (discarding the factor $L\left(\hat{X}\right)$):

$$\left\|\Psi\left(\hat{X}\right)\right\|^2 \simeq D - \left(\nabla_X R\left(\hat{X}\right)\right)^2 K_{\hat{X}}^{\alpha} << 1 \tag{335}$$

This corresponds to a very high level of capital. Consequently, equation (329) implies that the function $f\left(\hat{X}\right)$ can be rewritten:

$$\begin{aligned} f\left(\hat{X}\right) &= \frac{1}{\varepsilon}\left(r\left(\hat{X}\right)K_{\hat{X}}^{\alpha-1}-\gamma\left\|\Psi\left(\hat{X}\right)\right\|^{2}+b\arctan\left(\frac{K_{\hat{X}}^{\alpha}R\left(\hat{X}\right)}{\langle K_{X}^{\alpha}\rangle\langle R\left(X\right)\rangle}-1\right)\right) \\ &\simeq b\left(\frac{\pi}{2}-\frac{\langle K_{X}^{\alpha}\rangle\langle R\left(X\right)\rangle}{K_{\hat{X}}^{\alpha}R\left(\hat{X}\right)}\right) \\ &\equiv c-\frac{d}{K_{\hat{X}}^{\alpha}R\left(\hat{X}\right)}\simeq c>0 \end{aligned}$$

Consequently, the expressions for $f'\left(\hat{X}\right)g\left(\hat{X}\right)$ and $\nabla_{\hat{X}}g\left(\hat{X}\right)$ (346) and (347) are still valid.

Two different cases arise in the resolution of (275).

First, we assume that $\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^{2}\neq 0$.

In this case, we will solve (275) by using (335) to replace $K_{\hat{X}}\simeq\left(\frac{D}{\left(\nabla_{X}R\left(\hat{X}\right)\right)^{2}}\right)^{\frac{1}{\alpha}}$. We also change the variable $\frac{D}{\left(\nabla_{X}R\left(\hat{X}\right)\right)^{2}}\rightarrow D$ temporarily for the sake of simplicity.

Inequality (335) along with $K_{\hat{X}}>>>1$ and (329) implies that only the case $f>0$ has to be considered.

Note that using our results about stability, it is easy to check that in that case, this solution is locally unstable. A very high level of capital has the tendency to attract more investments.

Given our assumptions, equation (279) becomes:

$$\left(\nabla_{X}R\left(\hat{X}\right)\right)^{2}D^{\frac{1}{\alpha}}\left(D-K_{\hat{X}}^{\alpha}\right)=C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}\right)\frac{\Gamma\left(p+\frac{3}{2}\right)}{\left|f\left(\hat{X}\right)\right|} \tag{336}$$

or equivalently:

$$K_{\hat{X}}^{\alpha}=D-\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}\right)\Gamma\left(p+\frac{3}{2}\right)}{\left(\nabla_{X}R\left(\hat{X}\right)\right)^{2}D^{\frac{1}{\alpha}}\left|f\left(\hat{X}\right)\right|} \tag{337}$$

Then, defining $V=\frac{1}{K_{\hat{X}}^{\alpha}}$ as in the first case, we can write (337) as an equation for $V<<1$ by replacing all quantities in term of V and then perform a first order expansion.

First, we write (337) as:

$$V-\frac{1}{D-\frac{C(\bar{p})\sigma_{\hat{K}}^{2}\exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)^{2}\left(f'(X)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}\right)\Gamma\left(p+\frac{3}{2}\right)}{\left(\nabla_{X}R\left(\hat{X}\right)\right)^{2}D^{\frac{1}{\alpha}}\left|f\left(\hat{X}\right)\right|}}=0 \tag{338}$$

As in the previous case, the first order expansion in V of $\Gamma\left(p+\frac{3}{2}\right)$ arising in (338) is given by:

$$\Gamma\left(p+\frac{3}{2}\right)\simeq\Gamma\left(\frac{M}{c}\right)+\frac{MV}{c}\left(\frac{\nabla_{\hat{X}}^{2}R\left(\hat{X}\right)f}{MR\left(\hat{X}\right)}+\frac{d}{cR\left(\hat{X}\right)}\right)\Gamma'\left(\frac{M}{c}\right) \tag{339}$$

Moreover, at the first order:

$$\exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)\simeq 1$$

and (338) becomes:

$$V-\frac{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}}\left|f\left(\hat{X}\right)\right|}{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}}\left|f\left(\hat{X}\right)\right|-C\left(\bar{p}\right)\sigma_{\hat{K}}^2\Gamma\left(p+\frac{3}{2}\right)}=0$$

that is:

$$V-\frac{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}}\left(c-\frac{dV}{R\left(\hat{X}\right)}\right)}{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}}\left(c-\frac{dV}{R\left(\hat{X}\right)}\right)-C\left(\bar{p}\right)\sigma_{\hat{K}}^2\Gamma\left(p+\frac{3}{2}\right)}=0 \tag{340}$$

Using (339) the first order expansion of the dominator in (340) is:

$$\begin{aligned}
&\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}}\left(c-\frac{dV}{R\left(\hat{X}\right)}\right)-C\left(\bar{p}\right)\sigma_{\hat{K}}^2\Gamma\left(p+\frac{3}{2}\right)\\
=\;&\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}}c-C\left(\bar{p}\right)\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)\\
&-\left(\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}}\frac{d}{R\left(\hat{X}\right)}+\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 M}{c}\left(\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)f}{MR\left(\hat{X}\right)}+\frac{d}{cR\left(\hat{X}\right)}\right)\Gamma'\left(\frac{M}{c}\right)\right)V
\end{aligned}$$

so that (340) writes:

$$\begin{aligned}
&\frac{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}}c}{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}}c-C\left(\bar{p}\right)\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)}\\
=\;&\left(1-\frac{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}}c\left(\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}}\frac{d}{R\left(\hat{X}\right)}+\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 M}{c}\left(\frac{\nabla_{\hat{X}}^2R\left(\hat{X}\right)f}{MR\left(\hat{X}\right)}+\frac{d}{cR\left(\hat{X}\right)}\right)\Gamma'\left(\frac{M}{c}\right)\right)}{\left(\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}}c-C\left(\bar{p}\right)\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)\right)^2}\right)V\\
&+\frac{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}}\frac{d}{R\left(\hat{X}\right)}}{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{1+\frac{1}{\alpha}}c-C\left(\bar{p}\right)\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)}V
\end{aligned} \tag{341}$$

Equation (341) can be solved for V with solution:

$$\frac{1}{V}=D-\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)}{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2 D^{\frac{1}{\alpha}}c}+\frac{d}{cR\left(\hat{X}\right)}\left(1-\frac{\left(1+\frac{C(\bar{p})\sigma_{\hat{K}}^2 M\Gamma\left(\frac{M}{c}\right)}{c\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2D^{1+\frac{1}{\alpha}}}\left(\frac{\nabla_{\hat{X}}^2R\left(\hat{X}\right)f}{Md}+\frac{1}{c}\right)\operatorname{Psi}\left(\frac{M}{c}\right)\right)}{\left(1-\frac{C(\bar{p})\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)}{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2D^{1+\frac{1}{\alpha}}c}\right)}\right)$$

Ultimatly, restoring the variable:

$$D \to \frac{D}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2}$$

we obtain the solution $K_{\hat{X}}^{\alpha} = \frac{1}{V}$:

$$\begin{aligned} K_{\hat{X}}^{\alpha} &= \frac{D}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2} - \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M}{c}\right)}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{2\left(1-\frac{1}{\alpha}\right)} D^{\frac{1}{\alpha}} c} \\ &\quad + \frac{d}{cR\left(\hat{X}\right)} \left(1 - \frac{\left(1 + \frac{C(\bar{p})\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}} \sigma_{\hat{K}}^2}{cD^{1+\frac{1}{\alpha}}} \left(\frac{M}{c} + \frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right) f}{d}\right) \Gamma'\left(\frac{M}{c}\right)\right)}{\left(1 - \frac{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}} C(\bar{p}) \sigma_{\hat{K}}^2}{cD^{1+\frac{1}{\alpha}}} \Gamma\left(\frac{M}{c}\right)\right)}\right) \end{aligned} \tag{342}$$

As stated in the text, this is increasing in c, i.e. in $f\left(\hat{X}\right)$ and in $R\left(\hat{X}\right)$. This corresponds to a stable level of capital.

Case 2. Very high capital, $K_{\hat{X}} >>> 1$, unstable case In this second case, we consider that $\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2 \to 0$ and formula (335)and (342) are not valid anymore. Coming back to (171) leads rather to replace $\left(\nabla_X R\left(X\right)\right)^2$:

$$\left(\nabla_X R\left(X\right)\right)^2 \to \left(\nabla_X R\left(X\right)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R\left(K_X, X\right)}{H\left(K_X\right)} = \sigma_X^2 \frac{\nabla_X^2 R\left(K_X, X\right)}{H\left(K_X\right)}$$

Thus, if $\nabla_X^2 R\left(K_X, X\right) < 0$, (336) is replaced by:

$$K_{\hat{X}}^{\alpha}\left(D + \sigma_X^2 \left|\nabla_X^2 R\left(K_X, X\right)\right| K_{\hat{X}}^{\frac{\alpha}{2}}\right) = C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \frac{\Gamma\left(p+\frac{3}{2}\right)}{\left|f\left(\hat{X}\right)\right|}$$

with:

$$p + \frac{3}{2} \simeq \frac{M - \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)}{f\left(\hat{X}\right)}$$

and the equation for K_X writes:

$$\sigma_X^2 \left|\nabla_X^2 R\left(K_X, X\right)\right| K_{\hat{X}}^{\frac{3}{2}\alpha} = \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2 \Gamma\left(\frac{M - \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)}{f\left(\hat{X}\right)}\right)}{\left|f\left(\hat{X}\right)\right|}$$

Since, given our assumptions $f\left(\hat{X}\right) \to c$ we find:

$$K_{\hat{X}} = \left(\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}{\left|\nabla_X^2 R\left(K_X, X\right)\right| c} \Gamma\left(\frac{M - \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right)}{c}\right)\right)^{\frac{2}{3\alpha}} \tag{343}$$

Note that given (343), an equilibrium in the range $K_{\hat{X}} >>> 1$ is only possible for $c << 1$ Otherwise, there is no equilibrium for a maximum of $R\left(K_X, X\right)$. This equilibrium value of $K_{\hat{X}}$ decreases with c, which corresponds to an unstable equilibrium, as detailed in the text.

On the other hand, if $\nabla_X R(X) = 0$ and $\nabla^2_X R(K_X, X) > 0$, expression (335) becomes:

$$\left\| \Psi \left(\hat{X} \right) \right\|^2 \simeq D - \sigma_X^2 \frac{\nabla^2_X R(K_X, X)}{H(K_X)} K_{\hat{X}}^{\alpha} = D - \sigma_X^2 \nabla^2_X R(K_X, X) K_{\hat{X}}^{\frac{\alpha}{2}}$$

and thus:

$$K_{\hat{X}}^{\alpha} \simeq \left(\frac{D}{\sigma_X^2 \nabla^2_X R(K_X, X)} \right)^2 \tag{344}$$

However, this solution with $K_X >> 1$ corresponds to points such that $\nabla^2_X R(K_X, X) > 0$ and $\nabla_X R(X) = 0$. Then, these points are minima of $R(X)$. This equilibrium may exist only if the level of capital (344) is high enough to compensate the weakness of the purely position dependent part of expected return and match the condition:

$$\frac{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)}{\langle K_X^{\alpha} \rangle \langle R(X) \rangle} - 1 > 0$$

This equilibrium is thus unlikely and may be discarded in general.

Case 3. High capital, $K_{\hat{X}} >> 1$ In that case, we assume $K_{\hat{X}}$ relatively large, but bounded, to ensure that the approximation:

$$\left\| \Psi \left(\hat{X} \right) \right\|^2 \simeq D \tag{345}$$

is still valid.

Equations (329) and (332) imply that the function $f\left(\hat{X}\right)$ is independent of $K_{\hat{X}}$ and that $g\left(\hat{X}\right)$ is proportional to $\nabla_{\hat{X}} R\left(\hat{X}\right)$. Given (329), the function $f\left(\hat{X}\right)$ can be rewritten:

$$\begin{aligned} f\left(\hat{X}\right) &= \frac{1}{\varepsilon}\left(r\left(\hat{X}\right) K_{\hat{X}}^{\alpha-1} - \gamma \left\| \Psi\left(\hat{X}\right) \right\|^2 + b \arctan\left(\frac{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)}{\langle K_X^{\alpha} \rangle \langle R(X) \rangle} - 1 \right) \right) \\ &\simeq b\left(\frac{\pi}{2} - \frac{\langle K_X^{\alpha} \rangle \langle R(X) \rangle}{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)} \right) - \gamma D \\ &\equiv c - \frac{d}{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)} - \gamma D \end{aligned}$$

Consequently, the expression for $f'\left(\hat{X}\right)$ is:

$$f'\left(\hat{X}\right) \simeq \frac{d \nabla_{\hat{X}} R\left(\hat{X}\right)}{K_{\hat{X}}^{\alpha} R^2\left(\hat{X}\right)} \tag{346}$$

Similarly,we can approximate (332) as:

$$\begin{aligned} g\left(\hat{X}\right) &\simeq -\frac{\nabla_{\hat{X}} R\left(\hat{X}\right) f}{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)} \\ \nabla_{\hat{X}} g\left(\hat{X}\right) &\simeq -\frac{\nabla^2_{\hat{X}} R\left(\hat{X}\right) f}{K_{\hat{X}}^{\alpha} R\left(\hat{X}\right)} \end{aligned} \tag{347}$$

Given (345), and including the constant α in the definition of $C(\bar{p})$, equation (279) is:

$$K_{\hat{X}} D\left|f\left(\hat{X}\right)\right| = C(\bar{p})\,\sigma_{\hat{K}}^2 \exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)\Gamma\left(p+\frac{3}{2}\right) \tag{348}$$

with:

$$p+\frac{1}{2} = \frac{M-\left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2}+\left(f\left(\hat{X}\right)+\nabla_{\hat{X}} g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)\right)}{\sqrt{f^2\left(\hat{X}\right)}}$$

Defining $V=\frac{1}{K_{\hat{X}}^{\alpha}}$, we can write (348) as an equation for $V<<1$ by replacing all quantities in term of V and then perform a first order expansion. To do so, we first, we write (348) as:

$$V-\frac{D\left|f\left(\hat{X}\right)\right|}{C(\bar{p})\,\sigma_{\hat{K}}^2 \exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)\Gamma\left(p+\frac{3}{2}\right)}=0 \tag{349}$$

and then find an expansion in V for $\Gamma\left(p+\frac{3}{2}\right)$.

The first order expansion in V of $p+\frac{3}{2}$ is:

$$\begin{aligned}
p+\frac{3}{2} &= \frac{M-\left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2}+\nabla_{\hat{X}} g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{f\left(\hat{X}\right)}\\
&\simeq \frac{M-\left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2}+\nabla_{\hat{X}} g\left(\hat{X},K_{\hat{X}}\right)\right)}{c-\frac{d}{K_{\hat{X}}^{\alpha}R\left(\hat{X}\right)}}\\
&= \frac{M-\left(\frac{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\left(-\frac{fV}{R\left(\hat{X}\right)}\right)\right)^2}{\sigma_{\hat{X}}^2}+\nabla_{\hat{X}}^2 R\left(\hat{X}\right)\left(-\frac{fV}{R\left(\hat{X}\right)}\right)\right)}{c-\frac{dV}{R\left(\hat{X}\right)}}\\
&= \frac{M}{c}+\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)\frac{fV}{R\left(\hat{X}\right)}}{c}+\frac{M\frac{dV}{cR\left(\hat{X}\right)}}{c}
\end{aligned}$$

Consequently, $\Gamma\left(p+\frac{3}{2}\right)$ arising in (338) is given by:

$$\begin{aligned}
\Gamma\left(\frac{M}{c}+\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)\frac{fV}{R\left(\hat{X}\right)}}{c}+\frac{M\frac{dV}{cR\left(\hat{X}\right)}}{c}\right) &= \Gamma\left(\frac{M}{c}\left(1+\nabla_{\hat{X}}^2 R\left(\hat{X}\right)\frac{fV}{MR\left(\hat{X}\right)}+\frac{dV}{cR\left(\hat{X}\right)}\right)\right)\\
&\simeq \Gamma\left(\frac{M}{c}\right)+\frac{MV}{c}\left(\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)f}{MR\left(\hat{X}\right)}+\frac{d}{cR\left(\hat{X}\right)}\right)\Gamma'\left(\frac{M}{c}\right)\\
&= \Gamma\left(\frac{M}{c}\right)\left(1+\frac{MV}{c}\left(\frac{\nabla_{\hat{X}}^2 R\left(\hat{X}\right)f}{MR\left(\hat{X}\right)}+\frac{d}{cR\left(\hat{X}\right)}\right)\mathrm{Psi}\left(\frac{M}{c}\right)\right)
\end{aligned}$$

Ultimately, using that at the first order:

$$\exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)\simeq 1$$

equation (349) for V becomes:

$$V-\frac{D\left|f\left(\hat{X}\right)\right|}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)\left(1+\frac{MV}{c}\left(\frac{\nabla_{\hat{X}}^2R(\hat{X})f}{MR(\hat{X})}+\frac{d}{cR(\hat{X})}\right)\text{Psi}\left(\frac{M}{c}\right)\right)}=0$$

that is:

$$V-\frac{D\left(c-\frac{dV}{R(\hat{X})}-\gamma D\right)}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)\left(1+\frac{MV}{c}\left(\frac{\nabla_{\hat{X}}^2R(\hat{X})f}{MR(\hat{X})}+\frac{d}{cR(\hat{X})}\right)\text{Psi}\left(\frac{M}{c}\right)\right)}=0$$

And a first order expansion yields:

$$V-\frac{D\left(c-\gamma D\right)}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)}\left(1-\frac{dV}{\left(c-\gamma D\right)R\left(\hat{X}\right)}-MV\left(\frac{\nabla_{\hat{X}}^2R\left(\hat{X}\right)f}{MR\left(\hat{X}\right)}+\frac{d}{cR\left(\hat{X}\right)}\right)\text{Psi}\left(\frac{M}{c}\right)\right)=0$$

with solution:

$$V=\left(K_{\hat{X}}^{\alpha}\right)^{-1}=\frac{\frac{D(c-\gamma D)}{C(\bar{p})\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)}}{1+\frac{D(c-\gamma D)}{C(\bar{p})\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)}\left(\frac{d}{(c-\gamma D)R(\hat{X})}+M\left(\frac{\nabla_{\hat{X}}^2R(\hat{X})f}{MR(\hat{X})}+\frac{d}{cR(\hat{X})}\right)\text{Psi}\left(\frac{M}{c}\right)\right)}$$

Coming back to $K_{\hat{X}}^{\alpha}$ we have:

$$\begin{aligned}K_{\hat{X}}^{\alpha} &= \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)}{D\left(c-\gamma D\right)}+\frac{d}{\left(c-\gamma D\right)R\left(\hat{X}\right)}+M\left(\frac{\nabla_{\hat{X}}^2R\left(\hat{X}\right)f}{MR\left(\hat{X}\right)}+\frac{d}{cR\left(\hat{X}\right)}\right)\text{Psi}\left(\frac{M}{c}\right) \qquad (350)\\ &= \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\Gamma\left(\frac{M}{c}\right)}{D\left(c-\gamma D\right)}+\frac{d}{\left(c-\gamma D\right)R\left(\hat{X}\right)}\left(1+M\,\text{Psi}\left(\frac{M}{c}\right)\left(1+\frac{\nabla_{\hat{X}}^2R\left(\hat{X}\right)f}{M\left(c-\gamma D\right)}\right)\right)\end{aligned}$$

This solution satisfies the condition $K_{\hat{X}}>>1$ only if $\frac{C(\bar{p})\sigma_{\hat{K}}^2\sqrt{\frac{M-c}{c}}}{Dc}>>1$: formula (350) thus shows that the dependency of $K_{\hat{X}}^{\alpha}$ in the return $R\left(\hat{X}\right)$ depends on the sign of $1+M\,\text{Psi}\left(\frac{M}{c}\right)\left(1+\frac{\nabla_{\hat{X}}^2R(\hat{X})f}{M(c-\gamma D)}\right)$. If:

$$1+M\,\text{Psi}\left(\frac{M}{c}\right)\left(1+\frac{\nabla_{\hat{X}}^2R\left(\hat{X}\right)f}{M\left(c-\gamma D\right)}\right)>0$$

then $K_{\hat{X}}^{\alpha}$ decreases with $R\left(\hat{X}\right)$. As stated in the text, this corresponds to an unstable equilibrium.

If:

$$1+M\,\text{Psi}\left(\frac{M}{c}\right)\left(1+\frac{\nabla_{\hat{X}}^2R\left(\hat{X}\right)f}{M\left(c-\gamma D\right)}\right)<0$$

a stable equilibrium is possible and $K_{\hat{X}}^{\alpha}$ is an increasing function of $R\left(\hat{X}\right)$ and $f\left(\hat{X}\right)$. This corresponds to $\nabla_{\hat{X}}^{2}R\left(\hat{X}\right)<<0$, which arises for instance for a maximum of $R\left(\hat{X}\right)$. In such case, an increase in $R\left(\hat{X}\right)$ allows for an increased number $\left\Vert \Psi\left(\hat{X}\right)\right\Vert ^{2}$ of firms, without reducing the average capital per firm.

