A Statistical Field Perspective on Capital Allocation and Accumulation: Individual dynamics

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Abstract

We have shown, in a series of articles, that a classical description of a large number of economic agents can be replaced by a statistical fields formalism.

To better understand the accumulation and allocation of capital among different sectors, the present paper applies this statistical fields description to a large number of heterogeneous agents divided into two groups. The first group is composed of a large number of firms in different sectors that collectively own the entire physical capital. The second group, investors, holds the entire financial capital and allocates it between firms across sectors according to investment preferences, expected returns, and stock prices variations on financial markets. In return, firms pay dividends to their investors. Financial capital is thus a function of dividends and stock valuations, whereas physical capital is a function of the total capital allocated by the financial sector. Whereas our previous work focused on the background fields that describe potential long-term equilibria, here we compute the transition functions of individual agents and study their probabilistic dynamics in the background field, as a function of their initial state. We show that capital accumulation depends on various factors. The probability associated with each firm's trajectories is the result of several contradictory effects: the firm tends to shift towards sectors with the greatest long-term return, but must take into account the impact of its shift on its attractiveness for investors throughout its trajectory. Since this trajectory depends largely on the average capital of transition sectors, a firm's attractiveness during its relocation depends on the relative level of capital in those sectors. Thus, an under-capitalized firm reaching a high-capital sector will experience a loss of attractiveness, and subsequently, in investors. Moreover, the firm must also consider the effects of competition in the intermediate sectors. An under-capitalized firm will tend to be ousted out towards sectors with lower average capital, while an over-capitalized firm will tend to shift towards higher average-capital sectors. For investors, capital allocation depends on their short and long-term returns. These returns are not independent: in the short-term, returns are composed of both the firm's dividends and the increase in its stock prices. In the long-term, returns are based on the firm's growth expectations, but also, indirectly, on expectations of higher stock prices. Investors' capital allocation directly depends on the volatility of stock prices and firms' dividends. Investors will tend to reallocate their capital to maximize their short and long-term returns. The higher their level of capital, the stronger the re-allocation will be.

Key words: Financial Markets, Real Economy, Capital Allocation, Statistical Field Theory, Background fields, Collective states, Multi-Agent Model, Interactions.

JEL Classification: B40, C02, C60, E00, E1, G10

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1 Introduction

We have shown in a series of articles that a classic description of a large number of economic agents can be replaced by a description in terms of statistical fields. This formalism is used to study both the collective behavior of the system, by finding the so called background field of the system, and the probabilistic behavior of agents in this collective state, through the transition functions of an arbitrary number of agents.

As in (Gosselin Lotz Wambst 2022), we apply this formalism to the accumulation and allocation of capital between different sectors. Specifically, we consider two groups of agents: producers representing the real economy and investors representing the financial markets. Producers collectively own the entire physical capital, whereas investors hold and allocate the entire financial capital between firms across sectors according to investment preferences, expected returns, and stock prices variations on financial markets. Firms pay back their investors through dividends. Financial capital is thus a function of dividends and stock valuations, and physical capital is a function of the overall capital allocated by the financial sector.

These two groups of agents are described by two different interacting fields and by the system action functional. The solutions to the minimization equations of the action functional are the background fields of the system. From a sector perspective, the background fields determine the average distribution of capital and the firm density within sectors, given external parameters, such as expected returns, technological advances, and their dynamics.

Whereas (Gosselin Lotz Wambst 2022) presented the background fields modeling the potential collective states, the present paper will study the transition functions of individual agents and their probabilistic dynamics in the background field, as a function of their initial state. We demonstrate that several factors influence the probability of each firm's relocation path.

First, firms tend to relocate in sectors with highest long-term returns. However, the path followed by the firm to realocate depends on the characteristics of the transition sectors, that are themselves determined by the collective state of the system. The attractiveness of the firm during its relocation process depends on the average capital of the transition sectors it stumbles into. Depending on the sector, investors may over or underinvest in the firm. An under-capitalized firm may fail to attract investors in and either end up being stuck in this sector or be repelled to a less attractive one.

Second, competition along the transition sectors, depends on the background state of the system and impact differently the firm's level or capital and attractiveness. An overcapitalized firm facing many less-endowed competitors, will oust them out of the sector. On the contrary, an under-capitalized firm will be ousted out from its own sector and move towards less-capitalized and denser sectors in average. A capital gain - or loss - may follow. Under-capitalized firms tend to move towards lower than average capitalized sectors, while over-capitalized firms tend to move towards higher than average-capitalized sectors.

Third, investors' capital allocation depends on short and long-term returns. Yet these returns are not independent: short-term returns, dividends and stock prices variations are correlated to the long-term that depend on growth expectations and stock prices expectations.

Changes in investors' capital allocation are therefore directly dependent on stock prices' volatility and firms' dividends. Changes in growth expectations impact stock prices and incite investors to reallocate capital to maximize their returns. The higher their level of capital, the stronger the reallocation will be.

The paper is organized as follows. The second section is a literature review. The first part of the paper presents the use of field theory in economic modeling. Section three presents the principles of this approach. In section four, we explain the translation of a classical framework into a field model and section five derives the transition functions of the model. These functions compute the probabilities of the model to evolve from an initial to a final state. In section 6, we introduce the notions of background field, effective action and their use in computing and interpreting the transition functions.

The second part of the paper applies the field approach to a model of capital accumulation. Section 7 presents the classical framework of the model and section 8 translates this framework into a field model. In section 9, we derive the transition functions for one and two agents in this particular model. The results are presented and discussed in section 10 while section 11 concludes.

2 Literature review

Several branches of the economic literature seek to replace the representative agent with a collection of heterogeneous ones. However, among other things, they differ in the way they model this collection of agents. The comparison between these approaches and our formalism has been detailed in (paper), where we have also compared the notion of collective states in these approaches to our own. In the present paper, we instead detail the differences concerning individual dynamics.

Several branches share a common approach to finding the probability density of agent behavior. This approach is followed by mean-field theory, heterogeneous agents, new Keynesian (HANK) models, and the information-theoretic approach to economics. In mean field theory (see (Bensoussan et al. 2018; Lasry et al. 2010a, b), (Gomes et al., 2015)), agents interact with the population as a whole, described by the mean field. However, this literature does not consider the direct interactions between agents.

Heterogeneous agents new Keynesian (HANK) models (see Kaplan and Violante 2018 for an account) consider probabilistic techniques close to mean field theory: an equilibrium probability distribution for each group of agents is derived from a set of optimizing heterogeneous agents. Information theory (see Yang 2018) considers probabilistic states around the equilibrium derived from an entropy maximization program. Lastly, the rational inattention theory (Sims 2006) derives non-gaussian density laws from limited information and constraints.

Our approach concerning differs from this mean field-like technique. We do not seek individual transition at equilibrium. We first instead consider the whole set of agents and their interactions. Our statistical weight is also a probabilistic approach. Nevertheless, it is directly built from the microeconomic dynamic equations of N-interacting agents. This corresponds to working with a much larger space of dynamic variables than the ones considered in the other papers. At a collective level, this allows us to describe the model with fields and find the background field that arises from the interactions at the individual level. Once the collective states have been found, we can recover both the types of individual dynamics depending on the initial conditions and the "effective" form of interactions between two or more agents.

At the individual level, which is at stake here, agents are distributed along some probability law. However, this probability law is directly conditioned by the collective state of the system and the effective interactions. Different collective states, given different parameters, yield different individual dynamics. This approach allows for coming back and forth between collective and individual aspects of the system.

Different categories of agents appear in the emerging collective state. Dynamics may present very different patterns, given the collective state's form and the agents' initial conditions.

A second branch of the literature is closest to our approach since it considers the interacting system of agents. Finally, it is the multi-agent systems literature, notably agent-based models (see Gaffard Napoletano 2012, Mandel et al. 2010 2012) and economic networks (Jackson 2010).

Agent-based models use microfounded general macroeconomics models, whereas network models lower-scale models such as contract theory, behaviour diffusion, information sharing, or learning. In both settings, agents are typically defined by and follow various rules, leading to equilibria and dynamics otherwise inaccessible to the representative agent setup. Both approaches are, however, highly numerical and model-dependent. As a consequence, ABM models tend to reveal collective states numerically. Unlike our approach, they do not seek to reveal these collective states' relation to individual dynamics. In other words, individual dynamics are considered together to produce a collective state, as in our approach. However, the non-analytical approach to this collective state prevents a precise inspection of the emergence of several phases and the impact of the collective state on individual trajectories. Statistical fields theory, on the contrary, accounts for transitions between scales. Macroeconomic patterns are grounded in behaviours and interaction structures. Describing these structures in terms of field theory allows for the emergence of collective states at the macro scale and, in turn, the study of their impact at the individual level.

A third branch of the literature, Econophysics, is also related to ours since it often considers the set of agents as a statistical system (for a review, see Abergel et al. 2011a,b and references therein; or Lux 2008, 2016). However, it tends to focus on empirical laws rather than apply the full potential of field theory to economic systems. Kleinert (2009) uses path integrals to model stock price dynamics, but our approach differs in that it keeps track of usual microeconomic concepts, such as utility functions and expectations, and includes them into the analytical treatment of multi-agent systems by translating the main characteristics of optimizing agents in terms of statistical systems.

The literature on interactions between finance and real economy or capital accumulation mainly occurs in the context of DGSE models. A review of the literature is provided by Cochrane (2006), while further developments can be found in Grassetti et al. (2022), Grosshans and Zeisberger (2018), Böhm et al. (2008), Caggese and Orive (2012), Bernanke et al. (1999), Campello et al. (2010), Holmstrom and Tirole (1997), Jermann and Quadrini (2012), Khan and Thomas (2013), and Monacelli et al. (2011). Theoretical models in this literature typically include several types of agents at the aggregated level, such as producers for possibly several sectors, consumers, and financial intermediaries. They aim to describe the interactions between a few representative agents and determine interest rates, levels of production, and asset pricing, all in the context of ad-hoc anticipations.

Our formalism differs from this literature in three ways. First, we consider several groups of a large number of agents to describe the emergence of collective states and study the continuous space of sectors. Second, we consider expected returns and the longer-term horizon somewhat exogenous or structural. Expected returns combine elements such as technology, returns, productivity, sectoral capital stock, expectations, and beliefs. These returns are also a function defined over the sectors' space: the system's background fields are functionals of these expected returns. Taken together, the background fields of a field model describe an economic environment for a given configuration of expected returns. As such, expected returns are at first seen as exogenous. In a second step, when we consider the dynamics between capital accumulation and expectations, expectations may be seen as endogenous. Even then, the relations between actual and expected variables specified are general enough to derive some possible dynamics.

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Our formalism differs from this literature in three ways. First, we model several groups of agents across a continuous space of economic sectors to capture the emergence of collective states. Second, we consider expected returns and the long-term horizon as somewhat exogenous or structural, incorporating elements such as technology, returns, productivity, sectoral capital stock, expectations, and beliefs. These returns are a function defined over the sectors' space, and the system's background fields are themselves functionals of these expected returns. Thus, the background fields of the system describe an economic environment for a given configuration of expected returns, which are first seen as exogenous. However, we later consider the dynamics between capital accumulation and expectations, allowing expectations to become endogenous.

Last, we do not seek aggregated dynamics, and the individual dynamics depend on a particular collective state. In such a background, agents' typical dynamics are computed through transition functions. These functions compute the transition probabilities from one capital-sector point to another. Since backgrounds may be considered dynamic quantities, structural or long-term variations in the returns' landscape may modify the background and, in turn, the individual dynamics. Expected returns themselves depend on and interact with capital accumulation.

Field theory and economic modeling

The field formalism combines the micro and macro aspects of an economic system by considering the interactions among the entire set of agents and their environment. This allows for a back-and-forth analysis between micro and macro scales: once the collective state of a system is determined, it enables a return to individual dynamics within that collective state. We will first translate a microeconomic model its field equivalent, then explain how this translation allows to retrieve collective or macro states and the agents transition functions.

3 Principles

The field-formalism used in this paper is rooted in a probabilistic description of economic systems with large number of agents. Classically, each agent's dynamics is described by an optimal path for some vector variable, say $A_i(t)$, from an initial to a final point, up to some fluctuations. However, the same system of agents can be viewed probabilistically. An agent can be described by a *probability density* that is, due to idiosyncratic uncertainties, centered around the classical optimal path[1] (see Gosselin, Lotz and Wambst 2017, 2020, 2021). In this probabilistic approach, each possible dynamics for the set of N agents must be taken into account and weighted by its probability. The system is then described by a *statistical weight*, the probability density for any configuration of N arbitrary individual paths. Once this statistical weight is found, we can compute the transition probabilities of the system which are the probabilities for any number of agents to evolve from an initial to a final state in A_i, B_i in a given time.

Because this probabilistic approach implies keeping track of the N agents' probability transitions, it is practically untractable for a large number of agents. However, this step is necessary since it can be translated into a more compact *field formalism* (see Gosselin, Lotz, and Wambst 2017, 2020, 2021), which preserves the essential information encoded in the model but implements a change in perspective. It does not keep track of the N-indexed agents but describes their dynamics and interactions as a collective thread of all possible unlabeled paths. This collective thread can be seen as an environment that conditions the dynamics of individual agents from one state to another. The field formalism eases the computation of transition functions. More importantly, it detects the collective states or phases encompassed in the field, that would otherwise remain indetectable using the probabilistic formulation.

To translate the probabilistic approach into a field model, the N agents' trajectories $\mathbf{A}_i(t)$ is replaced by a field Ψ, a complex-valued function that solely depends on a single set of variables, $\mathbf{A}$. The statistical weight of the probabilistic approach is translated into a probability density on the space of complex-valued functions of the variables $\mathbf{A}$. For the configuration $\Psi(\mathbf{A})$, this probability density has the form $\exp(-S(\Psi))$. The functional $S(\Psi)$ is called the *field action functional.* It captures the microscopic features of individual agents' dynamics and interactions. The idea is that of a dictionary that would translate the various terms of the classical description in terms of their field equivalent. The integral of $\exp(-S(\Psi))$ over the configurations Ψ is the *partition function* of the system. The fields that minimize the action functional are the *classical background fields,* or more simply the *background fields.* They encapsulate the collective states of the system.

For several types of agents, the generalisation is straightforward. Each type α is described by a field $\Psi_\alpha(\mathbf{A}_\alpha)$. The field action depends on the whole set of fields $\{\Psi_\alpha\}$. It accounts for all types of agents and their interactions, and writes $S(\{\Psi_\alpha\})$. The form of $S(\{\Psi_\alpha\})$ is obtained directly from the classical description of our model.

In the following, we will detail a shortcut of this translation method and apply it to the microeconomic framework below.

4 From a classical framework to its field expression

To translate a classical economic framework into a field model, we must first consider the various types of agents in the model. We will rewrite their dynamics as the minimization equations of some initial functions, in the same way as, for instance, consumption dynamics could be derived from an utility function. These minimization functions will then be translated into field functionals of several independent fields[2], one for each type of agent. The sum of these functionals, the "field action functional", describes the whole system in terms of fields[3]. We will detail the process for a classical and relatively general model.

[1] Due to the infinite number of possible paths, each individual path has a null probability to exist. We, therefore, use the word "probability density" rather than "probability".

[2] The term functional refers to a function of a function, i.e. a function whose argument is itself a function.

[3] Details about the probabilistic step will be given as a reminder along the text and in appendix 1.

4.1 A general type of classical framework

We consider two types of agents, characterized by vector-variables $\{\mathbf{A}_i(t)\}_{i=1,\ldots N}$, and $\left\{\hat{\mathbf{A}}_l(t)\right\}_{l=1,\ldots\hat{N}}$... respectively, where N and $\hat{N}$ are the number of agents of each type.

In a classical model, the optimization by agents of their characteristic function, such as utility, consumption, production, ..., leads to a system of dynamic equations.

Our approach is the opposite: using dynamic equations that could result from the optimization of agents, we reconstruct what we will call a "minimization function," $s\left(\{\mathbf{A}_l\},\left\{\hat{\mathbf{A}}_l\right\},t\right)$, that depends on the entire system and whose minimization would yield the dynamic equations of the system. We will use in the following the general form of $s\left(\{\mathbf{A}_l\},\left\{\hat{\mathbf{A}}_l\right\},t\right)$ provided in (Gosselin Lotz Wambst 2022) [4]:

$$\begin{aligned}
s\left(\{\mathbf{A}_i\},\left\{\hat{\mathbf{A}}_l\right\},t\right) &= \sum_i\left(\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l\ldots}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)\right)^2 \\
&+\sum_i\left(\frac{d\hat{\mathbf{A}}_i(t)}{dt}-\sum_{j,k,l\ldots}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)\right)^2 \\
&+\sum_i\sum_{j,k,l\ldots}g\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)
\end{aligned} \tag{1}$$

Note incidentally that, setting $g=0$, the minimization of $s\left(\{\mathbf{A}_l\},\left\{\hat{\mathbf{A}}_l\right\},t\right)$ would indeed yield:

$$\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l\ldots}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)=0 \tag{2}$$

and:

$$\frac{d\hat{\mathbf{A}}_i(t)}{dt}-\sum_{j,k,l\ldots}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)=0 \tag{3}$$

that would be the usual type of dynamic equations for agents $\mathbf{A}_i$ and $\hat{\mathbf{A}}_l$.

It is specifically the last term in formula (1) that is a specific feature of our large number of agents approach. It represents the whole set of interactions both among and between two groups of agents, modifying the standard equations (2) and (3).

In a classical approach, the model is solved by numerically solving equation (2) for the n agents. Minimizing the function $s\left(\{\mathbf{A}_l\},\left\{\hat{\mathbf{A}}_l\right\},t\right)$ would not yield any new insight compared to the dynamic equations we started with if we consider that they are satisfied exactly. The key feature of introducing the minimization function $s\left(\{\mathbf{A}_l\},\left\{\hat{\mathbf{A}}_l\right\},t\right)$ only arises when the standard dynamic equations such as (2) and (3) are satisfied only up to error terms. These errors represent the fluctuations of the agents around an optimal dynamics.

$$\frac{d\mathbf{A}_i(t)}{dt}-\sum_{j,k,l\ldots}f\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)=\varepsilon_i \tag{4}$$

and:

$$\frac{d\hat{\mathbf{A}}_i(t)}{dt}-\sum_{j,k,l\ldots}\hat{f}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)\ldots\right)=\hat{\varepsilon}_l \tag{5}$$

[4] A generalisation of equation (1) in which agents interact at different times is presented in Appendix 1 of (Gosselin Lotz Wambst 2022), along with its translation in term of field.

These errors confer a probabilistic nature to the system, for they imply a change in perspective: we do no longer consider solely optimal trajectories. Rather, we deem any trajectory as possible, with a probability decreasing with the distance of the trajectory from the optimum. This allows to compute the probabilities for the entire system to evolve from any initial state to any final state.

This is the main purpose of the function $s\left(\{\mathbf{A}_l\},\left\{\hat{\mathbf{A}}_l\right\},t\right)$ defined in (1). By construction, it is minimal for the optimal path and in general, it increases with the distance to this optimal path. To gain a better insight, consider a case without interactions, i.e. $g=0$. In this particular case, the first two terms of equation (1) represent the squared deviation from the dynamics of an agent of each type. The higher this deviation, the higher s. We will associate to each trajectory a probability, its *statistical weight*, that decreases with the deviation, written:

$$\exp\left(-W\left(\{\mathbf{A}_l\},\left\{\hat{\mathbf{A}}_l\right\}\right)\right)=\exp\left(-\int s\left(\{\mathbf{A}_l\},\left\{\hat{\mathbf{A}}_l\right\},t\right)dt\right) \tag{6}$$

The integral over time accounts for summing over all deviations along the trajectory.

By assigning a probability to each trajectory of the system, the statistical weight allows us to describe all evolutions of the entire agent ensemble from one state to another by summing the probabilities of the trajectories connecting these two states. However, because this calculation requires considering the entire agent ensemble, it is generally intractable.

On the other hand, the statistical weight can be replaced by an equivalent in terms of fields that contains the same information and also allows the detection of collective states that are undetectable by the statistical weight.

4.2 Field Translation of the framework

We will translate the classical model described above into field terms. A field is an object that takes into account all possible states of the system for all variables, independently of individual agents. For example, if a set of agents is described by their individual consumptions, the field associated with this number of agents will overlook the details of the agents but describe all possible states of consumption within the system.

Mathematically, a field is a random variable whose realisations run over a space of complex-valued functions that depend on the variables of the system. The realizations of the field represent the various states of the system, and the associated probability for each state is given by a statistical weight. This weight is a function of the field and is constructed by translating $\exp\left(-W\left(\{\mathbf{A}_l\},\left\{\hat{\mathbf{A}}_l\right\}\right)\right)$ given by (6) into field terms. To obtain the equivalent of (6) in terms of fields, we proceed as follows:

For each function that characterize agents in a classical system, we write a function whose variables are the fields. This function translates the properties of the initial function but incorporates the change in perspective associated with the translation. It describes the range of possibilities for the function for each system state.

To translate formula (1) and (6) into field terms, recall that formula (1) includes three terms that correspond to three functions of the field. The last term in formula (1) describes the whole set of interactions both among and between two groups of agents, and is of the form:

$$\sum_i\sum_{j,k,l,m...}g\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right) \tag{7}$$

It involves terms with indexed variables but no temporal derivative terms.

These terms are the easiest to translate. Here, agents are characterized by their variables $\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t)...$ and $\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...$ respectively, for instance in our model firms and investors.

In the field translation, agents of type $\mathbf{A}_i(t)$ and $\hat{\mathbf{A}}_l(t)$ are described by a field $\Psi(\mathbf{A})$ and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$, respectively. The variables indexed j, k, l, m..., such as $\mathbf{A}_j(t)$, $\mathbf{A}_k(t)$, $\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)$... are replaced by $\mathbf{A}',\mathbf{A}''$, $\hat{\mathbf{A}}$, $\hat{\mathbf{A}}'$, and so on for all the indices in the function. The translation of (7) in terms of fields is:

$$\int g\left(\mathbf{A},\mathbf{A}',\mathbf{A}'',\hat{\mathbf{A}},\hat{\mathbf{A}}'...\right)|\Psi(\mathbf{A})|^2|\Psi(\mathbf{A}')|^2|\Psi(\mathbf{A}'')|^2\times...d\mathbf{A}d\mathbf{A}'d\mathbf{A}''... \tag{8}$$
$$\times\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2\left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2\times...d\hat{\mathbf{A}}d\hat{\mathbf{A}}'...$$

where the dots stand for the products of square fields and the necessary integration symbols.

In formula (1), the terms that imply a variable temporal derivative are of the form:

$$\sum_i \left(\frac{d\mathbf{A}_i^{(\alpha)}(t)}{dt} - \sum_{j,k,l,m\ldots} f^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)\ldots\right)\right)^2 \tag{9}$$

This particular form represents the dynamics of the α-th coordinate of a variable $\mathbf{A}_i(t)$ as a function of the other agents.

