Financial Markets and the Real Economy: A Statistical Field Perspective on Capital Allocation and Accumulation

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Abstract

This paper provides a general method to translate a classical economic framework with a large number of agents into a field-formalism model. This type of formalism allows the analytical treatment of economic models with an arbitrary number of agents while preserving the system's interactions and microeconomic features at the individual level.

We apply this methodology to model the interactions between financial markets and the real economy.

We start with a classical framework of a large number of heterogeneous agents, investors, and firms. Firms are spread among sectors but may shift between sectors to improve their returns. They compete by producing differentiated goods, and they reward their investors through dividends and better stock valuation. Investors invest in firms based on firms' expected long-run returns. They may also gradually reallocate their capital along the sectors space.

From this framework, we derive a field-formalism model in which collective states emerge. We show that the number of firms in each sector depends on the aggregate financial capital invested in the sector, and its firms' expected long-term returns. Capital accumulation in each sector depends both on the sector's short-term returns and relative expected long-term returns.

For each sector, three patterns of accumulation emerge. In the first pattern, sectors with a relatively large number of low-capitalized firms woo investors with dividends. In the second pattern, both short and long-term returns in the sector drive intermediate-to-high capital. In the third pattern, higher expectations of long-term returns drive massive inputs of capital.

Since instability in capital accumulation may arise among and within sectors, we widen our study to the dynamics of the collective configurations, in particular interactions between average capital and expected long-term returns, and show that the expectations formation process is crucial to overall stability.

Expectations highly reactive to capital variations stabilize high capital configurations. Depending on their initial capital, they may drive low-to-moderate capital sectors towards zero or a higher level of capital. Inversely, expectations moderately reactive to capital variations stabilize low-to-moderate capital configurations and drive high capital sectors towards a more moderate level of capital equilibria.

Eventually, expectations that are both highly sensitive to exogenous conditions and highly reactive to variations in capital induce large fluctuations of capital in the system, possibly at the expense of the real economy.

Key words: Financial Markets, Real Economy, Capital Allocation, Statistical Field Theory, Background fields, Collective states, Multi-Agent Model, Interactions.

JEL Classification: B40, C02, C60, E00, E1, G10

1 Introduction

This paper applies a statistical field-theoretic approach to systems with a large number of heterogeneous agents to study the interactions between financial and physical capital and the determinants of capital

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allocation.

Two groups of agents, producers and investors, represent the real economy and the financial markets respectively. The first group, producers, is composed of a large number of firms in different sectors that collectively own the entire physical capital. The second group, investors, holds and allocates the entire financial capital between firms across sectors according to investment preferences, expected returns paid back by dividends, and stock prices variations. Thus financial capital is a function of dividends and stocks' valuations, whereas physical capital is a function of the overall capital allocated by the financial sector.

We have shown in previous papers how, using a two-step process, a classical economic framework with a large number of heterogeneous agents could be translated into a field model. The present paper develops and applies a shortcut of this method to our system. Producers and investors are described by two interacting fields and the system action functional which encodes the whole set of agents' actions and interactions. The solutions to the minimization equations of the action functional are called the *background fields* of the system. They characterize the collective states of the system, structure interactions between both types of agents and condition individual dynamics. From a sector perspective, the collective states determine the average capital and the density, or number, of firms within sectors, given external parameters such as changes in expected returns, technological advances and their variations.

We first show that the number of firms per sector depends on the average capital invested in this sector, and on the sector's expected long-term return relative to neighbouring sectors. Sectoral capital accumulation itself depends on short-term returns, and on expected long-term returns, both absolute and relative.

The equilibrium capital at each point of the sectors space characteristises a collective configuration of the system. However, there are several possible equilibrium values of average capital for each sector and thus multiple collective states. Three patterns of accumulation per sector emerge, from low to high capital, but some may be unstable: due to the limited number of agents, changes in parameters or expectations may induce changes in portfolio allocation, and favour or deplete some sectors. At a macro-timescale, any deviation from an equilibrium drives the sector towards the next stable equilibrium, zero included, and if there is none, towards infinity. Note that this looming potential instability in sectors depends on the position of the sector relative to its neighbours. This notion of instability is thus relative and context-dependent: variations of parameters may propagate from one sector to others. Sectors may change pattern which induces transitions between collective states.