Case 4. Intermediate capital, $\infty>K_{\hat{X}}>1$: We start with asymptotic form of (275):

$$K_{\hat{X}}\left\Vert \Psi\left(\hat{X}\right)\right\Vert ^{2}\left|f\left(\hat{X}\right)\right|=C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}\right)\Gamma\left(p+\frac{3}{2}\right) \tag{351}$$

Up to a constant that can be absorbed in the definition of $C\left(\bar{p}\right)$, we have:

$$\Gamma\left(p+\frac{3}{2}\right)\sim_{\infty}\sqrt{p+\frac{1}{2}}\exp\left(\left(p+\frac{1}{2}\right)\left(\ln\left(p+\frac{1}{2}\right)-1\right)\right)$$

and (351) can be rewritten as:

$$K_{\hat{X}}\left\Vert \Psi\left(\hat{X}\right)\right\Vert ^{2}\left|f\left(\hat{X}\right)\right|=C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\sqrt{p+\frac{1}{2}}\exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}+\left(p+\frac{1}{2}\right)\left(\ln\left(p+\frac{1}{2}\right)-1\right)\right) \tag{352}$$

Since we are in an intermediate range for the parameters, we can replace, in first approximation, $\ln\left(p+\frac{1}{2}\right)$ by its average over this range: $\ln\left(\bar{p}+\frac{1}{2}\right)$. The exponential in (352) thus becomes:

$$\exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}-\frac{48\left|f\left(\hat{X}\right)\right|^{3}}{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(X\right)\right)^{2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}+\frac{24\left|f\left(\hat{X}\right)\right|^{3}}{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(X\right)\right)^{2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^{2}\right)$$

and equation (352) rewrites:

$$\begin{aligned}
&K_{\hat{X}}\left\Vert \Psi\left(\hat{X}\right)\right\Vert ^{2}\left|f\left(\hat{X}\right)\right|\left(\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(X\right)\right)^{2}\exp\left(-\frac{96\left|f\left(\hat{X}\right)\right|^{3}}{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(X\right)\right)^{2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^{2}\right)}{96\left|f\left(\hat{X}\right)\right|^{3}}\right)^{\frac{1}{4}} \qquad (353)\\
&= C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}-\frac{48\left|f\left(\hat{X}\right)\right|^{3}}{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(X\right)\right)^{2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}\right)\sqrt{\left(p+\frac{1}{2}\right)\sqrt{\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}}}
\end{aligned}$$

To solve (353) for $K_{\hat{X}}$, we proceed in two steps.

We first introduce an intermediate variable W and rewrite (353) as an equation for $K_{\hat{X}}$ and W. We set:

$$\sqrt{\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}}\left(p+\frac{1}{2}-\frac{48\left|f\left(\hat{X}\right)\right|^{3}}{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(X\right)\right)^{2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)\right)=W \tag{354}$$

and rewrite equation (353) partly in terms of W:

$$\begin{aligned}
& K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^{2}\left(\frac{\sigma_{X}^{2}\left(f'\left(X\right)\right)^{2}\left|f\left(\hat{X}\right)\right|\exp\left(-\frac{96\left|f\left(\hat{X}\right)\right|^{3}}{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(f'(X)\right)^{2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^{2}\right)}{96\left(\sigma_{\hat{K}}^{2}\right)^{3}}\right)^{\frac{1}{4}} \\
= \; & C\left(\bar{p}\right)\exp\left(-W^{2}\right)\sqrt{W+2\sqrt{\frac{96\left|f\left(\hat{X}\right)\right|^{3}}{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(X\right)\right)^{2}}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}
\end{aligned} \tag{355}$$

Note that, as seen from (354), W is a function of p and as such can be seen as a parameter depending on the shape of the sectors space.

Equation (355) both depends on $K_{\hat{X}}$ and W, and in a second step, we use (354) to write $K_{\hat{X}}$ as a function of W. To do so, we use that in the intermediate case $\infty > K_{\hat{X}} > 1$, we can assume that:

$$f\left(\hat{X}\right)=B_{1}\left(X\right)K_{\hat{X}}^{\alpha-1}+B_{2}\left(X\right)K_{\hat{X}}^{\alpha}-C\left(\hat{X}\right)\simeq B_{2}\left(X\right)K_{\hat{X}}^{\alpha} \tag{356}$$

and that:

$$\frac{M-\left(\frac{\left(\nabla_{\hat{X}}R(\hat{X})A(\hat{X})K_{\hat{X}}^{\alpha}\right)^{2}}{\sigma_{\hat{X}}^{2}}+\nabla_{\hat{X}}^{2}R\left(\hat{X}\right)A\left(\hat{X}\right)K_{\hat{X}}^{\alpha}\right)}{B_{1}\left(X\right)K_{\hat{X}}^{\alpha-1}+B_{2}\left(X\right)K_{\hat{X}}^{\alpha}-C\left(\hat{X}\right)}-\frac{3}{2}\simeq\frac{M-\left(\frac{\left(\nabla_{\hat{X}}R(\hat{X})A(\hat{X})K_{\hat{X}}^{\alpha}\right)^{2}}{\sigma_{\hat{X}}^{2}}+\nabla_{\hat{X}}^{2}R\left(\hat{X}\right)A\left(\hat{X}\right)K_{\hat{X}}^{\alpha}\right)}{B_{2}\left(X\right)K_{\hat{X}}^{\alpha}}-\frac{3}{2} \tag{357}$$

Moreover, we can approximate $\left\|\Psi\left(\hat{X}\right)\right\|^{2}$:

$$\left\|\Psi\left(\hat{X}\right)\right\|^{2}\simeq D \tag{358}$$

Our assumptions (356), (357) and (358) allow to rewrite the relation (354) between $K_{\hat{X}}^{\alpha}$ and W as:

$$\sqrt{\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(B_{2}'\left(X\right)\right)^{2}}{96B_{2}^{3}\left(X\right)K_{\hat{X}}^{\alpha}}}\left(p+\frac{1}{2}-\frac{48\left|f\left(\hat{X}\right)\right|^{3}}{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(X\right)\right)^{2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)\right)=W$$

that is:

$$\begin{aligned}
W \; = \; & \sqrt{\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(B_{2}'\left(X\right)\right)^{2}}{96B_{2}^{5}\left(X\right)K_{\hat{X}}^{3\alpha}}} \\
& \times\left(M-\left(\frac{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^{2}A\left(\hat{X}\right)}{\sigma_{\hat{X}}^{2}}+\frac{48B_{2}^{4}\left(X\right)\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(B_{2}'\left(X\right)\right)^{2}}\right)K_{\hat{X}}^{2\alpha}-\left(\nabla_{\hat{X}}^{2}R\left(\hat{X}\right)+B_{2}\left(X\right)\right)K_{\hat{X}}^{\alpha}\right)
\end{aligned} \tag{359}$$

To solve this equation for $K_{\hat{X}}^{\alpha}$, we consider M as the dominant parameter and find an approximate solution of (359). At the lowest order, we write:

$$\sqrt{\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(B_{2}'\left(X\right)\right)^{2}}{96B_{2}^{5}\left(X\right)K_{\hat{X}}^{3\alpha}}}M=W$$

with solution:

$$K_{\hat{X}}^{\alpha}=\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'\left(X\right)\right)^2M^2}{96B_2^5\left(X\right)W^2}\right)^{\frac{1}{3}}$$

Considering corrections to this result, the solution to (359) is decomposed as:

$$K_{\hat{X}}^{\alpha}=\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'\left(X\right)\right)^2M^2}{96B_2^5\left(X\right)W^2}\right)^{\frac{1}{3}}+\chi \tag{360}$$

and using the following intermediate results:

$$K_{\hat{X}}^{2\alpha}=\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'\left(X\right)\right)^2M^2}{96B_2^5\left(X\right)W^2}\right)^{\frac{2}{3}}\left(1+2\chi\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'\left(X\right)\right)^2M^2}{96B_2^5\left(X\right)W^2}\right)^{-\frac{1}{3}}\right)$$

$$K_{\hat{X}}^{3\alpha}=\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'\left(X\right)\right)^2M^2}{96B_2^5\left(X\right)W^2}\right)\left(1+3\chi\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'\left(X\right)\right)^2M^2}{96B_2^5\left(X\right)W^2}\right)^{-\frac{1}{3}}\right)$$

we are led to rewrite (359) as an equation for χ at first order:

$$\begin{aligned}
&\chi\left(\frac{3}{2}\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'\left(X\right)\right)^2M^2}{96B_2^5\left(X\right)W^2}\right)^{-\frac{1}{3}}W\right.\\
&+2\frac{W}{M}\left(\frac{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2A\left(\hat{X}\right)}{\sigma_{\hat{X}}^2}+\frac{48B_2^4\left(X\right)\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'\left(X\right)\right)^2}\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'\left(X\right)\right)^2M^2}{96B_2^5\left(X\right)W^2}\right)^{\frac{1}{3}}+\frac{W}{M}\left(\nabla_{\hat{X}}^2R\left(\hat{X}\right)+B_2\left(X\right)\right)\\
=\;&-\frac{W}{M}\left(\frac{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2A\left(\hat{X}\right)}{\sigma_{\hat{X}}^2}+\frac{48B_2^4\left(X\right)\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'\left(X\right)\right)^2}\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'\left(X\right)\right)^2M^2}{96B_2^5\left(X\right)W^2}\right)^{\frac{2}{3}}\\
&-\frac{W}{M}\left(\nabla_{\hat{X}}^2R\left(\hat{X}\right)+B_2\left(X\right)\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'\left(X\right)\right)^2M^2}{96B_2^5\left(X\right)W^2}\right)^{\frac{1}{3}}
\end{aligned}$$

whose solution is:

$$\chi=-\frac{\left(\frac{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2A\left(\hat{X}\right)}{\sigma_{\hat{X}}^2}+\frac{48B_2^4(X)\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2}\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2M^2}{96B_2^5(X)W^2}\right)+\left(\nabla_{\hat{X}}^2R\left(\hat{X}\right)+B_2\left(X\right)\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2M^2}{96B_2^5(X)W^2}\right)^{\frac{2}{3}}}{\frac{3}{2}M+2\left(\frac{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2A\left(\hat{X}\right)}{\sigma_{\hat{X}}^2}+\frac{48B_2^4(X)\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2}\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2M^2}{96B_2^5(X)W^2}\right)^{\frac{2}{3}}+\left(\nabla_{\hat{X}}^2R\left(\hat{X}\right)+B_2\left(X\right)\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2M^2}{96B_2^5(X)W^2}\right)}$$

so that (360) yields $K_{\hat{X}}^{\alpha}$:

$$K_{\hat{X}}^{\alpha}-\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2M^2}{96B_2^5(X)W^2}\right)^{\frac{1}{3}}$$

$$=-\frac{\left(\frac{\left(\nabla_{\hat{X}}R(\hat{X})\right)^2A(\hat{X})}{\sigma_{\hat{X}}^2}+\frac{48B_2^4(X)\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2}\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2M^2}{96B_2^5(X)W^2}\right)+\left(\nabla_{\hat{X}}^2R\left(\hat{X}\right)+B_2(X)\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2M^2}{96B_2^5(X)W^2}\right)^{\frac{2}{3}}}{\frac{3}{2}M+2\left(\frac{\left(\nabla_{\hat{X}}R(\hat{X})\right)^2A(\hat{X})}{\sigma_{\hat{X}}^2}+\frac{48B_2^4(X)\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2}\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2M^2}{96B_2^5(X)W^2}\right)^{\frac{2}{3}}+\left(\nabla_{\hat{X}}^2R\left(\hat{X}\right)+B_2(X)\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2M}{96B_2^5(X)W^2}\right.}$$

$$\frac{\left(\frac{\left(\nabla_{\hat{X}}R(\hat{X})\right)^2A(\hat{X})}{\sigma_{\hat{X}}^2}+\frac{48B_2^4(X)\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2}\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2M^2}{96B_2^5(X)W^2}\right)+\left(\nabla_{\hat{X}}^2R\left(\hat{X}\right)+B_2(X)\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2M^2}{96B_2^5(X)W^2}\right)^{\frac{2}{3}}}{\frac{3}{2}M+2\left(\frac{\left(\nabla_{\hat{X}}R(\hat{X})\right)^2A(\hat{X})}{\sigma_{\hat{X}}^2}+\frac{48B_2^4(X)\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2}\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2M^2}{96B_2^5(X)W^2}\right)^{\frac{2}{3}}+\left(\nabla_{\hat{X}}^2R\left(\hat{X}\right)+B_2(X)\right)\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2M^2}{96B_2^5(X)W^2}\right.}$$

In a third step, we can use equation (361) to rewrite (355) in an approximate form. Actually, expression (361) implies that in the intermediate case, where $K_{\hat{X}}^{\alpha}$ is of finite magnitude, we have $W^2\sim\sigma_X^2\sigma_{\hat{K}}^2M^2$ and:

$$\begin{aligned}&\exp\left(-W^2+\frac{24\left|B_2(X)\right|^3K_{\hat{X}}^{\alpha}}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right)\\ \simeq\quad&\exp\left(\frac{24\left|B_2(X)\right|^3K_{\hat{X}}^{\alpha}}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right)\end{aligned}$$

Moreover using that:

$$W+2\sqrt{\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\sigma_{\hat{K}}^2\left(f'(X)\right)^2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)\simeq2\sqrt{\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\sigma_{\hat{K}}^2\left(f'(X)\right)^2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)$$

and that ultimately the left hand side of equation (355) writes at the first order:

$$K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left(\frac{\sigma_X^2\left(f'(X)\right)^2\left|f\left(\hat{X}\right)\right|\exp\left(-\frac{96\left|f(\hat{X})\right|^3}{\sigma_X^2\sigma_{\hat{K}}^2\left(f'(X)\right)^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right)}{96\left(\sigma_{\hat{K}}^2\right)^3}\right)^{\frac{1}{4}}$$

$$=\left(\frac{\sigma_X^2\left(B_2'(X)\right)^2\left|B_2(X)\right|}{96\left(\sigma_{\hat{K}}^2\right)^3}\right)^{\frac{1}{4}}K_{\hat{X}}^{1+\frac{3\alpha}{4}}\left\|\Psi\left(\hat{X}\right)\right\|^2\exp\left(-\frac{24\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\sigma_{\hat{K}}^2\left(f'(X)\right)^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right)$$

$$\simeq D\left(\frac{\sigma_X^2\left(B_2'(X)\right)^2\left|B_2(X)\right|}{96\left(\sigma_{\hat{K}}^2\right)^3}\right)^{\frac{1}{4}}K_{\hat{X}}^{1+\frac{3\alpha}{4}}\exp\left(-\frac{24\left|B_2(X)\right|^3K_{\hat{X}}^{\alpha}}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right)$$

equation (355) writes:

$$\begin{aligned}
&D\left(\frac{\sigma_X^2\left(B_2'(X)\right)^2|B_2(X)|}{96\left(\sigma_{\hat{K}}^2\right)^3}\right)^{\frac{1}{4}}K_{\hat{X}}^{1+\frac{3\alpha}{4}}\exp\left(-\frac{24|B_2(X)|^3K_{\hat{X}}^{\alpha}}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right)\\
&= C(\bar{p})\sqrt{2\sqrt{\frac{96|B_2(X)|^3K_{\hat{X}}^{\alpha}}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)}
\end{aligned}$$

that is:

$$K_{\hat{X}}^{1+\frac{\alpha}{2}}\exp\left(-\frac{24|B_2(X)|^3K_{\hat{X}}^{\alpha}}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\right)=\frac{8C(\bar{p})}{D}\sqrt{\frac{3\sigma_{\hat{K}}^2|B_2(X)|}{\sigma_X^2\left(B_2'(X)\right)^2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right) \tag{362}$$

Equation (362) has the form:

$$x^d\exp(-ax)=c$$

with solution:

$$x=c^{\frac{1}{d}}\exp\left(-W_0\left(-\frac{a}{d}c^{\frac{1}{d}}\right)\right)$$

where W_0 is the Lambert W function with parameter 0. Applying this result to our case with:

$$\begin{aligned}
d &= \frac{1+\alpha}{2\alpha}\\
x &= K_{\hat{X}}^{\alpha}\\
a &= \frac{24|B_2(X)|^3}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^2}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^2\\
c &= \frac{8C(\bar{p})}{D}\sqrt{\frac{3\sigma_{\hat{K}}^2|B_2(X)|}{\sigma_X^2\left(B_2'(X)\right)^2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)
\end{aligned}$$

we obtain:

$$\begin{aligned}
K_{\hat{X}}^{\alpha} &= \left(\frac{8C(\bar{p})}{D}\sqrt{\frac{3\sigma_{\hat{K}}^2|B_2(X)|}{\sigma_X^2\left(B_2'(X)\right)^2}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)\right)^{\frac{2\alpha}{1+\alpha}}\\
&\times\exp\left(-W_0\left(-\frac{48\alpha}{1+\alpha}\left(\sqrt{\frac{3\sigma_{\hat{K}}^2}{\sigma_X^2}}\frac{8C(\bar{p})}{D}\right)^{\frac{2\alpha}{1+\alpha}}\frac{|B_2(X)|^{3+\frac{\alpha}{1+\alpha}}}{\sigma_X^2\sigma_{\hat{K}}^2\left(B_2'(X)\right)^{2+\frac{2\alpha}{1+\alpha}}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^{2+\frac{\alpha}{1+\alpha}}\right)\right)
\end{aligned}$$

As stated in the text, this is an increasing function of $B_2(X)$. Moreover, the corrections to this formula, given in (361) show that $K_{\hat{X}}^{\alpha}$ is a decreasing function of $\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2$ and $\nabla_{\hat{X}}^2R\left(\hat{X}\right)$.

Case 5. Low capital, $K_{\hat{X}}<<1$**:** When average physical capital per firm in sector $\hat{X}$ is very low, we can use our assumptions about $g\left(\hat{X},\Psi,\hat{\Psi}\right)$ and $\nabla_{\hat{X}}g\left(\hat{X},\Psi,\hat{\Psi}\right)$, equations (293) and (**??**), and assume that:

$$f\left(\hat{X}\right)\simeq B_1\left(\hat{X}\right)K_{\hat{X}}^{\alpha-1}>>1 \tag{363}$$

and:

$$g\left(\hat{X}\right)\simeq 0$$

and moreover that:

$$\left\|\Psi\left(\hat{X}\right)\right\|^{2}=D-L\left(X\right)\left(\nabla_{X}R\left(X\right)\right)^{2}K_{\hat{X}}^{\alpha}\simeq D$$

For these conditions, the solution of (**??**) is locally stable.

Moreover, the conditions $K_{\hat{X}}<<1$ and the defining equation (330) for f imply that $f>0$, and that for $\alpha<1$:

$$\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}<<1$$

Under these assumptions, equation (**??**) reduces to:

$$K_{\hat{X}}D\left|f\left(\hat{X}\right)\right|\simeq C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\hat{\Gamma}\left(p+\frac{1}{2}\right)\tag{364}$$

This equation (364) can be approximated. Actually, using formula (**??**) for p yields:

$$p+\frac{1}{2}=\frac{M-\left(\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)}{\sqrt{f^{2}\left(\hat{X}\right)}}-1\simeq-1$$

and an expansion of $\hat{\Gamma}\left(p+\frac{1}{2}\right)$ around the value $p+\frac{1}{2}=-1$ writes:

$$\hat{\Gamma}\left(p+\frac{1}{2}\right)\simeq\hat{\Gamma}\left(-1\right)+\hat{\Gamma}'\left(-1\right)\frac{M-\left(\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)}{\sqrt{f^{2}\left(\hat{X}\right)}}$$

Consequently, when returns are large, i.e. $f\left(\hat{X}\right)>>1$, equation (**??**) writes:

$$K_{\hat{X}}\left(B_{1}\left(\hat{X}\right)K_{\hat{X}}^{\alpha-1}\right)\simeq\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}}{D}\left(\hat{\Gamma}\left(-1\right)+\hat{\Gamma}'\left(-1\right)\frac{M-\left(\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)}{B_{1}\left(\hat{X}\right)K_{\hat{X}}^{\alpha-1}}\right)$$

with first order solution[70]:

$$K_{\hat{X}}=\left(\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\hat{\Gamma}\left(-1\right)}{DB_{1}\left(\hat{X}\right)}\right)^{\frac{1}{\alpha}}+\frac{\frac{C(\bar{p})\sigma_{\hat{K}}^{2}}{D}\hat{\Gamma}'\left(-1\right)\left(M-\left(\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\right)}{B_{1}^{\frac{1}{\alpha}}\left(\hat{X}\right)\left(\frac{C(\bar{p})\sigma_{\hat{K}}^{2}\hat{\Gamma}(-1)}{D}\right)^{1-\frac{1}{\alpha}}}\tag{365}$$

Equation (365) shows that average capital $K_{\hat{X}}$ increases with $M-\left(\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)$: when expected long-term returns increase, more capital is allocated to the sector. Equation (302) also shows that average capital $K_{\hat{X}}$ is maximal when returns $R\left(\hat{X}\right)$ are at a local maximum, i.e. when $\frac{\left(g\left(\hat{X}\right)\right)^{2}}{\sigma_{\hat{X}}^{2}}=0$ and $\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)<0$.