Their translation is of the form:

$$\int \Psi^{\dagger}(\mathbf{A}) \left(-\nabla_{\mathbf{A}^{(\alpha)}} \left(\frac{\sigma^2_{\mathbf{A}^{(\alpha)}}}{2} \nabla_{\mathbf{A}^{(\alpha)}} + \Lambda(\mathbf{A}) \right) \right) \Psi(\mathbf{A})\, d\mathbf{A} \tag{10}$$

with:

$$\Lambda(\mathbf{A}) = \int f^{(\alpha)}\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'\ldots\right) |\Psi(\mathbf{A}')|^2 |\Psi(\mathbf{A}'')|^2 d\mathbf{A}' d\mathbf{A}'' \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}} d\hat{\mathbf{A}}' \tag{11}$$

The variance $\sigma^2_{\mathbf{A}^{(\alpha)}}$ depicts the probabilistic nature of the model hidden behind the field formalism. This variance represents the characteristic level of uncertainty of the system's dynamics. It is a parameter of the model.

Ultimately, the field description is obtained by summing all the terms translated above and introducing a time dependency. This sum is called the action functional, denoted $S\left(\Psi, \Psi^{\dagger}\right)$. It is the sum of terms of the form (8) and (10). We obtain:

$$\begin{aligned} S\left(\Psi, \hat{\Psi}\right) = & \int \Psi^{\dagger}(\mathbf{A}) \left(-\nabla_{\mathbf{A}^{(\alpha)}} \left(\frac{\sigma^2_{\mathbf{A}^{(\alpha)}}}{2} \nabla_{\mathbf{A}^{(\alpha)}} + \Lambda_1(\mathbf{A}) \right) \right) \Psi(\mathbf{A})\, d\mathbf{A} \\ & + \int \hat{\Psi}^{\dagger}\left(\hat{\mathbf{A}}\right) \left(-\nabla_{\hat{\mathbf{A}}^{(\alpha)}} \left(\frac{\sigma^2_{\hat{\mathbf{A}}^{(\alpha)}}}{2} \nabla_{\hat{\mathbf{A}}^{(\alpha)}} + \Lambda_2(\hat{\mathbf{A}}) \right) \right) \hat{\Psi}\left(\hat{\mathbf{A}}\right) d\hat{\mathbf{A}} \\ & + \sum_m \int g_m\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'\ldots\right) |\Psi(\mathbf{A})|^2 |\Psi(\mathbf{A}')|^2 |\Psi(\mathbf{A}'')|^2 \times \ldots d\mathbf{A} d\mathbf{A}' d\mathbf{A}''\ldots \\ & \times \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 \times \ldots d\hat{\mathbf{A}} d\hat{\mathbf{A}}'\ldots \end{aligned} \tag{12}$$

where the sequence of functions g_m describes the various types of interactions in the system.

The field formalism allows to compute the transition functions of the system, i.e. the probability for any set of agent to evolve from a state to an other in a given time.

Several results can be derived from the field action, $S(\Psi)$, and its statistical weight, $\exp(-S(\Psi))$. The background states that minimize $S(\Psi)$ can be found, which in turn allows to compute the average quantities of the system, its associated effective action, and the transition functions. We will focus in the following specifically on the transition functions[5].

5 Transition functions

The transition functions describe the probabilities of agents to evolve within the system. They calculate the probability of any group of agents to transition from any initial state to any final state. Consequently, there exists an infinite number of transition functions, one for each possible group of agents. For instance there can be a transition function for a single agent of a certain type, two agents of the same type, an agent of one type and an agent of another type, multiple agents of the same type, and a single agent of another type, etc.

[5] The concept of background state and its corresponding average quantities have been presented in (Gosselin Lotz Wambst 2022).

The transition functions depend on the initial and final positions of each agent, the agent's type, and their interactions with others in the group. Furthermore, we will see that the transition functions also depend on the collective state of the system defined by the background field, which reflects the dependence of the group of agents on the entire macroeconomic system.

In principle, the transition functions could be computed using the classical system and the classical weight W by directly summing the probabilities of some trajectories. However, this approach is intractable and lacks interpretability. We will therefore present an alternative method to compute these transition functions using the field model.

5.1 Transition functions in a classical framework

In a classical perspective, the statistical weight (6) can be used to compute the transition probabilities of the system, i.e. the probabilities for any number of agents of both types to evolve from an initial state $\{\mathbf{A}_l\}_{l=1,\ldots}, \left\{\hat{\mathbf{A}}_l\right\}_{l=1,\ldots}$ to a final state in a given timespan. These transition functions describe the dynamic of the agents of the system.

To do so, we first compute the integral of equation (6) over all paths between the initial and the final points considered. Defining $\{\mathbf{A}_l(s)\}_{l=1,\ldots,N}$ and $\left\{\hat{\mathbf{A}}_l(s)\right\}_{l=1,\ldots,\hat{N}}$ the sets of paths for agents of each type, where N and $\hat{N}$ are the numbers of agents of each type, we consider the set of $N+\hat{N}$ independent paths written:

$$\mathbf{Z}(s) = \left(\{\mathbf{A}_l(s)\}_{l=1,\ldots,N}, \left\{\hat{\mathbf{A}}_l(s)\right\}_{l=1,\ldots,\hat{N}}\right)$$

The weight (6) can now be written $\exp(-W(\mathbf{Z}(s)))$.

The transition functions $T_t\left(\underline{(\mathbf{Z})}, \overline{(\mathbf{Z})}\right)$ compute the probability for the $(N, \hat{N}\ )$ agents to evolve from the initial points $Z(0) \equiv \underline{\mathbf{Z}}$ to the final points $Z(t) \equiv \overline{(\mathbf{Z})}$ during a time span t. This probability is defined by:

$$T_t\left(\underline{\mathbf{Z}}, \overline{(\mathbf{Z})}\right) = \frac{1}{\mathcal{N}} \int_{\substack{\mathbf{Z}(0)\equiv\underline{\mathbf{Z}} \\ \mathbf{Z}(t)\equiv\overline{(\mathbf{Z})}}} \exp(-W(\mathbf{Z}(s)))\, \mathcal{D}(\mathbf{Z}(s)) \tag{13}$$

The integration symbol $D\mathbf{Z}(s)$ covers all sets of $N \times \hat{N}$ paths constrained by $\mathbf{Z}(0) \equiv \underline{\mathbf{Z}}$ and $\mathbf{Z}(t) \equiv \overline{(\mathbf{Z})}$. The normalisation factor sets the total probability defined by the weight (6) to 1 and is equal to:

$$\mathcal{N} = \int \exp(-W(\mathbf{Z}(s)))\, \mathcal{D}\mathbf{Z}(s)$$

The interpretation of (13) is straightforward. Instead of studying the full trajectory of one or several agents, we compute their probability to evolve from one configuration to another, and in average, the usual trajectory approach remains valid.

Equation (13) can be generalized to define the transition functions for $k \leqslant N$ and $\hat{k} \leqslant \hat{N}$ agents of each type. The initial and final points respectively for this set of $k+\hat{k}$ agents are written:

$$\mathbf{Z}(0)^{[k,\hat{k}]} \equiv \underline{\mathbf{Z}}^{[k,\hat{k}]}$$

and:

$$\mathbf{Z}(t)^{[k,\hat{k}]} \equiv \overline{(\mathbf{Z})}^{[k,\hat{k}]}$$

The transition function for these agents is written:

$$T_t\left(\underline{(\mathbf{Z})}^{[k,\hat{k}]}, \overline{(\mathbf{Z})}^{[k,\hat{k}]}\right)$$

and the generalization of equation (13) is:

$$T_t\left(\underline{(\mathbf{Z})}^{[k,\hat{k}]}, \overline{(\mathbf{Z})}^{[k,\hat{k}]}\right) = \frac{1}{\mathcal{N}} \int_{\substack{\mathbf{Z}(0)^{[k,\hat{k}]}=\underline{(\mathbf{Z})}^{[k,\hat{k}]} \\ \mathbf{Z}(t)^{[k,\hat{k}]}=\overline{(\mathbf{Z})}^{[k,\hat{k}]}}} \exp(-W((\mathbf{Z}(s))))\, \mathcal{D}((\mathbf{Z}(s))) \tag{14}$$

The difference with (13) is that only k paths are constrained by their initial and final points.

Ultimately, the Laplace transform of $T_t\left(\underline{(Z)}^{[k,\hat{k}]},\overline{(Z)}^{[k,\hat{k}]}\right)$ computes the - time averaged - transition function for agents with random lifespan of mean $\frac{1}{\alpha}$, up to a factor $\frac{1}{\alpha}$, and is given by:

$$G_\alpha\left(\underline{(\mathbf{Z})}^{[k,\hat{k}]},\overline{(\mathbf{Z})}^{[k,\hat{k}]}\right)=\int_0^\infty \exp(-\alpha t)\,T_t\left(\underline{(\mathbf{Z})}^{[k,\hat{k}]},\overline{(\mathbf{Z})}^{[k,\hat{k}]}\right)dt \tag{15}$$

This formulation of the transition functions is relatively intractable. Therefore, we will now propose an alternative method based on the field model.

5.2 Field-theoretic expression

The transition functions (14) and (15) can be retrieved using the field theory transition functions - or Green functions, which compute the probability for a variable number $\left(k,\hat{k}\right)$ of agents to transition from an initial state $\underline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]}$ to a final state $\overline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]}$, where $\underline{(\boldsymbol{\theta})}^{[k,\hat{k}]}$ and $\overline{(\boldsymbol{\theta})}^{[k,\hat{k}]}$ are vectors of initial and final times for $k+\hat{k}$ agents respectively.

We will write:

$$T_t\left(\underline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]},\overline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]}\right)$$

the transition function between $\underline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]}$ and $\overline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]}$ with $\overline{(\boldsymbol{\theta})}_i<t$, $\forall i$, and:

$$G_\alpha\left(\underline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]},\overline{(\mathbf{Z},\boldsymbol{\theta})}^{[k,\hat{k}]}\right)$$

its Laplace transform. Setting $\underline{(\boldsymbol{\theta})}_i=0$ and $\overline{(\boldsymbol{\theta})}_i=t$ for $i=1,...,k+\hat{k}$, these functions reduce to (14) or (15): the probabilistic formalism of the transition functions is thus a particular case of the field formalism definition. In the sequel we therefore will use the term transition function indiscriminately.

The computation of the transition functions relies on the fact that $\exp(-S(\Psi))$ itself represents a statistical weight for the system. Gosselin, Lotz, Wambst (2020) showed that $S(\Psi)$ can be modified in a straightforward manner to include source terms:

$$S(\Psi,J)=S(\Psi)+\int\left(J(Z,\theta)\,\Psi^\dagger(Z,\theta)+J^\dagger(Z,\theta)\,\Psi(Z,\theta)\right)d(Z,\theta) \tag{16}$$

where $J(Z,\theta)$ is an arbitrary complex function, or auxiliary field.

Introducing $J(Z,\theta)$ in $S(\Psi,J)$ allows to compute the transition functions by successive derivatives. Actually, we can show that:

$$G_\alpha\left(\underline{(\mathbf{Z},\theta)}^{[k,\hat{k}]},\overline{(\mathbf{Z},\theta)}^{[k,\hat{k}]}\right)=\left[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{(\mathbf{Z},\theta)}_{i_l}\right)}\frac{\delta}{\delta J^\dagger\left(\overline{(\mathbf{Z},\theta)}_{i_l}\right)}\right)\int\exp(-S(\Psi,J))\,\mathcal{D}\Psi\mathcal{D}\Psi^\dagger\right]_{J=J^\dagger=0} \tag{17}$$

where the notation $\mathcal{D}\Psi\mathcal{D}\Psi^\dagger$ denotes an integration over the space of functions $\Psi(Z,\theta)$ and $\Psi^\dagger(Z,\theta)$, i.e. an integral in an infinite dimensional space. Even though these integrals can only be computed in simple cases, a series expansion of $G_\alpha\left(\underline{(\mathbf{Z},\theta)}^{[k,\hat{k}]},\overline{(\mathbf{Z},\theta)}^{[k,\hat{k}]}\right)$ can be found using Feynman graphs techniques.

Once $G_\alpha\left(\underline{(\mathbf{Z},\theta)}^{[k,\hat{k}]},\overline{(\mathbf{Z},\theta)}^{[k,\hat{k}]}\right)$ is computed, the expression of $T_t\left(\underline{(\mathbf{Z},\theta)}^{[k,\hat{k}]},\overline{(\mathbf{Z},\theta)}^{[k,\hat{k}]}\right)$ can be retrieved in principle by an inverse Laplace transform. In field theory, formula (17) shows that the transition functions (15) are correlation functions of the field theory with action $S(\Psi)$.

6 Field-theoretic computations of transition functions

The formula (17) provides a precise and compact definition of the transition functions for multiple agents in the system. However, in practice, this formula is not directly applicable and does not shed much light on the connection between the collective and microeconomic aspects of the considered system. To calculate the dynamics of the agents, we will proceed in three steps.

Firstly, we will minimize the system's action functional and determine the background field, which represents the collective state of the system. Once the background field is found, we will perform a series expansion of the action functional around this background field, referred to as the effective action of the system. It is with this effective action that we can compute the transition functions for the state defined by the background field. We will discover that each term in this expansion has an interpretation in terms of a transition function.

Instead of directly computing the transition functions, we can consider a series expansion of the action functional around a specific background field of the system.

6.1 Step 1: finding the background field

For a particular type of agent, background fields are defined as the fields $\Psi_0(Z,\theta)$ that maximize the statistical weight $\exp(-S(\Psi))$ or, alternatively, minimize $S(\Psi)$:

$$\frac{\delta S(\Psi)}{\delta\Psi}|_{\Psi_0(Z,\theta)}=0,\ \frac{\delta S(\Psi^{\dagger})}{\delta\Psi^{\dagger}}|_{\Psi_0^{\dagger}(Z,\theta)}=0$$

The field $\Psi_0(Z,\theta)$ represents the most probable configuration, a specfic state of the entire system that influences the dynamics of agents. It serves as the background state from which probability transitions and average values can be computed. As we will see, the agents' transitions explicitely depend on the chosen background field $\Psi_0(Z,\theta)$, which represents the macroeconomic state in which the agents evolve.

When considering two or more types of agents, the background field satisfies the following condition:

$$\begin{aligned}
\frac{\delta S\left(\Psi,\hat{\Psi}\right)}{\delta\Psi} & |_{\Psi_0(Z,\theta)}=0, & \frac{\delta S\left(\Psi,\hat{\Psi}\right)}{\delta\Psi^{\dagger}}|_{\Psi_0^{\dagger}(Z,\theta)}=0\\
\frac{\delta S\left(\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}} & |_{\hat{\Psi}_0(Z,\theta)}=0, & \frac{\delta S\left(\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}^{\dagger}}|_{\hat{\Psi}_0^{\dagger}(Z,\theta)}=0
\end{aligned}$$

6.2 Step 2: Series expansion around the background field

In a given background state, the *effective action*[6] is the series expansion of the field functional $S(\Psi)$ around $\Psi_0(Z,\theta)$. We will present the expansion for one type of agent, but generalizing it to two or several agents is straightforward.

The series expansion around the background field simplifies the computations **of transition functions** and provides an interpretation of these functions in terms of individual interactions within the collective state. To perform this series expansion, we decompose Ψ as:

$$\Psi=\Psi_0+\Delta\Psi$$

and write the series expansion of the action functional:

$$\begin{aligned}
S(\Psi) &= S(\Psi_0)+\int\Delta\Psi^{\dagger}(Z,\theta)\,O(\Psi_0(Z,\theta))\,\Delta\Psi(Z,\theta) \\
&\quad+\sum_{k>2}\int\prod_{i=1}^{k}\Delta\Psi^{\dagger}(Z_i,\theta)\,O_k(\Psi_0(Z,\theta),(Z_i))\prod_{i=1}^{k}\Delta\Psi(Z_i,\theta)
\end{aligned} \tag{18}$$

[6]Actually, this paper focuses on the *classical effective action*, which is an approximation sufficient for the computations at hand.

The series expansion can be interpreted economically as follows. The first term, $S(\Psi_0)$, describes the system of all agents in a given macroeconomic state, Ψ_0. The other terms potentially describe all the fluctuations or movements of the agents around this macroeconomic state considered as given. Therefore, the expansion around the background field represents the microeconomic reality of a historical macroeconomic state. More precisely, it describes the range of microeconomic possibilities allowed by a macroeconomic state.

The quadratic approximation is the first term of the expansion and can be written as:

$$S(\Psi) = S(\Psi_0) + \int \Delta\Psi^{\dagger}(Z,\theta)\, O(\Psi_0(Z,\theta))\, \Delta\Psi(Z,\theta) \tag{19}$$

It will allow us to find the transition functions of agents in the historical macro state, where all interactions are averaged. The other terms of the expansion allow us to detail the interactions within the nebula, and are written as follows:

$$\sum_{k>2} \int \prod_{i=1}^{k} \Delta\Psi^{\dagger}(Z_i,\theta)\, O_k(\Psi_0(Z,\theta),(Z_i)) \prod_{i=1}^{k} \Delta\Psi(Z_i,\theta)$$

They detail, given the historical macroeconomic state, how the interactions of two or more agents can impact the dynamics of these agents. Mathematically, this corresponds to correcting the transition probabilities calculated in the quadratic approximation.

Here, we provide an interpretation of the third and fourth-order terms.

The third order introduces possibilities for an agent, during its trajectory, to split into two, or conversely, for two agents to merge into one. In other words, the third-order terms take into account or reveal, in the historical macroeconomic environment, the possibilities for any agent to undergo modifications along its trajectory. However, this assumption has been excluded from our model.

The fourth order reveals that in the macroeconomic environment, due to the presence of other agents and their tendency to occupy the same space, all points in space will no longer have the same probabilities for an agent. In fact, the fourth-order terms reveal the notion of geographical or sectoral competition and potentially intertemporal competition. These terms describe the interaction between two agents crossing paths, which leads to a deviation of their trajectories due to the interaction.

We do not interpret higher-order terms, but the idea is similar. The even-order terms (2n) describe interactions among n agents that modify their trajectories. The odd-order terms modify the trajectories but also include the possibility of agents reuniting or splitting into multiple agents. We will see in more detail how these terms come into play in the transition functions.

6.3 Step 3: Computation of the transition functions

6.3.1 Quadratic approximation

In the first approximation, transition functions in a given background field $\Psi_0(Z,\theta)$ can be computed by replacing $S(\Psi)$ in (17), with its quadratic approximation (19). In formula (19), $O(\Psi_0(Z,\theta))$ is a differential operator of second order. This operator depends explicitly on $\Psi_0(Z,\theta)$. As a consequence, transition functions and agent dynamics explicitly depend on the collective state of the system. In this approximation, the formula for the transition functions (17) becomes:

$$\begin{aligned} G_{\alpha}\left(\underline{(Z,\theta)}^{[k]}, \overline{(Z,\theta)}^{[k]}\right) &= \left[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{(Z,\theta)}_{i_l}\right)}\frac{\delta}{\delta J^{\dagger}\left(\overline{(Z,\theta)}_{i_l}\right)}\right)\right. \\ &\quad \left.\times \int \exp\left(-\int \Delta\Psi^{\dagger}(Z,\theta)\, O(\Psi_0(Z,\theta))\, \Delta\Psi(Z,\theta)\right) \mathcal{D}\Psi\mathcal{D}\Psi^{\dagger}\right]_{J=J^{\dagger}=0} \end{aligned} \tag{20}$$

Using this formula, we can show that the one-agent transition function is given by:

$$G_{\alpha}\left(\underline{(Z,\theta)}^{[1]}, \overline{(Z,\theta)}^{[1]}\right) = O^{-1}(\Psi_0(Z,\theta))\left(\underline{(Z,\theta)}^{[1]}, \overline{(Z,\theta)}^{[1]}\right) \tag{21}$$

where:

$$O^{-1}(\Psi_0(Z,\theta))\left(\underline{(Z,\theta)}^{[1]}, \overline{(Z,\theta)}^{[1]}\right)$$

is the kernel of the inverse operator $O^{-1}(\Psi_0(Z,\theta))$. It can be seen as the $\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right)$ matrix element of $O^{-1}(\Psi_0(Z,\theta))$[7].

More generally, the k-agents transition functions are the product of individual transition functions:

$$G_\alpha\left(\underline{(Z,\theta)}^{[k]},\overline{(Z,\theta)}^{[k]}\right)=\prod_{i=1}^{k}G_\alpha\left(\underline{(Z,\theta)_i}^{[1]},\overline{(Z,\theta)_i}^{[1]}\right) \tag{22}$$

The above formula shows that in the quadratic approximation, the transition probability from one state to another for a group is the product of individual transition probabilities. In this approximation, the trajectories of these agents are therefore independent. The agents do not interact with each other and only interact with the environment described by the background field.

The quadratic approximation must be corrected to account for individual interactions within the group, by including higher-order terms in the expansion of the action.

6.3.2 Higher-order corrections

To compute the effects of interactions between agents of a given group, we consider terms of order greater than 2 in the effective action. These terms modify the transition functions. Writing the expansion:

$$\exp(-S(\Psi))=\exp\left(-\left(S(\Psi_0)+\int\Delta\Psi^\dagger(Z,\theta)\,O(\Psi_0(Z,\theta))\right)\right)\left(1+\sum_{n\geqslant 1}\frac{A^n}{n!}\right)$$

where:

$$A=\sum_{k>2}\int\prod_{i=1}^{k}\Delta\Psi^\dagger(Z_i,\theta)\,O(\Psi_0(Z,\theta),(Z_i))\prod_{i=1}^{k}\Delta\Psi(Z_i,\theta)$$

is the sum of all possible interaction terms, leads to the series expansion of (17):

$$\begin{aligned}G_\alpha\left(\underline{(Z,\theta)}^{[k]},\overline{(Z,\theta)}^{[k]}\right)&=\Bigg[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{(Z,\theta)}_{i_l}\right)}\frac{\delta}{\delta J^\dagger\left(\overline{(Z,\theta)}_{i_l}\right)}\right)\\&\int\exp\left(-\int\Delta\Psi^\dagger(Z,\theta)\,O(\Psi_0(Z,\theta))\,\Delta\Psi(Z,\theta)\right)\left(1+\sum_{n\geqslant 1}\frac{A^n}{n!}\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger\Bigg]_{J=J^\dagger=0}\end{aligned} \tag{23}$$

These corrections can be computed using graphs' expansion.

More precisely, the first term of the series:

$$\left[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{(Z,\theta)}_{i_l}\right)}\frac{\delta}{\delta J^\dagger\left(\overline{(Z,\theta)}_{i_l}\right)}\right)\int\exp\left(-\int\Delta\Psi^\dagger(Z,\theta)\,O(\Psi_0(Z,\theta))\,\Delta\Psi(Z,\theta)\right)\mathcal{D}\Psi\mathcal{D}\Psi^\dagger\right]_{J=J^\dagger=0} \tag{24}$$

is a transition function in the quadratic approximation. The other contributions of the series expansion correct the approximated n agents transtns functions (22).