To account for this systemic instability, we widen our model and consider a dynamic system involving average capital and endogenized long-term expected returns, which is the most volatile parameter of our model. In such a dynamic system, average capital per sector interacts with one another, but also with long-term expected returns.

This dynamic system differs from those in standard economics: whereas in economics the dynamics are usually studied around static equilibria, we consider the dynamic interactions between potential equilibria and expected long-term returns.

Some solutions to this dynamic system are oscillatory: changes in one or several sectors may propagate over the whole sectors' space. We find, for each sector, the conditions of stable or unstable oscillations for the system. Depending on the sector's specific characteristics, oscillations in average capital and expected long-term returns may dampen or increase. Some characteristics of the system discriminate between stable and unstable oscillations: some formation of expectations favour overall stability in equilibria, and others deter it.

Eventually, fluctuations in financial expectations impose their pace to the real economy. The combination of expectations both highly sensitive to exogenous conditions and highly reactive to variations in capital implies that large fluctuations of capital in the system, at the possible expense of the real economy.

The paper is organized as follows. The second section is a literature review. Part one is then devoted to presenting the model: section three presents the principles of field-theoretic modeling of a system with a large number of agents. Section four details the microeconomic framework on which our field model is based. Section five presents the general method of translation of a model with a large number of agents into a field-theoretic model. This method is applied in section six to our microeconomic framework to derive its field-theoretic representation. Section seven exposes the use of the field model in our context, and the various averages it allows to compute. Part two presents the resolution of the model: in sections eight and nine, we present the minimization of the field action functional for producers and investors respectively. We derive the background field for the real economy and the density of firms par sector. We then compute the background field for the financial agents and find the density of investors per sector and the defining equation for average capital per firm per sector. Section ten investigates the solutions to this equation. It studies its differential form, expands it around some particular solutions, and finds directly the solutions for some particular forms of the parameter functions defining the system. In section eleven, the model is extended to a dynamic system at the macro-time scale by endogenizing the expected long-term revenue. This dynamic system presents some oscillatory solutions whose stability depends on the various patterns of accumulation. Part three focuses on the results and discussion: Section twelve gathers the interpretations of the model. The results are presented in sections thirteen to eighteen. Section nineteen discusses the results and their interpretations. Section twenty concludes.

2 Literature review

Several branches of the economic literature seek to replace the representative agent with a collection of heterogeneous ones. Among other things, they differ in the way they model this collection of agents.

The first branch of the literature represents this collection of agents by probability densities. This is the approach followed by mean field theory, heterogeneous agents new Keynesian (HANK) models, and the information-theoretic approach to economics.

Mean field theory studies the evolution of agents' density in the state space of economic variables. It includes the interactions between agents and the population as a whole but does not consider the direct interactions between agents. This approach is thus at an intermediate scale between the macro and micro scale: it does not aggregate agents but replaces them with an overall probability distribution. Mean field theory has been applied to game theory (Bensoussan et al. 2018, Lasry et al. 2010a, b) and economics (Gomes et al. 2015). However, these mean fields are actually probability distributions. In our formalism, the notion of fields refers to some abstract complex functions defined on the state space and is similar to the second-quantized-wave functions of quantum theory. Interactions between agents are included at the individual level. Densities of agents are recovered from these fields and depend directly on interactions.

Heterogeneous agents' new Keynesian (HANK) models use a probabilistic treatment similar to mean fields theory. An equilibrium probability distribution is derived from a set of optimizing heterogeneous agents in a new Keynesian context (see Kaplan and Violante 2018 for an account). Our approach, on the contrary, focuses on the direct interactions between agents at the microeconomic level. We do not look for an equilibrium probability distribution for each agent, but rather directly build a probability density for the system of N agents seen as a whole, that includes interactions, and then translate this probability density in terms of fields. The states' space we consider is thus much larger than those considered in the above approaches. Because it is the space of all paths for a large number of agents, it allows studying the agents' economic structural relations and the emergence of the particular phases or collective states induced by these specific micro-relations, that will in turn impact each agent's stochastic dynamics at the microeconomic level. Other differences are worth mentioning. While HANK models stress the role of an infinite number of heterogeneously-behaved consumers, our formalism dwells on the relations between physical and financial capital[1]. Besides, our formalism does not rely on agents' rationality assumptions, since for a large number of agents, behaviours, be they fully or partly rational, can be modeled as random.