Inversely, the same equations (365) and (302) show that average capital $K_{\hat{X}}$ is decreasing in $f\left(\hat{X}\right)$. The equilibrium is unstable. When average capital is very low, i.e. $K_{\hat{X}}<<1$, which is the

[70] Given our hypotheses, $D>>1$, which implies that $K_{\hat{X}}<<1$, as needed.

case studied here, marginal returns are high. Any increase in capital above the threshold widely increases returns, which drives capital towards the next stable equilibrium, with higher $K_{\hat{X}}$. Recall that in this unstable equilibrium, $K_{\hat{X}}$ must be seen as a threshold. The rise in $f\left(\hat{X}\right)$ reduces the threshold $K_{\hat{X}}$, which favours capital accumulation and increases the average capital $K_{\hat{X}}$.

This case is thus an exception: the dependency of $K_{\hat{X}}$ in $R\left(\hat{X}\right)$ is stable, but the dependency in $f\left(\hat{X}\right)$ is unstable. This saddle path type of instability may lead the sector, either towards a higher level of capital (case 4 below) or towards 0. where the sector disappears.

A 4.3 Instability and modification of sectors' space

A 4.3.1 Disappearance of Low average capital sectors

Average capital is unstable when $B\left(\hat{X}\right) < -1$. A shock on average capital can either drive the equilibrium to some stable value, or worsen the sector's capital landscape.

In the latter case, investors tend to desert the sector, so that both the average capital and the density of investors tend to 0: $K_{\hat{X}} \to 0$ and $\left|\hat{\Psi}\left(\hat{X},\hat{K}\right)\right|^2 \to 0$. Producers remain in the sector but with a very low capital on average. The very lack of capital prevents these firms to shift towards more attractive sectors in the long run. Assuming physical capital returns are Cobb-Douglas, marginal productivity is mathematically high for a very low capital. Thus, short-term returns are very large: $f\left(\hat{X}\right) \to \infty$.

Note that this type of instability only applies to very low level of average capital, so that the total capital involved is negligible, and this instability does not impact the system globally.

A 4.3.2 Very high level of average capital and modification of space

Average capital is also unstable when $B\left(\hat{X}\right) > 1$. However, in this case investors are lured in the sector, so that average capital in the sector increases quickly $K_{\hat{X}} \to \infty$, and short-term returns tend to be small: $f\left(\hat{X}\right) \to c$ for some constant $c << 1$. Consequently, for $K_{\hat{X}} \to \infty$, $\frac{\partial f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}} \to 0$, which translates decreasing marginal returns. Similarly, the expected long-term returns will be caped, and $\frac{\partial p}{\partial K_{\hat{X}}} \to 0$, and $l\left(\hat{X},K_{\hat{X}}\right) \to 0$.

The instability condition (107) turns out to be a lower bound for the sensitivity of firms density relative to average capital:

$$\frac{\partial \ln\left|\Psi\left(\hat{X},K_{\hat{X}}\right)\right|^2}{\partial K_{\hat{X}}} > 1 \tag{366}$$

This lower bound creates a herd effect: the number of firms in sector $\hat{X}$ could grow indefinitely with capital: $\left|\Psi\left(\hat{X},K_{\hat{X}}\right)\right|^2 \to \infty$.

However, the fixed number of firms implies that this shift towards sector $\hat{X}$ will necessarily reach a maximum $\left|\hat{\Psi}\left(\hat{X},\hat{K}\right)\right|^2_{\max} >> 1$. For this maximum density, the corresponding level of average capital at sector $\hat{X}$ will be approximatively:

$$K_{\hat{X}} \simeq K_{\max} = \frac{\ln\left(\left|\hat{\Psi}\left(\hat{X},\hat{K}\right)\right|^2_{\max}\right)}{r}$$

This concentration of capital in some sectors directly impacts the amount of disposable capital along with the instability condition (107) for the rest of the system. This occurs in several steps.

First, the disposable average capital for the rest of the system reduces to $\langle K\rangle - \frac{K_{\max}}{V}$, with V, the volume of the sector space and $\langle K\rangle$, the average physical capital in the whole space.

Second, this reduction of average capital negatively impacts the growth prospects $R\left(\hat{X}\right)$, the stock prices $F_1(X)$, and consequently the short term returns $f\left(\hat{X}\right)$.

In turn, this modifies the stability condition $\left|B\left(\hat{X}\right)\right|$ over the whole space. Consequently, some sectors will move over the instability threshold $B\left(\hat{X}\right) > 1$, while others will move below $B\left(\hat{X}\right) < -1$. Some sectors will experience a capital increase, others will disappear.

If a stable situation finally emerges, the resulting sectors' space will be reduced: some sectors will have disappeared, and only sectors with positive capital will have remained.

A.4.4 Global instability

This appendix completes the analysis of the solutions of (99) for average capital. We have studied the local instability of solutions previously. However, a second source of instability of the system arises outside of the equations for average capital per firm per sector, (99), and its differential version, (280). It stems from the sectors' space expected long-term returns. It is induced by the minimization equations (88) and (89), and is a source of global instability for the background field.

A 4.4.1 Mechanism of global instability:

In these equations, the Lagrange multiplier $\hat{\lambda}$ is the eigenvalue of a second-order differential equation. Because there exist an infinite number of eigenvalues $\hat{\lambda}$, there are an infinite number of local minimum background fields $\Psi\left(\hat{X}, K_{\hat{X}}\right)$. But the most likely minimum, given in (**??**), is obtained for $\hat{\lambda} = M$ (see appendix 2).

Yet $\hat{\lambda}$ is also the Lagrange multiplier that implements the constraint of a fixed number N of agents.

Since the number of investors is computed by:

$$\int \left|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right|^2 d\left(\hat{X}, K_{\hat{X}}\right)$$

the constraint implemented by $\hat{\lambda}$ is:

$$\hat{N} = \int \left|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right|^2 d\left(\hat{X}, K_{\hat{X}}\right) \tag{367}$$

since this constraint runs over the whole space, it is a global property of the system.

Yet equations (88) and (89), the minimization equations defining $\Psi\left(\hat{X}, K_{\hat{X}}\right)$, may also be viewed as a set of local minimization equations at each point $\hat{X}$ of the sector space. Considered individually, each provide a lower minimum that could be reached separately for each $\hat{X}$. In other words, provided each sector's number of agents is fixed independently from the rest of the system, a stable background field could be reached at every point.

However, our global constraint rules out this set of local minimizations. The solutions of (88) and (89) are thus a local minimum for the sole points $\hat{X}$ such that the lowest value of $\hat{\lambda}$ is reached at $\hat{X}$, i.e. points such that[71]:

$$A\left(\hat{X}\right) \equiv \frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + f\left(\hat{X}\right) + \frac{1}{2}\sqrt{f^2\left(\hat{X}\right)} + \nabla_{\hat{X}} g\left(\hat{X}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}\right)}{2f^2\left(\hat{X}\right)} = M \tag{368}$$

For points $\hat{X}$ that do not satisfy (368), the solutions $\Psi\left(\hat{X}, K_{\hat{X}}\right)$ and $\Psi^{\dagger}\left(\hat{X}, K_{\hat{X}}\right)$ of (88) and (89), with $\hat{\lambda} = -M$ are not global minima, but merely a local one. Any perturbation $\delta\Psi\left(\hat{X}, K_{\hat{X}}\right)$ due to a change of parameters destabilizes the whole system: the equilibrium is unstable.

The stability of both the background field and the potential equilibria are thus determined by $A\left(\hat{X}\right)$, the sector space's overall shape of returns and expectations. An homogeneous shape, a space such that $A\left(\hat{X}\right)$, presents small deviations around M and is more background-stable than an heterogeneous space.

More importantly, the background fields and associated average capital must be understood as potential, not actual long-run equilibria: the whole system is better described as a dynamical system, which is defined in section 5 of the text, between potential backgrounds where time enters as a macro-variable. We consider the results of the background field's dynamical behavior in section 7.

Removing global instability As mentioned above, an homogeneous shape is a space such that the parameter $A\left(\hat{X}\right)$ presents small deviations around M. In an heterogeneous shape, the space presents large differences in $A\left(\hat{X}\right)$. We find that homogeneous shapes are more background-stable than heterogeneous ones. This partly results from the global constraint (367) imposed on the number of agents in the model, which ensures that the number of financial agents in the system is fixed over the whole sector space.

Relaxing this constraint fully would render the number of agents in sectors independent. The associated background field of each sector could, at each point, adjust to be minimum and stabilize the system.

To do so, we replace equation (88), the minimization equation, by a set of independent equations with independent Lagrange multipliers $\hat{\lambda}_{\hat{X}}$ for each sector $\hat{X}$, so that for each $\hat{X}$, the minimum configuration is reached by setting:

$$\hat{\lambda}_{\hat{X}} = \frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + f\left(\hat{X}\right) + \frac{1}{2}\sqrt{f^2\left(\hat{X}\right)} + \nabla_{\hat{X}} g\left(\hat{X}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}\right)}{2f^2\left(\hat{X}\right)}$$

This is similar to the Lagrange multiplier of the minimization equation for the background field, stripped of the maximum condition $\hat{\lambda} = -M$[72]. This $\hat{X}$ dependency of the Lagrange multiplier implies that the average capital equation (99) is replaced by[73]:

$$K_{\hat{X}} \left\|\Psi\left(\hat{X}\right)\right\|^2 \left|f\left(\hat{X}\right)\right| = C(\bar{p})\,\sigma_{\hat{K}}^2 \hat{\Gamma}\left(\frac{1}{2}\right) = C(\bar{p})\,\sigma_{\hat{K}}^2 \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(f'(X)\right)^2}{384\left|f\left(\hat{X}\right)\right|^3}\right) \tag{369}$$

[71] See definition (285) and section 8.2. for a study of such points.

[72] see discussion following equation (93).

[73] Expression (317) is used to compute $\hat{\Gamma}\left(\frac{1}{2}\right)$.

This equation is identical to (286) and has thus at least one locally stable solution. The solutions are computed in (288) and (289).

Solutions to (369) do no longer directly depend on the relative characteristics of a particular sector, but rather on the returns at point $f\left(\hat{X}\right)$ and on the number of firms in the sector, $\left\|\Psi\left(\hat{X}\right)\right\|^2$. Yet this dependency is only indirect, through the firms' density at sector $\hat{X}$, $\left\|\Psi\left(\hat{X}\right)\right\|^2$, and this quantity does not vary much in the sector space.

An intermediate situation between (99) and (369) could also be considered: it would be to assume a constant number of agents in some regions of the sector space.

Alternatively, limiting the number of investors per sector can be achieved through some public regulation to maintain a constant flow of investment in the sector.

Appendix 5. Dynamics for $K_{\hat{X}}$

A 5.1 Variation of the defining equation for $K_{\hat{X}}$

A 5.1.1 Compact formulation

As claimed in the text, we consider the dynamics for $K_{\hat{X}}$ generated by modification of the parameters. To do so, we compute the variation of equation (279). We need the variations of the functions involved in (279) with respect to two dynamical variables $K_{\hat{X}}$ and $R(X)$. Starting with (279):

$$K_{\hat{X}}\left(D-L\left(\hat{X}\right)K_{\hat{X}}^{\eta}\right)=\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}{\left|f\left(\hat{X}\right)\right|}\hat{\Gamma}\left(p+\frac{1}{2}\right) \tag{370}$$

where:

$$\begin{aligned}\hat{\Gamma}\left(p+\frac{1}{2}\right) &= \exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right) \\ &\times\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right)\end{aligned} \tag{371}$$

We first compute the variations of the right hand side and use that, in first approximation:

$$\frac{d}{dp}\left(\ln\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{-p}{2}\right)}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\Gamma\left(-\frac{p-1}{2}\right)\Gamma\left(-\frac{p-1}{2}\right)}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right)\right)\simeq\ln\left(p+\frac{1}{2}\right) \tag{372}$$

and:

$$\frac{d}{dp}\left(-\frac{\sigma_X^2\left(p+\frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)=-\frac{\sigma_X^2\left(p+\frac{1}{2}\right)\left(f'(X)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}$$

so that:

$$\frac{\frac{d}{dp}\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\hat{\Gamma}\left(p+\frac{1}{2}\right)}\simeq\ln\left(p+\frac{1}{2}\right)-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'(X)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3} \tag{373}$$

Assuming that $C(\bar{p})$ is constant, (373) allows to rewrite the variation of of equation (370):

$$\begin{aligned}\nabla_\theta\left(K_{\hat{X}}\left(D-L\left(\hat{X}\right)K_{\hat{X}}^\eta\right)\right) &= K_{\hat{X}}\left(D-L\left(\hat{X}\right)K_{\hat{X}}^\eta\right)\\ &\times\left(-\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|}+\left(\ln\left(p+\frac{1}{2}\right)-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}\right)\nabla_\theta p\right.\\ &\left.+\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\left(\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|}-\frac{\nabla_\theta\left(\frac{f'(X)}{f(\hat{X})}\right)^2}{\left(\frac{f'(X)}{f(\hat{X})}\right)^2}\right)\right)\end{aligned}$$

and we deduce from this equation, that the dynamic version of equation (370) is:

$$\begin{aligned}\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}-\frac{\nabla_\theta\left(L\left(\hat{X}\right)K_{\hat{X}}^\eta\right)}{D-L\left(\hat{X}\right)K_{\hat{X}}^\eta} &= \left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}-1\right)\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|}\\ &+\left(\ln\left(p+\frac{1}{2}\right)-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}\right)\nabla_\theta p\\ &-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2}{96\left|f\left(\hat{X}\right)\right|}\left(\nabla_\theta\left(\frac{f'\left(X\right)}{f\left(\hat{X}\right)}\right)^2\right)\end{aligned} \tag{374}$$

A5.1.2 Expanded form of (374)

To find the dynamic equation for $K_{\hat{X}}$ we expand each side of (374).

The left hand side of (374) can be developed as:

$$\begin{aligned}&\left(1-\eta\frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{D-L\left(\hat{X}\right)K_{\hat{X}}^\eta}\right)\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}-\frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{D-L\left(\hat{X}\right)K_{\hat{X}}^\eta}\frac{\nabla_\theta L\left(\hat{X}\right)}{L\left(\hat{X}\right)}\\ =&\left(1-\eta\frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\right)\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}-\frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\frac{\nabla_\theta L\left(\hat{X}\right)}{L\left(\hat{X}\right)}\\ =&\left(1-\eta\frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\right)\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}-\frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\frac{\nabla_\theta\left(\left(\nabla_X R\left(X\right)\right)^2+\sigma_X^2\frac{\nabla_X^2 R(K_X,X)}{H(K_X)}\right)}{\left(\nabla_X R\left(X\right)\right)^2+\sigma_X^2\frac{\nabla_X^2 R(K_X,X)}{H(K_X)}}\\ \simeq&\left(1-\eta\frac{L\left(\hat{X}\right)K_{\hat{X}}^\eta}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\right)\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}-2\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\frac{\nabla_\theta\left(\nabla_X R\left(X\right)\right)}{\nabla_X R\left(X\right)}\end{aligned}$$

and (374) becomes:

$$\begin{aligned}
&\left(1-\eta\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\right)\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}-2\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\frac{\nabla_X\left(\nabla_\theta R\left(X\right)\right)}{\nabla_X R\left(X\right)} \qquad (375)\\
=&\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}-1\right)\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|}+\left(\ln\left(p+\frac{1}{2}\right)-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}\right)\nabla_\theta p\\
&-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2}{96\left|f\left(\hat{X}\right)\right|}\nabla_\theta\left(\frac{f'\left(X\right)}{f\left(\hat{X}\right)}\right)^2
\end{aligned}$$

To compute the right hand side of (375). We use that:

$$p=-\frac{M-\left(g\left(\hat{X}\right)\right)^2+\sigma_{\hat{X}}^2\left(\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)}{\sigma_{\hat{X}}^2 f\left(\hat{X}\right)}-\frac{3}{2}$$

so that, the variation $\nabla_\theta p$ is given by:

$$\nabla_\theta p=-\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|}\left(p+\frac{3}{2}\right)-\left(2\frac{g\left(\hat{X}\right)\nabla_\theta g\left(\hat{X}\right)}{\sigma_{\hat{X}}^2}+\nabla_\theta\nabla_{\hat{X}}g\left(\hat{X}\right)\right)$$

To compute $\nabla_\theta p$ we must use the form of the functions defined in Appendix 2. We thus obtain:

$$\begin{aligned}
\frac{\nabla_\theta g\left(\hat{X}\right)}{g\left(\hat{X}\right)}&=\frac{\nabla_\theta\nabla_{\hat{X}}R\left(\hat{X}\right)}{\nabla_{\hat{X}}R\left(\hat{X}\right)}+\alpha\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}\\
\frac{\nabla_\theta\nabla_{\hat{X}}g\left(\hat{X}\right)}{\nabla_{\hat{X}}g\left(\hat{X}\right)}&=\frac{\nabla_\theta\nabla_{\hat{X}}^2R\left(\hat{X}\right)}{\nabla_{\hat{X}}^2R\left(\hat{X}\right)}+\alpha\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}
\end{aligned}$$

and as a consequence:

$$\nabla_\theta p=-\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|}\left(p+\frac{3}{2}\right)-\left(2\frac{g^2\left(\hat{X}\right)\left(\frac{\nabla_\theta\nabla_{\hat{X}}R\left(\hat{X}\right)}{\nabla_{\hat{X}}R\left(\hat{X}\right)}+\alpha\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}\right)}{\sigma_{\hat{X}}^2\left|f\left(\hat{X}\right)\right|}+\frac{\nabla_{\hat{X}}g\left(\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}\left(\frac{\nabla_\theta\nabla_{\hat{X}}^2R\left(\hat{X}\right)}{\nabla_{\hat{X}}^2R\left(\hat{X}\right)}+\alpha\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}\right)\right)$$

Ultimately, the right hand side of (375) is given by:

$$\left(\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}-1\right)\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|}+\left(\ln\left(p+\frac{1}{2}\right)-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}\right)\nabla_\theta p$$
$$-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2}{96\left|f\left(\hat{X}\right)\right|}\nabla_\theta\left(\frac{f'\left(X\right)}{f\left(\hat{X}\right)}\right)^2$$
$$=-\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|}\left(\left(1-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)+\left(p+\frac{3}{2}\right)\left(\ln\left(p+\frac{1}{2}\right)-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}\right)\right)$$
$$-\left(2\frac{g^2\left(\hat{X}\right)}{\sigma_{\hat{X}}^2\left|f\left(\hat{X}\right)\right|}\frac{\nabla_\theta\nabla_{\hat{X}}R\left(\hat{X}\right)}{\nabla_{\hat{X}}R\left(\hat{X}\right)}+\frac{\nabla_{\hat{X}}g\left(\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}\frac{\nabla_\theta\nabla_{\hat{X}}^2R\left(\hat{X}\right)}{\nabla_{\hat{X}}^2R\left(\hat{X}\right)}+\frac{\alpha\left(2\frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2}+\nabla_{\hat{X}}g\left(\hat{X}\right)\right)}{\left|f\left(\hat{X}\right)\right|}\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}\right)$$
$$\times\left(\ln\left(p+\frac{1}{2}\right)-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}\right)-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2}{96\left|f\left(\hat{X}\right)\right|}\nabla_\theta\left(\frac{f'\left(X\right)}{f\left(\hat{X}\right)}\right)^2$$

so that the variational equation for $K_{\hat{X}}$ (375) writes:

$$\left(1-\eta\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\right)\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}-2\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\frac{\nabla_X\left(\nabla_\theta R\left(X\right)\right)}{\nabla_X R\left(X\right)} \tag{376}$$
$$=-C_3\left(p,\hat{X}\right)\frac{\nabla_\theta\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|}-C_1\left(p,\hat{X}\right)\frac{\nabla_\theta\left(\frac{f'(X)}{f(\hat{X})}\right)^2}{\left(\frac{f'(X)}{f(\hat{X})}\right)^2}$$
$$-C_2\left(p,\hat{X}\right)\left(2\frac{g^2\left(\hat{X}\right)}{\sigma_{\hat{X}}^2\left|f\left(\hat{X}\right)\right|}\frac{\nabla_\theta\nabla_{\hat{X}}R\left(\hat{X}\right)}{\nabla_{\hat{X}}R\left(\hat{X}\right)}+\frac{\nabla_{\hat{X}}g\left(\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}\frac{\nabla_\theta\nabla_{\hat{X}}^2R\left(\hat{X}\right)}{\nabla_{\hat{X}}^2R\left(\hat{X}\right)}+\frac{\alpha\left(2\frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2}+\nabla_{\hat{X}}g\left(\hat{X}\right)\right)}{\left|f\left(\hat{X}\right)\right|}\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}\right)$$

with:

$$C_1\left(p,\hat{X}\right)=\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(p+\frac{1}{2}\right)^2\left(f'\left(X\right)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3} \tag{377}$$
$$C_2\left(p,\hat{X}\right)=\ln\left(p+\frac{1}{2}\right)-\frac{2C_1\left(p,\hat{X}\right)}{p+\frac{1}{2}}$$
$$C_3\left(p,\hat{X}\right)=1-C_1\left(p,\hat{X}\right)+\left(p+\frac{3}{2}\right)C_2\left(p,\hat{X}\right)$$