Typically a contribution:

$$\begin{aligned}G_\alpha\left(\underline{(Z,\theta)}^{[k]},\overline{(Z,\theta)}^{[k]}\right)&=\Bigg[\prod_{l=1}^{k}\left(\frac{\delta}{\delta J\left(\underline{(Z,\theta)}_{i_l}\right)}\frac{\delta}{\delta J^\dagger\left(\overline{(Z,\theta)}_{i_l}\right)}\right)\\&\int\exp\left(-\int\Delta\Psi^\dagger(Z,\theta)\,O(\Psi_0(Z,\theta))\,\Delta\Psi(Z,\theta)\right)\frac{A^n}{n!}\mathcal{D}\Psi\mathcal{D}\Psi^\dagger\Bigg]_{J=J^\dagger=0}\end{aligned} \tag{25}$$

[7] The differential operator $O(\Psi_0(Z,\theta))$ can be seen as an infinite dimensional matrix indexed by the double (infinite) entries $\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right)$. With this description, the kernel $O^{-1}(\Psi_0(Z,\theta))\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right)$ is the $\left(\underline{(Z,\theta)}^{[1]},\overline{(Z,\theta)}^{[1]}\right)$ element of the inverse matrix.

can be depicted by a graph. The power $\frac{A^n}{n!}$ translates that agents interact n times along their path. The trajectories of each agent of the group is broken n times between its initial and final points. At each time of interaction the trajectories of agents are deviated. To such a graph is associated a probility that modifies the quadratic approximation transition functions.

In the sequel we will only focus on the first order corrections to the two-agents transition functions.

Application to a model of capital accumulation

7 The microeconomic framework

The microeconomic framework presented in this section will be turned into a field model. Since our goal is to picture the interactions between the real and the financial economy, we consider two groups of agents, producers, and investors. We will refer to producers or firms i indistinctively in the following sections, and use the upper script $\hat{}$ for variables describing investors. This microframework is similar to the one developed in (Gosselin, Lotz, Wambst 2022), with the exception of two minor adjustments that do not affect the collective level, but that will provide further insight into individual dynamics.

7.1 Producers

Producers are modeled by firms evolving in a sector space composed of an infinite number of sectors. In this framework, the notions of firm and sector are flexible. A single firm with subsidiaries in different countries or offering differentiated products can be modeled as independent firms. Similarly, a sector refers to a group of firms with similar activities, but this criterion is loose: sectors can be decomposed into sectors per country, to account for local specificities, or in several sectors for that matter.

Producers move across sectors described by a vector space of arbitrary dimension. The position of producer i in this space is denoted X_i and his physical capital, K_i. Producers are defined by these two variables, which are both subject to dynamic changes. Producers may change their capital stocks over time or altogether shift sectors.

Each firm produces a single differentiated good. However, in the following, we will merely consider the return each producer may provide to its investors.

The return of producer i at time t, denoted r_i, depends on K_i, X_i and on the level of competition in the sector. It is written:

$$r_i = r\left(K_i, X_i\right) - \gamma \sum_j \delta\left(X_i - X_j\right) \frac{K_j}{K_i} \tag{26}$$

The first term is an arbitrary function that depends on the sector and the level of capital per firm in this sector. It represents the return of capital in a specific sector X_i under no competition. We deliberately keep the form of $r\left(K_i, X_i\right)$ unspecified, since most of the results of the model rely on the general properties of the functions involved. When needed, we will give a standard Cobb-Douglas form to the returns $r\left(K_i, X_i\right)$. The second term in (26) is the decreasing return of capital. In any given sector, it is proportional to both the number of competitors and the specific level of capital per firm used.

We also assume that, for all i, firm i has a market valuation defined by both its price, P_i, and the variation of this price on financial markets, $\dot{P}_i$. This variation is itself assumed to be a function of an expected long-term return denoted $R\left(K_i, X_i\right)$, and more precisely the relative return $\bar{R}\left(K_i, X_i\right)$ of firm i against the whole set of firms in its sector:

$$\frac{\dot{P}_i}{P_i} = F_1\left(\bar{R}\left(K_i, X_i\right)\right) \tag{27}$$

with:

$$\bar{R}\left(K_i, X_i\right) = \frac{R\left(K_i, X_i\right)}{\sum_l R\left(K_l, X_l\right)} \tag{28}$$

The function F_1 is arbitrary and reflects the preferences of the market relatively to the firm's relative returns.

We assume that firms shift their production in the sector space according to returns, in the direction of the gradient of the expected long-term return $R(K_i, X_i)$. Yet, the accumulation of agents at any point of the space creates a repulsive force, so that the evolution of X_i minimizes, up to some shocks, the following quantity:

$$L_i\left(X_i, \frac{dX_i}{dt}\right) = \left(\frac{dX_i}{dt} - \nabla_X R(K_i, X_i) H(K_i)\right)^2 + \tau \frac{K_{X_i}}{K_i} \sum_j \delta(X_i - X_j) \tag{29}$$

where K_{X_i} is the average capital per firm in sector X_i.

When $\tau = 0$, there are no repulsive forces and the move towards the gradient of R is given by the expression:

$$\frac{dX_i}{dt} = \nabla_X R(K_i, X_i) H(K_i)$$

When $\tau \neq 0$, repulsive forces deviate the trajectory. The dynamic equation associated to the minimization of (29) is given by the general formula of the dynamic optimization:

$$\frac{d}{dt} \frac{\partial}{\partial \frac{dX_i}{dt}} L_i\left(X_i, \frac{dX_i}{dt}\right) = \frac{\partial}{\partial X_i} L_i\left(X_i, \frac{dX_i}{dt}\right) \tag{30}$$

This last equation does not need to be developed further, since formula (29) is sufficient to switch to the field description of the system. Note for later purpose that the expression $\frac{dX_i}{dt}$ stands for the continuous version of a discrete variation, $X_i(t+1) - X_i(t)$.

Finally, it should be noted that we have introduced a difference from our previous work. In (Gosselin, Lotz, Wambst, 2022), we assumed a constant competition term τ between firms for simplicity. However, in this work, we focus on individual dynamics within the background field, and depart slightly from this assumption. We will consider that the strength of competition depends on the value of the firm's capital, and replace:

$$\tau \to \tau \frac{K_{X_i}}{K_i} \tag{31}$$

This modification does not affect the system on average, i.e. at the collective level, but it allows for more precise results about the effect of interactions at the individual level. Formula (31) models that the lower a firm's capital is compared to the sector average, the stronger the effect of competition.

7.2 Investors

Each investor j is defined by his level of capital $\hat{K}_j$ and his position $\hat{X}_j$ in the sector space. Investors can invest in the entire sector space, but tend to invest in sectors close to their position.

Besides, investors tend to diversify their capital: each investor j chose to allocate parts of his entire capital $\hat{K}_j$ between various firms i. The capital allocated by investor j to firm i is denoted $\hat{K}_j^{(i)}$, and given by:

$$\hat{K}_j^{(i)}(t) = \left(\hat{F}_2(s_i, R(K_i, X_i)) \hat{K}_j\right)(t) \tag{32}$$

where:

$$\hat{F}_2\left(s_i, R(K_i, X_i), \hat{X}_j\right) = \frac{F_2(s_i, R(K_i, X_i)) G\left(X_i - \hat{X}_j\right)}{\sum_l F_2(s_l, R(K_l, X_l)) G\left(X_l - \hat{X}_j\right)} \tag{33}$$

The function F_2 is arbitrary. It depends on the expected return of firm i and on the distance between sectors X_i and $\hat{X}_j$. The function $\hat{F}_2\left(s_i, R(K_i, X_i), \hat{X}_j\right)$ is the relative version of F_2 that reflects the dependence of investments on the relative attractiveness of firms.

Compared to our previous work, we introduced a variable s_i that individualize the shape of $\hat{F}_2$ for the firm, whereas in (Gosselin, Lotz, Wambst 2022) $\hat{F}_2$ was only capital and sector-dependent. On average, this modification does not change our previous results regarding the background field. However, it enables us to understand the role of investors' perception of a particular firm in individual transitions. Mathematically, s is a variable that determines the shape of the attractiveness function $\hat{F}_2$ for the firm. While a dynamics equation for this variable could be introduced, treating s as static is sufficient for studying the influence of this shape parameter on capital dynamics, without specifying its interaction with thecapital level and sector shifts.

We now define ε as the time scale for capital accumulation. The variation in capital of investor j between t and $t+\varepsilon$ is the sum of two terms: the short-term returns r_i of the firms in which j invested, and the stock price variations of these same firms:

$$\hat{K}_j(t+\varepsilon) - \hat{K}_j(t) = \sum_i \left(r_i + \frac{\dot{P}_i}{P_i} \right) \hat{K}_j^{(i)} = \sum_i \left(r_i + F_1 \left(\bar{R}(K_i, X_i), \frac{\dot{K}_i(t)}{K_i(t)} \right) \right) \hat{K}_j^{(i)} \tag{34}$$

Note that in equation (29), the time scale of firms' displacement within the sectors space is normalized to one. On the contrary, in equation (34) we assume that the time scale for investors shifts is ε, where $\varepsilon << 1$, meaning that capital dynamics is faster than firms' mobility within the sectors space. To rewrite (34) on the same time-span as $\frac{dX_i}{dt}$, we write:

$$\begin{aligned}
\hat{K}_j(t+1) - \hat{K}_j(t) &= \sum_{k=1}^{\frac{1}{\varepsilon}} \hat{K}_j(t+k\varepsilon) - \hat{K}_j(t) \\
&= \sum_{k=1}^{\frac{1}{\varepsilon}} \sum_i \left(r_i + \frac{\dot{P}_i}{P_i} \right) \hat{K}_j^{(i)}(t+k\varepsilon) \\
&\simeq \frac{1}{\varepsilon} \sum_i \left(r_i + F_1 \left(\bar{R}(K_i, X_i), \frac{\dot{K}_i(t)}{K_i(t)} \right) \right) \hat{K}_j^{(i)}
\end{aligned}$$

where the quantities in the sum have to be understood as averages over the time span $[t, t+1]$. Using equation (27), equation (34) becomes in the continuous approximation:

$$\frac{d}{dt}\hat{K}_j(t) = \frac{1}{\varepsilon} \sum_i \left(r_i + F_1 \left(\frac{R(K_i, X_i)}{\sum_l \delta(X_l - X_i) R(K_l, X_l)}, \frac{\dot{K}_i(t)}{K_i(t)} \right) \right) \hat{F}_2\left(s_i, R(K_i, X_i), \hat{X}_j \right) \hat{K}_j \tag{35}$$

where $\frac{d}{dt}\hat{K}_j(t) = \hat{K}_j(t+1) - \hat{K}_j(t)$ is now normalized to the time scale of $\frac{dX_i}{dt}$, i.e. 1.

7.3 Interactions between financial and physical capital

The entire financial capital is, at any time, completely allocated by investors between firms. For producers, there is no alternative source of financing: self-financing is discarded, since it amounts to consider two agents, a producer and an investor, as one. The physical capital of a any given firm is thus the sum of all capital allocated to this firm by all its investors. Physical capital entirely depends on the arbitrage and allocations of the financial sector. Firms do not possess their capital: they return it fully at the end of each period with a dividend, though possibly negative. Investors then entirely reallocate their capital between firms at the beginning of the next period.

This set up may not be fully accurate in the short-run but, since physical capital cannot subsist without investment, it holds in the long-run. When investors choose not to finance a firm, this firm is bound to disappear in the long run. Under these assumptions, the following identity holds:

$$K_i(t+\varepsilon) = \sum_j \hat{K}_j^{(i)} = \sum_j \hat{F}_2\left(s_i, R(K_i(t), X_i(t)), \hat{X}_j(t) \right) \hat{K}_j(t) \tag{36}$$

where K_i stands for the physical capital of firm i at time t, and $\sum_j \hat{K}_j^{(i)}$ for the sum of capital invested in firm i by investors j. Recall that the parameter ε accounts for the specific time scale of capital accumulation. It differs from that of mobility within the sector space (29), which is normalized to one.

The dynamics (36) rewrites:

$$\frac{K_i(t+\varepsilon)-K_i(t)}{\varepsilon}=\frac{1}{\varepsilon}\left(\sum_j \hat{F}_2\left(s_i,R\left(K_i(t),X_i(t)\right),\hat{X}_j(t)\right)\hat{K}_j(t)-K_i(t)\right) \tag{37}$$

Using the same token as in the derivation of (35), we obtain in the continuous approximation:

$$\frac{d}{dt}K_i(t)+\frac{1}{\varepsilon}\left(K_i(t)-\sum_j \hat{F}_2\left(s_i,R\left(K_i(t),X_i(t)\right),\hat{X}_j(t)\right)\hat{K}_j(t)\right)=0 \tag{38}$$

where $\frac{d}{dt}K_i(t)$ stands for $K_i(t+1)-K_i(t)$.

7.4 Capital allocation dynamics

Investors choose to allocate their capital within sectors, and may modify their portfolio according to the returns of the sector or firms they invest in. This is modelled by a move along the sectors' space in the direction of the gradient of $R(K_i,X_i)$ Investors located within a particular sector shift to neighbouring sectors that offer higher expected long-term returns. The shift of $\hat{X}_j$ is described by a dynamic equation that models its movement over time.

$$\frac{d}{dt}\hat{X}_j-\frac{1}{\sum_i \delta\left(X_i-\hat{X}_j\right)}\sum_i\left(\nabla_{\hat{X}}F_0\left(R\left(K_i,\hat{X}_j\right)\right)+\nu\nabla_{\hat{X}}F_1\left(\bar{R}\left(K_i,\hat{X}_j\right)\right)\right)=0 \tag{39}$$

where the factor $\sum_i \delta\left(X_i-\hat{X}_j\right)$ is the agents' density in the sector $\hat{X}_j$, so that the more competitors in a sector, the slower the move.

In equation (39), $F_0\left(R\left(K_i,\hat{X}_j\right)\right)$ is a function of long-term returns representing the tendency of investors to invest in sectors with the highest returns. The gradient $\nabla_{\hat{X}}F_0\left(R\left(K_i,\hat{X}_j\right)\right)$ induces a shift of investors towards sectors with the highest returns, as described in equation (39).

The function $F_1\left(\bar{R}\left(K_i,\hat{X}_j\right)\right)$ measures the expected variations in the stock prices of the firm, so that $\nu\nabla_{\hat{X}}F_1\left(\bar{R}\left(K_i,\hat{X}_j\right)\right)$ describes the investors' tendency to shift towards stocks with the highest price-dividend ratio.

8 Translation of the framework

Let us now use the method summed-up above to translate the classical framework developed in section 3. The details are given in (Gosselin Lotz Wambst 2022).

There are two types of agents in our model, firms and investors. Each type of agent is described by two dynamic equations, thus, there are four minimization functions to find. These minimization functions will be translated into four functionals of two independent fields[8], one for producers $\Psi(K,X)$[9] and one for investors $\hat{\Psi}\left(\hat{K},\hat{X}\right)$. The sum of these four functionals will be the "field action functional" that describes the whole system in terms of fields[10].

8.0.1 The Real Economy

We show in (Gosselin Lotz Wambst 2022) that the capital allocation dynamics (29) and the capital accumulation dynamics (38) have the following field equivalents.

[8] The term functional refers to a function of a function, i.e. a function whose argument is itself a function.

[9] Note that the field should be writen $\Psi(K,X,s)$ to include the variable s. However, since s_i is static, we can omit it when writing the fields. It will only appear as a parameter in the formula for $\hat{F}_2$ and in the transition functions.

[10] Details about the probabilistic step will be given as a reminder along the text and in appendix 1.

Field action functional for physical capital allocation

$$S_1 = -\int \Psi^{\dagger}(K,X)\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X-\nabla_X R(K,X)H(K)\right)\Psi(K,X)\,dKdX \tag{40}$$
$$+\tau\int\left|\Psi(K',X)\right|^2\left|\Psi(K,X)\right|^2 dK'dKdX$$

Field action functional for physical capital accululation

$$S_2=-\int\Psi^{\dagger}(K,X)\nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K+\frac{1}{\varepsilon}\left(K-\int\hat{F}_2\left(s,R\left(K,\hat{X}\right),\hat{X}\right)\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\hat{K}d\hat{X}\right)\right)\Psi(K,X) \tag{41}$$

with:

$$\hat{F}_2\left(s,R(K,),\hat{X}\right)=\frac{F_2\left(s,R(K,X),\hat{X}\right)G\left(X-\hat{X}\right)}{\int F_2\left(R(K',X'),\hat{X}\right)G\left(X'-\hat{X}\right)\left|\Psi(K',X')\right|^2 d(K',X')} \tag{42}$$

Here:

$$F_2\left(R(K',X'),\hat{X}\right)$$

is the average over the parameter s' of:

$$F_2\left(s',R(K',X'),\hat{X}\right)$$

8.0.2 Financial markets

Financial capital dynamics and financial capital allocation are given by equations (35) and (39) respectively. Both expressions include a time derivative and are thus of type (9). Their translation is straightforward.

Field action functional for financial capital dynamics

$$S_3 = -\int\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}}-\frac{\hat{K}}{\varepsilon}\int\left(r(K,X)-\gamma\frac{\int K'\left\|\Psi(K',X)\right\|^2}{K}\right.\right. \tag{43}$$
$$\left.\left.+F_1\left(\bar{R}(K,X),\Gamma(K,X)\right)\right)\hat{F}_2\left(s,R(K,X),\hat{X}\right)\left\|\Psi(K,X)\right\|^2 d(K,X)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right)$$

where:

$$\bar{R}(K,X) = \frac{R(K,X)}{\int R(K',X')\left\|\Psi(K',X')\right\|^2 d(K',X')} \tag{44}$$

$$\Gamma(K,X) = \frac{\int\hat{F}_2\left(s,R(K,X),\hat{X}\right)\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\hat{K}d\hat{X}}{K}-1 \tag{45}$$

Field action for financial capital allocation

$$S_4 = -\int\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right) \tag{46}$$
$$\times\left(\nabla_{\hat{X}}\sigma_{\hat{X}}^2\left(\nabla_{\hat{X}}-\int\left(\frac{\nabla_{\hat{X}}F_0\left(R\left(K,\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_1\left(\bar{R}(K,X),\Gamma(K,X)\right)}{\int\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\right)\left\|\Psi\left(K,\hat{X}\right)\right\|^2 dK\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right)$$

8.0.3 Gathering contributions: the action functional

Once these translations are performed, the action functional of the system is described by the sum of all the contributions, i.e. (40),(41),(6),(46):

$$S = S_1 + S_2 + S_3 + S_4$$

We can assume to simplify that investors invest in only one sector, which translates into the following condition:

$$G\left(X - \hat{X}\right) = \delta\left(X - \hat{X}\right) \tag{47}$$

This simplification does not reduce the generality of our model since an investor acting in several sectors can be modelled as an aggregation of several investors possibly moving from one sector to another.

A compact form for the action functional S can be written:

$$\begin{aligned} S \;=\; & -\int \Psi^{\dagger}(K,X)\left(\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X - \nabla_X R(K,X)\,H(K)\right) - \tau\left(\int \left|\Psi(K',X)\right|^2 dK'\right)\right. \qquad (48)\\ & \left. + \nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K + u\left(K,X,\Psi,\hat{\Psi}\right)\right)\right)\Psi(K,X)\,dK\,dX \\ & -\int \hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} - \hat{K} f\left(\hat{X},\Psi,\hat{\Psi}\right)\right) + \nabla_{\hat{X}}\left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}} - g\left(K,X,\Psi,\hat{\Psi}\right)\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right) \end{aligned}$$

where we have defined:

$$u\left(K,X,\Psi,\hat{\Psi}\right) = \frac{1}{\varepsilon}\left(K - \int \hat{F}_2\left(s,R(K,X)\right)\hat{K}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 d\hat{K}\right) \tag{49}$$

$$f\left(\hat{X},\Psi,\hat{\Psi}\right) = \frac{1}{\varepsilon}\int\left(r(K,X) - \frac{\gamma\int K'\left|\Psi(K,X)\right|^2}{K} + F_1\left(\bar{R}(K,X),\Gamma(K,X)\right)\right) \tag{50}$$

$$\times \hat{F}_2\left(R(K,X)\right)\left|\Psi\left(K,\hat{X}\right)\right|^2 dK \tag{51}$$

$$g\left(K,\hat{X},\Psi,\hat{\Psi}\right) = \int \frac{\nabla_{\hat{X}} F_0\left(R\left(K,\hat{X}\right)\right) + \nu\nabla_{\hat{X}} F_1\left(\bar{R}\left(K,\hat{X}\right),\Gamma(K,X)\right)}{\int\left|\Psi\left(K',\hat{X}\right)\right|^2 dK'}\left|\Psi\left(K,\hat{X}\right)\right|^2 dK \tag{52}$$

and where $\bar{R}(K,X)$ is given by (44). Under assumption (47), functions $\hat{F}_2$ and Γ write:

$$\hat{F}_2\left(s,R(K,X)\right) = \frac{F_2\left(s,R(K,X)\right)}{\int F_2\left(R(K',X)\right)\left|\Psi(K',X)\right|^2 dK'} \tag{53}$$

$$\Gamma(K,X) = \frac{\int \hat{F}_2\left(s,R(K,X)\right)\hat{K}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 d\hat{K}}{K} - 1 \tag{54}$$

where:

$$F_2\left(R(K',X)\right)$$

is the average over the parameter s' of:

$$F_2\left(s',R(K',X)\right)$$

Recall that $H(K_X)$ is a function that encompasses the determinants of the firms' mobility across the sector space. We will specify later this function as a function of expected long term-returns and capital.

Function u describes the evolution of the capital of a firm, that is located at X. Its dynamics depends on the relative value of a function F_2 that is itself a function of the firms' expected returns $R(K,X)$. Investors allocate their capital based on their expectations of the firms' long-term returns.

Function f describes the returns of investors located at $\hat{X}$ investing in sector X a capital K. These returns depend on short-term dividends $r(K,X)$, the field-equivalent cost of capital $\frac{\gamma\int K'\|\Psi(K,X)\|^2}{K}$, and firms' expected long-term stock valuations through the function F_1.. These returns themselves depend on the relative attractiveness of a firm expected long-term returns vis-a-vis its competitors.

Function g describes investors' reallocation of capital across the sectors' space. These reallocation are driven by the gradient of expected long-term returns and stocks valuations.

Recall that here we depart from the general formalism. We do not introduce a time variable in the present model. Our purpose is to find collective, or characteristic, configurations of the system that can, as such, be considered static. It is only when we will derive these configurations that a macro time scale will be introduced to study how the background states evolve over time.

9 Computation of transition functions

We use the results of section 5.2 to compute the agents' transition functions. To do so we need to derive the background fields of the system and then compute the effective action of the system by expanding the action functional around these background fields.

9.1 Background fields and averages

Minimizing the field action functional S yields the background field of the system. In turn, the background fields allow to compute the average quantities of the system in that particular state. As shown in (Gosselin Lotz Wambst 2022) both averages and background field are interrelated, and that the system is solved by finding a system of equations for the fields and the averages.

We first recall the definitions of the averages and then the solutions for the background field and the defining equation for average capital that define the system at the collective level.

9.1.1 Averages

At the collective level, the squared background fields of the system $|\Psi(K,X)|^2$ and $\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2$, represent the density of firms and the density of investors per sector and for a given capital K, in the collective state defined by $\Psi(K,X)$ and $\hat{\Psi}\left(\hat{K},\hat{X}\right)$. Thus, the collective state determines, for each sector and for a given capital K, the density of firms and the density of investors.

Moreover, these two functions squared allow to compute various global quantities of the system in the collective state $\Psi(K,X)$ and $\hat{\Psi}\left(\hat{K},\hat{X}\right)$.