The information theoretic approach to economics (see Yang 2018) considers probabilistic states around the equilibrium. It is close to our methodological stance: it replaces the Walrasian equilibrium with a statistical equilibrium derived from an entropy maximisation program. Our statistical weight is similar to the one they use, but is directly built from microeconomic dynamic equations. The same difference stands for the rational inattention theory (Sims 2006) in which non-gaussian density laws are derived from limited information and constraints: our setting directly includes constraints in the random description of an agent (Gosselin, Lotz, Wambst 2020).

A second branch of the literature is closest to our approach since it considers the interacting system of agents in itself. It is the multi-agent systems literature, notably agent-based models (see Gaffard Napoletano 2012, Mandel et al. 2010 2012) and economic networks (Jackson 2010).

Agent-based models use general macroeconomics models, whereas network models lower-scale models such as contract theory, behaviour diffusion, information sharing, or learning. In both settings, agents are

[1] Note that our formalism could also include heterogeneous consumers (see Gosselin, Lotz, Wambst 2020).

typically defined by and follow various sets of rules, leading to the emergence of equilibria and dynamics otherwise inaccessible to the representative agent setup. Both approaches are however highly numerical and model-dependent and rely on microeconomic relations - such as ad-hoc reaction functions - that may be too simplistic. Statistical fields theory on the contrary accounts for transitions between scales. Macroeconomic patterns do not emerge from the sole dynamics of a large set of agents: they are grounded in behaviours and interaction structures. Describing these structures in terms of field theory allows for the emergence of phases at the macro scale, and the study of their impact at the individual level.

A third branch of the literature, Econophysics, is also related to ours since it often considers the set of agents as a statistical system (for a review, see Abergel et al. 2011a,b and references therein; or Lux 2008, 2016). But it tends to focus on empirical laws, rather than apply the full potential of field theory to economic systems. In the same vein, Kleinert (2009) uses path integrals to model stock prices' dynamics. Our approach, in contrast, keeps track of usual microeconomic concepts, such as utility functions, expectations, and forward-looking behaviours, and includes these behaviours into the analytical treatment of multi-agent systems by translating the main characteristics of optimizing agents in terms of statistical systems.

The literature on interactions between finance and real economy or capital accumulation takes place mainly in the context of DGSE models. (for a review of the literature, see Cochrane 2006; for further developments see Grassetti et al. 2022, Grosshans and Zeisberger 2018, Böhm et al. 2008, Caggese and Orive, Bernanke e al. 1999, Campello et al. 2010, Holmstrom and Tirole 1997, Jermann, and Quadrini 2012, Khan Thomas 2013, Monacelli et al. 2011). Theoretical models include several types of agents at the aggregated level. They describe the interactions between a few representative agents such as producers for possibly several sectors, consumers, financial intermediaries, etc. to determine interest rates, levels of production, and asset pricing, in a context of ad-hoc anticipations.

Our formalism differs from this literature in three ways. First, we consider several groups of a large number of agents to describe the emergence of collective states and study the continuous space of sectors. Second, we consider expected returns and the longer-term horizon as somewhat exogenous or structural. Expected returns are a combination of elements, such as technology, returns, productivity, sectoral capital stock, expectations, and beliefs. These returns are also a function defined over the sectors' space: the system's background fields are functionals of these expected returns. Taken together, the background fields of a field model describe an economic configuration for a given environment of expected returns. As such, expected returns are at first seen as exogenous functions. It is only in the second step, when we consider the dynamics between capital accumulation and expectations, that expectations may themselves be seen as endogenous. Even then, the form of relations between actual and expected variables specified are general enough to derive some types of possible dynamics.

Last but not least, we do not seek individual or even aggregated dynamics, but rather background fields that describe potential long-term equilibria and may evolve with the structural parameters. For such a background, agents' individual typical dynamics may nevertheless be retrieved through Green functions (see GLW). These functions compute the transition probabilities from one capital-sector point to another. But backgrounds themselves may be considered as dynamical quantities. Structural or long-term variations in the returns' landscape may modify the background and in turn the individual dynamics. Expected returns themselves depend on and interact with, capital accumulation.