These term can be reordered and the general dynamic equation for $K_{\hat{X}}$ is ultimately written as:

$$\left(1-\eta\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^{2}}{\left\|\Psi\left(\hat{X}\right)\right\|^{2}}+\frac{\alpha\left(2\frac{g^{2}(\hat{X})}{\sigma_{\hat{X}}^{2}}+\nabla_{\hat{X}}g\left(\hat{X}\right)\right)}{\left|f\left(\hat{X}\right)\right|}C_{2}\left(p,\hat{X}\right)\right)\frac{\nabla_{\theta}K_{\hat{X}}}{K_{\hat{X}}} \tag{378}$$
$$+2\left(\frac{g^{2}\left(\hat{X}\right)C_{2}\left(p,\hat{X}\right)}{\sigma_{\hat{X}}^{2}\left|f\left(\hat{X}\right)\right|}-\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^{2}}{\left\|\Psi\left(\hat{X}\right)\right\|^{2}}\right)\frac{\nabla_{\theta}\nabla_{\hat{X}}R\left(\hat{X}\right)}{\nabla_{\hat{X}}R\left(\hat{X}\right)}+\frac{\nabla_{\hat{X}}g\left(\hat{X}\right)C_{2}\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}\frac{\nabla_{\theta}\nabla_{\hat{X}}^{2}R\left(\hat{X}\right)}{\nabla_{\hat{X}}^{2}R\left(\hat{X}\right)}$$
$$=\quad -C_{3}\left(p,\hat{X}\right)\frac{\nabla_{\theta}\left|f\left(\hat{X}\right)\right|}{\left|f\left(\hat{X}\right)\right|}-C_{1}\left(p,\hat{X}\right)\frac{\nabla_{\theta}\left(\frac{f'(X)}{f(\hat{X})}\right)^{2}}{\left(\frac{f'(X)}{f(\hat{X})}\right)^{2}}$$

A5.1.3 Dynamic equation for particular forms of $f\left(\hat{X},K_{\hat{X}}\right)$ and $\left\|\Psi\left(\hat{X}\right)\right\|^{2}$

We can put equation (378) in a specific form, by using the explicit formula for $f\left(\hat{X},K_{\hat{X}}\right)$ and $\left\|\Psi\left(\hat{X}\right)\right\|^{2}$ given in appendix 2. We have:

$$\begin{aligned}
\frac{\nabla_{\theta}f\left(\hat{X},K_{\hat{X}}\right)}{f\left(\hat{X},K_{\hat{X}}\right)} &\simeq \frac{r\left(K_{\hat{X}},\hat{X}\right)\left(\frac{\nabla_{\theta}r\left(K_{\hat{X}},\hat{X}\right)}{r\left(K_{\hat{X}},\hat{X}\right)}+(\alpha-1)\frac{\nabla_{\theta}K_{\hat{X}}}{K_{\hat{X}}}\right)}{f\left(\hat{X}\right)} \\
&+\frac{\gamma\left(\eta L\left(\hat{X}\right)K_{\hat{X}}^{\eta}\frac{\nabla_{\theta}K_{\hat{X}}}{K_{\hat{X}}}+2L\left(\hat{X}\right)K_{\hat{X}}^{\eta}\frac{\nabla_{\theta}(\nabla_{X}R(X))}{\nabla_{X}R(X)}\right)+F_{1}'\left(R\left(K_{\hat{X}},\hat{X}\right)\right)\frac{\nabla_{\theta}R\left(K_{\hat{X}},\hat{X}\right)}{R\left(K_{\hat{X}},\hat{X}\right)}}{f\left(\hat{X}\right)} \\
&\simeq \frac{r\left(K_{\hat{X}},\hat{X}\right)\left(\frac{\nabla_{\theta}r\left(\hat{X}\right)}{r\left(K_{\hat{X}},\hat{X}\right)}+(\alpha-1)\frac{\nabla_{\theta}K_{\hat{X}}}{K_{\hat{X}}}\right)}{f\left(\hat{X}\right)} \\
&+\frac{\gamma\left(\eta L\left(\hat{X}\right)K_{\hat{X}}^{\eta}\frac{\nabla_{\theta}K_{\hat{X}}}{K_{\hat{X}}}+2L\left(\hat{X}\right)K_{\hat{X}}^{\eta}\frac{\nabla_{\theta}(\nabla_{X}R(X))}{\nabla_{X}R(X)}\right)+\varsigma F_{1}\left(R\left(K_{\hat{X}},\hat{X}\right)\right)\frac{\nabla_{\theta}R\left(K_{\hat{X}},\hat{X}\right)}{R\left(K_{\hat{X}},\hat{X}\right)}}{f\left(\hat{X}\right)} \\
&= \frac{r\left(K_{\hat{X}},\hat{X}\right)\frac{\nabla_{\theta}r\left(\hat{X}\right)}{r\left(K_{\hat{X}},\hat{X}\right)}}{f\left(\hat{X}\right)} \\
&+\frac{\left(\gamma\eta L\left(\hat{X}\right)K_{\hat{X}}^{\eta}+(\alpha-1)\right)\frac{\nabla_{\theta}K_{\hat{X}}}{K_{\hat{X}}}+\varsigma F_{1}\left(R\left(K_{\hat{X}},\hat{X}\right)\right)\frac{\nabla_{\theta}R\left(K_{\hat{X}},\hat{X}\right)}{R\left(K_{\hat{X}},\hat{X}\right)}+2\gamma L\left(\hat{X}\right)K_{\hat{X}}^{\eta}\frac{\nabla_{\theta}(\nabla_{X}R(X))}{\nabla_{X}R(X)}}{f\left(\hat{X}\right)}
\end{aligned}$$

To compute $\nabla_{\theta}\ln\left(\frac{f'(X)}{f(\hat{X})}\right)^{2}$ arising in (378), we use that in first approximation, for relatively large $K_{\hat{X}}$:

$$\left(\frac{f'(X)}{f\left(\hat{X}\right)}\right)^{2}\simeq\left(\frac{F_{1}'\left(R\left(K_{\hat{X}},\hat{X}\right)\right)\nabla_{\hat{X}}R\left(K_{\hat{X}},\hat{X}\right)}{F_{1}\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}\right)^{2}\simeq\left(\varsigma\nabla_{\hat{X}}R\left(K_{\hat{X}},\hat{X}\right)\right)^{2}$$

that can be considered int the sequel negligible at the first order.

Consequently, for the chosen forms of the parameter functions, the dynamics equation (378) becomes ultimately:

$$k\frac{\nabla_{\theta}K_{\hat{X}}}{K_{\hat{X}}}+l\frac{\nabla_{\theta}R\left(\hat{X}\right)}{R\left(\hat{X}\right)}-2m\frac{\nabla_{\hat{X}}\nabla_{\theta}R\left(\hat{X}\right)}{\nabla_{\hat{X}}R\left(\hat{X}\right)}+n\frac{\nabla_{\hat{X}}^{2}\nabla_{\theta}R\left(\hat{X}\right)}{\nabla_{\hat{X}}^{2}R\left(\hat{X}\right)}=-C_{3}\left(p,\hat{X}\right)\frac{\nabla_{\theta}r\left(\hat{X}\right)}{f\left(\hat{X}\right)} \tag{379}$$

with:

$$\begin{aligned}
k &= 1-\eta\left(1-\frac{\gamma C_{3}\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}\right)\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^{2}}{\left\|\Psi\left(\hat{X}\right)\right\|^{2}} \\
&\quad+\frac{\alpha\left(2\frac{g^{2}\left(\hat{X}\right)}{\sigma_{\hat{X}}^{2}}+\nabla_{\hat{X}}g\left(\hat{X}\right)\right)C_{2}\left(p,\hat{X}\right)-\left(1-\alpha\right)C_{3}\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \\
l &= \frac{\varsigma F_{1}\left(R\left(K_{\hat{X}},\hat{X}\right)\right)C_{3}\left(p,\hat{X}\right)}{f\left(\hat{X}\right)} \\
m &= \left(1-\frac{\gamma C_{3}\left(p,\hat{X}\right)}{f\left(\hat{X}\right)}\right)\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^{2}}{\left\|\Psi\left(\hat{X}\right)\right\|^{2}}-\frac{g^{2}\left(\hat{X}\right)C_{2}\left(p,\hat{X}\right)}{\sigma_{\hat{X}}^{2}} \\
n &= \frac{\nabla_{\hat{X}}g\left(\hat{X}\right)C_{2}\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}
\end{aligned} \tag{380}$$

A5.2 Full dynamical system

To make the system self-consistent, we introduce also a dynamics for R.

We assume that R depends on $K_{\hat{X}}, \hat{X}$ and $\nabla_{\theta}K_{\hat{X}}$, that leads to write: $R\left(K_{\hat{X}},\hat{X},\nabla_{\theta}K_{\hat{X}}\right)$. The variation is assumed to follow a diffusion process:

$$\begin{aligned}
\nabla_{\theta}R\left(\theta,\hat{X}\right) &= \int_{\theta'<\theta}G_{1}\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\theta'}R\left(\theta',\hat{X}'\right)d\left(\theta',\hat{X}'\right) \\
&\quad+\int_{\theta'<\theta}G_{2}\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\theta'}K_{\hat{X}'}d\left(\theta',\hat{X}'\right)
\end{aligned}$$

The first orders expansion of the right hand side leads to the following form for $\nabla_{\theta}R\left(\theta,\hat{X}\right)$:

$$\begin{aligned}
\nabla_{\theta}R\left(\theta,\hat{X}\right) &= \int\left(\hat{X}-\hat{X}'\right)\left(G_{1}\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\hat{X}}\nabla_{\theta}R\left(\theta,\hat{X}\right)+G_{2}\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\hat{X}}\nabla_{\theta}K_{\hat{X}}\right) \\
&\quad+\frac{1}{2}\int\left(\hat{X}-\hat{X}'\right)^{2}\left(G_{1}\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\hat{X}}^{2}\nabla_{\theta}R\left(\theta,\hat{X}\right)+G_{2}\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\hat{X}}^{2}\nabla_{\theta}K_{\hat{X}}\right) \\
&\quad+\int\left(\theta-\theta'\right)\left(G_{1}\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\theta}\nabla_{\theta}R\left(\theta,\hat{X}\right)+G_{2}\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\theta}\nabla_{\theta}K_{\hat{X}}\right) \\
&\quad+\frac{1}{2}\int\left(\theta-\theta'\right)^{2}\left(G_{1}\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\theta}^{2}\nabla_{\theta}R\left(\theta,\hat{X}\right)+G_{2}\left(\left(\theta,\hat{X}\right),\left(\theta',\hat{X}'\right)\right)\nabla_{\theta}^{2}\nabla_{\theta}K_{\hat{X}}\right) \\
&\quad+\ldots
\end{aligned} \tag{381}$$

where the crossed derivatives have been discarded for the sake of simplicity. We assume $G_1\left(\left(\theta,\hat{X}\right),\left(\theta,\hat{X}\right)\right)=0$ to avoid auto-interaction.

Performing the integrals yields:

$$\begin{aligned}\nabla_\theta R\left(\theta,\hat{X}\right) &= a_0\left(\hat{X}\right)\nabla_\theta K_{\hat{X}}+a\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta K_{\hat{X}}+b\left(\hat{X}\right)\nabla_{\hat{X}}^2\nabla_\theta K_{\hat{X}} \\ &+c\left(\hat{X}\right)\nabla_\theta\left(\nabla_\theta K_{\hat{X}}\right)+d\left(\hat{X}\right)\nabla_\theta^2\left(\nabla_\theta K_{\hat{X}}\right) \\ &+e\left(\hat{X}\right)\nabla_{\hat{X}}\left(\nabla_\theta R\left(\theta,\hat{X}\right)\right)+f\left(\hat{X}\right)\nabla_{\hat{X}}^2\left(\nabla_\theta R\left(\theta,\hat{X}\right)\right) \\ &+g\left(\hat{X}\right)\nabla_\theta\left(\nabla_\theta R\left(\theta,\hat{X}\right)\right)+h\left(\hat{X}\right)\nabla_\theta^2\left(\nabla_\theta R\left(\theta,\hat{X}\right)\right) \\ &+u\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta\left(\nabla_\theta K_{\hat{X}}\right)+v\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta\left(\nabla_\theta R\left(\theta,\hat{X}\right)\right)\end{aligned} \tag{382}$$

We ssume that the coefficients are slowly varying, since their are obtained by averages.

Gathering the dynamics (379) and (382) for $\nabla_\theta K_{\hat{X}}$ and $\nabla_\theta R\left(\theta,\hat{X}\right)$ leads to a matricial system:

$$\begin{aligned}0=&\begin{pmatrix}\frac{k}{K_{\hat{X}}} & \frac{l}{R(\hat{X})} \\ -a_0\left(\hat{X}\right) & 1\end{pmatrix}\begin{pmatrix}\nabla_\theta K_{\hat{X}} \\ \nabla_\theta R\end{pmatrix} \\ &-\begin{pmatrix}0 & \frac{2m}{\nabla_{\hat{X}}R(\hat{X})}\nabla_{\hat{X}} \\ a\left(\hat{X}\right)\nabla_{\hat{X}}+c\left(\hat{X}\right)\nabla_\theta & e\left(\hat{X}\right)\nabla_{\hat{X}}+g\left(\hat{X}\right)\nabla_\theta\end{pmatrix}\begin{pmatrix}\nabla_\theta K_{\hat{X}} \\ \nabla_\theta R\end{pmatrix} \\ &-\begin{pmatrix}0 & -\frac{n}{\nabla_{\hat{X}}^2 R(\hat{X})}\nabla_{\hat{X}}^2 \\ d\left(\hat{X}\right)\nabla_\theta^2+b\left(\hat{X}\right)\nabla_{\hat{X}}^2+u\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta & e\left(\hat{X}\right)\nabla_\theta^2+f\left(\hat{X}\right)\nabla_{\hat{X}}^2+v\nabla_{\hat{X}}\nabla_\theta\end{pmatrix}\begin{pmatrix}\nabla_\theta K_{\hat{X}} \\ \nabla_\theta R\end{pmatrix}\end{aligned} \tag{383}$$

A5.3 Oscillatory solutions

We look for a solution of (384) of the form:

$$\begin{pmatrix}\nabla_\theta K_{\hat{X}} \\ \nabla_\theta R\left(\hat{X}\right)\end{pmatrix}=\exp\left(i\Omega\left(\hat{X}\right)\theta+iG\left(\hat{X}\right)\hat{X}\right)\begin{pmatrix}\nabla_\theta K_0 \\ \nabla_\theta R_0\end{pmatrix}$$

with $G\left(\hat{X}\right)$ and $\Omega\left(\hat{X}\right)$ slowly varying. Consequently, the system (383) writes:

$$\begin{pmatrix}\frac{k}{K_{\hat{X}}} & \frac{l}{R(\hat{X})}-i\frac{2m}{\nabla_{\hat{X}}R(\hat{X})}G-\frac{n}{\nabla_{\hat{X}}^2R(\hat{X})}G^2 \\ \begin{array}{c}-a_0\left(\hat{X}\right)-ia\left(\hat{X}\right)G-ic\left(\hat{X}\right)\Omega \\ +d\Omega^2+bG^2+u\Omega G\end{array} & \begin{array}{c}1-ie\left(\hat{X}\right)G-ig\left(\hat{X}\right)\Omega+e\Omega^2 \\ +fG^2+u\Omega G\end{array}\end{pmatrix}\begin{pmatrix}\nabla_\theta K_{\hat{X}} \\ \nabla_\theta R\end{pmatrix}=0 \tag{384}$$

By canceling the determinant of the system, we are led to the following relation between $\Omega\left(\hat{X}\right)$ and $G\left(\hat{X}\right)$:

$$\begin{aligned}0 &= \frac{k}{K_{\hat{X}}}\left(1-ieG-ig\Omega\right)+\left(\frac{l}{R\left(\hat{X}\right)}-i\frac{2m}{\nabla_{\hat{X}}R\left(\hat{X}\right)}G\right)\left(a_0+iaG+ic\Omega\right) \\ &-\frac{l}{R\left(\hat{X}\right)}\left(d\Omega^2+bG^2+u\Omega G\right)+\frac{k}{K_{\hat{X}}}\left(e\Omega^2+fG^2+v\Omega G\right)\end{aligned}$$

In the sequel, we restrict to the first order terms, which yields the expression for Ω:

$$\begin{aligned}\Omega &= \frac{i}{\left(\frac{lc}{R(\hat{X})}-i\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)-\frac{kg}{K_{\hat{X}}}}\left(\frac{k}{K_{\hat{X}}}(1-ieG)+\left(\frac{l}{R\left(\hat{X}\right)}-i\frac{2m}{\nabla_{\hat{X}}R\left(\hat{X}\right)}G\right)(a_0+iaG)\right)\\ &= \frac{\left(\frac{lc}{R(\hat{X})}-\frac{kg}{K_{\hat{X}}}\right)+i\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G}{\left(\frac{lc}{R(\hat{X})}-\frac{kg}{K_{\hat{X}}}\right)^2+\left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2}\\ &\quad\times\left(\left(\frac{ke}{K_{\hat{X}}}+\left(\frac{2ma_0}{\nabla_{\hat{X}}R\left(\hat{X}\right)}-\frac{la}{R\left(\hat{X}\right)}\right)\right)G+i\left(\frac{k}{K_{\hat{X}}}+\frac{a_0 l}{R\left(\hat{X}\right)}+\frac{2ma}{\nabla_{\hat{X}}R\left(\hat{X}\right)}G^2\right)\right)\end{aligned}$$

Or equivalently:

$$\begin{aligned}\Omega &= \frac{\left(\frac{lc}{R(\hat{X})}-\frac{kg}{K_{\hat{X}}}\right)\left(\frac{ke}{K_{\hat{X}}}+\left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})}-\frac{la}{R(\hat{X})}\right)\right)G-\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\left(\frac{k}{K_{\hat{X}}}+\frac{a_0 l}{R(\hat{X})}+\frac{2ma}{\nabla_{\hat{X}}R(\hat{X})}G^2\right)}{\left(\frac{lc}{R(\hat{X})}-\frac{kg}{K_{\hat{X}}}\right)^2+\left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2}\\ &\quad+i\frac{\left(\frac{lc}{R(\hat{X})}-\frac{kg}{K_{\hat{X}}}\right)\left(\frac{k}{K_{\hat{X}}}+\frac{a_0 l}{R(\hat{X})}+\frac{2ma}{\nabla_{\hat{X}}R(\hat{X})}G^2\right)+\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}\left(\frac{ke}{K_{\hat{X}}}+\left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})}-\frac{la}{R(\hat{X})}\right)\right)G^2}{\left(\frac{lc}{R(\hat{X})}-\frac{kg}{K_{\hat{X}}}\right)^2+\left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2}\end{aligned}$$

We focus on the influence of time variations of $\nabla_\theta K_{\hat{X}}$ on $\nabla_\theta R$, and we can assume $g\simeq 0$ so that there is no self influence of $\nabla_\theta R$ on itself: $\nabla_\theta R$ depends on the variations of $\nabla_\theta K_{\hat{X}}$ as well as the neighboorhood sectors variations of $\nabla_\theta R$. Moreover, the coefficients e and a, being obtained by integration or first order expansion, can be considered as nul.

Consequently, the equation for Ω reduces to:

$$\Omega=\frac{\frac{lc}{R(\hat{X})}\left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})}\right)G-\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\left(\frac{k}{K_{\hat{X}}}+\frac{a_0 l}{R(\hat{X})}\right)}{\left(\frac{lc}{R(\hat{X})}\right)^2+\left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2}+i\frac{\frac{lc}{R(\hat{X})}\left(\frac{k}{K_{\hat{X}}}+\frac{a_0 l}{R(\hat{X})}\right)+\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}\left(\frac{2ma_0}{\nabla_{\hat{X}}R(\hat{X})}\right)G^2}{\left(\frac{lc}{R(\hat{X})}\right)^2+\left(\frac{2mc}{\nabla_{\hat{X}}R(\hat{X})}G\right)^2}$$

A5.4 Stability

The system is stable and the dynamics is dampening if:

$$\frac{lc}{R\left(\hat{X}\right)}\left(\frac{k}{K_{\hat{X}}}+\frac{a_0 l}{R\left(\hat{X}\right)}\right)+\frac{4m^2 c a_0}{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2}G^2>0 \tag{385}$$

To study the sign of (385) we need to estimate the coefficient k.

A5.4.1 Estimation of the coefficients k, l and m

We can estimate k and l by computing the factors $C_i\left(p,\hat{X}\right)$, for $i=1,2,3$.

This is done by estimating $p+\frac{1}{2}$. We start with the asymptotic form of $\hat{\Gamma}\left(p+\frac{1}{2}\right)$:

$$\hat{\Gamma}\left(p+\frac{1}{2}\right)\simeq\sqrt{p+\frac{1}{2}}\exp\left(-\frac{\sigma_X^2\left(p+\frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)$$

and rewriting the equation for $K_{\hat{X}}$ as:

$$K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|\left(\frac{\sigma_X^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)^{\frac{1}{4}}=C\left(\bar{p}\right)\sigma_{\hat{K}}^2\exp\left(-\frac{\sigma_X^2\left(p+\frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)\sqrt{\left(p+\frac{1}{2}\right)\sqrt{\frac{\sigma_X^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}}} \tag{386}$$

Then, using (377), we set:

$$\left(p+\frac{1}{2}\right)\sqrt{\frac{\sigma_X^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}}=\sqrt{C_1\left(p,\hat{X}\right)} \tag{387}$$

Equation (386) writes:

$$\frac{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|\left(\frac{\sigma_X^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\right)^{\frac{1}{4}}}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}=\exp\left(-C_1\left(p,\hat{X}\right)\right)\left(C_1\left(p,\hat{X}\right)\right)^{\frac{1}{4}} \tag{388}$$

and the solution to (388) is:

$$\begin{aligned}C_1\left(p,\hat{X}\right)&=\frac{\sigma_X^2\left(p+\frac{1}{2}\right)^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}\\&=C_0\left(\hat{X},K_{\hat{X}}\right)\exp\left(-W\left(k,-4C_0\left(\hat{X},K_{\hat{X}}\right)\right)\right)\end{aligned} \tag{389}$$

with:

$$C_0\left(\hat{X},K_{\hat{X}}\right)=\left(\frac{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}\right)^4\frac{\sigma_X^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}$$

and where $W(k,x)$ is the Lambert W function. The parameter $k=0$ for the stable case with low $K_{\hat{X}}$ and $k=-1$ for the unstable case with $K_{\hat{X}}$ large.