The number of producers in sector X, $N(X)$ and the number of investors in sector $\hat{X}$, $\hat{N}\left(\hat{X}\right)$ are computed using the formula:

$$N(X) = \int |\Psi(K,X)|^2 dK \tag{55}$$

$$\hat{N}\left(\hat{X}\right) = \int \left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\hat{K} \tag{56}$$

The average values of total invested capital $\hat{K}_X$ for each sector X is:

$$\hat{K}_{\hat{X}} = \int \hat{K}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 d\hat{K}$$

and the average invested capital per firm for sector X is:

$$K_X = \frac{\int \hat{K}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 d\hat{K}}{N(X)} \tag{57}$$

Note that this K_X is also equal to the average physical capital per firm for sector X, i.e. :

$$K_X = \frac{\int K |\Psi(K,X)|^2 dK}{N(X)} \tag{58}$$

Indeed, given our assumptions, the total physical capital is equal to the total capital invested:

$$\int K \left|\Psi(K,X)\right|^2 dK = \int \hat{K} \left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\hat{K}$$

Ultimately, a collective state determines the distribution of invested capital and capital per firm across sectors, which are given by $\frac{\left|\hat{\Psi}(\hat{K},X)\right|^2}{\hat{N}(X)}$ and $\frac{|\Psi(K,X)|^2}{N(X)}$ respectively.

Equations (55), (56) and (57) show that each collective state defined by $\Psi(K,X)$ and $\hat{\Psi}\left(\hat{K},\hat{X}\right)$ is determined by the collection of data that characterizes each sector: the number of firms for each sector, the number of investors for each sector, the average capital for each sector and the density of distribution of capital in each sector.

The above quantities allow for the study of capital allocation among sectors and how it depends on system parameters such as expected long-term return, short-term return, and any other parameters involved in the model. These quantities shape the landscape in which individual agents operate.

9.1.2 Background fields

In (Gosselin Lotz Wambst 2022) we derived expressions for the background fields and the densities of agents (firms and investors). This ultimately led us to find a defining equation for the average capital invested per sector.

Background field $\Psi(K,X)$ In (Gosselin Lotz Wambst 2022), we computed the density of firms in a given sector as a function of the average capital in that sector. This density is given by:

$$\|\Psi(X)\|^2 = \mathcal{N} \exp\left(-\left(K - \hat{F}_2\left(s, R(K,X)\right) K_X\right)^2\right) \|\Psi(X)\|^2$$

Here, $\mathcal{N}$ is a normalization factor and K_X represents the average invested capital per firm in sector X. This quantity is given by equations (57) or (58), and:

$$\|\Psi(X)\|^2 = \frac{D\left(\|\Psi\|^2\right)}{2\tau} - \frac{1}{4\tau}\left(\left(\nabla_X R(X)\right)^2 + \frac{\sigma_X^2 \nabla_X^2 R(K_X,X)}{H(K_X)}\right)\left(1 - \frac{H'\left(\hat{K}_X\right) K_X}{H\left(\hat{K}_X\right)}\right) H^2(K_X) \tag{59}$$

The constant $D\left(\|\Psi\|^2\right)$ depends on the norm of the background field.

Formula (59) will be useful below while computing the transition functions.

Background fields $\hat{\Psi}\left(\hat{X},\hat{K}\right)$ and average capital per sector The density of investors per sectors in the background field has the form:

$$\left|\hat{\Psi}_{-M}\left(\hat{K},\hat{X}\right)\right|^2 \simeq C(\bar{p}) \exp\left(-\frac{\sigma_X^2 \hat{K}^4 \left(f'(X)\right)^2}{96\sigma_{\hat{K}}^2 \left|f\left(\hat{X}\right)\right|}\right) D^2_{p(\hat{X})}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^2}\right)^{\frac{1}{2}}\left(\hat{K} + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}\right)}{f^2\left(\hat{X}\right)}\right)\right) \tag{60}$$

where D_p is the parabolic cylinder function with parameter $p\left(\hat{X}\right)$ and:

$$p\left(\hat{X}\right) = \frac{M - A\left(\hat{X}\right)}{\sqrt{f^2\left(\hat{X}\right)}} \tag{61}$$

The constant $C(\bar{p})$ ensures that the constraint given by equation (56) is satisfied.

Although the background field for the financial sector does not directly appear in the transition functions, it is central to computing the average quantities for each sector. In fact, it directly determines $K_{\hat{X}}$, the average capital per firm in sector $\hat{X}$, in this environment. Indeed, the defining equation of $K_{\hat{X}}$, given in (**??**), writes:

$$K_{\hat{X}} \left\| \Psi\left(\hat{X}\right) \right\|^2 = \int \hat{K} \left\| \hat{\Psi}\left(\hat{K}, \hat{X}\right) \right\|^2 d\hat{K} \tag{62}$$

This equation is actually an equation for $K_{\hat{X}}$. Actually, we have expressed in equation (59) the squared background field $\left\| \Psi\left(\hat{X}\right) \right\|^2$ as a function of $K_{\hat{X}}$, and the field $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$, and $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$ is itself a function of $K_{\hat{X}}$ through equation (60).

We showed that depending on the model parameters, several possible patterns of accumulation exist in each sector.

9.2 Effective action expansion

9.2.1 Second-order expansion of effective action

Consider the field action:

$$S = S_1 + S_2 + S_3 + S_4$$

where the S_i are defined by equations (40),(41),(6) and (46). Expanding the action S to the second-order around the background field will allow us to compute the transition functions of individual agents in the background, without taking into account individual interactions. We can rewrite the fields as follows:

$$\begin{aligned} \Psi(K, X) &= \Psi_0(K, X) + \Delta\Psi(Z, \theta) \\ \hat{\Psi}\left(\hat{K}, \hat{X}\right) &= \hat{\Psi}_0\left(\hat{K}, \hat{X}\right) + \Delta\hat{\Psi}(Z, \theta) \end{aligned}$$

where $\Psi_0(K, X), \hat{\Psi}_0\left(\hat{K}, \hat{X}\right)$ are the background fields. This yields the quadratic approximation:

$$S\left(\Psi, \hat{\Psi}\right) = S\left(\Psi_0, \hat{\Psi}_0\right) + \int \left(\Delta\Psi^{\dagger}(Z, \theta), \Delta\hat{\Psi}^{\dagger}(Z, \theta)\right)(Z, \theta)\, O\left(\Psi_0(Z, \theta)\right) \begin{pmatrix} \Delta\Psi(Z, \theta) \\ \Delta\hat{\Psi}(Z, \theta) \end{pmatrix} \tag{63}$$

with:

$$O\left(\Psi_0(Z, \theta)\right) = \begin{pmatrix} \frac{\delta^2 S(\Psi)}{\delta\Psi^{\dagger}\delta\Psi} & \frac{\delta^2 S(\Psi)}{\delta\Psi^{\dagger}(Z,\theta)\delta\hat{\Psi}} \\ \frac{\delta^2 S(\Psi)}{\delta\hat{\Psi}^{\dagger}\delta\Psi} & \frac{\delta^2 S(\Psi)}{\delta\hat{\Psi}^{\dagger}\delta\hat{\Psi}} \end{pmatrix}_{\substack{\Psi(Z,\theta)=\Psi_0(Z,\theta) \\ \hat{\Psi}(Z,\theta)=\hat{\Psi}_0(Z,\theta)}} \tag{64}$$

The anti-diagonal terms in equation (64) involve crossed derivatives with respect to both the fields of the real economy and the financial economy. These terms represent the interactions between the two economies. However, as explained in (Gosselin Lotz Wambst 2022), the cross-dependency between $\Psi(Z, \theta)$ and $\hat{\Psi}(Z, \theta)$ is relatively weak, since these interactions are taken into account by the background fields. In first approximation, the minimization of $S(\Psi)$ can be separated between $S_1 + S_2$ and $S_3 + S_4$. Therefore, we can write:

$$O\left(\Psi_0(Z, \theta)\right) \simeq \begin{pmatrix} \frac{\delta^2(S_1+S_2)}{\delta\Psi^{\dagger}\delta\Psi} & 0 \\ 0 & \frac{\delta^2(S_3(\Psi)+S_4(\Psi))}{\delta\hat{\Psi}^{\dagger}\delta\hat{\Psi}} \end{pmatrix}_{\substack{\Psi(Z,\theta)=\Psi_0(Z,\theta) \\ \hat{\Psi}(Z,\theta)=\hat{\Psi}_0(Z,\theta)}} \tag{65}$$

The second-order expansion then becomes:

$$\begin{aligned} S\left(\Psi, \hat{\Psi}\right) &= S\left(\Psi_0, \hat{\Psi}_0\right) + \Delta S_2\left(\Psi, \hat{\Psi}\right) \\ &= S\left(\Psi_0, \hat{\Psi}_0\right) + \int \Delta\Psi^{\dagger}(K, X) \frac{\delta^2(S_1 + S_2)}{\delta\Psi^{\dagger}(Z, \theta)\,\delta\Psi(Z, \theta)} \Delta\Psi(K, \theta) \\ &\quad + \int \Delta\hat{\Psi}^{\dagger}(Z, \theta) \frac{\delta^2(S_3(\Psi) + S_4(\Psi))}{\delta\hat{\Psi}^{\dagger}(Z, \theta)\,\delta\hat{\Psi}(Z, \theta)} \Delta\hat{\Psi}(Z, \theta) \end{aligned} \tag{66}$$

Computing the second order derivatives involved in (66), and using the definition of the background fields (see appendix 1) leads to the formulas:

$$\frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^{\dagger}\left(Z,\theta\right)\delta\Psi\left(Z,\theta\right)} = -\frac{\sigma_X^2}{2}\nabla_X^2-\frac{\sigma_K^2}{2}\nabla_K^2+\left(D\left(\left\Vert\Psi\right\Vert^2\right)+2\tau\frac{\left|\Psi\left(X\right)\right|^2\left(K_X-K\right)}{K}\right) +\frac{1}{2\sigma_K^2}\left(K-\hat{F}_2\left(R\left(K,X\right)\right)K_X\right)^2+\frac{1-\nabla_K\hat{F}_2\left(R\left(K,X\right)\right)K_X}{2}$$

$$\frac{\delta^2\left(S_3\left(\Psi\right)+S_4\left(\Psi\right)\right)}{\delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\delta\hat{\Psi}\left(Z,\theta\right)} = \left(-\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2+\frac{\left(g\left(\hat{X}\right)\right)^2+\sigma_{\hat{X}}^2\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}} -\frac{\sigma_{\hat{K}}^2}{2\sqrt{f^2\left(\hat{X}\right)}}\nabla_{\hat{K}}^2+\left(\frac{\sqrt{f^2\left(\hat{X}\right)}\left(\hat{K}+\frac{\sigma_{\hat{K}}^2F\left(\hat{X},K_{\hat{X}}\right)}{f^2\left(\hat{X}\right)}\right)^2}{4\sigma_{\hat{K}}^2}\right)\right)$$

9.2.2 Fourth-order corrections

Calculating the fourth-order corrections to the effective action is sufficient for deriving the main aspects of the interactions in a given background field. We show in appendix 2 that the third-order terms can be neglected, and that the series expansion of the action to the fourth-order writes:

$$\begin{aligned} S\left(\Psi,\hat{\Psi}\right) &= S\left(\Psi_0,\hat{\Psi}_0\right) \\ &+\int\Delta\Psi^{\dagger}\left(K,X\right)\frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^{\dagger}\left(Z,\theta\right)\delta\Psi\left(Z,\theta\right)}\Delta\Psi\left(K,\theta\right) \\ &+\int\Delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\frac{\delta^2\left(S_3\left(\Psi\right)+S_4\left(\Psi\right)\right)}{\delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\delta\hat{\Psi}\left(Z,\theta\right)}\Delta\hat{\Psi}\left(Z,\theta\right)+\Delta S_4\left(\Psi,\hat{\Psi}\right)\end{aligned} \tag{67}$$

with:

$$\begin{aligned} &\Delta S_4\left(\Psi,\hat{\Psi}\right) \\ \simeq\ & 2\tau\int\left|\Delta\Psi\left(K',X\right)\right|^2dK'\left|\Delta\Psi\left(K,X\right)\right|^2dKdX \\ &-\Delta\Psi^{\dagger}\left(K,X\right)\Delta\Psi^{\dagger}\left(K',X'\right)\nabla_K\frac{\delta^2u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}\Delta\Psi\left(K',X'\right)\Delta\Psi\left(K,X\right) \\ &-\Delta\Psi^{\dagger}\left(K,\theta\right)\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\theta\right)\nabla_K\frac{\delta^2u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}\Delta\hat{\Psi}\left(\hat{K},\theta\right)\Delta\Psi\left(K,\theta\right) \\ &-\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\Delta\Psi^{\dagger}\left(K',\theta\right)\left\{\nabla_{\hat{K}}\frac{\hat{K}\delta^2f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}+\nabla_{\hat{X}}\frac{\delta^2g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}\right\}\Delta\Psi\left(K',X'\right)\Delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\end{aligned} \tag{68}$$

Computing the terms involved in (68) (see appendix 2) allows us to interpret the various terms arising in the correction to the action.

The first term in the right-hand side of (68) describes the direct repulsive interaction between firms due to competition in a given sector.

The second term describes the indirect competition between firms through capital allocation by investors, since:

$$\frac{\delta^2 u\left(K, X, \Psi, \hat{\Psi}\right)}{\delta\Psi\left(K', X\right)\delta\Psi^{\dagger}\left(K', X\right)} = -\frac{1}{\varepsilon}\hat{F}_2\left(s, R\left(K, X\right)\right)\hat{F}_2\left(s, R\left(K', X'\right)\right)\hat{K}_X \tag{69}$$

and this term involves the relative attractiveness of two firms with capital K and K' respectively in sector X.

The third term represents the firms-investors direct interactions through investment, since:

$$\frac{\delta^2 u\left(K, X, \Psi, \hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K}, \hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K}, \hat{X}\right)} = \frac{1}{\varepsilon}\hat{F}_2\left(s, R\left(K, X\right)\right)\hat{K} \tag{70}$$

is the relative attractiveness of a firm with capital K' at sector X.

The last term describes the variation of investement due to the relative short-term and long-term return of a given firm. Specifically, we have:

$$\frac{\delta^2 f\left(\hat{X}, \Psi, \hat{\Psi}\right)}{\delta\Psi\left(K', X\right)\delta\Psi^{\dagger}\left(K', X\right)} \simeq \frac{1}{\varepsilon}\left(\Delta f\left(K', \hat{X}, \Psi, \hat{\Psi}\right) - \gamma\frac{K'}{K_X}\right) \tag{71}$$

and:

$$\frac{\delta^2 g\left(\hat{X}, \Psi, \hat{\Psi}\right)}{\delta\Psi\left(K', X\right)\delta\Psi^{\dagger}\left(K', X\right)} = \frac{1}{\int\left\|\Psi\left(K', \hat{X}\right)\right\|^2 dK'}\Delta\left(g\left(K', \hat{X}, \Psi, \hat{\Psi}\right)\right) \tag{72}$$

with:

$$\Delta f\left(K', \hat{X}, \Psi, \hat{\Psi}\right) = f\left(K', \hat{X}, \Psi, \hat{\Psi}\right) - f\left(\hat{X}, \Psi, \hat{\Psi}\right)$$

and:

$$\Delta g\left(K', \hat{X}, \Psi, \hat{\Psi}\right) = g\left(K', \hat{X}, \Psi, \hat{\Psi}\right) - g\left(\hat{X}, \Psi, \hat{\Psi}\right)$$

are the relative short-term return and long-term return for firm with capital K' at sector $\hat{X}$ respectively.

9.3 One agent transition functions

Following section 5.2.4, we consider first the "free" transition functions that are given by the inverse operator of:

$$\left(O\left(\Psi_0\left(Z, \theta\right)\right) + \alpha\right)^{-1} \tag{73}$$

Given (65), the inverse (73) reduces to:

$$\begin{pmatrix} \left(\frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^{\dagger}\delta\Psi} + \alpha\right)^{-1} & 0 \\ 0 & \left(\frac{\delta^2\left(S_3(\Psi)+S_4(\Psi)\right)}{\delta\hat{\Psi}^{\dagger}\delta\hat{\Psi}} + \alpha\right)^{-1} \end{pmatrix}_{\substack{\Psi(Z,\theta)=\Psi_0(Z,\theta) \\ \hat{\Psi}(Z,\theta)=\hat{\Psi}_0(Z,\theta)}}$$

This implies that the transition functions can be computed independently for the individual firms and investors. We will write:

$$G_1\left(\left(K_f, X_f\right), \left(X_i, K_i\right), \alpha\right)$$

the transition probability for a firm between an initial state (X_i, K_i) and a final state (K_f, X_f) during an average timespan α^{-1} and:

$$G_2\left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right), \alpha\right)$$

the transition probability for a firm between an initial state $\left(\hat{X}_i, \hat{K}_i\right)$ and a final state $\left(\hat{K}_f, \hat{X}_f\right)$ average timespan α^{-1}. Appendix 3 computes these transition functions. We find the following results.

One firm transition function

$$\begin{aligned} & G_1\left(\left(K_f, X_f\right), \left(X_i, K_i\right)\right) \\ = \quad & \exp\left(D\left(\left(K_f, X_f\right), \left(X_i, K_i\right)\right) - \alpha_{eff}\left(\Psi, \left(K_f, X_f\right), \left(X_i, K_i\right)\right)\sqrt{\frac{\left(X_f - X_i\right)^2}{2\sigma_X^2} + \frac{\left(K_f' - K_i'\right)^2}{2\sigma_K^2}}\right) \end{aligned} \tag{74}$$

where:

$$D\left(\left(K_f, X_f\right), \left(X_i, K_i\right)\right) = D_1 + D_2 + D_3 \tag{75}$$

with:

$$D_1 = \int_{X_i}^{X_f} \frac{\nabla_X R\left(K_X, X\right)}{\sigma_X^2} H\left(K_X\right) \tag{76}$$

$$D_2 = -\int_{K_i}^{K_f} \left(K - \hat{F}_2\left(s, R\left(K, \bar{X}\right)\right) K_{\bar{X}}\right) dK \tag{77}$$

$$D_3 = \int_{K_i}^{K_f} \left(\left(\frac{X_f - X_i}{2}\right) \nabla_X \hat{F}_2\left(s, R\left(K, \bar{X}\right)\right) K_{\bar{X}}\right) dK \tag{78}$$

$$\begin{aligned} & \alpha_{eff}\left(\Psi, \left(K_f, X_f\right), \left(X_i, K_i\right)\right) \\ = \quad & \alpha + D\left(\|\Psi\|^2\right) + \tau\left(\frac{\left|\Psi\left(X_f\right)\right|^2\left(K_{X_f} - K_f\right)}{K_f} - \frac{\left|\Psi\left(X_i\right)\right|^2\left(K_{X_i} - K_i\right)}{K_i}\right) + \frac{\sigma_K^2}{2} K_f' K_i' \end{aligned} \tag{79}$$

and:

$$\begin{aligned} K_i' &= K_i - \hat{F}_2\left(s, R\left(K_{X_i}, X_i\right)\right) K_{X_i} \\ K_f' &= K_f - \hat{F}_2\left(s, R\left(K_{X_f}, X_f\right)\right) K_{X_f} \end{aligned}$$

One investor transition function

$$\begin{aligned} & G_2\left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right)\right) \\ = \quad & \exp\left(D'\left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right)\right)\right) \\ & \times \exp\left(-\alpha_{eff}'\left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right)\right)\left|\left(\hat{K}_f + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}_f, K_{\hat{X}_f}\right)}{f^2\left(\hat{X}_f\right)}\right) - \left(\hat{K}_i + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}_i, K_{\hat{X}_i}\right)}{f^2\left(\hat{X}_i\right)}\right)\right|\right) \end{aligned} \tag{80}$$

with:

$$D'\left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right)\right) = \frac{1}{\sigma_{\hat{X}}^2}\int_{\hat{X}_i}^{\hat{X}_f} g\left(\hat{X}\right) d\hat{X} + \frac{\hat{K}_f^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}_f\right) - \frac{\hat{K}_i^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}_i\right) \tag{81}$$

and:

$$\begin{aligned} & \alpha_{eff}'\left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right)\right) \\ = \quad & \left(\alpha + \frac{\sigma_{\hat{X}}^2}{2}\left(\hat{K}_f + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}_f, K_{\hat{X}_f}\right)}{f^2\left(\hat{X}_f\right)}\right)\left(\hat{K}_i + \frac{\sigma_{\hat{K}}^2 F\left(\hat{X}_i, K_{\hat{X}_i}\right)}{f^2\left(\hat{X}_i\right)}\right)\right)\sqrt{\frac{\left|f\left(\frac{\hat{X}_f + \hat{X}_i}{2}\right)\right|}{2\sigma_{\hat{X}}^2}} + g^{(R)}\left(\hat{X}\right) \end{aligned} \tag{82}$$

with:

$$g^{(R)}\left(\hat{X}\right) = \int_{\hat{X}_i}^{\hat{X}_f} \frac{\left(g\left(\hat{X}\right)\right)^2 + \sigma_{\hat{X}}^2\left(f\left(\hat{X}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{\left\|\hat{X}_f - \hat{X}_i\right\| \sigma_{\hat{X}}^2 \sqrt{f^2\left(\hat{X}\right)}}$$

9.4 Two agents transition functions and Interactions between agents

To study the agents interactions within the background field we consider the two-agent transition functions. There are three of them. One for two firms:

$$G_{11}\left(\left[\left(K_f,X_f\right),\left(K_f,X_f\right)'\right],\left[\left(X_i,K_i\right),\left(X_i,K_i\right)'\right]\right)$$

one for one firm and one investor:

$$G_{12}\left(\left[\left(K_f,X_f\right),\left(\hat{K}_f,\hat{X}_f\right)\right],\left[\left(X_i,K_i\right),\left(\hat{X}_i,\hat{K}_i\right)\right]\right)$$

and one for for two investors:

$$G_{22}\left(\left[\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{K}_f,\hat{X}_f\right)'\right],\left[\left(\hat{X}_i,\hat{K}_i\right),\left(\hat{X}_i,\hat{K}_i\right)'\right]\right)$$

If we neglect the terms of order greater than 2 in the effective action, the transition functions reduce to simple products:

$$\begin{aligned}&G_{11}\left(\left[\left(K_f,X_f\right),\left(K_f,X_f\right)'\right],\left[\left(X_i,K_i\right),\left(X_i,K_i\right)'\right]\right)\\=\;&G_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)G_1\left(\left(K_f,X_f\right)',\left(X_i,K_i\right)'\right)\end{aligned}$$

$$\begin{aligned}&G_{12}\left(\left[\left(K_f,X_f\right),\left(\hat{K}_f,\hat{X}_f\right)\right],\left[\left(X_i,K_i\right),\left(\hat{X}_i,\hat{K}_i\right)\right]\right)\\=\;&G_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)G_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)\end{aligned}$$

$$\begin{aligned}&G_{22}\left(\left[\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{K}_f,\hat{X}_f\right)'\right],\left[\left(\hat{X}_i,\hat{K}_i\right),\left(\hat{X}_i,\hat{K}_i\right)'\right]\right)\\=\;&G_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)G_2\left(\left(\hat{K}_f,\hat{X}_f\right)',\left(\hat{X}_i,\hat{K}_i\right)'\right)\end{aligned}$$

In first approximation, agents behave independently, solely influenced by the given background state.