Part I: Setup

3 From a microeconomic to a field formalism

The field-formalism used in this paper is rooted in a probabilistic description of economic systems with large number of agents. Classically, each agent's dynamics is described by an optimal path for some vector variable, say $A_i(t)$, from an initial to a final point, up to some fluctuations. The same system of agents can however be seen as probabilistic: indeed an agent can be described by a *probability density* that is, due to idiosyncratic uncertainties, centered around the classical optimal path[2] (see Gosselin, Lotz and Wambst 2017, 2020, 2021). In this probabilistic approach, each possible dynamics for the set of N agents must be taken into account and weighted by its probability. The system is then described by a *statistical weight*, the probability density for any configuration of N arbitrary individual paths. Once this statistical weight is found, we can compute the transition probabilities of the system, i.e. the probabilities for any number of agents to evolve from an initial to a final state in A_i, B_i in a given time.

Because this probabilistic approach implies keeping track of the N agents' probability transitions, it is practically untractable for a large number of agents. It remains however a necessary step since it can conveniently be translated into a more compact *field formalism* (see Gosselin, Lotz, and Wambst 2017, 2020, 2021). This field formalism preserves the essential information encoded in the model but implements a change in perspective. It does not keep track of the N-indexed agents but describes their dynamics and interactions as a collective thread of all possible anonymous paths. This collective thread can be seen as an environment that conditions the dynamics of individual agents from one state to another. The field formalism eases the computation of transition functions. More importantly, it detects the collective states or phases encompassed in the field, that would otherwise remain indetectable using the probabilistic formulation.

To translate the probabilistic approach into a field model, the N agents' trajectories $\mathbf{A}_i(t)$ is replaced by a field Ψ, which is a complex-valued function that solely depends on a single set of variables, $\mathbf{A}$. The statistical weight of the probabilistic approach is translated into a probability density on the space of complex-valued functions of the variables $\mathbf{A}$. For the configuration $\Psi(\mathbf{A})$, this probability density has the form $\exp(-S(\Psi))$. The functional $S(\Psi)$ is called the *field action functional.* It encodes the microscopic features of individual agents' dynamics and interactions. The idea is that of a dictionary that would translate the various terms of the classical description in terms of their field equivalent. The integral of $\exp(-S(\Psi))$ over the configurations Ψ is the *partition function* of the system. The fields that minimize the action functional are the *classical background fields,* or more simply the *background fields.* They encapsulate the collective states of the system.

For several types of agents, the generalisation is straightforward. Each type α is described by a field $\Psi_\alpha(\mathbf{A}_\alpha)$. The field action depends on the whole set of fields $\{\Psi_\alpha\}$. It accounts for all types of agents and their interactions, and writes $S(\{\Psi_\alpha\})$. The form of $S(\{\Psi_\alpha\})$ is obtained directly from the classical description of our model.

In the following, we will detail a shortcut of this translation method and apply it to the microeconomic framework below.

4 The microeconomic framework

This section develops a microeconomic framework that will be turned into a field model. Since our goal is to picture the interactions between the real and the financial economy, we consider two groups of agents, producers, and investors. In the following, we will refer to producers or firms i indistinctively, and use the upper script $\hat{}$ for variables describing investors.

4.1 Producers

Producers are modeled as firms that belong to sectors. Here, both the notions of firm and sector are versatile: a single firm with subsidiaries in different countries and/or offering differentiated products can be modeled

[2] Due to the infinite number of possible paths, each individual path has a null probability to exist. We, therefore, use the word "probability density" rather than "probability".

as a set of independent firms. Similarly, a sector refers to a group of firms with similar activities, but this criterion is loose: sectors can be decomposed into sectors per country, to account for local specificities, or in several sectors for that matter.

Producers move across sectors described by a vector space of arbitrary dimension. The position of producer i in this space is denoted X_i and his physical capital, K_i. Producers are defined by these two variables, which are both subject to dynamic changes. Producers may change their capital stocks over time or altogether shift sectors.

Each firm produces a single differentiated good. However, in the following, we will merely consider the return each producer may provide to its investors.