We can deduce $p+\frac{1}{2}$ from (389):

$$p+\frac{1}{2}=\frac{\sqrt{C_1\left(p,\hat{X}\right)}}{\sqrt{\frac{\sigma_X^2\left(f'(X)\right)^2}{96\left|f\left(\hat{X}\right)\right|^3}}} \tag{390}$$

and $2\frac{C_1\left(p,\hat{X}\right)}{p+\frac{1}{2}}$:

$$2\frac{C_1\left(p,\hat{X}\right)}{p+\frac{1}{2}}=\sqrt{C_1\left(p,\hat{X}\right)\frac{\sigma_X^2\left(f'(X)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}}$$

From (390) and (**??**) we deduce:

$$C_2\left(p,\hat{X}\right) = \ln\left(p+\frac{1}{2}\right)-\frac{2C_1\left(p,\hat{X}\right)}{p+\frac{1}{2}} \tag{391}$$
$$= \frac{1}{2}\ln\frac{C_1\left(p,\hat{X}\right)}{\frac{\sigma_X^2(f'(X))^2}{96\left|f(\hat{X})\right|^3}}-\sqrt{C_1\left(p,\hat{X}\right)\frac{\sigma_X^2\left(f'\left(X\right)\right)^2}{48\left|f\left(\hat{X}\right)\right|^3}}\simeq\frac{1}{2}\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}$$

We can also compute:

$$\left(p+\frac{3}{2}\right)\ln\left(p+\frac{1}{2}\right) \simeq \frac{48\left|f\left(\hat{X}\right)\right|^3 C_1\left(p,\hat{X}\right)}{\sigma_X^2\left(f'\left(X\right)\right)^2}\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}$$
$$= \frac{1}{2}\left(\frac{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}\right)^4\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}$$
$$\times\exp\left(-W\left(k,-4C_0\left(\hat{X},K_{\hat{X}}\right)\right)\right)$$

so that:

$$C_3\left(p,\hat{X}\right) = 1-C_1\left(p,\hat{X}\right)+\left(p+\frac{3}{2}\right)C_2\left(p,\hat{X}\right) \tag{392}$$
$$= 1-C_1\left(p,\hat{X}\right)+\left(p+\frac{3}{2}\right)\left(\ln\left(p+\frac{1}{2}\right)-\frac{2C_1\left(p,\hat{X}\right)}{p+\frac{1}{2}}\right)$$
$$\simeq 1+\left(\frac{48\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}-1\right)C_1\left(p,\hat{X}\right)$$
$$\simeq 1+\frac{1}{2}\left(\frac{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}\right)^4\exp\left(-W\left(k,-4C_0\left(\hat{X},K_{\hat{X}}\right)\right)\right)\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_X^2\left(f'\left(X\right)\right)^2}$$

Given that our assumptions $\sigma_X^2<1$ and in most cases $\frac{96\left|f(\hat{X})\right|^3}{\sigma_X^2(f'(X))^2}>>1$, then $\frac{96\left|f(\hat{X})\right|^3}{\sigma_X^2(f'(X))^2}>>1$ and $C_3\left(p,\hat{X}\right)>>1$.

These computations allow to estimate k and l. We start with k. Given that (see (380)):

$$k = 1-\eta\left(1-\frac{\gamma C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}\right)\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}$$
$$+\frac{\alpha\left(2\frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2}+\nabla_{\hat{X}}g\left(\hat{X}\right)\right)C_2\left(p,\hat{X}\right)-\left(1-\alpha\right)C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}$$
$$l = \frac{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)}$$
$$m = \left(1-\frac{\gamma C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)}\right)\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}$$

the sign of k and l depend on the magnitude of $K_{\hat{X}}$.

A5.4.1.1 $K_{\hat{X}} >> 1$ For $K_{\hat{X}} >> 1$, using (??) and:

$$\left\|\Psi\left(\hat{X}\right)\right\|^2 = D - \left(\nabla_X R\left(\hat{X}\right)\right)^2 K_{\hat{X}}^{\alpha}$$

we have:

$$K_{\hat{X}}^{\alpha} \simeq \frac{D}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2} - \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\sqrt{\frac{M-c}{c}}}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{2\left(1-\frac{1}{\alpha}\right)} D^{\frac{1}{\alpha}} c}$$

and:

$$\frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} \simeq \frac{D^{1+\frac{1}{\alpha}} c}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\sqrt{\frac{M-c}{c}}\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}}}$$

The constant c has been defined in appendix 3, and satisfies $c << 1$. As a consequence:

$$\begin{aligned} k &\simeq \eta \frac{\gamma C_3\left(p, \hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} - (1-\alpha)\frac{C_3\left(p, \hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \\ &\simeq \left(\frac{\eta\gamma D^{1+\frac{1}{\alpha}} c}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\sqrt{\frac{M-c}{c}}\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}}} - (1-\alpha)\right)\frac{C_3\left(p, \hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \end{aligned}$$

This may be negative or positive depending on the relative magnitude of $\frac{\eta\gamma D^{1+\frac{1}{\alpha}} c}{C(\bar{p})\sigma_{\hat{K}}^2\sqrt{\frac{M-c}{c}}(\nabla_{\hat{X}} R(\hat{X}))^{\frac{2}{\alpha}}}$ and $(1-\alpha)$. The first case correspond to the stable equilibrium with large $K_{\hat{X}}$ and the second case to the stable case with large $K_{\hat{X}}$ studied in appendix 2.

Unstable case This case corresponds to:

$$\frac{D^{1+\frac{1}{\alpha}} c}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\sqrt{\frac{M-c}{c}}\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}}} >> 1$$

Moreover, using (392) and the following estimation, we have:

$$k \simeq \eta \frac{\gamma C_3\left(p, \hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \frac{\eta\gamma D^{1+\frac{1}{\alpha}} c}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\sqrt{\frac{M-c}{c}}\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}}} >> 1 \tag{393}$$

We can also estimate $\left|\frac{k}{K_{\hat{X}}}\right|$. In this case:

$$\frac{k}{K_{\hat{X}}} \simeq \eta \frac{\gamma C_3\left(p, \hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \frac{\eta\gamma D^{\frac{1}{\alpha}} c}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\sqrt{\frac{M-c}{c}}\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}}} >> 1 \tag{394}$$

We can estimate l by the same token:

$$l=\frac{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)}>>1$$

and using (394) we have:

$$\left|\frac{k}{K_{\hat{X}}}\right|>>l$$

The coefficient m is obtained by using that in this case:

$$m\simeq\left(1-\frac{\gamma C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)}\right)\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}\simeq-\frac{1}{\eta}k$$

Stable case For the stable case we have:

$$\frac{\eta\gamma D^{1+\frac{1}{\alpha}}c}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2\sqrt{\frac{M-c}{c}}\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}}}-(1-\alpha)<0$$

and we write:

$$k\simeq-(1-\alpha)\frac{C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}<0$$

We have:

$$|k|>>1$$

and moreover:

$$\left|\frac{k}{K_{\hat{X}}}\right|\simeq(1-\alpha)\frac{C_3\left(p,\hat{X}\right)}{K_{\hat{X}}\left|f\left(\hat{X}\right)\right|}=\frac{1-\alpha}{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)K_{\hat{X}}}l<<l \tag{395}$$

The coefficient m is obtained by using that in the stable case:

$$m\simeq-\frac{\gamma}{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}l$$

A5.4.1.2 $K_{\hat{X}}<<1$ On the other hand, for $K_{\hat{X}}\leqslant 1$, we have:

$$\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}<<1 \tag{396}$$

so that:

$$k\simeq 1+\frac{\alpha\left(2\frac{g^2(\hat{X})}{\sigma_{\hat{X}}^2}+\nabla_{\hat{X}}g\left(\hat{X}\right)\right)C_2\left(p,\hat{X}\right)-(1-\alpha)C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}$$

Given (391) and (392), this yields:

$$k\simeq-\frac{(1-\alpha)C_3\left(p,\hat{X}\right)}{\left|f\left(\hat{X}\right)\right|}<0 \tag{397}$$

and, as in the previous case:

$$\begin{aligned}|k| &> > 1\\ l &> > 1\end{aligned}$$

Moreover, given that $K_{\hat{X}} << 1$:

$$\left|\frac{k}{K_{\hat{X}}}\right| >> 1 \tag{398}$$

and:

$$\left|\frac{k}{K_{\hat{X}}}\right| >> l \tag{399}$$

Moreover, given (396):

$$|m| = \left|1 - \frac{\gamma C_3\left(p, \hat{X}\right)}{f\left(\hat{X}\right)}\right| \frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} << \left|\frac{\gamma C_3\left(p, \hat{X}\right)}{f\left(\hat{X}\right)}\right|$$

and:

$$|m| << l$$

A5.4.1.3 Intermediate case In this case, we can consider that $\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2}$ is of order 1:

$$\frac{D - \left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} = O(1) \tag{400}$$

Assuming that $\gamma << 1$ we have:

$$\begin{aligned} k &\simeq 1 + \frac{\alpha\left(2\frac{g^2\left(\hat{X}\right)}{\sigma^2_{\hat{X}}} + \nabla_{\hat{X}} g\left(\hat{X}\right)\right) C_2\left(p, \hat{X}\right) - (1-\alpha) C_3\left(p, \hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \\ &\simeq 1 + \frac{\frac{\alpha}{2}\left(2\frac{g^2\left(\hat{X}\right)}{\sigma^2_{\hat{X}}} + \nabla_{\hat{X}} g\left(\hat{X}\right)\right) - \frac{1-\alpha}{2}\left(\frac{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|}{C(\bar{p})\sigma^2_{\hat{K}}}\right)^4 \exp\left(-W\left(k, -4C_0\left(\hat{X}, K_{\hat{X}}\right)\right)\right)}{\left|f\left(\hat{X}\right)\right|} \ln \frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma^2_X\left(f'(X)\right)^2} \end{aligned}$$

Given that the intermediate case is stable (see appendix 2), the relation between $K_{\hat{X}}$ and $R\left(\hat{X}\right)$ is positive, we can assume that $k < 0$ and:

$$\begin{aligned} k &\simeq 1 + \frac{\alpha\left(2\frac{g^2\left(\hat{X}\right)}{\sigma^2_{\hat{X}}} + \nabla_{\hat{X}} g\left(\hat{X}\right)\right) C_2\left(p, \hat{X}\right) - (1-\alpha) C_3\left(p, \hat{X}\right)}{\left|f\left(\hat{X}\right)\right|} \\ &\simeq -\frac{\frac{1-\alpha}{2}\left(\frac{K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^2\left|f\left(\hat{X}\right)\right|}{C(\bar{p})\sigma^2_{\hat{K}}}\right)^4 \exp\left(-W\left(0, -4C_0\left(\hat{X}, K_{\hat{X}}\right)\right)\right)}{\left|f\left(\hat{X}\right)\right|} \ln \frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma^2_X\left(f'(X)\right)^2} \end{aligned}$$

and:

$$
\begin{aligned}
l &= \frac{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)C_3\left(p,\hat{X}\right)}{f\left(\hat{X}\right)} \simeq l \\
&= \frac{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)\left(\frac{K_{\hat{X}}\left\|\Psi(\hat{X})\right\|^2\left|f(\hat{X})\right|}{C(\bar{p})\sigma_{\hat{K}}^2}\right)^4\exp\left(-W\left(0,-4C_0\left(\hat{X},K_{\hat{X}}\right)\right)\right)}{f\left(\hat{X}\right)}\ln\frac{96\left|f\left(\hat{X}\right)\right|^3}{\sigma_{\hat{X}}^2\left(f'\left(X\right)\right)^2}
\end{aligned}
$$

Note that in this case:

$$k \simeq -\frac{1-\alpha}{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}l$$

and, given (400):

$$m \simeq -\gamma\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}l$$

A5.4.2 Stability conditions

This appendix presents the computations leading to the stability conditions for the three ranges of capital considered. Apart from the intermediate case, interpretations are detailed in the text.

A5.4.2.1 Case $K_{\hat{X}} >> 1$

Stable case As shown above, $k<0$, $|k|>>1$, $l>>1$ and $\left|\frac{k}{K_{\hat{X}}}\right|<<l$. Coefficients l and m are of the same order. Thus (385) becomes:

$$\frac{l^2a_0c}{\left(R\left(\hat{X}\right)\right)^2}+\frac{4m^2ca_0}{\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2}G^2>0$$

That is, for $c>0$ the oscillations are stable, whereas for $c<0$ they are unstable.

Unstable case In this case, $k>0$, $|k|>>1$, $l>>1$ and $\left|\frac{k}{K_{\hat{X}}}\right|>>l$. We have also $m\simeq-\frac{1}{\eta}k$ and (385) writes:

$$\frac{cl}{R\left(\hat{X}\right)}\frac{k}{K_{\hat{X}}}+\frac{4k^2ca_0}{\eta^2\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2}G^2>0 \tag{401}$$

That is, for $c>0$ the oscillations are stable, whereas for $c<0$ they are unstable.

A5.4.2.1.2 Case $K_{\hat{X}} << 1$ Equations (397) and (398) show that $k<0$, $|k|>>1$, $l>>1$, $|m|<<l$ and $\left|\frac{k}{K_{\hat{X}}}\right|>>l$. Equation (385) thus writes:

$$\frac{cl}{R\left(\hat{X}\right)}\frac{k}{K_{\hat{X}}}>0 \tag{402}$$

That is, for $c>0$ the oscillations are unstable, whereas for $c<0$ they are stable.

A5.4.2.3 Intermediate case In this case, we have seen above that $k<0$:

$$k\simeq-\frac{1-\alpha}{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}l$$

and:

$$m\simeq-\gamma\frac{D-\left\|\Psi\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}$$

Consequently, equation (385) particularizes as:

$$\frac{l^2c}{R\left(\hat{X}\right)}\left(\frac{a_0}{R\left(\hat{X}\right)}-\frac{1-\alpha}{\varsigma K_{\hat{X}}F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}\right)+4ca_0\left(\frac{\gamma\left(D-\left\|\Psi\left(\hat{X}\right)\right\|^2\right)}{\varsigma\nabla_{\hat{X}}R\left(\hat{X}\right)F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)\left\|\Psi\left(\hat{X}\right)\right\|^2}\right)^2G^2>0$$

Given the definition of a_0 and the stability of the intermediate case, we assume $a_0>0$. Thus, 2 possibilities arise.

Coefficient $c>0$ In this case, the oscillations are stable if:

$$\frac{a_0}{R\left(\hat{X}\right)}-\frac{1-\alpha}{\varsigma K_{\hat{X}}F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}>0$$

or if:

$$\frac{a_0}{R\left(\hat{X}\right)}-\frac{1-\alpha}{\varsigma K_{\hat{X}}F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}<0$$

and:

$$G^2>\frac{l^2\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2}{4a_0R\left(\hat{X}\right)}\left(\frac{\varsigma\left(\nabla_{\hat{X}}R\left(\hat{X}\right)F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)\left\|\Psi\left(\hat{X}\right)\right\|^2\right)}{\gamma\left(D-\left\|\Psi\left(\hat{X}\right)\right\|^2\right)}\right)^2\left|\frac{a_0}{R\left(\hat{X}\right)}-\frac{1-\alpha}{\varsigma F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}\right|$$

Otherwise, the oscillations are unstable.

The constant ς is irrelevant here, although it arises in appendix 3 to estimate short-term returns. The function F_1, defined in (35), determines the stock's prices evolution. The coefficient α is the Cobb-Douglas power arising in the dividend part of short-term returns. The constant D, defined in (81), determines the relation between number of firms and average capital at sector $\hat{X}$.

We recover the large average capital case. A relatively high reactivity of expectations to fluctuations in capital allows to maintain the capital at its equilibrium value. This stability is favoured for sectors with large average capital when G is relatively large, i.e. when this sectors present large discrepancies in capital with their neighbours.

Coefficient $c<0$ The oscillations are stable if:

$$\frac{a_0}{R\left(\hat{X}\right)}-\frac{1-\alpha}{\varsigma K_{\hat{X}}F_1\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}<0 \tag{403}$$

and:

$$G^2 < \frac{l^2 \left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2}{4a_0 R\left(\hat{X}\right)} \left(\frac{\varsigma \left(\nabla_{\hat{X}} R\left(\hat{X}\right) F_1\left(R\left(K_{\hat{X}}, \hat{X}\right)\right) \left\Vert\Psi\left(\hat{X}\right)\right\Vert^2\right)}{\gamma\left(D - \left\Vert\Psi\left(\hat{X}\right)\right\Vert^2\right)}\right)^2 \left|\frac{a_0}{R\left(\hat{X}\right)} - \frac{1-\alpha}{\varsigma F_1\left(R\left(K_{\hat{X}}, \hat{X}\right)\right)}\right| \tag{404}$$

Conditions (403) and (404) correspond to the case of relatively low capital for which a stability in the oscillations may be reached when expectations are moderately reactive to variation in capital. The condition (404) shows that the stability in oscillations is reached for moderate values of G, i.e. relatively small discrepancy between neighbouring sectors.

We recover the large average capital case. A relatively high reactivity of expectations to fluctuations in capital allows to maintain the capital at its equilibrium value. This stability is favoured for sectors with large average capital when G is relatively large, i.e. when this sectors present large discrepancies in capital with their neighbours.

Appendix 6 Computation of effective action at the second order

We compute the second-order derivatives for the real and the financial economy respectively.

A6.1 Real economy

In first approximation:

$$\begin{aligned} & \frac{\delta^2 \left(S_1 + S_2\right)}{\delta\Psi^{\dagger}\left(Z,\theta\right)\delta\Psi\left(Z,\theta\right)} \\ \simeq\ & -\int \delta\Psi^{\dagger}\left(K,X\right)\left(\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X - \nabla_X R\left(K,X\right)H\left(K\right)\right) - 4\tau\left(\left|\Psi_0\left(X\right)\right|^2\right)\right. \\ & \left. + \nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K + u\left(K,X,\Psi_0,\hat{\Psi}_0\right)\right)\right)\delta\Psi\left(K,X\right)dKdX \end{aligned} \tag{405}$$

where:

$$\left|\Psi_0\left(X\right)\right|^2 = \int \left|\Psi_0\left(K',X\right)\right|^2 dK'$$

and:

$$u\left(K,X,\Psi_0,\hat{\Psi}_0\right) \rightarrow \frac{1}{\varepsilon}\left(K - \int \hat{F}_2\left(s,R\left(K,X\right)\right)\hat{K}\left\Vert\hat{\Psi}_0\left(\hat{K},X\right)\right\Vert^2 d\hat{K}\right) = \frac{1}{\varepsilon}\left(K - \hat{F}_2\left(s,R\left(K,X\right)\right)K_X d\hat{K}\right) \tag{406}$$

In equation (406), we used the notation:

$$\int \hat{F}_2\left(s,R\left(K,X\right)\right)\hat{K}\left\Vert\hat{\Psi}_0\left(\hat{K},X\right)\right\Vert^2 d\hat{K} = \hat{F}_2\left(s,R\left(K,X\right)\right)\hat{K}_X$$

We perform a change of variables in (405):

$$\Delta\Psi(K,X) = \exp\left(\int^{X}\frac{\nabla_X R(X)}{\sigma_X^2}H\left(\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right) \tag{407}$$
$$\times\exp\left(\int\left(K-\frac{F_2(R(K,X))K_X}{F_2(R(K_X,X))}\right)dK\right)\delta\Psi(K,X)$$
$$\Delta\Psi^{\dagger}(K,X) = \exp\left(-\int^{X}\frac{\nabla_X R(X)}{\sigma_X^2}H\left(\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2}\right)\right)$$
$$\times\exp\left(-\int\left(K-\frac{F_2(R(K,X))K_X}{F_2(R(K_X,X))}\right)dK\right)\delta\Psi^{\dagger}(K,X)$$

where K_X, the average invested capital per firm in sector X:

$$K_X=\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}}{\|\Psi(X)\|^2} \tag{408}$$

so that the effective action (405) for the real economy becomes:

$$\Delta\Psi^{\dagger}(Z,\theta)\left(\frac{\delta^2(S_1+S_2)}{\delta\Psi^{\dagger}(Z,\theta)\,\delta\Psi(Z,\theta)}\right)_{\Psi(Z,\theta)=\Psi_0(Z,\theta)}\Delta\Psi(Z,\theta) \tag{409}$$
$$=\int\Delta\Psi^{\dagger}(Z,\theta)\left(-\frac{\sigma_X^2}{2}\nabla_X^2+\frac{(\nabla_X R(K,X)H(K_X))^2}{2\sigma_X^2}+\frac{\nabla_X^2 R(K,X)}{2}H(K)+4\tau|\Psi(X)|^2\right)\Delta\Psi(Z,\theta)$$
$$+\int\Delta\Psi^{\dagger}(Z,\theta)\left(-\frac{\sigma_K^2}{2}\nabla_K^2+\frac{1}{2\sigma_K^2}\left(K-\hat{F}_2(s,R(K,X))K_X\right)^2+\frac{1-\nabla_K\hat{F}_2(s,R(K,X))K_X}{2}\right)\Delta\Psi(Z,\theta)$$

As explained in section 10.1.2, the effects of competition can be refined by considering repulsive forces that are capital dependent. It amounts to replace in (409), the term:

$$\int\Delta\Psi^{\dagger}(Z,\theta)\left(2\tau|\Psi(X)|^2\right)\Delta\Psi(Z,\theta)$$

by the term:

$$\int\Delta\Psi^{\dagger}(K,X)\left(2\tau\frac{\int K'|\Psi(K',X)|^2 dK'}{K}\right)\Delta\Psi(K,\theta) \tag{410}$$
$$=\int\Delta\Psi^{\dagger}(K,X)\left(2\tau\frac{|\Psi(X)|^2 K_X}{K}\right)\Delta\Psi(K,\theta)$$

with:

$$|\Psi(X)|^2 = \int|\Psi(K',X)|^2 dK'$$
$$K_X = \frac{\int K'|\Psi(K',X)|^2 dK'}{|\Psi(X)|^2}$$

This models repulsive forces that are inversely proportional to capital and mainly affect low-capital firms. Note that this change in the interaction does not modify the collective state, since by setting $K = K_X$, we recover the previous repulsive term. Ultimately, using:

$$\|\Psi(X)\|^2 = (2\tau)^{-1} \left(D\left(\|\Psi\|^2\right) - \frac{1}{2\sigma_X^2} \left((\nabla_X R(X))^2 + \frac{\sigma_X^2 \nabla_X^2 R(K_X, X)}{H(K_X)} \right) H^2(K_X) \left(1 - \frac{H'\left(\hat{K}_X\right) K_X}{H\left(\hat{K}_X\right)} \right) \right) \tag{411}$$

the interaction term (410) becomes:

$$\begin{aligned} & \int \Delta\Psi^{\dagger}(K, X) \frac{1}{2} \left((\nabla_X R(X))^2 + \frac{\sigma_X^2 \nabla_X^2 R(K_X, X)}{H(K_X)} \right) \\ & \times H^2(K_X) \left(1 - \frac{H'\left(\hat{K}_X\right) K_X}{H\left(\hat{K}_X\right)} \right) + 2\tau \frac{|\Psi(X)|^2 K_X}{K} \Delta\Psi(K, \theta) \\ = & \int \Delta\Psi^{\dagger}(K, X) \left(D\left(\|\Psi\|^2\right) + 2\tau \frac{|\Psi(X)|^2 (K_X - K)}{K} \right) \Delta\Psi(K, \theta) \end{aligned}$$

When the above expression is used to rewrite (409), it yields the formula:

$$\begin{aligned} \frac{\delta^2 (S_1 + S_2)}{\delta\Psi^{\dagger}(Z, \theta)\, \delta\Psi(Z, \theta)} = & -\frac{\sigma_X^2}{2} \nabla_X^2 - \frac{\sigma_K^2}{2} \nabla_K^2 + \left(D\left(\|\Psi\|^2\right) + 2\tau \frac{|\Psi(X)|^2 (K_X - K)}{K} \right) \\ & + \frac{1}{2\sigma_K^2} \left(K - \hat{F}_2(s, R(K, X)) K_X \right)^2 + \frac{1 - \nabla_K \hat{F}_2(s, R(K, X)) K_X}{2} \end{aligned} \tag{412}$$

as stated in the text.