To take into account agents interactions we write the expansion:

$$\exp\left(-S\left(\Psi\right)\right)=\exp\left(-\left(S\left(\Psi_0,\hat{\Psi}_0\right)+\Delta S_2\left(\Psi,\hat{\Psi}\right)\right)\right)\left(1+\sum_{n\geqslant 1}\frac{\left(-\Delta S_4\left(\Psi,\hat{\Psi}\right)\right)^n}{n!}\right)$$

as explained in section 5.2.5, the series produces corrective terms to the transition functions. Appendix 4 presents the computations and compute the transitions in the approximations of paths that cross each other one time at some X. In this approximation, we find:

9.4.1 Firm-firm transition function:

$$\begin{aligned}&G_{11}\left(\left[\left(K_f,X_f\right),\left(K_f,X_f\right)'\right],\left[\left(X_i,K_i\right),\left(X_i,K_i\right)'\right]\right)\\\simeq\;&G_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)G_1\left(\left(K_f,X_f\right)',\left(X_i,K_i\right)'\right)\\&-\left(2\tau-\nabla_K\frac{\delta^2 u\left(\bar{K},\bar{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(\bar{K}',\bar{X}\right)\delta\Psi^{\dagger}\left(K',\bar{X}\right)}\right)\hat{G}_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)\hat{G}_1\left(\left(K_f,X_f\right)',\left(X_i,K_i\right)'\right)\end{aligned}\tag{83}$$

9.4.2 Firm-investor transition function:

$$
\begin{aligned}
&G_{12}\left(\left[\left(K_f,X_f\right),\left(\hat{K}_f,\hat{X}_f\right)'\right],\left[\left(X_i,K_i\right),\left(\hat{X},\hat{K}_i\right)'\right]\right) \\
\simeq\ &G_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)G_2\left(\left(\hat{K}_f,\hat{X}_f\right)',\left(\hat{X},\hat{K}_i\right)'\right) \\
&+\left(\nabla_K\frac{\delta^2 u\left(\bar{K},\bar{X},\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\overline{\hat{K},\hat{X}}\right)\delta\hat{\Psi}^{\dagger}\left(\overline{\hat{K},\hat{X}}\right)}+\nabla_{\hat{K}}\frac{\hat{K}\delta^2 f\left(\overline{\hat{X}},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(\overline{K',X}\right)\delta\Psi^{\dagger}\left(\overline{K',X}\right)}+\nabla_{\hat{X}}\frac{\delta^2 g\left(\overline{\hat{X}},\Psi,\hat{\Psi}\right)}{\delta\Psi\overline{(K',X)}\delta\Psi^{\dagger}\left(\overline{K',X}\right)}\right) \\
&\times\hat{G}_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)\hat{G}_2\left(\left(\hat{K}_f,\hat{X}_f\right)',\left(\hat{X},\hat{K}_i\right)'\right)
\end{aligned}
\tag{84}
$$

9.4.3 Investor-investor transition function:

$$
\begin{aligned}
&G_{22}\left(\left[\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{K}_f,\hat{X}_f\right)'\right],\left[\left(\hat{X},\hat{K}_i\right),\left(\hat{X},\hat{K}_i\right)'\right]\right) \\
\simeq\ &G_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X},\hat{K}_i\right)\right)G_2\left(\left(\hat{K}_f,\hat{X}_f\right)',\left(\hat{X},\hat{K}_i\right)'\right)
\end{aligned}
\tag{85}
$$

with:

$$
\begin{aligned}
\left(\bar{X},\bar{K}\right) &= \frac{\left(K_f,X_f\right)+\left(X_i,K_i\right)}{2} \\
\left(\bar{X},\bar{K}\right)' &= \frac{\left(K_f,X_f\right)'+\left(X_i,K_i\right)'}{2}
\end{aligned}
$$

The derivatives are given in (69), (70), (71), (72) and:

$$
\hat{G}_i\left(\left(K_f,X_f\right),\left(X,K\right)\right)\hat{G}_j\left(\left(K_f,X_f\right)',\left(X,K\right)'\right)
$$

is the transition function computed on paths that cross once.

10 Results and interpretations

10.1 One-agent transition functions

We present a synthesis of the results for firms and investors transition functions. Some technical details are given in appendix 5.

10.1.1 Firms transition function

For a given background state, the probability of transition for a firm between two states K_i, X_i and K_f, X_f, over an average time of $1/\alpha$, is given by G_1 (see 74). This formula computes the probability that a firm initialy endowed with a capital K_i in sector X_i will relocate to sector X_f with capital K_f. The transition probability is the result of competing effects, as it is composed of several interdependent terms of similar magnitude. Firm transitions occur over the medium to long term but at a slower time scale than transitions for investors. Firms remain in each transitory sector long enough to resettle, and for investors to adjust the capital allocated between firms. Thus, in each transitory sector, firm capital evolves depending on the characteristics of the firm, the sector, and investors expectations.

Attractiveness and sectors shifts The drift term D in formula (75) is the average transition of a firm between its initial and final points (X_i, K_i) and (K_f, X_f), respectively. This term is usually different from zero because firms tend to shift sectors, and their capital evolves. This tendency for a firm to evolve depends both on the transitory sectors and the background field, i.e., the entire landscape in which the transition occurs. In addition, fluctuations around the drift term can alter a firm's trajectory, contributing to the probabilistic nature of the transition.

The drift term of equation (75) is composed of three interacting contributions, D_1, D_2 and D_3.

The first component D_1 shows that firms tend to relocate to sectors with higher long-term returns, shifts which in turn modify their present and future attractiveness to investors.

The second component D_2 shows that the shift alters the capital of the firm. Specifically, the amount of investment that investors are willing to make in the firm, $\hat{F}_2\left(R\left(K,\bar{X}\right)\right)K_{\bar{X}}$ depends on three key parameters: the average capital of the new sector, $K_{\bar{X}}$, the absolute average return on capital in the sector, $R\left(K,\bar{X}\right)$, and the propensity of investors, $\hat{F}_2$ to invest in the firm based on its given capital compared to the average capital of firms in the sector.

When a firm begins the process of relocating to a nearby sector, its capitalization may differ from that of firms already present in that sector, which in turn affects its attractiveness to investors, represented by $\hat{F}_2$. The shape of $\hat{F}_2$ reflects the propensity of investors to invest in the firm. When $\hat{F}_2$ is concave, this propensity marginally decreases, while a convex shape results in a marginal increase.

The equilibrium capital of the firm in the new sector is $\hat{F}_2\left(s,R\left(K,\bar{X}\right)\right)K_{\bar{X}}$. When a firm relocates, its capital may turn out to be below or above this equilibrium level. For each of these cases, two possibilities arise depending on the shape s of $\hat{F}_2$.

If $\hat{F}_2$ is concave, the marginal propensity of investors to invest is decreasing: once the firm has entered the sector, its capital will converge towards the sector's average capital. It will either increase or decrease, depending on whether its initial level of capital is above or below the equilibrium capital, respectively. If $\hat{F}_2\left(s,R\left(K,\bar{X}\right)\right)$ is convex, the marginal propensity of investors to invest is increasing: the dynamics of its capital accumulation is unstable. Investors will tend to over or underinvest in the firm.

The third contribution D_3 reflects the firm's relative attractiveness in different transition sectors. If the the firm's relative attractiveness is reduced during the shift, such that it attracts less capital than the average capital of the transition sector, it may become stuck in an intermediate sector.

Impact of competition The coefficient α_{eff} defined in equation (79) represents the inverse of the average mobility of a firm. This mobility depends on the competition in transitional sectors which is captured by the two first terms on the right-hand side of (79).

The first term, $D\left(\|\Psi\|^2\right)$ is a constant that characterizes the background state of the firms and is correlated with the total number of firms in the space of sectors. As competition increases, α_{eff} rises and firms' mobility decreases.

The second term measures the local competition that firms face as they move through the sector space. It is determined by the density of agents in the sector, multiplied by the variation, along the path, of the firm's excess capital with respect to the average capital of the sector. A well-capitalized firm facing many less-capitalized competitors will repel them and create its own market share. Relocation will occur towards sectors that are denser and less capitalized. Under-capitalized firms will be forced out of their sectors and into denser, less capitalized sectors. The relocation process may result in a capital gain or loss. However, holding capital constant, initially under-capitalized firms will tend to move towards sectors with lower average capital, whereas over-capitalized firms tend to move towards sectors with higher average capital.

Stabilization terms: The square-rooted term multiplying α_{eff} is written:

$$\sqrt{\frac{(X_f - X_i)^2}{2\sigma_X^2} + \frac{\left(\tilde{K}_f - \tilde{K}_i\right)^2}{2\sigma_K^2}} \tag{86}$$

and the last term in the right-hand side of equation (75) both describe random oscillations around a path of zero marginal capital demand. Changes in equity, investments, for instance, may modify (86). and the oscillations are of magnitude $\frac{\sigma_K^2}{2}$. However, these oscillations do not necessarily imply a return to the initial point. The larger the deviation from the average, the more likely firms are to deviate from the average, and possibly shift to a new trajectory. Therefore, a capital increase above the average may induce a shift in sector, which in turn may modify the firm's accumulation and prospects. Thus, oscillations do not prevent trends and may even initiate them. However, such "random shifts" may prove disadvantageous as they could harm the firm's position and reduce its capital compared to the sector.

Possible paths Overall, what are the possible dynamics for a firm in terms of capital and sector? If a firm experiences capital growth in a sector where the investor propensity, F_2, is concave, the accumulation of its capital could cause the firm to shift to a higher-return sector, but this may result in the firm being perceived as less attractive by investors in this new sector.

This shift can lead to a change in the firm's attractiveness to investors, F_2. The growth or decline of the firm in the new sector will depend on both its capital level and the shape of F_2. These factors will also determine the speed of this change. If the firm's capital level gradually declines, it may have time to react and reposition itself. However, if the decline in capital is sudden, the firm may not have enough resources to reposition itself. The new sector may turn out to be a capital trap.

The patterns of possible trajectories are various and may be irregular, with some transitions occurring at a constant rate, while others may involve discontinuities and sudden increases or reductions in capital, depending on the characteristics of the landscape such as expected returns in sectors, density of firms, and other background factors.

10.1.2 Investors transition functions

Drift term Short-term and long-term returns are the two parameters that determine investors' capital allocation. Short-term returns include the firm's dividends and increase with the value of its shares, while long-term returns reflect the market's expectations for the firm's future growth potential, which in turn affect expectations for higher dividends and share price appreciation. Both types of returns are captured in the drift term D', which is defined in equation (81). The most likely paths are those that maximize both short-term and long-term returns.

However, these returns are not independent, since faltering share prices in the short-term impact long-term returns expectations, and vice versa.

Ideally, to maximize their capital, investors seek both higher short-term and long-term returns. Therefore, capital allocation within and across sectors will depend on firms share prices volatility and dividends.

A sector in which share prices increase tends to attract capital, since investors can maximize both short-term and long-term returns: an increase in share prices sustains the firm's growth expectations. Investors tend to move towards the next local maximum of long-term returns while also maximizing their short-term return. In this case, there is no trade-off between the two objectives.

In a sector where stock prices fall or remain stagnant, investors are faced with a trade-off between short- and long-term returns. When stock prices no longer support long-term earnings expectations, capital allocation is determined by short-term dividends. Capital reallocation will depend on the level of capital held by investors. While investors may consider long-term expectationsthey must also generate short-term returns to maintain their capital. An investor who ignores dividends in a context of falling share prices would eventually see his capital depleted, which could hinder or impair his ability to reallocate capital in the long term.

Stabilization terms: Similarly to firms, investors have an effective inverse mobility α'_{eff}, defined in equation (82). This formula shows that mobility $\frac{1}{\alpha'_{eff}}$ decreases with the average short-term return along the path : the higher the returns, the lower the incentive to switch from one sector to another. Similarly, mobility increases with $g^{(R)}\left(\hat{X}\right)$, which measures the relative long-term return of the sectors along the path. The higher this relative return, the lower the incentive to switch to another sector.

Moreover, $\frac{1}{\alpha'_{eff}}$ decreases with the final level of capital $\hat{K}_f$ increases, impairing the firm's capacity to reach high levels of capital. Conversely, $\frac{1}{\alpha'_{eff}}$ decreases with the initial capital $\hat{K}_i$ decreases, indicating that investors with high capitalization are less likely to experience significant capital losses. This is supported by the factor multiplying α'_{eff}:

$$\left|\left(\hat{K}_f + \frac{\sigma^2_{\hat{K}} F\left(\hat{X}_f, K_{\hat{X}_f}\right)}{f^2\left(\hat{X}_f\right)}\right) - \left(\hat{K}_i + \frac{\sigma^2_{\hat{K}} F\left(\hat{X}_i, K_{\hat{X}_i}\right)}{f^2\left(\hat{X}_i\right)}\right)\right|$$

As a result, the probability for an investor to deviate significantly from its initial capital value, apart from the smoothing term which can be neglected, is relatively low.

10.2 Two-agent transition functions

First, it should be noted that the transition function G_{22}, as defined in equation (85), does not include any interaction corrections. Specifically, the transition probability for two investors is simply the product of their individual transition probabilities. In our model, investors do not directly interact with each other, but only through their investments in various firms. Only two transition functions are affected by these indirect interactions.

10.2.1 Firm-firm interactions

First the transition G_{11} is modified by the term:

$$I = 2\tau - \nabla_K \frac{\delta^2 u\left(\bar{K}, \bar{X}, \Psi, \hat{\Psi}\right)}{\delta\Psi\left(\bar{K}', \bar{X}\right)\delta\Psi^{\dagger}\left(K', \bar{X}\right)}$$

The interaction I measures the interactions between two firms in the same sector. The first contribution to I describes a direct competition between firms in a given sector, whereas the second term describes the competition of the firms to attract investors that share their investments between the two firms. Given that the 2-agents transition functions are modified by (see (83)):

$$\left(2\tau - \nabla_K \frac{\delta^2 u\left(\bar{K}, \bar{X}, \Psi, \hat{\Psi}\right)}{\delta\Psi\left(\bar{K}', \bar{X}\right)\delta\Psi^{\dagger}\left(K', \bar{X}\right)}\right)\hat{G}_1\left(\left(K_f, X_f\right), \left(X_i, K_i\right)\right)\hat{G}_1\left(\left(K_f, X_f\right)', \left(X_i, K_i\right)'\right)$$

and since $I > 0$, the contribution to the green function of paths crossing at some point are underweighted. The competition between the two firms repell them from the sector where they interact. If we consider that the competion factor τ is capital-dependent (see (31)), the less capitalized firm is relatively more repelled than the more capitalized one.

10.2.2 Firm-investor interactions

Second, the firm-investor transition function G_{12} is modified by the term:

$$\left(\nabla_K \frac{\delta^2 u\left(\bar{K}, \bar{X}, \Psi, \hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K}, \hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K}, \hat{X}\right)} + \left\{\nabla_{\hat{K}} \frac{\hat{K}\delta^2 f\left(\overline{\hat{X}}, \Psi, \hat{\Psi}\right)}{\delta\Psi\left(\overline{K', X}\right)\delta\Psi^{\dagger}\left(\overline{K', X}\right)} + \nabla_{\hat{X}} \frac{\delta^2 g\left(\overline{\hat{X}}, \Psi, \hat{\Psi}\right)}{\delta\Psi\overline{(K', X)}\delta\Psi^{\dagger}\left(\overline{K', X}\right)}\right\}\right)$$

Given (70), (**??**), (72), this term depends mainly on three contributions:

$$\nabla_K \frac{F_2\left(s, R\left(K', X\right)\right)\hat{K}}{\int F_2\left(s, R\left(K', X\right)\right)\left\|\Psi\left(K', X\right)\right\|^2 dK'}$$

$$\nabla_{\hat{K}} \Delta f\left(K', \hat{X}, \Psi, \hat{\Psi}\right)$$

$$\nabla_{\hat{X}} \Delta\left(g\left(K', \hat{X}, \Psi, \hat{\Psi}\right)\right)$$

each of this contribution describes the relative perspectives of the firm in his path through the sectors.

The first one represents the gradient of firm"s attractiveness with respect to capital. The investor willl decide to invest or not depending on the marginal gain of long term returns of the firm.

The second term represents the marginal short-term return of an investment of the firm, and the third one measures the reltive attractiveness of the firm with respect to his neighbours (see Gosselin Lotz Wambst 2022).

The interaction between the firm and the investor is a combination of these three quantities.

When the combination of these term is positive, the firm has positive perspective either in terms of short or long term returns, or relatively to his neighbors.

In this case the associated corrections to the path crossing at some points is positive and this paths will be overweighted: in probability, this translates by the fact that paths in which a firm presents above average perspectives in his capital accumaltion and shift in sectors, will be favoured by an increase in investment. The firm will take advantage from its interaction with the investor, except if this one experiences, for any reason, an decrease of capital. On the contrary, a firm perceived as moving toward lower perspective will experience in average a decrease in investment. This decrease will be dampened if its investor has itself low capital to invest. Some mixed situation may arise: good short term perspective, but uncertain long term expectations may cancel or compensate each other.

10.3 Discussion

The central feature of the formalism presented and applied in this paper is to encompass macro and microeconomic elements: the macro scale keeps track of the entire set of agents and, in turn, influences the microeconomic scale, allowing for two-level interpretations.

The macroeconomic scale was studied in (Gosselin Lotz Wambst 2022). We showed that the underlying macroeconomic state of the model reveals disparities among sectors and instabilities in capital accumulation. Different sectors can exhibit distinct accumulation patterns. Some sectors may attract significant capital, while others may experience depletion. Moreover, these accumulation patterns undergo changes in perspectives or expectations, leading to potential large fluctuations in capital allocations. These fluctuations can even result in a change of accumulation pattern. Furthermore, individual dynamics heavily rely on the underlying macroeconomic state. Some parameters governing these dynamics depend on the average capital and the number of firms per sector, which are both characteristics of the collective or macroeconomic state.

The present paper studies the microeconomic scale and mechanisms resulting from macroeconomic states and fluctuations.

In the face of these fluctuations, investors may experience capital losses. However they can always shield their capital by reallocating it to more profitable or stable sectors. In doing so, they may amplify global capital fluctuations for firms, which are unable to react at the same pace. Financial risk is therefore limited in our model. Investors can always reposition themselves and, as a result, do not bear the same risk as firms that move to attract investors. The primary burden of risk falls on the firms themselves, not the financial sector. Our model demonstrates that investors do not experience the eviction phenomenon that firms do. However, investors may face eviction from certain investment sectors if their capital no longer allows them to invest in sectors perceived as the most promising, based on returns and share prices.

We posit that firms have a natural inclination to switch sectors. Indeed, firms tend to change due to the continuous evolution and transformation of sectors and the changing economic environment. In our model, the historical development of a sector is not depicted by a specific variable, but rather by firms shifts between closely related sectors. In the shift, the initial sector is the past state of the sector, and the final sector its present state. Thus, firms transitions captures both firm reorientations and their adaptation to an evolving environment.

Attracting investors is crucial to firms and can be achieved through continuous expansion. However, firms face higher uncertainty and risk than investors. Specifically, firms face two distinct risks:

First, the individual risk associated with seeking higher returns. Switching to more attractive sectors may expose firms to higher competition and faltering investors sentiment. For example, a firm shifting to a high-capitalized sector will experience a stronger competition and weaker prospects, potentially deterring any present or additional investment. When these two phenomena combine, they may induce a substantial loss of capital, and trap the firm in the sector, evict it towards less-capitalized and less-attractive ones, and impact its ability to position itself for future sectoral changes and transformations.

Second, the global risk, caused by exogenous and macro fluctuations. This risk can alter sectoral growth prospects and, consequently, affect individual dynamics. Our model captures these potential instabilities at the individual level. Within a sector, sub-sectors may emerge, some presenting more promising opportunities than others. However, the entire sector can be impacted. Even though, on average, the collective state may exhibit some stability, fluctuations among a set of similar firms can be substantial at the individual dynamics level. Consequently, fluctuations in this context magnify the uncertainty at the individual level, making it difficult to identify and capitalize on profitable shifts while also increasing the risk of making detrimental moves. To sum up, both collective and individual results suggest that firms with high initial capitalization are generally less exposed to market fluctuations. Note incidentally that these risks may be amplified by swift financial reallocation in the face of global uncertainties.

Therefore, firms can undergo sharp changes in dynamics due to variations in the landscape of expected returns, reactivity of expectations, relative attractiveness compared to neighboring sectors, or the number of competing firms.

The present paper also advocates that field formalism, in addition to mixing macro and micro analysis, provides some precise insights about the structures of interactions inside the macroeconomic state. The technique of series expansion of the effective action induces emerging interactions that are not detected in the classical formalism, such as indirect emerging competition among agents. More precisely, interactions between firms within a sector reveal phenomena of specialization and eviction. Competition is at first determined by the firms relative levels of capital. This is the direct form of competition. The firm with the highest capital is more likely to evict its competitors. However, field formalism reveals that competition also revolves around attracting investor capital. This is the indirect form of competition. A firm that successfully differentiates itself within a sector, through specialization, has the potential to attract capital and mitigate or reverse a possible eviction. However, specialization makes the firm dependent on its investors. If investors suffer capital losses, the firm is directly impacted.

Interactions between firms and their investors detail the impact of investment at the individual level. A firm that attracts more investors will be better positioned in the sector, as it enjoys a stronger position, whereas its competitors will be compelled to reorient themselves. To attract investors, a firm needs to demonstrate a high growth potential, which may favor new entrants in a sector, provided they have the necessary capital to position themselves, or better growth prospects.

To conclude, note that the concept of comparative advantage is not relevant in our model. Indeed, given that changes are inevitable within sectors, any comparative advantage is bound to be swept away, potentially even by relatively distant and unexpected causes. Actually, exogenous fluctuations, such as the perception of the sector and the firm within it (by investors), as well as competition among firms to retain their position and attract investors, create inherent instability within a specific sector. Specializing in a single sector exposes firms to the risk of eventual eviction, forming a trap.

11 Conclusion

We have studied the impact of financial capital on physical capital allocation in a field-formalism setting. In a previous paper, we introduced the concept of background or collective state of the system as a modeling of a macroeconomic historical state. This paper presents the probabilistic dynamics of agents in this environment. We have identified several types of dynamics for producers, depending on the firms' landscape, returns, and the firms' and sectors' relative attractiveness. A firm's dynamics depends on these parameters as well as its initial sector and level of capital, and may exhibit turning points. Increased competition in transitory sectors may reduce the capital of the firm and impair its dynamics. The dependence of the firm on investors and their expectations, as well as on modifications of the macroeconomic state may lead to significant fluctuations in a firm's growth trajectory.

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Appendix 1 Computation of effective action at the second order

We compute the second-order derivatives for the real and the financial economy respectively.