The return of producer i at time t, denoted r_i, depends on K_i, X_i and on the level of competition in the sector. It is written:

$$r_i = r\left(K_i, X_i\right) - \gamma \sum_j \delta\left(X_i - X_j\right) \frac{K_j}{K_i} \tag{1}$$

The first term is an arbitrary function that depends on the sector and the level of capital per firm in this sector. It represents the return of capital in a specific sector X_i under no competition. We deliberately keep the form of $r\left(K_i, X_i\right)$ unspecified, since most of the results of the model rely on the general properties of the functions involved. When needed, we will give a standard Cobb-Douglas form to the returns $r\left(K_i, X_i\right)$. The second term in (1) is the decreasing return of capital. In any given sector, it is proportional to both the number of competitors and the specific level of capital per firm used.

We also assume that, for all i, firm i has a market valuation defined by both its price, P_i, and the variation of this price on financial markets, $\dot{P}_i$. This variation is itself assumed to be a function of an expected long-term return denoted $R\left(K_i, X_i\right)$, and more precisely the relative return $\bar{R}\left(K_i, X_i\right)$ of firm i against the whole set of firms in its sector:

$$\frac{\dot{P}_i}{P_i} = F_1\left(\bar{R}\left(K_i, X_i\right)\right) \tag{2}$$

with:

$$\bar{R}\left(K_i, X_i\right) = \frac{R\left(K_i, X_i\right)}{\sum_l R\left(K_l, X_l\right)} \tag{3}$$

The function F_1 is arbitrary and reflects the preferences of the market relatively to the firm's relative returns.

We assume that firms shift their production in the sector space according to returns, in the direction of the gradient of the expected long-term return $R\left(K_i, X_i\right)$. Yet, the accumulation of agents at any point of the space creates a repulsive force, so that the evolution of X_i minimizes, up to some shocks, the following quantity:

$$L_i\left(X_i, \frac{dX_i}{dt}\right) = \left(\frac{dX_i}{dt} - \nabla_X R\left(K_i, X_i\right) H\left(K_i\right)\right)^2 + \tau \sum_j \delta\left(X_i - X_j\right) \tag{4}$$

When $\tau = 0$, there are no repulsive forces and the move towards the gradient of R is given by the expression:

$$\frac{dX_i}{dt} = \nabla_X R\left(K_i, X_i\right) H\left(K_i\right)$$

When $\tau \neq 0$, repulsive forces deviate the trajectory. The dynamic equation associated to the minimization of (4) is given by the general formula of the dynamic optimization:

$$\frac{d}{dt} \frac{\partial}{\partial \frac{dX_i}{dt}} L_i\left(X_i, \frac{dX_i}{dt}\right) = \frac{\partial}{\partial X_i} L_i\left(X_i, \frac{dX_i}{dt}\right) \tag{5}$$

This last equation does not need to be developed further, since formula (4) is sufficient to switch to the field description of the system. Note for later purpose that the expression $\frac{dX_i}{dt}$ stands for the continuous version of a discrete variation, $X_i\left(t+1\right) - X_i\left(t\right)$.

4.2 Investors

Each investor j is defined by his level of capital $\hat{K}_j$ and his position $\hat{X}_j$ in the sector space. Investors can invest in the entire sector space, but tend to invest in sectors close to their position.

Besides, investors tend to diversify their capital: each investor j chose to allocate parts of his entire capital $\hat{K}_j$ between various firms i. The capital allocated by investor j to firm i is denoted $\hat{K}_j^{(i)}$, and given by:

$$\hat{K}_j^{(i)}(t) = \left(\hat{F}_2\left(R\left(K_i, X_i\right)\right)\hat{K}_j\right)(t) \tag{6}$$

where:

$$\hat{F}_2\left(R\left(K_i, X_i\right), \hat{X}_j\right) = \frac{F_2\left(R\left(K_i, X_i\right)\right) G\left(X_i - \hat{X}_j\right)}{\sum_l F_2\left(R\left(K_l, X_l\right)\right) G\left(X_l - \hat{X}_j\right)} \tag{7}$$

The function F_2 is arbitrary. It depends on the expected return of firm i and on the distance between sectors X_i and $\hat{X}_j$. The function $\hat{F}_2\left(R\left(K_i, X_i\right), \hat{X}_j\right)$ is the relative version of F_2 and translates the dependency of investments on firms' relative attractivity.