A6.2 Financial economy

For the financial sector, we consider the field-action for $\hat{\Psi}^{\dagger}\left(\hat{K}, \hat{X}\right)$:

$$S_3 + S_4 = -\int \hat{\Psi}^{\dagger}\left(\hat{K}, \hat{X}\right) \left(\nabla_{\hat{K}} \left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} - \hat{K} f\left(\hat{X}, K_{\hat{X}}\right) \right) + \nabla_{\hat{X}} \left(\frac{\sigma_{\hat{X}}^2}{2} \nabla_{\hat{X}} - g\left(\hat{X}, K_{\hat{X}}\right) \right) \right) \hat{\Psi}\left(\hat{K}, \hat{X}\right) \tag{413}$$

with:

$$f\left(\hat{X}, K_{\hat{X}}\right) = \frac{1}{\varepsilon} \left(r\left(K_{\hat{X}}, \hat{X}\right) - \gamma \left\|\Psi\left(\hat{X}\right)\right\|^2 + F_1 \left(\frac{R\left(K_{\hat{X}}, \hat{X}\right)}{\int R\left(K'_{X'}, X'\right) \|\Psi(X')\|^2 dX'} \right) \right) \tag{414}$$

$$g\left(\hat{X}, K_{\hat{X}}\right) = \left(\frac{\nabla_{\hat{X}} F_0\left(R\left(K_{\hat{X}}, \hat{X}\right)\right)}{\left\|\nabla_{\hat{X}} R\left(K_{\hat{X}}, \hat{X}\right)\right\|} + \nu \nabla_{\hat{X}} F_1 \left(\frac{R\left(K_{\hat{X}}, \hat{X}\right)}{\int R\left(K'_{X'}, X'\right) \|\Psi(X')\|^2 dX'} \right) \right) \tag{415}$$

Using a change of variable (see appendix 3.1):

$$\begin{aligned} \hat{\Psi} & \rightarrow \exp\left(\frac{1}{\sigma_{\hat{X}}^2} \int g\left(\hat{X}\right) d\hat{X} + \frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}\right) \right) \hat{\Psi} \\ \hat{\Psi}^{\dagger} & \rightarrow \exp\left(\frac{1}{\sigma_{\hat{X}}^2} \int g\left(\hat{X}\right) d\hat{X} + \frac{\hat{K}^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}\right) \right) \hat{\Psi}^{\dagger} \end{aligned} \tag{416}$$

the action (413) becomes:

$$S_3+S_4=-\int\hat{\Psi}^{\dagger}\left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2-\frac{1}{2\sigma_{\hat{X}}^2}\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2-\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\hat{\Psi} \tag{417}$$
$$-\int\hat{\Psi}^{\dagger}\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}-\hat{K}f\left(\hat{X},K_{\hat{X}}\right)\right)\right)\hat{\Psi}$$

To obtain the second-order expansion of the field's action, we start by the first derivative of (417) arising in the minimization equation in (Gosselin Lotz Wambst 2022):

$$\begin{aligned}\frac{\delta\left(S_3\left(\Psi\right)+S_4\left(\Psi\right)\right)}{\delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)} &= -\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2\hat{\Psi}-\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}^2\hat{\Psi}+\frac{1}{2\sigma_{\hat{X}}^2}\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\hat{\Psi} \\ &\quad +\frac{\hat{K}^2}{2\sigma_{\hat{K}}^2}f^2\left(\hat{X}\right)+\frac{1}{2}f\left(\hat{X},K_{\hat{X}}\right)\hat{\Psi}+F\left(\hat{X},K_{\hat{X}}\right)\hat{K}\hat{\Psi}\end{aligned} \tag{418}$$

with:

$$\begin{aligned}F\left(\hat{X},K_{\hat{X}}\right) &= \nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)+f\left(\hat{X},K_{\hat{X}}\right)\right)\frac{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} \\ &\quad +\frac{\nabla_{K_{\hat{X}}}f^2\left(\hat{X},K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2\left\|\Psi\left(\hat{X}\right)\right\|^2}\left\langle\hat{K}^2\right\rangle_{\hat{X}}\end{aligned} \tag{419}$$

so that :

$$\begin{aligned}\frac{\delta^2\left(S_3\left(\Psi\right)+S_4\left(\Psi\right)\right)}{\delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\delta\hat{\Psi}\left(Z,\theta\right)} &= -\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2-\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}^2+\frac{1}{2\sigma_{\hat{X}}^2}\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right) \\ &\quad +\frac{\hat{K}^2}{2\sigma_{\hat{K}}^2}f^2\left(\hat{X}\right)+\frac{1}{2}f\left(\hat{X},K_{\hat{X}}\right)+F\left(\hat{X},K_{\hat{X}}\right)\hat{K}-\hat{\Psi}^{\dagger}\frac{\delta F\left(\hat{X},K_{\hat{X}}\right)}{\delta\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^2}\hat{K}\hat{\Psi}\end{aligned}$$

where the last term is given by:

$$\begin{aligned}\frac{\delta F\left(\hat{X},K_{\hat{X}}\right)}{\delta\hat{\Psi}\left(Z,\theta\right)} &\simeq \nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^2}{2\sigma_{\hat{X}}^2}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)+f\left(\hat{X},K_{\hat{X}}\right)\right)\frac{\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}{\left\|\Psi\left(\hat{X}\right)\right\|^2} \\ &\quad +\frac{\nabla_{K_{\hat{X}}}f^2\left(\hat{X},K_{\hat{X}}\right)}{\sigma_{\hat{K}}^2\left\|\Psi\left(\hat{X}\right)\right\|^2}\hat{K}^2\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\end{aligned}$$

Following (Gosselin Lotz Wambst 2022) we neglect in first approximation the derivatives with respect to $K_{\hat{X}}$, and define the new variable:

$$y=\frac{\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}}{\sqrt{\sigma_{\hat{K}}^2}}\left(f^2\left(\hat{X}\right)\right)^{\frac{1}{4}} \tag{420}$$

$$\frac{\delta^2 (S_3 (\Psi) + S_4 (\Psi))}{\delta \hat{\Psi}^\dagger (Z, \theta) \delta \hat{\Psi} (Z, \theta)} = -\frac{\sigma^2_{\hat{X}}}{2} \nabla^2_{\hat{X}} - \nabla^2_y + \left(\frac{y^2}{4} + \frac{\left(g\left(\hat{X}\right)\right)^2 + \sigma^2_{\hat{X}} \left(f\left(\hat{X}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma^2_{\hat{K}} F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} \right)}{\sigma^2_{\hat{X}} \sqrt{f^2\left(\hat{X}\right)}} \right) \tag{421}$$

This leads to:

$$\begin{aligned}
&\Delta \hat{\Psi}^\dagger (Z, \theta) \frac{\delta^2 (S_3 (\Psi) + S_4 (\Psi))}{\delta \hat{\Psi}^\dagger (Z, \theta) \delta \hat{\Psi} (Z, \theta)} \Delta \hat{\Psi} (Z, \theta) \\
= \; & \Delta \hat{\Psi}^\dagger (Z, \theta) \left(-\frac{\sigma^2_{\hat{X}}}{2} \nabla^2_{\hat{X}} + \frac{\left(g\left(\hat{X}\right)\right)^2 + \sigma^2_{\hat{X}} \left(f\left(\hat{X}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma^2_{\hat{K}} F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)} \right)}{\sigma^2_{\hat{X}} \sqrt{f^2\left(\hat{X}\right)}} \right. \\
&\left. - \frac{\sigma^2_{\hat{K}}}{2\sqrt{f^2\left(\hat{X}\right)}} \nabla^2_{\hat{K}} + \left(\frac{\sqrt{f^2\left(\hat{X}\right)} \left(\hat{K} + \frac{\sigma^2_{\hat{K}} F\left(\hat{X}, K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)} \right)^2}{4\sigma^2_{\hat{K}}} \right) \right) \Delta \hat{\Psi} (Z, \theta)
\end{aligned}$$

Appendix 7 Higher order corrections to the effective action

The higher-order corrections are obtained by expanding at higher-orders in $\Delta\Psi (Z, \theta)$ and $\Delta\hat{\Psi} (Z, \theta)$. These variations around the background fields can be considered to be orthogonals to $\Psi_0 (Z, \theta)$ and $\hat{\Psi}_0 (Z, \theta)$.

A7.1 Third order terms

The orthogonality condition implies that the third-order terms in the expansion can be neglected. Actually, in first approximation the third-order terms arising in the expansion of S have the form:

$$\begin{aligned}
&2\tau \int \Delta\Psi (K', X) \Psi^\dagger_0 (K', X') \, dK' \, |\Delta\Psi (K, X)|^2 \, dK dX \qquad (422) \\
&- \int \Delta\Psi^\dagger (K, X) \Psi^\dagger_0 (K', X') \nabla_K \frac{\delta u\left(K, X, \Psi, \hat{\Psi}\right)}{\delta |\Psi (K', X)|^2} \Delta\Psi (K', X') \Delta\Psi (K, X) \\
&- \int \Delta\Psi^\dagger (K, \theta) \hat{\Psi}^\dagger_0 \left(\hat{K}, \theta\right) \nabla_K \frac{\delta u\left(K, X, \Psi, \hat{\Psi}\right)}{\delta \left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2} \Delta\hat{\Psi}\left(\hat{K}, \theta\right) \Delta\Psi (K, \theta) \\
&- \int \Delta\hat{\Psi}^\dagger \left(\hat{K}, \hat{X}\right) \Psi^\dagger_0 (K', \theta) \left\{ \nabla_{\hat{K}} \frac{\hat{K} \delta^2 f\left(\hat{X}, \Psi, \hat{\Psi}\right)}{\delta |\Psi (K', X)|^2} + \nabla_{\hat{X}} \frac{\delta g\left(\hat{X}, \Psi, \hat{\Psi}\right)}{\delta |\Psi (K', X)|^2} \right\} \Delta\Psi (K', X') \Delta\hat{\Psi}\left(\hat{K}, \hat{X}\right) \\
&+ H.C.
\end{aligned}$$

where the notation $H.C.$ stands for the hermitian conjugate of the expression. Replacing the terms:

$$\nabla_K \frac{\delta u\left(K, X, \Psi, \hat{\Psi}\right)}{\delta |\Psi (K', X)|^2} \;, \nabla_K \frac{\delta u\left(K, X, \Psi, \hat{\Psi}\right)}{\delta \left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2}$$

and:

$$\nabla_{\hat{K}} \frac{\hat{K} \delta^2 f\left(\hat{X}, \Psi, \hat{\Psi}\right)}{\delta |\Psi (K', X)|^2} + \nabla_{\hat{X}} \frac{\delta g\left(\hat{X}, \Psi, \hat{\Psi}\right)}{\delta |\Psi (K', X)|^2}$$

by their averages in (422), and using the orthogonality conditions:

$$\int \hat{\Psi}_0^{\dagger}\left(\hat{K},\theta\right)\Delta\hat{\Psi}\left(\hat{K},\theta\right)=\int \Psi_0^{\dagger}\left(K',X'\right)\Delta\Psi\left(K',X'\right)=0$$

leads to neglect the third-order terms in first approximation.

A7.2 Fourth order terms

A7.2.1 General formula

Considering the fourth-order in the action expansion yields quartic corrections. Using that in average:

$$\frac{\delta^2 f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}\simeq 0$$

$$\frac{\delta^2 g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}\simeq 0$$

the fourth-order terms in the fields' action become:

$$\begin{aligned}
&2\tau\int\left|\Delta\Psi\left(K',X\right)\right|^2 dK'\left|\Delta\Psi\left(K,X\right)\right|^2 dKdX \\
&-\Delta\Psi^{\dagger}\left(K,X\right)\Delta\Psi^{\dagger}\left(K',X'\right)\nabla_K\frac{\delta^2 u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}\Delta\Psi\left(K',X'\right)\Delta\Psi\left(K,X\right) \\
&-\Delta\Psi^{\dagger}\left(K,\theta\right)\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\theta\right)\nabla_K\frac{\delta^2 u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}\Delta\hat{\Psi}\left(\hat{K},\theta\right)\Delta\Psi\left(K,\theta\right) \\
&-\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\Delta\Psi^{\dagger}\left(K',\theta\right)\left\{\nabla_{\hat{K}}\frac{\hat{K}\delta^2 f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}+\nabla_{\hat{X}}\frac{\delta^2 g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}\right\}\Delta\Psi\left(K',X'\right)\Delta\hat{\Psi}\left(\hat{K},\hat{X}\right)
\end{aligned}\tag{423}$$

A72.2 Estimation of the various terms

The three last terms in the rhs of (423) can be evaluated. The second term is given by:

$$\begin{aligned}
&\Delta\Psi^{\dagger}\left(K,X\right)\Delta\Psi^{\dagger}\left(K',X'\right)\frac{\delta^2 u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}\Delta\Psi\left(K,X\right)\Delta\Psi\left(K',\theta\right) \\
=\;&\Delta\Psi^{\dagger}\left(K,\theta\right)\Delta\Psi^{\dagger}\left(K',\theta\right)\left\{\int\hat{F}_2\left(s,R\left(K,X\right)\right)\hat{F}_2\left(s',R\left(K',X'\right)\right)\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}\right\}\Delta\Psi\left(K,\theta\right)\Delta\Psi\left(K',\theta\right) \\
&-2\Delta\Psi^{\dagger}\left(K,\theta\right)\Delta\Psi^{\dagger}\left(K',\theta\right) \\
&\times\left\{\int\Psi_0^{\dagger}\left(K',X\right)\frac{\hat{F}_2\left(s,R\left(K,X\right)\right)\hat{F}_2\left(s',R\left(K',X'\right)\right)}{\int F_2\left(s',R\left(K',X\right)\right)\left\|\Psi\left(K',X\right)\right\|^2 dK'}\Psi_0\left(K,\hat{X}\right)\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}\right\}\Delta\Psi\left(K,\theta\right)\Delta\Psi\left(K',\theta\right) \\
\simeq\;&\Delta\Psi^{\dagger}\left(K,\theta\right)\Delta\Psi^{\dagger}\left(K',\theta\right)\left\{\int\hat{F}_2\left(s,R\left(K,X\right)\right)\hat{F}_2\left(s',R\left(K',X'\right)\right)\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2 d\hat{K}\right\}\Delta\Psi\left(K,\theta\right)\Delta\Psi\left(K',\theta\right)
\end{aligned}$$

The second term in the rhs of (423) is equal to:

$$\begin{aligned}
&\Delta\Psi^{\dagger}(K,\theta)\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\theta\right)\frac{\delta^{2}u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}\Delta\Psi(K,\theta)\Delta\hat{\Psi}\left(\hat{K},\theta\right)\\
=&-\Delta\Psi^{\dagger}(K,\theta)\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\theta\right)\frac{1}{\varepsilon}\hat{F}_{2}(s,R(K,X))\hat{K}\Delta\Psi(K,\theta)\Delta\hat{\Psi}\left(\hat{K},\theta\right)
\end{aligned}$$

Ultimately, the last term in the rhs of (423):

$$\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\Delta\Psi^{\dagger}(K',\theta)\left\{\nabla_{\hat{K}}\frac{\hat{K}\delta^{2}f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\delta\Psi^{\dagger}(K',X)}+\nabla_{\hat{X}}\frac{\delta^{2}g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\delta\Psi^{\dagger}(K',X)}\right\}\Delta\Psi(K',X')\Delta\hat{\Psi}\left(\hat{K},\hat{X}\right)$$

is obtained by using the expressions of $f\left(\hat{X},\Psi,\hat{\Psi}\right)$ and $g\left(\hat{X},\Psi,\hat{\Psi}\right)$ that compute short-term and long-term returns, respectively:

$$\begin{aligned}
f\left(\hat{X},\Psi,\hat{\Psi}\right)&=\frac{1}{\varepsilon}\int\left(r(K,X)-\gamma\frac{\int K'\|\Psi(K',X)\|^{2}}{K}+F_{1}\left(\frac{R(K,X)}{\int R(K',X')\|\Psi(K',X')\|^{2}d(K',X')},\Gamma(K,X)\right)\right)\\
&\times\hat{F}_{2}(s,R(K,X))\left\|\Psi\left(K,\hat{X}\right)\right\|^{2}dK\\
g\left(K,\hat{X},\Psi,\hat{\Psi}\right)&=\int\left(\nabla_{\hat{X}}F_{0}\left(R\left(K,\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_{1}\left(\frac{R\left(K,\hat{X}\right)}{\int R(K',X')\|\Psi(K',X')\|^{2}d(K',X')}\right)\right)\\
&\times\frac{\left\|\Psi\left(K,\hat{X}\right)\right\|^{2}dK}{\int\left\|\Psi\left(K',\hat{X}\right)\right\|^{2}dK'}
\end{aligned}$$

We find:

$$\begin{aligned}
&\frac{\delta^{2}f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\delta\Psi^{\dagger}(K',X)}\\
=&\frac{1}{\varepsilon}\Delta\left(r(K',X)-\gamma\frac{\int K'\|\Psi(K',X)\|^{2}}{K'}+F_{1}\left(\frac{R(K',X)}{\int R(K',X')\|\Psi(K',X')\|^{2}d(K',X')},\Gamma(K,X)\right)\right)\\
&\times\frac{F_{2}\left(s',R\left(K',\hat{X}\right)\right)}{\int F_{2}\left(s'R\left(K',\hat{X}\right)\right)\left\|\Psi\left(K',\hat{X}\right)\right\|^{2}dK'}\\
&-\frac{1}{\varepsilon}\int\left(\gamma\frac{K'}{K}+\frac{R(K',X)R\left(K_{\hat{X}},X\right)}{\left(\int R(K',X')\|\Psi(K',X')\|^{2}d(K',X')\right)^{2}}F_{1}'\left(\frac{R(K,X)}{\int R(K',X')\|\Psi(K',X')\|^{2}d(K',X')},\Gamma(K,X)\right)\right)\\
&\times\hat{F}_{2}(s,R(K,X))\left\|\Psi\left(K,\hat{X}\right)\right\|^{2}
\end{aligned}\tag{424}$$

where we define the deviation ΔY of a quantity by the difference:

$$\Delta Y=Y-\langle Y\rangle\tag{425}$$

with $\langle Y\rangle$, the average of Y:

$$\langle Y\rangle=\int Y(K,X)\,dKdX$$

Thus we write:

$$\begin{aligned}
&\Delta\left(r(K',X)-\gamma\frac{\int K'\left\|\Psi(K',X)\right\|^2}{K'}+F_1\left(\frac{R(K',X)}{\int R(K',X')\left\|\Psi(K',X')\right\|^2 d(K',X')},\Gamma(K,X)\right)\right)\\
=&\left(r(K',X)-\gamma\frac{\int K'\left\|\Psi(K',X)\right\|^2}{K'}+F_1\left(\frac{R(K',X)}{\int R(K',X')\left\|\Psi(K',X')\right\|^2 d(K',X')},\Gamma(K,X)\right)\right)\\
&-\left\langle\left(r(K',X)-\gamma\frac{\int K'\left\|\Psi(K',X)\right\|^2}{K'}+F_1\left(\frac{R(K',X)}{\int R(K',X')\left\|\Psi(K',X')\right\|^2 d(K',X')},\Gamma(K,X)\right)\right)\right\rangle
\end{aligned}$$

and in first approximation, (424) reduces to:

$$\begin{aligned}
&\frac{\delta^2 f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\,\delta\Psi^\dagger(K',X)}\\
\simeq&\frac{1}{\varepsilon}\left(\Delta\left(r(K',X)-\gamma\frac{K_X}{K'}+F_1\left(\frac{R(K',X)}{\int R(K',X')\left\|\Psi(K',X')\right\|^2 d(K',X')},\Gamma(K,X)\right)\right)-\gamma\frac{K'}{K_X}\right)\\
\simeq&\frac{1}{\varepsilon}\left(\Delta f\left(K',\hat{X},\Psi,\hat{\Psi}\right)-\gamma\frac{K'}{K_X}\right)
\end{aligned}$$

where:

$$\Delta f\left(K',\hat{X},\Psi,\hat{\Psi}\right)=f\left(K',\hat{X},\Psi,\hat{\Psi}\right)-f\left(K_{\hat{X}},\hat{X},\Psi,\hat{\Psi}\right)$$

is the relative short-term return for firm with capital K' at sector $\hat{X}$.