Real economy

In first approximation:

$$\begin{aligned}
&\frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^\dagger\left(Z,\theta\right)\delta\Psi\left(Z,\theta\right)} \\
\simeq\; & -\int \delta\Psi^\dagger\left(K,X\right)\left(\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X-\nabla_X R\left(K,X\right)H\left(K\right)\right)-4\tau\left(\left|\Psi_0\left(X\right)\right|^2\right)\right. \\
& \left.+\nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K+u\left(K,X,\Psi_0,\hat{\Psi}_0\right)\right)\right)\delta\Psi\left(K,X\right)dKdX
\end{aligned} \tag{87}$$

where:

$$\left|\Psi_0\left(X\right)\right|^2=\int\left|\Psi_0\left(K',X\right)\right|^2dK'$$

and:

$$u\left(K,X,\Psi_0,\hat{\Psi}_0\right)\rightarrow\frac{1}{\varepsilon}\left(K-\int\hat{F}_2\left(s,R\left(K,X\right)\right)\hat{K}\left\|\hat{\Psi}_0\left(\hat{K},X\right)\right\|^2d\hat{K}\right)=\frac{1}{\varepsilon}\left(K-\hat{F}_2\left(s,R\left(K,X\right)\right)K_Xd\hat{K}\right) \tag{88}$$

In equation (88), we used the notation:

$$\int\hat{F}_2\left(s,R\left(K,X\right)\right)\hat{K}\left\|\hat{\Psi}_0\left(\hat{K},X\right)\right\|^2d\hat{K}=\hat{F}_2\left(s,R\left(K,X\right)\right)\hat{K}_X$$

We perform a change of variables in (87):

$$\begin{aligned}
\Delta\Psi\left(K,X\right) &= \exp\left(\int^X\frac{\nabla_X R\left(X\right)}{\sigma_X^2}H\left(\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2d\hat{K}}{\left\|\Psi\left(X\right)\right\|^2}\right)\right) \\
&\quad\times\exp\left(\int\left(K-\frac{F_2\left(R\left(K,X\right)\right)K_X}{F_2\left(R\left(K_X,X\right)\right)}\right)dK\right)\delta\Psi\left(K,X\right) \\
\Delta\Psi^\dagger\left(K,X\right) &= \exp\left(-\int^X\frac{\nabla_X R\left(X\right)}{\sigma_X^2}H\left(\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2d\hat{K}}{\left\|\Psi\left(X\right)\right\|^2}\right)\right) \\
&\quad\times\exp\left(-\int\left(K-\frac{F_2\left(R\left(K,X\right)\right)K_X}{F_2\left(R\left(K_X,X\right)\right)}\right)dK\right)\delta\Psi^\dagger\left(K,X\right)
\end{aligned} \tag{89}$$

where K_X, the average invested capital per firm in sector X:

$$K_X=\frac{\int\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^2d\hat{K}}{\left\|\Psi\left(X\right)\right\|^2} \tag{90}$$

so that the effective action (87) for the real economy becomes:

$$\begin{aligned}
&\Delta\Psi^\dagger\left(Z,\theta\right)\left(\frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^\dagger\left(Z,\theta\right)\delta\Psi\left(Z,\theta\right)}\right)_{\Psi(Z,\theta)=\Psi_0(Z,\theta)}\Delta\Psi\left(Z,\theta\right) \\
=\;&\int\Delta\Psi^\dagger\left(Z,\theta\right)\left(-\frac{\sigma_X^2}{2}\nabla_X^2+\frac{\left(\nabla_X R\left(K,X\right)H\left(K_X\right)\right)^2}{2\sigma_X^2}+\frac{\nabla_X^2 R\left(K,X\right)}{2}H\left(K\right)+4\tau\left|\Psi\left(X\right)\right|^2\right)\Delta\Psi\left(Z,\theta\right) \\
&+\int\Delta\Psi^\dagger\left(Z,\theta\right)\left(-\frac{\sigma_K^2}{2}\nabla_K^2+\frac{1}{2\sigma_K^2}\left(K-\hat{F}_2\left(s,R\left(K,X\right)\right)K_X\right)^2+\frac{1-\nabla_K\hat{F}_2\left(s,R\left(K,X\right)\right)K_X}{2}\right)\Delta\Psi\left(Z,\theta\right)
\end{aligned} \tag{91}$$

As explained in section 7.2.1, the effects of competition can be refined by considering repulsive forces that are capital dependent. It amounts to replace in (91), the term:

$$\int \Delta\Psi^{\dagger}(Z,\theta)\left(2\tau\left|\Psi(X)\right|^{2}\right)\Delta\Psi(Z,\theta)$$

by the term:

$$\begin{aligned}
&\int \Delta\Psi^{\dagger}(K,X)\left(2\tau\frac{\int K'\left|\Psi(K',X)\right|^{2}dK'}{K}\right)\Delta\Psi(K,\theta) \qquad (92)\\
=&\int \Delta\Psi^{\dagger}(K,X)\left(2\tau\frac{\left|\Psi(X)\right|^{2}K_{X}}{K}\right)\Delta\Psi(K,\theta)
\end{aligned}$$

with:

$$\begin{aligned}
\left|\Psi(X)\right|^{2} &= \int\left|\Psi(K',X)\right|^{2}dK' \\
K_{X} &= \frac{\int K'\left|\Psi(K',X)\right|^{2}dK'}{\left|\Psi(X)\right|^{2}}
\end{aligned}$$

This models repulsive forces that are inversely proportional to capital and mainly affect low-capital firms. Note that this change in the interaction does not modify the collective state, since by setting $K=K_X$, we recover the previous repulsive term. Ultimately, using:

$$\left\|\Psi(X)\right\|^{2}=(2\tau)^{-1}\left(D\left(\left\|\Psi\right\|^{2}\right)-\frac{1}{2\sigma_{X}^{2}}\left(\left(\nabla_{X}R(X)\right)^{2}+\frac{\sigma_{X}^{2}\nabla_{X}^{2}R(K_{X},X)}{H(K_{X})}\right)H^{2}(K_{X})\left(1-\frac{H'\left(\hat{K}_{X}\right)K_{X}}{H\left(\hat{K}_{X}\right)}\right)\right) \qquad (93)$$

the interaction term (92) becomes:

$$\begin{aligned}
&\int \Delta\Psi^{\dagger}(K,X)\left(\frac{1}{2}\left(\left(\nabla_{X}R(X)\right)^{2}+\frac{\sigma_{X}^{2}\nabla_{X}^{2}R(K_{X},X)}{H(K_{X})}\right)H^{2}(K_{X})\left(1-\frac{H'\left(\hat{K}_{X}\right)K_{X}}{H\left(\hat{K}_{X}\right)}\right)+2\tau\frac{\left|\Psi(X)\right|^{2}K_{X}}{K}\right)\Delta\Psi(K,\theta)\\
=&\int \Delta\Psi^{\dagger}(K,X)\left(D\left(\left\|\Psi\right\|^{2}\right)+2\tau\frac{\left|\Psi(X)\right|^{2}(K_{X}-K)}{K}\right)\Delta\Psi(K,\theta)
\end{aligned}$$

When the above expression is used to rewrite (91), it yields the formula:

$$\begin{aligned}
\frac{\delta^{2}(S_{1}+S_{2})}{\delta\Psi^{\dagger}(Z,\theta)\,\delta\Psi(Z,\theta)} =& -\frac{\sigma_{X}^{2}}{2}\nabla_{X}^{2}-\frac{\sigma_{K}^{2}}{2}\nabla_{K}^{2}+\left(D\left(\left\|\Psi\right\|^{2}\right)+2\tau\frac{\left|\Psi(X)\right|^{2}(K_{X}-K)}{K}\right) \qquad (94)\\
&+\frac{1}{2\sigma_{K}^{2}}\left(K-\hat{F}_{2}(s,R(K,X))K_{X}\right)^{2}+\frac{1-\nabla_{K}\hat{F}_{2}(s,R(K,X))K_{X}}{2}
\end{aligned}$$

as stated in the text.

Financial economy

For the financial sector, we consider the field-action for $\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)$:

$$S_{3}+S_{4}=-\int\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^{2}}{2}\nabla_{\hat{K}}-\hat{K}f\left(\hat{X},K_{\hat{X}}\right)\right)+\nabla_{\hat{X}}\left(\frac{\sigma_{\hat{X}}^{2}}{2}\nabla_{\hat{X}}-g\left(\hat{X},K_{\hat{X}}\right)\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right) \qquad (95)$$

with:

$$f\left(\hat{X},K_{\hat{X}}\right) = \frac{1}{\varepsilon}\left(r\left(K_{\hat{X}},\hat{X}\right)-\gamma\left\|\Psi\left(\hat{X}\right)\right\|^{2}+F_{1}\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\int R\left(K'_{X'},X'\right)\left\|\Psi\left(X'\right)\right\|^{2}dX'}\right)\right) \tag{96}$$

$$g\left(\hat{X},K_{\hat{X}}\right) = \left(\frac{\nabla_{\hat{X}}F_{0}\left(R\left(K_{\hat{X}},\hat{X}\right)\right)}{\left\|\nabla_{\hat{X}}R\left(K_{\hat{X}},\hat{X}\right)\right\|}+\nu\nabla_{\hat{X}}F_{1}\left(\frac{R\left(K_{\hat{X}},\hat{X}\right)}{\int R\left(K'_{X'},X'\right)\left\|\Psi\left(X'\right)\right\|^{2}dX'}\right)\right) \tag{97}$$

Using a change of variable (see appendix 3.1.2):

$$\begin{aligned}\hat{\Psi} &\rightarrow \exp\left(\frac{1}{\sigma_{\hat{X}}^{2}}\int g\left(\hat{X}\right)d\hat{X}+\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)\hat{\Psi} \\ \hat{\Psi}^{\dagger} &\rightarrow \exp\left(\frac{1}{\sigma_{\hat{X}}^{2}}\int g\left(\hat{X}\right)d\hat{X}+\frac{\hat{K}^{2}}{\sigma_{\hat{K}}^{2}}f\left(\hat{X}\right)\right)\hat{\Psi}^{\dagger}\end{aligned} \tag{98}$$

the action (95) becomes:

$$\begin{aligned}S_{3}+S_{4}= &-\int\hat{\Psi}^{\dagger}\left(\frac{\sigma_{\hat{X}}^{2}}{2}\nabla_{\hat{X}}^{2}-\frac{1}{2\sigma_{\hat{X}}^{2}}\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}-\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\right)\hat{\Psi} \\ &-\int\hat{\Psi}^{\dagger}\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^{2}}{2}\nabla_{\hat{K}}-\hat{K}f\left(\hat{X},K_{\hat{X}}\right)\right)\right)\hat{\Psi}\end{aligned} \tag{99}$$

To obtain the second-order expansion of the field's action, we start by the first derivative of (99) arising in the minimization equation in (Gosselin Lotz Wambst 2022):

$$\begin{aligned}\frac{\delta\left(S_{3}\left(\Psi\right)+S_{4}\left(\Psi\right)\right)}{\delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)} = &-\frac{\sigma_{\hat{X}}^{2}}{2}\nabla_{\hat{X}}^{2}\hat{\Psi}-\frac{\sigma_{\hat{K}}^{2}}{2}\nabla_{\hat{K}}^{2}\hat{\Psi}+\frac{1}{2\sigma_{\hat{X}}^{2}}\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)\hat{\Psi} \\ &+\frac{\hat{K}^{2}}{2\sigma_{\hat{K}}^{2}}f^{2}\left(\hat{X}\right)+\frac{1}{2}f\left(\hat{X},K_{\hat{X}}\right)\hat{\Psi}+F\left(\hat{X},K_{\hat{X}}\right)\hat{K}\hat{\Psi}\end{aligned} \tag{100}$$

with:

$$\begin{aligned}F\left(\hat{X},K_{\hat{X}}\right) = &\nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}}{2\sigma_{\hat{X}}^{2}}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)+f\left(\hat{X},K_{\hat{X}}\right)\right)\frac{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^{2}}{\left\|\Psi\left(\hat{X}\right)\right\|^{2}} \\ &+\frac{\nabla_{K_{\hat{X}}}f^{2}\left(\hat{X},K_{\hat{X}}\right)}{\sigma_{\hat{K}}^{2}\left\|\Psi\left(\hat{X}\right)\right\|^{2}}\left\langle\hat{K}^{2}\right\rangle_{\hat{X}}\end{aligned} \tag{101}$$

so that :

$$\begin{aligned}\frac{\delta^{2}\left(S_{3}\left(\Psi\right)+S_{4}\left(\Psi\right)\right)}{\delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\delta\hat{\Psi}\left(Z,\theta\right)} = &-\frac{\sigma_{\hat{X}}^{2}}{2}\nabla_{\hat{X}}^{2}-\frac{\sigma_{\hat{K}}^{2}}{2}\nabla_{\hat{K}}^{2}+\frac{1}{2\sigma_{\hat{X}}^{2}}\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right) \\ &+\frac{\hat{K}^{2}}{2\sigma_{\hat{K}}^{2}}f^{2}\left(\hat{X}\right)+\frac{1}{2}f\left(\hat{X},K_{\hat{X}}\right)+F\left(\hat{X},K_{\hat{X}}\right)\hat{K}-\hat{\Psi}^{\dagger}\frac{\delta F\left(\hat{X},K_{\hat{X}}\right)}{\delta\left\|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right\|^{2}}\hat{K}\hat{\Psi}\end{aligned}$$

where the last term is given by:

$$\frac{\delta F\left(\hat{X},K_{\hat{X}}\right)}{\delta\hat{\Psi}\left(Z,\theta\right)} \simeq \nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X},K_{\hat{X}}\right)\right)^{2}}{2\sigma_{\hat{X}}^{2}}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)+f\left(\hat{X},K_{\hat{X}}\right)\right)\frac{\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}{\left\|\Psi\left(\hat{X}\right)\right\|^{2}} +\frac{\nabla_{K_{\hat{X}}}f^{2}\left(\hat{X},K_{\hat{X}}\right)}{\sigma_{\hat{K}}^{2}\left\|\Psi\left(\hat{X}\right)\right\|^{2}}\hat{K}^{2}\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)$$

Following (Gosselin Lotz Wambst 2022) we neglect in first approximation the derivatives with respect to $K_{\hat{X}}$, and define the new variable:

$$y=\frac{\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}}{\sqrt{\sigma_{\hat{K}}^{2}}}\left(f^{2}\left(\hat{X}\right)\right)^{\frac{1}{4}} \tag{102}$$

$$\frac{\delta^{2}\left(S_{3}\left(\Psi\right)+S_{4}\left(\Psi\right)\right)}{\delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\delta\hat{\Psi}\left(Z,\theta\right)}=-\frac{\sigma_{\hat{X}}^{2}}{2}\nabla_{\hat{X}}^{2}-\nabla_{y}^{2}+\left(\frac{y^{2}}{4}+\frac{\left(g\left(\hat{X}\right)\right)^{2}+\sigma_{\hat{X}}^{2}\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}\right)}{\sigma_{\hat{X}}^{2}\sqrt{f^{2}\left(\hat{X}\right)}}\right) \tag{103}$$

This leads to:

$$\begin{aligned}
&\Delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\frac{\delta^{2}\left(S_{3}\left(\Psi\right)+S_{4}\left(\Psi\right)\right)}{\delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\delta\hat{\Psi}\left(Z,\theta\right)}\Delta\hat{\Psi}\left(Z,\theta\right)\\
=\;&\Delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\left(-\frac{\sigma_{\hat{X}}^{2}}{2}\nabla_{\hat{X}}^{2}+\frac{\left(g\left(\hat{X}\right)\right)^{2}+\sigma_{\hat{X}}^{2}\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X},K_{\hat{X}}\right)}{2f^{2}\left(\hat{X}\right)}\right)}{\sigma_{\hat{X}}^{2}\sqrt{f^{2}\left(\hat{X}\right)}}\right.\\
&\left.-\frac{\sigma_{\hat{K}}^{2}}{2\sqrt{f^{2}\left(\hat{X}\right)}}\nabla_{\hat{K}}^{2}+\left(\frac{\sqrt{f^{2}\left(\hat{X}\right)}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X},K_{\hat{X}}\right)}{f^{2}\left(\hat{X}\right)}\right)^{2}}{4\sigma_{\hat{K}}^{2}}\right)\right)\Delta\hat{\Psi}\left(Z,\theta\right)
\end{aligned}$$

Appendix 2 Higher order corrections to the effective action

The higher-order corrections are obtained by expanding at higher-orders in $\Delta\Psi\left(Z,\theta\right)$ and $\Delta\hat{\Psi}\left(Z,\theta\right)$. These variations around the background fields can be considered to be orthogonals to $\Psi_{0}\left(Z,\theta\right)$ and $\hat{\Psi}_{0}\left(Z,\theta\right)$.

Third order terms

The orthogonality condition implies that the third-order terms in the expansion can be neglected. Actually, in first approximation the third-order terms arising in the expansion of S have the form:

$$\begin{aligned}
&2\tau\int\Delta\Psi\left(K',X\right)\Psi_0^{\dagger}\left(K',X'\right)dK'\left|\Delta\Psi\left(K,X\right)\right|^2dKdX \qquad (104)\\
&-\int\Delta\Psi^{\dagger}\left(K,X\right)\Psi_0^{\dagger}\left(K',X'\right)\nabla_K\frac{\delta u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\left|\Psi\left(K',X\right)\right|^2}\Delta\Psi\left(K',X'\right)\Delta\Psi\left(K,X\right)\\
&-\int\Delta\Psi^{\dagger}\left(K,\theta\right)\hat{\Psi}_0^{\dagger}\left(\hat{K},\theta\right)\nabla_K\frac{\delta u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}\Delta\hat{\Psi}\left(\hat{K},\theta\right)\Delta\Psi\left(K,\theta\right)\\
&-\int\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\Psi_0^{\dagger}\left(K',\theta\right)\left\{\nabla_{\hat{K}}\frac{\hat{K}\delta^2 f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\left|\Psi\left(K',X\right)\right|^2}+\nabla_{\hat{X}}\frac{\delta g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\left|\Psi\left(K',X\right)\right|^2}\right\}\Delta\Psi\left(K',X'\right)\Delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\\
&+H.C.
\end{aligned}$$

where the notation $H.C.$ stands for the hermitian conjugate of the expression. Replacing the terms:

$$\nabla_K\frac{\delta u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\left|\Psi\left(K',X\right)\right|^2}\ ,\ \nabla_K\frac{\delta u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2}$$

and:

$$\nabla_{\hat{K}}\frac{\hat{K}\delta^2 f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\left|\Psi\left(K',X\right)\right|^2}+\nabla_{\hat{X}}\frac{\delta g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\left|\Psi\left(K',X\right)\right|^2}$$

by their averages in (104), and using the orthogonality conditions:

$$\int\hat{\Psi}_0^{\dagger}\left(\hat{K},\theta\right)\Delta\hat{\Psi}\left(\hat{K},\theta\right)=\int\Psi_0^{\dagger}\left(K',X'\right)\Delta\Psi\left(K',X'\right)=0$$

leads to neglect the third-order terms in first approximation.

Fourth order terms

General formula

Considering the fourth-order in the action expansion yields quartic corrections. Using that in average:

$$\frac{\delta^2 f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}\simeq 0$$

$$\frac{\delta^2 g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}\simeq 0$$

the fourth-order terms in the fields' action become:

$$
\begin{aligned}
&2\tau \int \left|\Delta\Psi\left(K',X\right)\right|^{2} dK' \left|\Delta\Psi\left(K,X\right)\right|^{2} dKdX \qquad (105)\\
&-\Delta\Psi^{\dagger}\left(K,X\right)\Delta\Psi^{\dagger}\left(K',X'\right)\nabla_{K}\frac{\delta^{2}u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}\Delta\Psi\left(K',X'\right)\Delta\Psi\left(K,X\right)\\
&-\Delta\Psi^{\dagger}\left(K,\theta\right)\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\theta\right)\nabla_{K}\frac{\delta^{2}u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}\Delta\hat{\Psi}\left(\hat{K},\theta\right)\Delta\Psi\left(K,\theta\right)\\
&-\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\Delta\Psi^{\dagger}\left(K',\theta\right)\left\{\nabla_{\hat{K}}\frac{\hat{K}\delta^{2}f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}+\nabla_{\hat{X}}\frac{\delta^{2}g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}\right\}\Delta\Psi\left(K',X'\right)\Delta\hat{\Psi}\left(\hat{K},\hat{X}\right)
\end{aligned}
$$

Estimation of the various terms

The three last terms in the rhs of (105) can be evaluated. The second term is given by:

$$
\begin{aligned}
&\Delta\Psi^{\dagger}\left(K,X\right)\Delta\Psi^{\dagger}\left(K',X'\right)\frac{\delta^{2}u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}\Delta\Psi\left(K,X\right)\Delta\Psi\left(K',\theta\right)\\
=\;&\Delta\Psi^{\dagger}\left(K,\theta\right)\Delta\Psi^{\dagger}\left(K',\theta\right)\left\{\int\hat{F}_{2}\left(s,R\left(K,X\right)\right)\hat{F}_{2}\left(s',R\left(K',X'\right)\right)\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^{2}d\hat{K}\right\}\Delta\Psi\left(K,\theta\right)\Delta\Psi\left(K',\theta\right)\\
&-2\Delta\Psi^{\dagger}\left(K,\theta\right)\Delta\Psi^{\dagger}\left(K',\theta\right)\left\{\int\Psi_{0}^{\dagger}\left(K',X\right)\frac{\hat{F}_{2}\left(s,R\left(K,X\right)\right)\hat{F}_{2}\left(s',R\left(K',X'\right)\right)}{\int F_{2}\left(s',R\left(K',X\right)\right)\left\|\Psi\left(K',X\right)\right\|^{2}dK'}\Psi_{0}\left(K,\hat{X}\right)\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^{2}d\hat{K}\right\}\Delta\Psi\left(\right.\\
\simeq\;&\Delta\Psi^{\dagger}\left(K,\theta\right)\Delta\Psi^{\dagger}\left(K',\theta\right)\left\{\int\hat{F}_{2}\left(s,R\left(K,X\right)\right)\hat{F}_{2}\left(s',R\left(K',X'\right)\right)\hat{K}\left\|\hat{\Psi}\left(\hat{K},X\right)\right\|^{2}d\hat{K}\right\}\Delta\Psi\left(K,\theta\right)\Delta\Psi\left(K',\theta\right)
\end{aligned}
$$

The second term in the rhs of (105) is equal to:

$$
\begin{aligned}
&\Delta\Psi^{\dagger}\left(K,\theta\right)\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\theta\right)\frac{\delta^{2}u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}\Delta\Psi\left(K,\theta\right)\Delta\hat{\Psi}\left(\hat{K},\theta\right)\\
=\;&-\Delta\Psi^{\dagger}\left(K,\theta\right)\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\theta\right)\frac{1}{\varepsilon}\hat{F}_{2}\left(s,R\left(K,X\right)\right)\hat{K}\Delta\Psi\left(K,\theta\right)\Delta\hat{\Psi}\left(\hat{K},\theta\right)
\end{aligned}
$$

Ultimately, the last term in the rhs of (105):

$$
\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\Delta\Psi^{\dagger}\left(K',\theta\right)\left\{\nabla_{\hat{K}}\frac{\hat{K}\delta^{2}f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}+\nabla_{\hat{X}}\frac{\delta^{2}g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}\right\}\Delta\Psi\left(K',X'\right)\Delta\hat{\Psi}\left(\hat{K},\hat{X}\right)
$$

is obtained by using the expressions of $f\left(\hat{X},\Psi,\hat{\Psi}\right)$ and $g\left(\hat{X},\Psi,\hat{\Psi}\right)$ that compute short-term and long-term returns, respectively:

$$
\begin{aligned}
f\left(\hat{X},\Psi,\hat{\Psi}\right) &= \frac{1}{\varepsilon}\int\left(r\left(K,X\right)-\gamma\frac{\int K'\left\|\Psi\left(K',X\right)\right\|^{2}}{K}+F_{1}\left(\frac{R\left(K,X\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^{2}d\left(K',X'\right)},\Gamma\left(K,X\right)\right)\right)\\
&\quad\times\hat{F}_{2}\left(s,R\left(K,X\right)\right)\left\|\Psi\left(K,\hat{X}\right)\right\|^{2}dK\\
g\left(K,\hat{X},\Psi,\hat{\Psi}\right) &= \int\left(\nabla_{\hat{X}}F_{0}\left(R\left(K,\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_{1}\left(\frac{R\left(K,\hat{X}\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^{2}d\left(K',X'\right)}\right)\right)\frac{\left\|\Psi\left(K,\hat{X}\right)\right\|^{2}dK}{\int\left\|\Psi\left(K',\hat{X}\right)\right\|^{2}dK'}
\end{aligned}
$$

We find:

$$\begin{aligned}
&\frac{\delta^2 f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)} \qquad (106)\\
=\;&\frac{1}{\varepsilon}\Delta\left(r\left(K',X\right)-\gamma\frac{\int K'\left\|\Psi\left(K',X\right)\right\|^2}{K'}+F_1\left(\frac{R\left(K',X\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)},\Gamma\left(K,X\right)\right)\right)\\
&\times\frac{F_2\left(s',R\left(K',\hat{X}\right)\right)}{\int F_2\left(s'R\left(K',\hat{X}\right)\right)\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\\
&-\frac{1}{\varepsilon}\int\left(\gamma\frac{K'}{K}+\frac{R\left(K',X\right)R\left(K_{\hat{X}},X\right)}{\left(\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)\right)^2}F_1'\left(\frac{R\left(K,X\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)},\Gamma\left(K,X\right)\right)\right)\\
&\times\hat{F}_2\left(s,R\left(K,X\right)\right)\left\|\Psi\left(K,\hat{X}\right)\right\|^2
\end{aligned}$$

where we define the deviation ΔY of a quantity by the difference:

$$\Delta Y = Y - \langle Y\rangle \qquad (107)$$

with $\langle Y\rangle$, the average of Y:

$$\langle Y\rangle = \int Y\left(K,X\right)dKdX$$

Thus we write:

$$\begin{aligned}
&\Delta\left(r\left(K',X\right)-\gamma\frac{\int K'\left\|\Psi\left(K',X\right)\right\|^2}{K'}+F_1\left(\frac{R\left(K',X\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)},\Gamma\left(K,X\right)\right)\right)\\
=\;&\left(r\left(K',X\right)-\gamma\frac{\int K'\left\|\Psi\left(K',X\right)\right\|^2}{K'}+F_1\left(\frac{R\left(K',X\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)},\Gamma\left(K,X\right)\right)\right)\\
&-\left\langle\left(r\left(K',X\right)-\gamma\frac{\int K'\left\|\Psi\left(K',X\right)\right\|^2}{K'}+F_1\left(\frac{R\left(K',X\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)},\Gamma\left(K,X\right)\right)\right)\right\rangle
\end{aligned}$$

and in first approximation, (106) reduces to:

$$\begin{aligned}
\frac{\delta^2 f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)} &\simeq \frac{1}{\varepsilon}\left(\Delta\left(r\left(K',X\right)-\gamma\frac{K_X}{K'}+F_1\left(\frac{R\left(K',X\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)},\Gamma\left(K,X\right)\right)\right)-\gamma\frac{K'}{K_X}\right)\\
&\simeq \frac{1}{\varepsilon}\left(\Delta f\left(K',\hat{X},\Psi,\hat{\Psi}\right)-\gamma\frac{K'}{K_X}\right)
\end{aligned}$$

where:

$$\Delta f\left(K',\hat{X},\Psi,\hat{\Psi}\right)=f\left(K',\hat{X},\Psi,\hat{\Psi}\right)-f\left(K_{\hat{X}},\hat{X},\Psi,\hat{\Psi}\right)$$

is the relative short-term return for firm with capital K' at sector $\hat{X}$.

Similarly, the second derivative for $g\left(\hat{X},\Psi,\hat{\Psi}\right)$ is:

$$\begin{aligned}
\frac{\delta^2 g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)} &= \frac{1}{\int\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\Delta\left(\nabla_{\hat{X}}F_0\left(R\left(K',\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K',\hat{X}\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right.}\right.\right.\\
&= \frac{1}{\int\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\Delta\left(g\left(K',\hat{X},\Psi,\hat{\Psi}\right)\right)
\end{aligned}$$

with:

$$
\begin{aligned}
&\Delta\left(\nabla_{\hat{X}}F_0\left(R\left(K',\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K',\hat{X}\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)}\right)\right)\\
=&\left(\nabla_{\hat{X}}F_0\left(R\left(K',\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K',\hat{X}\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)}\right)\right)\\
&-\left\langle\nabla_{\hat{X}}F_0\left(R\left(K',\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_1\left(\frac{R\left(K',\hat{X}\right)}{\int R\left(K',X'\right)\left\|\Psi\left(K',X'\right)\right\|^2 d\left(K',X'\right)}\right)\right\rangle
\end{aligned}
$$

in other words:

$$\Delta g\left(K',\hat{X},\Psi,\hat{\Psi}\right)=g\left(K',\hat{X},\Psi,\hat{\Psi}\right)-g\left(\hat{X},\Psi,\hat{\Psi}\right)$$

is the relative long-term return for firm with capital K' at sector $\hat{X}$.

Appendix 3: "free" transition functions

Given the second-order operator arising in the expansion for the fields' action:

$$O\left(\Psi_0\left(Z,\theta\right)\right)\simeq\left(\begin{array}{cc}\frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^{\dagger}(Z,\theta)\delta\Psi(Z,\theta)} & 0\\ 0 & \frac{\delta^2\left(S_3(\Psi)+S_4(\Psi)\right)}{\delta\hat{\Psi}^{\dagger}(Z,\theta)\delta\hat{\Psi}(Z,\theta)}\end{array}\right)_{\substack{\Psi(Z,\theta)=\Psi_0(Z,\theta)\\ \hat{\Psi}(Z,\theta)=\hat{\Psi}_0(Z,\theta)}} \tag{108}$$

The transition functions for the individual firms:

$$G_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),\alpha\right)$$

and investors:

$$G_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right),\alpha\right)$$

satisfy:

$$
\begin{aligned}
\left(\frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^{\dagger}\left(Z,\theta\right)\delta\Psi\left(Z,\theta\right)}+\alpha\right)G_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),\alpha\right) &= \delta\left(\left(K_f,X_f\right)-\left(X_i,K_i\right)\right)\\
\left(\frac{\delta^2\left(S_3\left(\Psi\right)+S_4\left(\Psi\right)\right)}{\delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\delta\hat{\Psi}\left(Z,\theta\right)}+\alpha\right)G_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right),\alpha\right) &= \delta\left(\left(\hat{K}_f,\hat{X}_f\right)-\left(\hat{X}_i,\hat{K}_i\right)\right)
\end{aligned}
$$

The functions $G_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),\alpha\right)$ and $G_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right),\alpha\right)$ are the Laplace transforms of the following transition functions:

$$T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),t\right)$$

$$T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right),t\right)$$

satisfying:

$$-\frac{\partial}{\partial t}T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),t\right)=\left(\frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^{\dagger}\left(Z,\theta\right)\delta\Psi\left(Z,\theta\right)}\right)T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),t\right) \tag{109}$$

$$-\frac{\partial}{\partial t}T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right),t\right)=\left(\frac{\delta^2\left(S_3\left(\Psi\right)+S_4\left(\Psi\right)\right)}{\delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\delta\hat{\Psi}\left(Z,\theta\right)}\right)T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right),t\right) \tag{110}$$

Approximations to and (109) and (110)

We consider some approximations to find the solutions of equations (94) and (103). We first assume that:

$$\frac{\nabla_K \frac{F_2(R(K,X))}{\langle F_2(R(K,X))\rangle_K} K_X}{2} << 1$$

so that:

$$\begin{aligned} K-\hat{F}_2\left(s,R\left(K,X\right)\right)K_X &\simeq K-\hat{F}_2\left(R\left(K_X,X\right)\right)K_X-\nabla_{K_X}\hat{F}_2\left(R\left(K_X,X\right)\right)\left(K-K_X\right) \\ &\simeq K-\hat{F}_2\left(R\left(K_X,X\right)\right)K_X \end{aligned}$$

Equation (94) then simplifies as:

$$\begin{aligned} \frac{\delta^2\left(S_1+S_2\right)}{\delta\Psi^\dagger\left(Z,\theta\right)\delta\Psi\left(Z,\theta\right)} &= -\frac{\sigma_X^2}{2}\nabla_X^2-\frac{\sigma_K^2}{2}\nabla_K^2+\left(D\left(\|\Psi\|^2\right)+2\tau\frac{|\Psi\left(X\right)|^2\left(K_X-K\right)}{K}\right) \\ &+\frac{1}{2\sigma_K^2}\left(K-\hat{F}_2\left(s,R\left(K_X,X\right)\right)K_X\right)^2 \end{aligned} \quad (111)$$

and equation (109) becomes:

$$\begin{aligned} -\frac{\partial}{\partial t}T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),t\right) &= \left(-\frac{\sigma_X^2}{2}\nabla_X^2+D\left(\|\Psi\|^2\right)+2\tau\frac{K_X-K}{K}\|\Psi\left(X\right)\|^2\right)T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),t\right) \\ &+\left(-\frac{\sigma_K^2}{2}\nabla_K^2+\frac{1}{2\sigma_K^2}\left(K-\hat{F}_2\left(R\left(K_X,X\right)\right)K_X\right)^2\right)T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right),t\right) \end{aligned} \quad (112)$$

Second, we assumed from the beginning that the motion of firms in the sectors space is at slower pace than capital fluctuations. Moreover, we may assume that in average $\left|\frac{K_X-K}{K}\right| << 1$. As a consequence, along the path from the initial point (X_i, K_i) to the final point (K_f, X_f), we can consider that:

$$\frac{K_X-K}{K}\|\Psi\left(X\right)\|^2$$

is slowly varying and can be replaced by its average.

The equation for T_1 thus rewrites:

$$\begin{aligned} &-\frac{\partial}{\partial t}T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right) \quad (113) \\ &= \left(-\frac{\sigma_X^2}{2}\nabla_X^2+D\left(\|\Psi\|^2\right)+\tau\left(\frac{|\Psi\left(X_f\right)|^2\left(K_{X_f}-K_f\right)}{K_f}+\frac{|\Psi\left(X_i\right)|^2\left(K_{X_i}-K_i\right)}{K_i}\right)\right)T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right) \\ &+\left(-\frac{\sigma_K^2}{2}\nabla_K^2+\frac{1}{2\sigma_K^2}\left(K-\hat{F}_2\left(R\left(K_X,X\right)\right)K_X\right)^2\right)T_1\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right) \end{aligned}$$

On the other hand, the derivation of the equation for T_2 yields directly:

$$\begin{aligned} &-\frac{\partial}{\partial t}T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right) \quad (114) \\ &= \left(-\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2-\nabla_y^2\right)T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right) \\ &+\left(\frac{y^2}{4}+\frac{\left(g\left(\hat{X}\right)\right)^2+\sigma_{\hat{X}}^2\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_K^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}}\right)T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right) \end{aligned}$$

Computation of T_1

11.0.1 Solution of (113)

We first rewrite the competition term in (113) as:

$$\begin{aligned}&\frac{\left|\Psi\left(X_{f}\right)\right|^{2}\left(K_{X_{f}}-K_{f}\right)}{K_{f}}+\frac{\left|\Psi\left(X_{i}\right)\right|^{2}\left(K_{X_{i}}-K_{i}\right)}{K_{i}}\\ =&\left(\frac{\left|\Psi\left(X_{f}\right)\right|^{2}\left(K_{X_{f}}-K_{f}\right)}{K_{f}}-\frac{\left|\Psi\left(X_{i}\right)\right|^{2}\left(K_{X_{i}}-K_{i}\right)}{K_{i}}\right)+\frac{\left|\Psi\left(X_{i}\right)\right|^{2}\left(K_{X_{i}}-K_{i}\right)}{K_{i}}\end{aligned}$$

Then, we normalize the transition functions by factoring the solution of (113):

$$\begin{aligned}&T_{1}\left(\left(K_{f},X_{f}\right),\left(X_{i},K_{i}\right)\right)\\ =&\exp\left(-t\left(D\left(\|\Psi\|^{2}\right)+\tau\left(\frac{\left|\Psi\left(X_{f}\right)\right|^{2}\left(K_{X_{f}}-K_{f}\right)}{K_{f}}+\frac{\left|\Psi\left(X_{i}\right)\right|^{2}\left(K_{X_{i}}-K_{i}\right)}{K_{i}}\right)\right)\right)\hat{T}_{1}\left(\left(K_{f},X_{f}\right),\left(X_{i},K_{i}\right)\right)\end{aligned}\tag{115}$$

so that the transition equation writes:

$$\begin{aligned}&-\frac{\partial}{\partial t}\hat{T}_{1}\left(\left(K_{f},X_{f}\right),\left(X_{i},K_{i}\right)\right)\\ =&\left(-\frac{\sigma_{X}^{2}}{2}\nabla_{X}^{2}-\frac{\sigma_{K}^{2}}{2}\nabla_{K}^{2}+\frac{1}{2\sigma_{K}^{2}}\left(K-\hat{F}_{2}\left(R\left(K,X\right)\right)K_{X}\right)^{2}\right)\hat{T}_{1}\left(\left(K_{f},X_{f}\right),\left(X_{i},K_{i}\right)\right)\end{aligned}\tag{116}$$

Note that, given the exponential factor, if $K_{i}<<K_{X_{i}}$, $\frac{|\Psi(X_{i})|^{2}\left(K_{X_{i}}-K_{i}\right)}{K_{i}}<0$ and the probability to move away from X_{i} is very low. The same applies for $\frac{|\Psi(X_{f})|^{2}\left(K_{X_{f}}-K_{f}\right)}{K_{f}}>0$.

The transition function $\hat{T}_{1}\left(\left(K_{f},X_{f}\right),\left(X_{i},K_{i}\right)\right)$ can be found by using our assumption that shifts in sectors space are slower than the fluctuations in capital. In (116) we can thus consider in first approximation that the term:

$$K-\hat{F}_{2}\left(R\left(K,X\right)\right)K_{X}$$

shifts the inital and final values of capital:

$$\begin{aligned}K_{i}&\rightarrow K_{i}-\hat{F}_{2}\left(s,R\left(K_{X_{i}},X_{i}\right)\right)K_{X_{i}}=K_{i}^{\prime}\\ K_{f}&\rightarrow K_{f}-\hat{F}_{2}\left(s,R\left(K_{X_{f}},X_{f}\right)\right)K_{X_{f}}=K_{f}^{\prime}\end{aligned}$$

So that we have:

$$\hat{T}_{1}\left(\left(K_{f},X_{f}\right),\left(X_{i},K_{i}\right)\right)\simeq\tilde{T}_{1}\left(\left(K_{f}-\hat{F}_{2}\left(s,R\left(K_{X_{f}},X_{f}\right)\right)K_{X_{f}},X_{f}\right),\left(K_{i}-\hat{F}_{2}\left(R\left(s,K_{X_{i}},X_{i}\right)\right)K_{X_{i}},K_{i}\right)\right)\tag{117}$$

where $\tilde{T}_{1}$ satisfies:

$$-\frac{\partial}{\partial t}\tilde{T}_{1}=\left(-\frac{\sigma_{X}^{2}}{2}\nabla_{X}^{2}-\frac{\sigma_{K}^{2}}{2}\nabla_{K}^{2}+\frac{1}{2\sigma_{K}^{2}}K^{2}\right)\tilde{T}_{1}\tag{118}$$

Up to a normalization factor, the solution of (118) is:

$$\tilde{T}_{1}\left(\left(K_{f}^{\prime},X_{f}\right),\left(X_{i},K_{i}^{\prime}\right)\right)=\exp\left(-\left(\frac{\left(X_{f}-X_{i}\right)^{2}}{2t\sigma_{X}^{2}}+\frac{\left(K_{f}^{\prime}-K_{i}^{\prime}\right)^{2}}{2t\sigma_{K}^{2}}-\frac{t\sigma_{K}^{2}}{2}K_{f}^{\prime}K_{f}^{\prime}\right)\right)$$

Using (117) and (115), we find the solution of (**??**):

$$\begin{aligned}&\exp\left(-t\left(D\left(\|\Psi\|^{2}\right)+\tau\left(\frac{\left|\Psi\left(X_{f}\right)\right|^{2}\left(K_{X_{f}}-K_{f}\right)}{K_{f}}+\frac{\left|\Psi\left(X_{i}\right)\right|^{2}\left(K_{X_{i}}-K_{i}\right)}{K_{i}}\right)\right)\right)\\ &\times\exp\left(-\left(\frac{\left(X_{f}-X_{i}\right)^{2}}{2t\sigma_{X}^{2}}+\frac{\left(K_{f}^{\prime}-K_{i}^{\prime}\right)^{2}}{2t\sigma_{K}^{2}}-\frac{t\sigma_{K}^{2}}{2}K_{f}^{\prime}K_{f}^{\prime}\right)\right)\end{aligned}\tag{119}$$

11.0.2 Full transition function

To obtain the full transition function, recall that (119) has been obtained by a change of variable (89). To come back to the initial variables we have to introduce an other exponential factor to account for the trend of the transition, and we find:

$$\begin{aligned}
&T_1\left(\left(K_f, X_f\right), \left(X_i, K_i\right)\right) \\
\simeq\ & \exp\left(\int_{X_i}^{X_f} \frac{\nabla_X R\left(K_X, X\right)}{\sigma_X^2} H\left(K_X\right) - t\left(D\left(\|\Psi\|^2\right) + \tau\left(\frac{\left|\Psi\left(X_f\right)\right|^2\left(K_{X_f} - K_f\right)}{K_f} + \frac{\left|\Psi\left(X_i\right)\right|^2\left(K_{X_i} - K_i\right)}{K_i}\right)\right)\right) \\
& \times \exp\left(-\int^{K_f}\left(K - \hat{F}_2\left(R\left(s, K, X_f\right)\right) K_{X_f}\right) dK + \int^{K_i}\left(K - \hat{F}_2\left(s, R\left(K, X_i\right)\right) K_{X_i}\right) dK\right) \exp\left(-\left(\frac{\left(X_f - X_i\right)^2}{2t\sigma_X^2} + \frac{(\ldots}{\ldots}\right.\right. \\
& \times \exp\left(-\frac{t\sigma_K^2}{2}\left(K_f - \hat{F}_2\left(s, R\left(K_f, \bar{X}\right)\right) K_{\bar{X}}\right)\left(K_i - \hat{F}_2\left(s, R\left(K_i, \bar{X}\right)\right) K_{\bar{X}}\right)\right)
\end{aligned}$$

$$\begin{aligned}
&T_1\left(\left(K_f, X_f\right), \left(X_i, K_i\right)\right) \qquad (120) \\
\simeq\ & \exp\left(\int_{X_i}^{X_f} \frac{\nabla_X R\left(K_X, X\right)}{\sigma_X^2} H\left(K_X\right) - t\left(D\left(\|\Psi\|^2\right) + \tau\left(\frac{\left|\Psi\left(X_f\right)\right|^2\left(K_{X_f} - K_f\right)}{K_f} + \frac{\left|\Psi\left(X_i\right)\right|^2\left(K_{X_i} - K_i\right)}{K_i}\right)\right)\right) \\
& \times \exp\left(-\int^{K_f}\left(K - \hat{F}_2\left(s, R\left(K, X_f\right)\right) K_{X_f}\right) dK + \int^{K_i}\left(K - \hat{F}_2\left(s, R\left(K, X_i\right)\right) K_{X_i}\right) dK\right) \\
& \times \exp\left(-\left(\frac{\left(X_f - X_i\right)^2}{2t\sigma_X^2} + \frac{\left(K'_f - K'_i\right)^2}{2t\sigma_K^2} - \frac{t\sigma_K^2}{2} K'_f K'_f\right)\right)
\end{aligned}$$

with:

$$\begin{aligned}
K'_i &= K_i - \hat{F}_2\left(R\left(K_{X_i}, X_i\right)\right) K_{X_i} \\
K'_f &= K_f - \hat{F}_2\left(R\left(K_{X_f}, X_f\right)\right) K_{X_f}
\end{aligned}$$

The Laplace transform of this function is the transition function given in the text.

Computation of T_2

11.0.3 Solution ot (114)

Solving (114) is straightforward, and similar to the derivation T_1.

We first introduce a change of variable:

$$\begin{aligned}
&T_2\left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right)\right) \\
=\ & \exp\left(-t\int_{\hat{X}_i}^{\hat{X}_f} \frac{\left(g\left(\hat{X}\right)\right)^2 + \sigma_{\hat{X}}^2\left(f\left(\hat{X}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{\left\|\hat{X}_f - \hat{X}_i\right\| \sigma_{\hat{X}}^2 \sqrt{f^2\left(\hat{X}\right)}}\right) \hat{T}_2\left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right)\right)
\end{aligned}$$

The term in the exponential is the average of the relative return:

$$\frac{\left(g\left(\hat{X}\right)\right)^2 + \sigma_{\hat{X}}^2\left(f\left(\hat{X}\right) + \nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) - \frac{\sigma_{\hat{K}}^2 F^2\left(\hat{X}, K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{\sigma_{\hat{X}}^2 \sqrt{f^2\left(\hat{X}\right)}}$$

along the average path, considered as a straight line, from $\hat{X}_i$ to $\hat{X}_f$. We have assumed slow shifts in the sectors space, so that $\hat{T}_2$ satisfies the following equation in first approximation:

$$\begin{aligned}&-\frac{\partial}{\partial t}\hat{T}_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right) \\ \simeq\quad &\left(-\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}}^2-\nabla_y^2\right)\hat{T}_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)+\frac{y^2}{4}\hat{T}_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)\end{aligned}\tag{121}$$

Given (102), we can assume that y is independent from $\hat{X}$ in first approximation. Thus, solving (121) yields:

$$\begin{aligned}&\hat{T}_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right) \\ \simeq\quad &\exp\left(-\left(\frac{\sigma_{\hat{X}}^2}{2}t\left(\hat{K}_f+\frac{\sigma_{\hat{K}}^2F\left(\hat{X}_f,K_{\hat{X}_f}\right)}{f^2\left(\hat{X}_f\right)}\right)\left(\hat{K}_i+\frac{\sigma_{\hat{K}}^2F\left(\hat{X}_i,K_{\hat{X}_i}\right)}{f^2\left(\hat{X}_i\right)}\right)\right)\right) \\ &\times\exp\left(-\frac{\sqrt{f^2\left(\frac{\hat{X}_f+\hat{X}_i}{2}\right)}}{2t\sigma_{\hat{X}}^2}\left(\left(\hat{K}_f+\frac{\sigma_{\hat{K}}^2F\left(\hat{X}_f,K_{\hat{X}_f}\right)}{f^2\left(\hat{X}_f\right)}\right)-\left(\hat{K}_i+\frac{\sigma_{\hat{K}}^2F\left(\hat{X}_i,K_{\hat{X}_i}\right)}{f^2\left(\hat{X}_i\right)}\right)\right)^2\right)\end{aligned}\tag{122}$$

11.0.4 Full transition function

Reintroducing the change of variables (98) amounts to introduce a factor:

$$\exp\left(\frac{1}{\sigma_{\hat{X}}^2}\int_{\hat{X}_i}^{\hat{X}_f}g\left(\hat{X}\right)d\hat{X}+\frac{\hat{K}_f^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}_f\right)-\frac{\hat{K}_i^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}_i\right)\right)$$

in the formula for T_2 and this leads to the full formula for the transition function:

$$\begin{aligned}&T_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right) \\ \simeq\quad &\exp\left(-t\int_{\hat{X}_i}^{\hat{X}_f}\frac{\left(g\left(\hat{X}\right)\right)^2+\sigma_{\hat{X}}^2\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{\left\Vert\hat{X}_f-\hat{X}_i\right\Vert\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}}\right) \\ &\times\exp\left(\frac{1}{\sigma_{\hat{X}}^2}\int_{\hat{X}_i}^{\hat{X}_f}g\left(\hat{X}\right)d\hat{X}+\frac{\hat{K}_f^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}_f\right)-\frac{\hat{K}_i^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}_i\right)\right)\hat{T}_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)\end{aligned}\tag{123}$$

with:

$$\hat{T}_2\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)$$

given by (122).