Similarly, the second derivative for $g\left(\hat{X},\Psi,\hat{\Psi}\right)$ is:

$$\begin{aligned}
&\frac{\delta^2 g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\,\delta\Psi^\dagger(K',X)}\\
=&\frac{1}{\int\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\Delta\left(\nabla_{\hat{X}}F_0\left(R\left(K',\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K',\hat{X}\right)}{\int R(K',X')\left\|\Psi(K',X')\right\|^2 d(K',X')}\right)\right)\\
=&\frac{1}{\int\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\Delta\left(g\left(K',\hat{X},\Psi,\hat{\Psi}\right)\right)
\end{aligned}$$

with:

$$\begin{aligned}
&\Delta\left(\nabla_{\hat{X}}F_0\left(R\left(K',\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K',\hat{X}\right)}{\int R(K',X')\left\|\Psi(K',X')\right\|^2 d(K',X')}\right)\right)\\
=&\left(\nabla_{\hat{X}}F_0\left(R\left(K',\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K',\hat{X}\right)}{\int R(K',X')\left\|\Psi(K',X')\right\|^2 d(K',X')}\right)\right)\\
&-\left\langle\nabla_{\hat{X}}F_0\left(R\left(K',\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K',\hat{X}\right)}{\int R(K',X')\left\|\Psi(K',X')\right\|^2 d(K',X')}\right)\right\rangle
\end{aligned}$$

in other words:

$$\Delta g\left(K',\hat{X},\Psi,\hat{\Psi}\right)=g\left(K',\hat{X},\Psi,\hat{\Psi}\right)-g\left(\hat{X},\Psi,\hat{\Psi}\right)$$

is the relative long-term return for firm with capital K' at sector $\hat{X}$.

Appendix 8: "free" transition functions

Given the second-order operator arising in the expansion for the fields' action:

$$O\left(\Psi_0\left(Z,\theta\right)\right)\simeq\left(\begin{array}{cc}\frac{\delta^2(S_1+S_2)}{\delta\Psi^\dagger(Z,\theta)\delta\Psi(Z,\theta)} & 0\\ 0 & \frac{\delta^2(S_3(\Psi)+S_4(\Psi))}{\delta\hat{\Psi}^\dagger(Z,\theta)\delta\hat{\Psi}(Z,\theta)}\end{array}\right)_{\substack{\Psi(Z,\theta)=\Psi_0(Z,\theta)\\ \hat{\Psi}(Z,\theta)=\hat{\Psi}_0(Z,\theta)}}\tag{426}$$

The transition functions for the individual firms:

$$G_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),\alpha\right)$$

and investors:

$$G_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right),\alpha\right)$$

satisfy:

$$\begin{aligned}\left(\frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^\dagger\left(Z,\theta\right)\delta\Psi\left(Z,\theta\right)}+\alpha\right)G_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),\alpha\right) &= \delta\left(\left(K_f,X_f\right)-\left(X_i,K_i\right)\right)\\ \left(\frac{\delta^2\left(S_3\left(\Psi\right)+S_4\left(\Psi\right)\right)}{\delta\hat{\Psi}^\dagger\left(Z,\theta\right)\delta\hat{\Psi}\left(Z,\theta\right)}+\alpha\right)G_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right),\alpha\right) &= \delta\left(\left(\hat{K}_f,\hat{X}_f\right)-\left(\hat{X}_i,\hat{K}_i\right)\right)\end{aligned}$$

The functions $G_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),\alpha\right)$ and $G_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right),\alpha\right)$ are the Laplace transforms of the following transition functions:

$$T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),t\right)$$

$$T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right),t\right)$$

satisfying:

$$-\frac{\partial}{\partial t}T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),t\right)=\left(\frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^\dagger\left(Z,\theta\right)\delta\Psi\left(Z,\theta\right)}\right)T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),t\right)\tag{427}$$

$$-\frac{\partial}{\partial t}T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right),t\right)=\left(\frac{\delta^2\left(S_3\left(\Psi\right)+S_4\left(\Psi\right)\right)}{\delta\hat{\Psi}^\dagger\left(Z,\theta\right)\delta\hat{\Psi}\left(Z,\theta\right)}\right)T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right),t\right)\tag{428}$$

A8.1 Approximations to and (427) and (428)

We consider some approximations to find the solutions of equations (412) and (421). We first assume that:

$$\frac{\nabla_K\frac{F_2(R(K,X))}{\langle F_2(R(K,X))\rangle_K}K_X}{2}<<1$$

so that:

$$\begin{aligned}K-\hat{F}_2\left(s,R\left(K,X\right)\right)K_X &\simeq K-\hat{F}_2\left(R\left(K_X,X\right)\right)K_X-\nabla_{K_X}\hat{F}_2\left(R\left(K_X,X\right)\right)\left(K-K_X\right)\\ &\simeq K-\hat{F}_2\left(R\left(K_X,X\right)\right)K_X\end{aligned}$$

Equation (412) then simplifies as:

$$\begin{aligned}\frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^\dagger\left(Z,\theta\right)\delta\Psi\left(Z,\theta\right)} &= -\frac{\sigma_X^2}{2}\nabla_X^2-\frac{\sigma_K^2}{2}\nabla_K^2+\left(D\left(\left\Vert\Psi\right\Vert^2\right)+2\tau\frac{\left|\Psi\left(X\right)\right|^2\left(K_X-K\right)}{K}\right)\\ &\quad+\frac{1}{2\sigma_K^2}\left(K-\hat{F}_2\left(s,R\left(K_X,X\right)\right)K_X\right)^2\end{aligned}\tag{429}$$

and equation (427) becomes:

$$\begin{aligned}
&-\frac{\partial}{\partial t}T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),t\right) &(430)\\
=\;&\left(-\frac{\sigma_X^2}{2}\nabla_X^2+D\left(\left\|\Psi\right\|^2\right)+2\tau\frac{K_X-K}{K}\left\|\Psi\left(X\right)\right\|^2\right)T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),t\right)\\
&+\left(-\frac{\sigma_K^2}{2}\nabla_K^2+\frac{1}{2\sigma_K^2}\left(K-\hat{F}_2\left(R\left(K_X,X\right)\right)K_X\right)^2\right)T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),t\right)
\end{aligned}$$

Second, we assumed from the beginning that the motion of firms in the sectors space is at slower pace than capital fluctuations. Moreover, we may assume that in average $\left|\frac{K_X-K}{K}\right| << 1$. As a consequence, along the path from the initial point (X_i,K_i) to the final point (K_f,X_f), we can consider that:

$$\frac{K_X-K}{K}\left\|\Psi\left(X\right)\right\|^2$$

is slowly varying and can be replaced by its average.

The equation for T_1 thus rewrites:

$$\begin{aligned}
&-\frac{\partial}{\partial t}T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right) &(431)\\
=\;&\left(-\frac{\sigma_X^2}{2}\nabla_X^2+D\left(\left\|\Psi\right\|^2\right)+\tau\left(\frac{\left|\Psi\left(X_f\right)\right|^2\left(K_{X_f}-K_f\right)}{K_f}+\frac{\left|\Psi\left(X_i\right)\right|^2\left(K_{X_i}-K_i\right)}{K_i}\right)\right)T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)\\
&+\left(-\frac{\sigma_K^2}{2}\nabla_K^2+\frac{1}{2\sigma_K^2}\left(K-\hat{F}_2\left(R\left(K_X,X\right)\right)K_X\right)^2\right)T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)
\end{aligned}$$

On the other hand, the derivation of the equation for T_2 yields directly:

$$\begin{aligned}
&-\frac{\partial}{\partial t}T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right) &(432)\\
=\;&\left(-\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2-\nabla_y^2\right)T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)\\
&+\left(\frac{y^2}{4}+\frac{\left(g\left(\hat{X}\right)\right)^2+\sigma_{\hat{X}}^2\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}}\right)T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)
\end{aligned}$$

A8.2 Computation of T_1

13.0.3 A8.2.1 Solution of (431)

We first rewrite the competition term in (431) as:

$$\begin{aligned}
&\frac{\left|\Psi\left(X_f\right)\right|^2\left(K_{X_f}-K_f\right)}{K_f}+\frac{\left|\Psi\left(X_i\right)\right|^2\left(K_{X_i}-K_i\right)}{K_i}\\
=\;&\left(\frac{\left|\Psi\left(X_f\right)\right|^2\left(K_{X_f}-K_f\right)}{K_f}-\frac{\left|\Psi\left(X_i\right)\right|^2\left(K_{X_i}-K_i\right)}{K_i}\right)+\frac{\left|\Psi\left(X_i\right)\right|^2\left(K_{X_i}-K_i\right)}{K_i}
\end{aligned}$$

Then, we normalize the transition functions by factoring the solution of (431):

$$\begin{aligned} & T_1\left(\left(K_f, X_f\right),\left(X_i, K_i\right)\right) && (433) \\ = & \exp\left(-t\left(D\left(\|\Psi\|^2\right)+\tau\left(\frac{\left|\Psi\left(X_f\right)\right|^2\left(K_{X_f}-K_f\right)}{K_f}+\frac{\left|\Psi\left(X_i\right)\right|^2\left(K_{X_i}-K_i\right)}{K_i}\right)\right)\right)\hat{T}_1\left(\left(K_f, X_f\right),\left(X_i, K_i\right)\right) \end{aligned}$$

so that the transition equation writes:

$$\begin{aligned} & -\frac{\partial}{\partial t}\hat{T}_1\left(\left(K_f, X_f\right),\left(X_i, K_i\right)\right) && (434) \\ = & \left(-\frac{\sigma_X^2}{2}\nabla_X^2-\frac{\sigma_K^2}{2}\nabla_K^2+\frac{1}{2\sigma_K^2}\left(K-\hat{F}_2\left(R\left(K, X\right)\right)K_X\right)^2\right)\hat{T}_1\left(\left(K_f, X_f\right),\left(X_i, K_i\right)\right) \end{aligned}$$

Note that, given the exponential factor, if $K_i << K_{X_i}$, $\frac{\left|\Psi(X_i)\right|^2\left(K_{X_i}-K_i\right)}{K_i}<0$ and the probability to move away from X_i is very low. The same applies for $\frac{\left|\Psi(X_f)\right|^2\left(K_{X_f}-K_f\right)}{K_f}>0$.

The transition function $\hat{T}_1\left(\left(K_f, X_f\right),\left(X_i, K_i\right)\right)$ can be found by using our assumption that shifts in sectors space are slower than the fluctuations in capital. In (434) we can thus consider in first approximation that the term:

$$K-\hat{F}_2\left(R\left(K, X\right)\right)K_X$$

shifts the inital and final values of capital:

$$\begin{aligned} K_i & \rightarrow K_i-\hat{F}_2\left(s, R\left(K_{X_i}, X_i\right)\right)K_{X_i}=K_i' \\ K_f & \rightarrow K_f-\hat{F}_2\left(s, R\left(K_{X_f}, X_f\right)\right)K_{X_f}=K_f' \end{aligned}$$

So that we have:

$$\hat{T}_1\left(\left(K_f, X_f\right),\left(X_i, K_i\right)\right)\simeq\tilde{T}_1\left(\left(K_f-\hat{F}_2\left(s, R\left(K_{X_f}, X_f\right)\right)K_{X_f}, X_f\right),\left(K_i-\hat{F}_2\left(R\left(s, K_{X_i}, X_i\right)\right)K_{X_i}, K_i\right)\right) \tag{435}$$

where $\tilde{T}_1$ satisfies:

$$-\frac{\partial}{\partial t}\tilde{T}_1=\left(-\frac{\sigma_X^2}{2}\nabla_X^2-\frac{\sigma_K^2}{2}\nabla_K^2+\frac{1}{2\sigma_K^2}K^2\right)\tilde{T}_1 \tag{436}$$

Up to a normalization factor, the solution of (436) is:

$$\tilde{T}_1\left(\left(K_f', X_f\right),\left(X_i, K_i'\right)\right)=\exp\left(-\left(\frac{\left(X_f-X_i\right)^2}{2t\sigma_X^2}+\frac{\left(K_f'-K_i'\right)^2}{2t\sigma_K^2}-\frac{t\sigma_K^2}{2}K_f'K_f'\right)\right)$$

Using (435) and (433), we find the solution of (165):

$$\begin{aligned} & \exp\left(-t\left(D\left(\|\Psi\|^2\right)+\tau\left(\frac{\left|\Psi\left(X_f\right)\right|^2\left(K_{X_f}-K_f\right)}{K_f}+\frac{\left|\Psi\left(X_i\right)\right|^2\left(K_{X_i}-K_i\right)}{K_i}\right)\right)\right) && (437) \\ & \times\exp\left(-\left(\frac{\left(X_f-X_i\right)^2}{2t\sigma_X^2}+\frac{\left(K_f'-K_i'\right)^2}{2t\sigma_K^2}-\frac{t\sigma_K^2}{2}K_f'K_f'\right)\right) \end{aligned}$$

13.0.4 A8.2.2 Full transition function

To obtain the full transition function, recall that (437) has been obtained by a change of variable (407). To come back to the initial variables we have to introduce an other exponential factor to account for the trend of the transition, and we find:

$$\begin{aligned}
&T_1\left(\left(K_f, X_f\right), \left(X_i, K_i\right)\right) \\
\simeq\ & \exp\left(\int_{X_i}^{X_f} \frac{\nabla_X R\left(K_X, X\right)}{\sigma_X^2} H\left(K_X\right) - t\left(D\left(\|\Psi\|^2\right) + \tau\left(\frac{\left|\Psi\left(X_f\right)\right|^2 \left(K_{X_f} - K_f\right)}{K_f} + \frac{\left|\Psi\left(X_i\right)\right|^2 \left(K_{X_i} - K_i\right)}{K_i}\right)\right)\right) \\
&\times \exp\left(-\int^{K_f}\left(K - \hat{F}_2\left(R\left(s, K, X_f\right)\right) K_{X_f}\right) dK + \int^{K_i}\left(K - \hat{F}_2\left(s, R\left(K, X_i\right)\right) K_{X_i}\right) dK\right) \\
&\times \exp\left(-\left(\frac{\left(X_f - X_i\right)^2}{2t\sigma_X^2} + \frac{\left(\tilde{K}_f - \tilde{K}_i\right)^2}{2t\sigma_K^2}\right)\right) \\
&\times \exp\left(-\frac{t\sigma_K^2}{2}\left(K_f - \hat{F}_2\left(s, R\left(K_f, \bar{X}\right)\right) K_{\bar{X}}\right)\left(K_i - \hat{F}_2\left(s, R\left(K_i, \bar{X}\right)\right) K_{\bar{X}}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&T_1\left(\left(K_f, X_f\right), \left(X_i, K_i\right)\right) \\
\simeq\ & \exp\left(\int_{X_i}^{X_f} \frac{\nabla_X R\left(K_X, X\right)}{\sigma_X^2} H\left(K_X\right) - t\left(D\left(\|\Psi\|^2\right) + \tau\left(\frac{\left|\Psi\left(X_f\right)\right|^2 \left(K_{X_f} - K_f\right)}{K_f} + \frac{\left|\Psi\left(X_i\right)\right|^2 \left(K_{X_i} - K_i\right)}{K_i}\right)\right)\right) \\
&\times \exp\left(-\int^{K_f}\left(K - \hat{F}_2\left(s, R\left(K, X_f\right)\right) K_{X_f}\right) dK + \int^{K_i}\left(K - \hat{F}_2\left(s, R\left(K, X_i\right)\right) K_{X_i}\right) dK\right) \\
&\times \exp\left(-\left(\frac{\left(X_f - X_i\right)^2}{2t\sigma_X^2} + \frac{\left(K'_f - K'_i\right)^2}{2t\sigma_K^2} - \frac{t\sigma_K^2}{2} K'_f K'_f\right)\right)
\end{aligned} \tag{438}$$

with:

$$\begin{aligned}
K'_i &= K_i - \hat{F}_2\left(R\left(K_{X_i}, X_i\right)\right) K_{X_i} \\
K'_f &= K_f - \hat{F}_2\left(R\left(K_{X_f}, X_f\right)\right) K_{X_f}
\end{aligned}$$

The Laplace transform of this function is the transition function given in the text.

A8.3 Computation of T_2

13.0.5 A8.3.1 Solution ot (432)

Solving (432) is straightforward, and similar to the derivation T_1.

We first introduce a change of variable:

$$\begin{aligned}
&T_2\left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right)\right) \\
=\ & \exp\left(-t\int_{\hat{X}_i}^{\hat{X}_f} \frac{\left(g\left(\hat{X}\right)\right)^2 + \sigma_{\hat{X}}^2\left(f\left(\hat{X}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{\left\|\hat{X}_f - \hat{X}_i\right\| \sigma_{\hat{X}}^2 \sqrt{f^2\left(\hat{X}\right)}}\right) \hat{T}_2\left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right)\right)
\end{aligned}$$

The term in the exponential is the average of the relative return:

$$\frac{\left(g\left(\hat{X}\right)\right)^2+\sigma_{\hat{X}}^2\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}}$$

along the average path, considered as a straight line, from $\hat{X}_i$ to $\hat{X}_f$. We have assumed slow shifts in the sectors space, so that $\hat{T}_2$ satisfies the following equation in first approximation:

$$\begin{aligned}&-\frac{\partial}{\partial t}\hat{T}_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right) \\ \simeq\ &\left(-\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2-\nabla_y^2\right)\hat{T}_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)+\frac{y^2}{4}\hat{T}_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)\end{aligned} \tag{439}$$

Given (420), we can assume that y is independent from $\hat{X}$ in first approximation. Thus, solving (439) yields:

$$\begin{aligned}&\hat{T}_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right) \\ \simeq\ &\exp\left(-\left(\frac{\sigma_{\hat{X}}^2}{2}t\left(\hat{K}_f+\frac{\sigma_{\hat{K}}^2F\left(\hat{X}_f,K_{\hat{X}_f}\right)}{f^2\left(\hat{X}_f\right)}\right)\left(\hat{K}_i+\frac{\sigma_{\hat{K}}^2F\left(\hat{X}_i,K_{\hat{X}_i}\right)}{f^2\left(\hat{X}_i\right)}\right)\right)\right) \\ &\times\exp\left(-\frac{\sqrt{f^2\left(\frac{\hat{X}_f+\hat{X}_i}{2}\right)}}{2t\sigma_{\hat{X}}^2}\left(\left(\hat{K}_f+\frac{\sigma_{\hat{K}}^2F\left(\hat{X}_f,K_{\hat{X}_f}\right)}{f^2\left(\hat{X}_f\right)}\right)-\left(\hat{K}_i+\frac{\sigma_{\hat{K}}^2F\left(\hat{X}_i,K_{\hat{X}_i}\right)}{f^2\left(\hat{X}_i\right)}\right)\right)^2\right)\end{aligned} \tag{440}$$

13.0.6 A8.3.2 Full transition function

Reintroducing the change of variables (416) amounts to introduce a factor:

$$\exp\left(\frac{1}{\sigma_{\hat{X}}^2}\int_{\hat{X}_i}^{\hat{X}_f}g\left(\hat{X}\right)d\hat{X}+\frac{\hat{K}_f^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}_f\right)-\frac{\hat{K}_i^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}_i\right)\right)$$

in the formula for T_2 and this leads to the full formula for the transition function:

$$\begin{aligned}&T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right) \\ \simeq\ &\exp\left(-t\int_{\hat{X}_i}^{\hat{X}_f}\frac{\left(g\left(\hat{X}\right)\right)^2+\sigma_{\hat{X}}^2\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{\left\Vert\hat{X}_f-\hat{X}_i\right\Vert\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}}\right) \\ &\times\exp\left(\frac{1}{\sigma_{\hat{X}}^2}\int_{\hat{X}_i}^{\hat{X}_f}g\left(\hat{X}\right)d\hat{X}+\frac{\hat{K}_f^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}_f\right)-\frac{\hat{K}_i^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}_i\right)\right)\hat{T}_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)\end{aligned} \tag{441}$$

with:

$$\hat{T}_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)$$

given by (440).