The Laplace transform of (123) is the formula presented in the text.

Appendix 4

We write the series expansion in $\Delta S_{\text{fourth order}}$ of $\exp\left(-S\left(\Psi\right)\right)$:

$$\begin{aligned}\exp\left(-S\left(\Psi\right)\right)\quad=\quad&\exp\left(-\left(S\left(\Psi_0,\hat{\Psi}_0\right)+\int\left(\Delta\Psi^{\dagger}\left(Z,\theta\right),\Delta\hat{\Psi}^{\dagger}\left(Z,\theta\right)\right)\left(Z,\theta\right)O\left(\Psi_0\left(Z,\theta\right)\right)\begin{pmatrix}\Delta\Psi\left(Z,\theta\right)\\ \Delta\hat{\Psi}\left(Z,\theta\right)\end{pmatrix}\right)\right) \\ &\left(1+\sum_{n\geqslant 1}\frac{\left(-\Delta S_{\text{fourth order}}\left(\Psi,\hat{\Psi}\right)\right)^n}{n!}\right)\end{aligned}$$

where $O\left(\Psi_0\left(Z,\theta\right)\right)$ is defined in (65).

Then, we decompose $\Delta S_{\text{fourth order}}\left(\Psi,\hat{\Psi}\right)$ as a sum of two combinations:

$$\begin{aligned}\Delta S_{\text{fourth order}}\left(\Psi,\hat{\Psi}\right) &= \int \Delta\Psi^{\dagger}\left(K,X\right)\Delta\Psi^{\dagger}\left(K',X'\right)\Delta S_{11}\Delta\Psi\left(K',X'\right)\Delta\Psi\left(K,X\right) \\ &\quad +\Delta\Psi^{\dagger}\left(K',X'\right)\Delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\Delta S_{12}\Delta\Psi\left(K',X'\right)\Delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\end{aligned}$$

with:

$$\Delta S_{11}=\left(2\tau-\nabla_K\frac{\delta^2 u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}\right)\delta\left(X-X'\right)$$

and:

$$\Delta S_{12}=-\left(\nabla_K\frac{\delta^2 u\left(K,X,\Psi,\hat{\Psi}\right)}{\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)\delta\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)}+\left\{\nabla_{\hat{K}}\frac{\hat{K}\delta^2 f\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}+\nabla_{\hat{X}}\frac{\delta^2 g\left(\hat{X},\Psi,\hat{\Psi}\right)}{\delta\Psi\left(K',X\right)\delta\Psi^{\dagger}\left(K',X\right)}\right\}\right)\delta\left(X-X'\right)$$

Application of (23) leads to the following form of the transition functions:

$$\begin{aligned}&G_{ij}\left(\left[\left(K_f,X_f\right),\left(K_f,X_f\right)'\right],\left[\left(X_i,K_i\right),\left(X_i,K_i\right)'\right]\right) \\ =\;& G_i\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)G_j\left(\left(K_f,X_f\right)',\left(X_i,K_i\right)'\right) \\ &+\sum_{p\geqslant 1}\frac{(-1)^p}{p!}\int G_i\left(\left(K_f,X_f\right),\left(X_p,K_p\right)\right)G_j\left(\left(K_f,X_f\right)',\left(X_p,K_p\right)'\right)\Delta S_{ij}\left(\left(X_p,K_p\right),\left(X_p,K_p\right)'\right) \\ &\times G_i\left(\left(X_p,K_p\right),\left(X_{p-1},K_{p-1}\right)\right)G_j\left(\left(X_p,K_p\right)',\left(X_{p-1},K_{p-1}\right)'\right)\Delta S_{ij}\left(\left(X_{p-1},K_{p-1}\right),\left(X_{p-1},K_{p-1}\right)'\right) \\ &...\times\Delta S_{ij}\left(\left(K_1,X_1\right),\left(K_1,X_1\right)'\right)G_1\left(\left(K_1,X_1\right),\left(X_i,K_i\right)\right)G_1\left(\left(K_1,X_1\right)',\left(X_i,K_i\right)'\right)\prod_{k\leqslant p}d\left(\left(X_k,K_k\right),\left(X_k,K_k\right)'\right)\end{aligned} \tag{124}$$

These corrections modify the n agents Green functions and can be computed using graphs expansion. In the sequel we will focus only on the first order corrections to the four agents Green functions. This is sufficient to stress the impact of interactions of agents in the background state.

The term ΔS_{11} measures the interaction between firms, and ΔS_{12} the firms-investors interactions. There is no term ΔS_{22} of investors-investors interaction. In our model all interactions depend on firms.

To estimate the impact of interactions, we can assume the paths from $\left(\left(X_i,K_i\right),\left(K_f,X_f\right)\right)$ to $\left(\left(X_i,K_i\right),\left(K_f,X_f\right)'\right)$ cross each other one time at some X and approximate the terms ΔS_{ij} by their average value estimated on the average paths from K_i,K_i' to K_f,K_f',

In this approximation, we find:

$$\begin{aligned}&G_{ij}\left(\left[\left(K_f,X_f\right),\left(K_f,X_f\right)'\right],\left[\left(X_i,K_i\right),\left(X_i,K_i\right)'\right]\right) \\ \simeq\;& G_i\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)G_j\left(\left(K_f,X_f\right)',\left(X_i,K_i\right)'\right) \\ &-G_i\left(\left(K_f,X_f\right),\left(X,K\right)\right)G_j\left(\left(K_f,X_f\right)',\left(X,K\right)'\right)\Delta S_{ij}\left(\left(X,\bar{K}\right),\left(X,\bar{K}\right)'\right)G_1\left(\left(X,K\right),\left(X_i,K_i\right)\right)G_1\left(\left(X,K\right)',\left(X_i,K_i\right)'\right) \\ \simeq\;& G_i\left(\left(K_f,X_f\right),\left(X_i,K_i\right)\right)G_j\left(\left(K_f,X_f\right)',\left(X_i,K_i\right)'\right)-\Delta S_{ij}\left(\left(\bar{X},\bar{K}\right),\left(\bar{X},\bar{K}\right)'\right)\hat{G}_i\left(\left(K_f,X_f\right),\left(X,K\right)\right)\hat{G}_j\left(\left(K_f,X_f\right)',\left(X,\right.\right.\end{aligned}$$

with:

$$\begin{aligned}\left(\bar{X},\bar{K}\right) &= \frac{\left(K_f,X_f\right)+\left(X_i,K_i\right)}{2} \\ \left(\bar{X},\bar{K}\right)' &= \frac{\left(K_f,X_f\right)'+\left(X_i,K_i\right)'}{2}\end{aligned}$$

and:

$$\hat{G}_i\left(\left(K_f,X_f\right),\left(X,K\right)\right)\hat{G}_j\left(\left(K_f,X_f\right)',\left(X,K\right)'\right)$$

is the transition function computed on path that cross once. Applied to the three transition functions for two agents yields the results of the text.

Appendix 5

One agent transition functions

Firms transition function

We interpret the various term involved in (74) and their influence on firms individual dynamics.

Drift term

The three contributions The first term in (74):

$$D\left(\left(K_f, X_f\right), \left(X_i, K_i\right)\right)$$

is a drift term between (X_i, K_i) and (K_f, X_f). It is composed of three contributions (see (75)):

The first term of (75):

$$\int_{X_i}^{X_f} \frac{\nabla_X R\left(K_X, X\right)}{\sigma_X^2} H\left(K_X\right)$$

models the shift of producers towards sectors that have the highest long-term returns.

To interpret the second contribution to $D\left(\left(K_f, X_f\right), \left(X_i, K_i\right)\right)$:

$$\int_{K_i}^{K_f} \left(K - \hat{F}_2\left(s, R\left(K, \bar{X}\right)\right) K_{\bar{X}}\right) dK \tag{125}$$

, recall that $\frac{F_2\left(R\left(K,\bar{X}\right)\right)}{F_2\left(R\left(K_{\bar{X}},\bar{X}\right)\right)}$ models the relative expectations of returns of the firm along its path from (X_i, K_i) to (K_f, X_f) based on their returns' expectation $R\left(K, \bar{X}\right)$ and that $\frac{F_2\left(R\left(K,\bar{X}\right)\right)K_{\bar{X}}}{F_2\left(R\left(K_{\bar{X}},\bar{X}\right)\right)}$ represents the capital investors are ready to invest in the firm along this path. Along the path from (X_i, K_i) to (K_f, X_f), the capital invested in the firm will increase as long as the investors expect growth and as long as additional investment is likely to increase the firm's returns. Once the level of capital reaches their expectations, that is

$$K_T\left(\bar{X}\right) - \hat{F}_2\left(R\left(K_T, \bar{X}\right)\right) K_{\bar{X}} = 0 \tag{126}$$

i.e., when the firm has reached the capital threshold, investment stops.

However, this condition is not always fulfilled. The shape of F_2 is critical. If F_2 is above the line $Y = K$, then for $K < K_T$, the threshold K_T will be reached gradually. In this case, K_T is an equilibrium point. If, on the contrary, F_2 is below the line $Y = K$, then for $K < K_T$, the threshold K_T will never be reached, and $K \to 0$. If $K > K_T$, K can increase indefinitely. This corresponds to firms whose profitability is perceived as boundless as long as more capital is invested in.

The third term in (75):

$$\int_{K_i}^{K_f} \left(\left(\frac{X_f - X_i}{2}\right) \nabla_X \hat{F}_2\left(s, R\left(K, \bar{X}\right)\right) K_{\bar{X}}\right) dK$$

induces firms to move towards more appropriate sectors, according to investors and given the capital of the firm. The firm does not solely move according to the new investors it could attract but must also take into account its current investors. If it moves, it risks losing the investors it has already attracted.

Trade-off between terms There is a trade-off between the first and the third terms: firms want to move towards sectors with higher returns, but differences in average capital between sectors could make a firm unattractive in a new sector. The loss of investors **incured** during a shift of sector must be compared with the number of investors possibly attracted in the new sector: the level of attractiveness may decrease for a given amount of capital.

The second contribution to D is an indicator of the firm's growth potential in a given sector. It depends on the firm's level of capital compared to the threshold capital requirement and its dynamics in this sector.

A move along sectors due to the terms 1 and 3 modifies the firm's relative capital, which is sector-dependant: F_2, measuring the firm attractiveness in the sector and indirectly the threshold of capital K_T defined in (126) are modified by the shift from one sector to another. Therefore, a firm could be below the value of K_T in one sector, then above in the next sector, which will reverse its capital dynamics. The firm's capital dynamics remains the same as long as its relative attractiveness in a sector does not change significantly.

Effective time of transition

The following term depends on the competition in a sector:

$$\begin{aligned}\alpha_{eff}\left(\Psi,\left(K_f,X_f\right),\left(X_i,K_i\right)\right) &= \alpha+D\left(\|\Psi\|^2\right)+\frac{\tau}{2}\left(\frac{\left|\Psi\left(X_f\right)\right|^2\left(K_{X_f}-K_f\right)}{K_f}-\frac{\left|\Psi\left(X_i\right)\right|^2\left(K_{X_i}-K_i\right)}{K_i}\right) \\ &+\frac{\sigma_K^2}{2}\left(K_f-\hat{F}_2\left(s,R\left(K_f,\bar{X}\right)\right)K_{\bar{X}}\right)\left(K_i-\hat{F}_2\left(s,R\left(K_i,\bar{X}\right)\right)K_{\bar{X}}\right)\end{aligned} \tag{127}$$

Recall that the constant α is the inverse of the average lifetime of the agents. The larger it is, the lower the probability of transition. In the transition functions, α is shifted by two path-dependent terms and replaced by α_{eff}. Thus, α_{eff} is the inverse mobility of the firm during its transition from one point to another.

Therefore, the likelihood of shifts in capital and sectors depends not only on the average lifespan of the firm, but also on terms that are directly related to the collective state.

The first correction to α is $D\left(\|\Psi\|^2\right)$ that is related to competition. We have shown in (Gosselin Lotz Wambst 2022):

$$D\left(\|\Psi\|^2\right)\simeq 2\tau\frac{N}{V-V_0}+\frac{1}{2\sigma_X^2}\left\langle\left(\nabla_X R\left(X\right)\right)^2\right\rangle_{V/V_0}H^2\left(\frac{\left\langle\hat{K}\right\rangle}{N}\right)\left(1-\frac{H'\left(\frac{\left\langle\hat{K}\right\rangle}{N}\right)}{H\left(\frac{\left\langle\hat{K}\right\rangle}{N}\right)}\frac{\left\langle\hat{K}\right\rangle}{N}\right)$$

where V is the volume of the sectors space and V_0 is the locus where $\|\Psi\left(X\right)\|^2=0$. As a consequence, the stronger the competition, i.e., the larger τ, the greater $D\left(\|\Psi\|^2\right)$, and the less possibilities of shifting from a sector to another.

The third term in (127):

$$\frac{\tau}{2}\left(\frac{\left|\Psi\left(X_f\right)\right|^2\left(K_{X_f}-K_f\right)}{K_f}-\frac{\left|\Psi\left(X_i\right)\right|^2\left(K_{X_i}-K_i\right)}{K_i}\right) \tag{128}$$

is also linked to the competition, but depends on the level of capital of the agent, and the number of d agents in the sectors crossed. This term measures the strength of agent's shift from one sector to another.

It is negative when $K_f>K_{X_f}$, and when $K_i<K_{X_i}$. In other words, when a firm has less capital than the average in its initial sector, and when, it ends up in a sector in which it has more capital than the average, the probability of transition from K_i to K_f is high. In other words, under-average capital favors the exit from a sector. Above-average capital promotes entry into the sector. Shifts from high average capital sectors to lower-average-capital sectors are favoured.

This phenomenon is amplified by the number of agents. The greater the competition in a sector, i.e., the more firms in the sector, the greater the probability for a lower-than average capitalized firm to be ousted from the sector by higher-than average capitalized firms that enter the sector. Thus, the density of producers $\left|\Psi\left(X\right)\right|^2$ along the movement enhances competition and favours high capitalized firm to move towards higher capitalized sectors, and drives low capitalized firms toward low capitalized sectors.

The last term in (127):

$$\frac{\sigma_K^2}{2}\left(K_f-\hat{F}_2\left(s,R\left(K_f,\bar{X}\right)\right)K_{\bar{X}}\right)\left(K_i-\hat{F}_2\left(s,R\left(K_i,\bar{X}\right)\right)K_{\bar{X}}\right)$$

shows that in average, shifts from the initial point to the final point is done respecting $K = \hat{F}_2 \left(s, R \left(K, \bar{X}\right)\right) K_{\bar{X}}$, the capital investors allocate to the firm. Actually, if $K_i - \hat{F}_2 \left(s, R \left(K_i, \bar{X}\right)\right) K_{\bar{X}} < 0$, there is a higher probability to reach $K_f - \hat{F}_2 \left(s, R \left(K_f, \bar{X}\right)\right) K_{\bar{X}} > 0$. If $K_i - \hat{F}_2 \left(s, R \left(K_i, \bar{X}\right)\right) K_{\bar{X}} > 0$, there is a higher probability to reach $K_f - \hat{F}_2 \left(s, R \left(K_f, \bar{X}\right)\right) K_{\bar{X}} < 0$.

If a firm starts with a capital lower than $\hat{F}_2 \left(s, R \left(K, \bar{X}\right)\right) K_{\bar{X}}$, it is more likely to end up with capital above the new $\hat{F}_2 \left(s, R \left(K, \bar{X}\right)\right) K_{\bar{X}}$, in another sector. The movement induces a change in the F_2, but firms tend to fluctuate around the F_2 of transition.

Fluctuation terms

The term:

$$\sqrt{\frac{\left(X_f - X_i\right)^2}{2\sigma_X^2} + \frac{\left(\tilde{K}_f - \tilde{K}_i\right)^2}{2\sigma_K^2}}$$

describes oscillations around:

$$K_f - \hat{F}_2 \left(s, R \left(K_f, \bar{X}\right)\right) K_{\bar{X}} + K_i - \hat{F}_2 \left(s, R \left(K_i, \bar{X}\right)\right) K_{\bar{X}} = 0$$

On average, during a transition from the initial point to the final point, the firm's capital is governed by the equation $K = K_T = \hat{F}_2 \left(R \left(K_i, \bar{X}\right)\right) K_{\bar{X}}$. If a firm starts with less capital than the threshold set by F_2, it is more likely to end up with a capital above the new F_2, in another sector. The transition from one sector to another involves oscillations around the sector-dependent threshold F_2. However, these oscillations may affect the final destination of the transition. Starting with a capital level above K_T, i.e. $K > K_T$, the firm may shift towards sectors with higher perspectives, i.e. with a higher threshold K_T. This favors in turn accumulation and faster transition to other sectors. Finally, if $\nabla_K \hat{F}_2 \left(s, R \left(\frac{K_f + K_i}{2}, \bar{X}\right)\right) > 0$, F_2 is highly responsive to changes in capital, and larger moves are favored.

Investors transition functions

We interpret the different contributions in (80).

Drift term

The term $D' \left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right)\right)$ is composed of two contributions. The first one:

$$\frac{1}{\sigma_{\hat{X}}^2} \int_{\hat{X}_i}^{\hat{X}_f} g\left(\hat{X}\right) d\hat{X} + \frac{\hat{K}_f^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}_f\right) - \frac{\hat{K}_i^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}_i\right)$$

is composed of two elements.

The first element is $\frac{1}{\sigma_{\hat{X}}^2} \int_{\hat{X}_i}^{\hat{X}_f} g\left(\hat{X}\right) d\hat{X}$. Since the function $g\left(\hat{X}\right)$ is the anticipation of higher returns and rising stock prices, investors move towards sectors were they anticipate the highest returns and stock prices increase.

The second element $\frac{\hat{K}_f^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}_f\right) - \frac{\hat{K}_i^2}{\sigma_{\hat{K}}^2} f\left(\hat{X}_i\right)$: this element can be rewritten as two bits:

$$\frac{\hat{K}_f^2 - \hat{K}_i^2}{\sigma_{\hat{K}}^2} f\left(\frac{\hat{X}_f + \hat{X}_i}{2}\right) + \frac{\left(\hat{K}_f^2 + \hat{K}_i^2\right)}{2\sigma_{\hat{K}}^2} \left(\hat{X}_f - \hat{X}_i\right) \nabla_X f\left(\frac{\hat{X}_f + \hat{X}_i}{2}\right)$$

The first bit, $\frac{\hat{K}_f^2 - \hat{K}_i^2}{\sigma_{\hat{K}}^2} f\left(\frac{\hat{X}_f + \hat{X}_i}{2}\right)$ shows that the highest the short-term return, the more probable is the increase in capital. The second bit, which is equal to $\frac{\left(\hat{K}_f^2 + \hat{K}_i^2\right)}{2\sigma_{\hat{K}}^2} \left(\hat{X}_f - \hat{X}_i\right) \nabla_X f\left(\frac{\hat{X}_f + \hat{X}_i}{2}\right)$ indicates that investors move towards sectors with highest returns. The more capital they have, the fastest the shift.

The second term arising in $D' \left(\left(\hat{K}_f, \hat{X}_f\right), \left(\hat{X}_i, \hat{K}_i\right)\right)$:

$$-\frac{1}{\sigma_{\hat{X}}^2}\int_{\hat{X}_i}^{\hat{X}_f}\frac{\left(g\left(\hat{X}\right)\right)^2+\sigma_{\hat{X}}^2\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}}\right)-\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}}d\hat{X}$$

is similar to the determinant of capital accumulation in a collective state and has the same interpretation. There is a tradeoff between long-term and short-term returns. It further shows the importance of relative long-term return. Investors move towards relative long-term returns. Mathematically, we can measure the dependence of agents' capital accumulation in neighboring sectors using the integrand:

$$p=\frac{-\left(\frac{\left(g\left(\hat{X},K_{\hat{X}_M}\right)\right)^2}{\sigma_{\hat{X}}^2}+\nabla_{\hat{X}}g\left(\hat{X},K_{\hat{X}_M}\right)-\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}\right)}{f\left(\hat{X}\right)} \tag{129}$$

The function p represents the relative attractivity of a sector vis-a-vis its neighbours and depends on the gradients of long-term returns $R\left(\hat{X}\right)$ through the function $g\left(\hat{X}\right)$, the capital mobility at sector $\hat{X}$. This function $g\left(\hat{X}\right)$, which depicts investors' propensity to seek higher returns across sectors,and is indeed proportional to $\nabla_{\hat{X}}R\left(\hat{X}\right)$. The gradient of g, $\nabla_{\hat{X}}g$, is proportional to $\nabla_{\hat{X}}^2R\left(\hat{X}\right)$: it measures the position of the sector relative to its neighbours. At a local maximum, the second derivative of $R\left(\hat{X}\right)$ is negative: $\nabla_{\hat{X}}^2R\left(\hat{X}\right)<0$. At a minimum, it is positive.

The last term, $\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X},K_{\hat{X}}\right)}{2f^2\left(\hat{X}\right)}$, involved in the definition of $Y(\hat{X})$ and p is a smoothing factor between neighbouring sectors. It reduces differences between sectors: it increases when the relative attractivity with respect to $K_{\hat{X}}$ decreases. The number of investors and capital will increase in sectors that are in the neighbourhood of significantly more attractive sectors, i.e. with higher average capital and number of investors. It slows down the transitions.

Fluctuation terms

The last term involved in (80):

$$\alpha_{eff}'\left(\left(\hat{K}_f,\hat{X}_f\right),\left(\hat{X}_i,\hat{K}_i\right)\right)\sqrt{\frac{\left|f\left(\frac{\hat{X}_f+\hat{X}_i}{2}\right)\right|}{2\sigma_{\hat{X}}^2}}\left|\left(\hat{K}_f+\frac{\sigma_{\hat{K}}^2F\left(\hat{X}_f,K_{\hat{X}_f}\right)}{f^2\left(\hat{X}_f\right)}\right)-\left(\hat{K}_i+\frac{\sigma_{\hat{K}}^2F\left(\hat{X}_i,K_{\hat{X}_i}\right)}{f^2\left(\hat{X}_i\right)}\right)\right|$$

depicts the possible oscillations around averages (that) occur (within)in a definite timespan, so(such) that the transition probability decrases with $\hat{K}_f-\hat{K}_i$ and X_f-X_i. However, this probability decreases with short-term returns: the higher the returns, the lower the incentive to switch from one sector to another.