The Laplace transform of (441) is the formula presented in the text.

Appendix 9

We write the series expansion in $\Delta S_{\text{fourth order}}$ of $\exp(-S(\Psi))$:

$$\begin{aligned}
\exp(-S(\Psi)) &= \exp\left(-\left(S\left(\Psi_0,\hat{\Psi}_0\right)+\int\left(\Delta\Psi^{\dagger}(Z,\theta),\Delta\hat{\Psi}^{\dagger}(Z,\theta)\right)(Z,\theta)\,O\left(\Psi_0(Z,\theta)\right)\begin{pmatrix}\Delta\Psi(Z,\theta)\\ \Delta\hat{\Psi}(Z,\theta)\end{pmatrix}\right)\right)\\
&\left(1+\sum_{n\geqslant 1}\frac{\left(-\Delta S_{\text{fourth order}}\left(\Psi,\hat{\Psi}\right)\right)^n}{n!}\right)
\end{aligned}$$

where $O(\Psi_0(Z,\theta))$ is defined in (112).

Then, we decompose $\Delta S_{\text{fourth order}}\left(\Psi,\hat{\Psi}\right)$ as a sum of two combinations:

$$\begin{aligned}
\Delta S_{\text{fourth order}}\left(\Psi,\hat{\Psi}\right) &= \int\Delta\Psi^{\dagger}(K,X)\,\Delta\Psi^{\dagger}(K',X')\,\Delta S_{11}\Delta\Psi(K',X')\,\Delta\Psi(K,X)\\
&+\Delta\Psi^{\dagger}(K',X')\,\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\Delta S_{12}\Delta\Psi(K',X')\,\Delta\hat{\Psi}\left(\hat{K},\hat{X}\right)
\end{aligned}$$

with:

$$\Delta S_{11}=\left(2\tau-\nabla_K\frac{\delta^2 u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\,\delta\Psi^{\dagger}(K',X)}\right)\delta(X-X')$$

and:

$$\Delta S_{12}=-\left(\nabla_K\frac{\delta^2 u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}+\left\{\nabla_{\hat{K}}\frac{\hat{K}\delta^2 f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\,\delta\Psi^{\dagger}(K',X)}+\nabla_{\hat{X}}\frac{\delta^2 g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi(K',X)\,\delta\Psi^{\dagger}(K',X)}\right\}\right)\delta(X-X')$$

Application of (31) leads to the following form of the transition functions:

$$\begin{aligned}
&G_{ij}\left(\left[(K_f,X_f),(K_f,X_f)'\right],\left[(X_i,K_i),(X_i,K_i)'\right]\right) \qquad (442)\\
=\;&G_i\left((K_f,X_f),(X_i,K_i)\right)G_j\left((K_f,X_f)',(X_i,K_i)'\right)\\
&+\sum_{p\geqslant 1}\frac{(-1)^p}{p!}\int G_i\left((K_f,X_f),(X_p,K_p)\right)G_j\left((K_f,X_f)',(X_p,K_p)'\right)\Delta S_{ij}\left((X_p,K_p),(X_p,K_p)'\right)\\
&\times G_i\left((X_p,K_p),(X_{p-1},K_{p-1})\right)G_j\left((X_p,K_p)',(X_{p-1},K_{p-1})'\right)\Delta S_{ij}\left((X_{p-1},K_{p-1}),(X_{p-1},K_{p-1})'\right)\\
&...\times\Delta S_{ij}\left((K_1,X_1),(K_1,X_1)'\right)G_1\left((K_1,X_1),(X_i,K_i)\right)G_1\left((K_1,X_1)',(X_i,K_i)'\right)\prod_{k\leqslant p}d\left((X_k,K_k),(X_k,K_k)'\right)
\end{aligned}$$

These corrections modify the n agents Green functions and can be computed using graphs expansion. In the sequel we will focus only on the first order corrections to the four agents Green functions. This is sufficient to stress the impact of interactions of agents in the background state.

The term ΔS_{11} measures the interaction between firms, and ΔS_{12} the firms-investors interactions. There is no term ΔS_{22} of investors-investors interaction. In our model all interactions depend on firms.

To estimate the impact of interactions, we can assume the paths from $((X_i,K_i),(K_f,X_f))$ to $((X_i,K_i),(K_f,X_f)')$ cross each other one time at some X and approximate the terms ΔS_{ij} by their average value estimated on the average paths from K_i,K_i' to K_f,K_f', In this approximation, we find:

$$\begin{aligned}
& G_{ij}\left(\left[(K_f,X_f),(K_f,X_f)'\right],\left[(X_i,K_i),(X_i,K_i)'\right]\right) \\
\simeq\ & G_i\left((K_f,X_f),(X_i,K_i)\right)G_j\left((K_f,X_f)',(X_i,K_i)'\right) \\
& -G_i\left((K_f,X_f),(X,K)\right)G_j\left((K_f,X_f)',(X,K)'\right) \\
& \times\Delta S_{ij}\left(\left(X,\bar{K}\right),\left(X,\bar{K}\right)'\right)G_1\left((X,K),(X_i,K_i)\right)G_1\left((X,K)',(X_i,K_i)'\right) \\
\simeq\ & G_i\left((K_f,X_f),(X_i,K_i)\right)G_j\left((K_f,X_f)',(X_i,K_i)'\right) \\
& -\Delta S_{ij}\left(\left(\bar{X},\bar{K}\right),\left(\bar{X},\bar{K}\right)'\right)\hat{G}_i\left((K_f,X_f),(X,K)\right)\hat{G}_j\left((K_f,X_f)',(X,K)'\right)
\end{aligned}$$

with:

$$\begin{aligned}
\left(\bar{X},\bar{K}\right) &= \frac{(K_f,X_f)+(X_i,K_i)}{2} \\
\left(\bar{X},\bar{K}\right)' &= \frac{(K_f,X_f)'+(X_i,K_i)'}{2}
\end{aligned}$$

and:

$$\hat{G}_i\left((K_f,X_f),(X,K)\right)\hat{G}_j\left((K_f,X_f)',(X,K)'\right)$$

is the transition function computed on path that cross once. Applied to the three transition functions for two agents yields the results of the text.

Appendix 10

One agent transition functions

A10.1 Firms transition function

We interpret the various term involved in (121) and their influence on firms individual dynamics.

A10.1.1 Drift term

The three contributions The first term in (121):

$$D\left((K_f,X_f),(X_i,K_i)\right)$$

is a drift term between (X_i,K_i) and (K_f,X_f). It is composed of three contributions (see (340)):

The first term of (340):

$$\int_{X_i}^{X_f}\frac{\nabla_X R\left(K_X,X\right)}{\sigma_X^2}H\left(K_X\right)$$

models the shift of producers towards sectors that have the highest long-term returns.

To interpret the second contribution to $D\left((K_f,X_f),(X_i,K_i)\right)$:

$$\int_{K_i}^{K_f}\left(K-\hat{F}_2\left(s,R\left(K,\bar{X}\right)\right)K_{\bar{X}}\right)dK \tag{443}$$

, recall that $\frac{F_2\left(R\left(K,\bar{X}\right)\right)}{F_2\left(R\left(K_{\bar{X}},\bar{X}\right)\right)}$ models the relative expectations of returns of the firm along its path from (X_i,K_i) to (K_f,X_f) based on their returns' expectation $R\left(K,\bar{X}\right)$ and that $\frac{F_2\left(R\left(K,\bar{X}\right)\right)K_{\bar{X}}}{F_2\left(R\left(K_{\bar{X}},\bar{X}\right)\right)}$ represents the capital investors are ready to invest in the firm along this path. Along the path from (X_i, K_i) to (K_f, X_f), the capital invested in the firm will increase as long as the investors expect growth and as long as additional investment is likely to increase the firm's returns. Once the level of capital reaches their expectations, that is

$$K_T\left(\bar{X}\right) - \hat{F}_2\left(R\left(K_T, \bar{X}\right)\right) K_{\bar{X}} = 0 \tag{444}$$

i.e., when the firm has reached the capital threshold, investment stops.

However, this condition is not always fulfilled. The shape of F_2 is critical. If F_2 is above the line $Y = K$, then for $K < K_T$, the threshold K_T will be reached gradually. In this case, K_T is an equilibrium point. If, on the contrary, F_2 is below the line $Y = K$, then for $K < K_T$, the threshold K_T will never be reached, and $K \to 0$. If $K > K_T$, K can increase indefinitely. This corresponds to firms whose profitability is perceived as boundless as long as more capital is invested in.

The third term in (340):

$$\int_{K_i}^{K_f} \left(\left(\frac{X_f - X_i}{2}\right) \nabla_X \hat{F}_2\left(s, R\left(K, \bar{X}\right)\right) K_{\bar{X}}\right) dK$$

induces firms to move towards more appropriate sectors, according to investors and given the capital of the firm. The firm does not solely move according to the new investors it could attract but must also take into account its current investors. If it moves, it risks losing the investors it has already attracted.

Trade-off between terms There is a trade-off between the first and the third terms: firms want to move towards sectors with higher returns, but differences in average capital between sectors could make a firm unattractive in a new sector. The loss of investors **incured** during a shift of sector must be compared with the number of investors possibly attracted in the new sector: the level of attractiveness may decrease for a given amount of capital.

The second contribution to D is an indicator of the firm's growth potential in a given sector. It depends on the firm's level of capital compared to the threshold capital requirement and its dynamics in this sector.

A move along sectors due to the terms 1 and 3 modifies the firm's relative capital, which is sector-dependant: F_2, measuring the firm attractiveness in the sector and indirectly the threshold of capital K_T defined in (444) are modified by the shift from one sector to another. Therefore, a firm could be below the value of K_T in one sector, then above in the next sector, which will reverse its capital dynamics. The firm's capital dynamics remains the same as long as its relative attractiveness in a sector does not change significantly.

A10.1.2 Effective time of transition

The following term depends on the competition in a sector:

$$\begin{aligned}\alpha_{eff}\left(\Psi, \left(K_f, X_f\right), \left(X_i, K_i\right)\right) &= \alpha + D\left(\|\Psi\|^2\right) + \frac{\tau}{2}\left(\frac{\left|\Psi\left(X_f\right)\right|^2\left(K_{X_f} - K_f\right)}{K_f} - \frac{\left|\Psi\left(X_i\right)\right|^2\left(K_{X_i} - K_i\right)}{K_i}\right) \\ &\quad + \frac{\sigma_K^2}{2}\left(K_f - \hat{F}_2\left(s, R\left(K_f, \bar{X}\right)\right) K_{\bar{X}}\right)\left(K_i - \hat{F}_2\left(s, R\left(K_i, \bar{X}\right)\right) K_{\bar{X}}\right)\end{aligned} \tag{445}$$

Recall that the constant α is the inverse of the average lifetime of the agents. The larger it is, the lower the probability of transition. In the transition functions, α is shifted by two path-dependent terms and replaced by α_{eff}. Thus, α_{eff} is the inverse mobility of the firm during its transition from one point to another.

Therefore, the likelihood of shifts in capital and sectors depends not only on the average lifespan of the firm, but also on terms that are directly related to the collective state.

The first correction to α is $D\left(\|\Psi\|^2\right)$ that is related to competition. We have shown in (Gosselin Lotz Wambst 2022):

$$D\left(\|\Psi\|^2\right) \simeq 2\tau \frac{N}{V-V_0} + \frac{1}{2\sigma_X^2}\left\langle\left(\nabla_X R\left(X\right)\right)^2\right\rangle_{V/V_0} H^2\left(\frac{\left\langle\hat{K}\right\rangle}{N}\right)\left(1-\frac{H'\left(\frac{\left\langle\hat{K}\right\rangle}{N}\right)}{H\left(\frac{\left\langle\hat{K}\right\rangle}{N}\right)}\frac{\left\langle\hat{K}\right\rangle}{N}\right)$$

where V is the volume of the sectors space and V_0 is the locus where $\|\Psi\left(X\right)\|^2 = 0$. As a consequence, the stronger the competition, i.e., the larger τ, the greater $D\left(\|\Psi\|^2\right)$, and the less possibilities of shifting from a sector to another.

The third term in (445):

$$\frac{\tau}{2}\left(\frac{\left|\Psi\left(X_f\right)\right|^2\left(K_{X_f}-K_f\right)}{K_f}-\frac{\left|\Psi\left(X_i\right)\right|^2\left(K_{X_i}-K_i\right)}{K_i}\right) \tag{446}$$

is also linked to the competition, but depends on the level of capital of the agent, and the number of d agents in the sectors crossed. This term measures the strength of agent's shift from one sector to another.

It is negative when $K_f > K_{X_f}$, and when $K_i < K_{X_i}$. In other words, when a firm has less capital than the average in its initial sector, and when, it ends up in a sector in which it has more capital than the average, the probability of transition from K_i to K_f is high. In other words, under-average capital favors the exit from a sector. Above-average capital promotes entry into the sector. Shifts from high average capital sectors to lower-average-capital sectors are favoured.

This phenomenon is amplified by the number of agents. The greater the competition in a sector, i.e., the more firms in the sector, the greater the probability for a lower-than average capitalized firm to be ousted from the sector by higher-than average capitalized firms that enter the sector. Thus, the density of producers $\left|\Psi\left(X\right)\right|^2$ along the movement enhances competition and favours high capitalized firm to move towards higher capitalized sectors, and drives low capitalized firms toward low capitalized sectors.

The last term in (445):

$$\frac{\sigma_K^2}{2}\left(K_f-\hat{F}_2\left(s,R\left(K_f,\bar{X}\right)\right)K_{\bar{X}}\right)\left(K_i-\hat{F}_2\left(s,R\left(K_i,\bar{X}\right)\right)K_{\bar{X}}\right)$$

shows that in average, shifts from the initial point to the final point is done respecting $K = \hat{F}_2\left(s,R\left(K,\bar{X}\right)\right)K_{\bar{X}}$, the capital investors allocate to the firm. Actually, if $K_i-\hat{F}_2\left(s,R\left(K_i,\bar{X}\right)\right)K_{\bar{X}} < 0$, there is a higher probability to reach $K_f-\hat{F}_2\left(s,R\left(K_f,\bar{X}\right)\right)K_{\bar{X}} > 0$. If $K_i-\hat{F}_2\left(s,R\left(K_i,\bar{X}\right)\right)K_{\bar{X}} > 0$, there is a higher probability to reach $K_f-\hat{F}_2\left(s,R\left(K_f,\bar{X}\right)\right)K_{\bar{X}} < 0$.

If a firm starts with a capital lower than $\hat{F}_2\left(s,R\left(K,\bar{X}\right)\right)K_{\bar{X}}$, it is more likely to end up with capital above the new $\hat{F}_2\left(s,R\left(K,\bar{X}\right)\right)K_{\bar{X}}$, in another sector. The movement induces a change in the F_2, but firms tend to fluctuate around the F_2 of transition.

A10.1.3 Fluctuation terms

The term:

$$\sqrt{\frac{(X_f - X_i)^2}{2\sigma_X^2} + \frac{\left(\tilde{K}_f - \tilde{K}_i\right)^2}{2\sigma_K^2}}$$

describes oscillations around:

$$K_f - \hat{F}_2\left(s, R\left(K_f, \bar{X}\right)\right) K_{\bar{X}} + K_i - \hat{F}_2\left(s, R\left(K_i, \bar{X}\right)\right) K_{\bar{X}} = 0$$

On average, during a transition from the initial point to the final point, the firm's capital is governed by the equation $K = K_T = \hat{F}_2\left(R\left(K_i, \bar{X}\right)\right) K_{\bar{X}}$. If a firm starts with less capital than the threshold set by F_2, it is more likely to end up with a capital above the new F_2, in another sector. The transition from one sector to another involves oscillations around the sector-dependent threshold F_2. However, these oscillations may affect the final destination of the transition. Starting with a capital level above K_T, i.e. $K > K_T$, the firm may shift towards sectors with higher perspectives, i.e. with a higher threshold K_T. This favors in turn accumulation and faster transition to other sectors. Finally, if $\nabla_K \hat{F}_2\left(s, R\left(\frac{K_f+K_i}{2}, \bar{X}\right)\right) > 0$, F_2 is highly responsive to changes in capital, and larger moves are favored.

A10.2 Investors transition functions

We interpret the different contributions in (127).

A10.2.1 Drift term

The term $D'\left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right)\right)$ is composed of two contributions. The first one:

$$\frac{1}{\sigma_{\hat{X}}^2} \int_{\hat{X}_i}^{\hat{X}_f} g\left(\hat{X}\right) d\hat{X} + \frac{\hat{K}_f^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}_f\right) - \frac{\hat{K}_i^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}_i\right)$$

is composed of two elements.

The first element is $\frac{1}{\sigma_{\hat{X}}^2} \int_{\hat{X}_i}^{\hat{X}_f} g\left(\hat{X}\right) d\hat{X}$. Since the function $g\left(\hat{X}\right)$ is the anticipation of higher returns and rising stock prices, investors move towards sectors were they anticipate the highest returns and stock prices increase.

The second element $\frac{\hat{K}_f^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}_f\right) - \frac{\hat{K}_i^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}_i\right)$: this element can be rewritten as two bits:

$$\frac{\hat{K}_f^2 - \hat{K}_i^2}{\sigma_{\hat{K}}^2} f\left(\frac{\hat{X}_f + \hat{X}_i}{2}\right) + \frac{\left(\hat{K}_f^2 + \hat{K}_i^2\right)}{2\sigma_{\hat{K}}^2} \left(\hat{X}_f - \hat{X}_i\right) \nabla_X f\left(\frac{\hat{X}_f + \hat{X}_i}{2}\right)$$

The first bit, $\frac{\hat{K}_f^2 - \hat{K}_i^2}{\sigma_{\hat{K}}^2} f\left(\frac{\hat{X}_f + \hat{X}_i}{2}\right)$ shows that the highest the short-term return, the more probable is the increase in capital. The second bit, which is equal to $\frac{\left(\hat{K}_f^2 + \hat{K}_i^2\right)}{2\sigma_{\hat{K}}^2} \left(\hat{X}_f - \hat{X}_i\right) \nabla_X f\left(\frac{\hat{X}_f + \hat{X}_i}{2}\right)$ indicates that investors move towards sectors with highest returns. The more capital they have, the fastest the shift.

The second term arising in $D'\left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right)\right)$:

$$-\frac{1}{\sigma_{\hat{X}}^2}\int_{\hat{X}_i}^{\hat{X}_f}\frac{\left(g\left(\hat{X}\right)\right)^2+\sigma_{\hat{X}}^2\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}}d\hat{X}$$

is similar to the determinant of capital accumulation in a collective state and has the same interpretation. There is a tradeoff between long-term and short-term returns. It further shows the importance of relative long-term return. Investors move towards relative long-term returns. Mathematically, we can measure the dependence of agents' capital accumulation in neighboring sectors using the integrand:

$$p=\frac{-\left(\frac{\left(g\left(\hat{X},K_{\hat{X}_M}\right)\right)^2}{\sigma_{\hat{X}}^2}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}_M}\right)-\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{f\left(\hat{X}\right)} \tag{447}$$

The function p represents the relative attractivity of a sector vis-a-vis its neighbours and depends on the gradients of long-term returns $R\left(\hat{X}\right)$ through the function $g\left(\hat{X}\right)$, the capital mobility at sector $\hat{X}$. This function $g\left(\hat{X}\right)$, which depicts investors' propensity to seek higher returns across sectors,and is indeed proportional to $\nabla_{\hat{X}}R\left(\hat{X}\right)$. The gradient of g, $\nabla_{\hat{X}}g$, is proportional to $\nabla_{\hat{X}}^2R\left(\hat{X}\right)$: it measures the position of the sector relative to its neighbours. At a local maximum, the second derivative of $R\left(\hat{X}\right)$ is negative: $\nabla_{\hat{X}}^2R\left(\hat{X}\right)<0$. At a minimum, it is positive.

The last term, $\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}$, involved in the definition of $Y(\hat{X})$ and p is a smoothing factor between neighbouring sectors. It reduces differences between sectors: it increases when the relative attractivity with respect to $K_{\hat{X}}$ decreases. The number of investors and capital will increase in sectors that are in the neighbourhood of significantly more attractive sectors, i.e. with higher average capital and number of investors. It slows down the transitions.

A10.2.2 Fluctuation terms

The last term involved in (127):

$$\alpha_{eff}'\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)\sqrt{\frac{\left|f\left(\frac{\hat{X}_f+\hat{X}_i}{2}\right)\right|}{2\sigma_{\hat{X}}^2}}\left|\left(\hat{K}_f+\frac{\sigma_{\hat{K}}^2F\left(\hat{X}_f,K_{\hat{X}_f}\right)}{f^2\left(\hat{X}_f\right)}\right)-\left(\hat{K}_i+\frac{\sigma_{\hat{K}}^2F\left(\hat{X}_i,K_{\hat{X}_i}\right)}{f^2\left(\hat{X}_i\right)}\right)\right|$$

depicts the possible oscillations around averages (that) occur (within)in a definite timespan, so(such) that the transition probability decrases with $\hat{K}_f-\hat{K}_i$ and X_f-X_i. However, this probability decreases with short-term returns: the higher the returns, the lower the incentive to switch from one sector to another.