We now define ε the time scale for capital accumulation. The variation of capital of investor j between t and $t+\varepsilon$ is the sum of two terms: the short-term returns r_i of the firms in which j invested, and the stock price variations of these same firms:

$$\hat{K}_j(t+\varepsilon) - \hat{K}_j(t) = \sum_i \left(r_i + \frac{\dot{P}_i}{P_i}\right)\hat{K}_j^{(i)} = \sum_i \left(r_i + F_1\left(\bar{R}\left(K_i, X_i\right), \frac{\dot{K}_i(t)}{K_i(t)}\right)\right)\hat{K}_j^{(i)} \tag{8}$$

Incidentally, note that in equation (4), the time scale of motions within the sectors space was normalized to one. Here, on the contrary, we define this motion time scale as ε, and assume $\varepsilon << 1$: the mobility in the sector space is lower than capital dynamics. To rewrite (8) on the same time-span as $\frac{dX_i}{dt}$, we write:

$$\begin{aligned}
\hat{K}_j(t+1) - \hat{K}_j(t) &= \sum_{k=1}^{\frac{1}{\varepsilon}} \hat{K}_j(t+k\varepsilon) - \hat{K}_j(t) \\
&= \sum_{k=1}^{\frac{1}{\varepsilon}} \sum_i \left(r_i + \frac{\dot{P}_i}{P_i}\right)\hat{K}_j^{(i)}(t+k\varepsilon) \\
&\simeq \frac{1}{\varepsilon}\sum_i \left(r_i + F_1\left(\bar{R}\left(K_i, X_i\right), \frac{\dot{K}_i(t)}{K_i(t)}\right)\right)\hat{K}_j^{(i)}
\end{aligned}$$

where the quantities in the sum have to be understood as averages over the time span $[t, t+1]$. Using equation (2), equation (8) becomes in the continuous approximation:

$$\frac{d}{dt}\hat{K}_j(t) = \frac{1}{\varepsilon}\sum_i \left(r_i + F_1\left(\frac{R\left(K_i, X_i\right)}{\sum_l \delta\left(X_l - X_i\right) R\left(K_l, X_l\right)}, \frac{\dot{K}_i(t)}{K_i(t)}\right)\right)\hat{F}_2\left(R\left(K_i, X_i\right), \hat{X}_j\right)\hat{K}_j \tag{9}$$

where $\frac{d}{dt}\hat{K}_j(t) = \hat{K}_j(t+1) - \hat{K}_j(t)$ is now normalized to the time scale of $\frac{dX_i}{dt}$, i.e. 1.

4.3 Link between financial and physical capital

The entire financial capital is, at any time, completely allocated by investors between firms. For producers, there is no alternative source of financing: self-financing is discarded, since it amounts to consider two agents, a producer and an investor, as one. The physical capital of a any given firm is thus the sum of all capital allocated to this firm by all its investors. Physical capital entirely depends on the arbitrage and allocations of the financial sector. Firms do not own their capital: they return it fully at the end of each period with a dividend, though possibly negative. Investors then entirely reallocate their capital between firms at the beginning of the next period.

This set up is a generalisation of the dividend irrelevance theory. It may not be fully accurate in the short-run but, since physical capital cannot subsist without investment, it holds in the long-run. When investors choose not to finance a firm, this firm is bound to disappear in the long run. Under these assumptions, the following identity holds:

$$K_i (t + \varepsilon) = \sum_j \hat{K}_j^{(i)} = \sum_j \hat{F}_2 \left(R \left(K_i (t), X_i (t) \right), \hat{X}_j (t) \right) \hat{K}_j (t) \tag{10}$$

where K_i stands for the physical capital of firm i at time t, and $\sum_j \hat{K}_j^{(i)}$ for the sum of capital invested in firm i by investors j. Recall that the parameter ε accounts for the specific time scale of capital accumulation. It differs from that of mobility within the sector space (4), which is normalized to one.

The dynamics (10) rewrites:

$$\frac{K_i (t + \varepsilon) - K_i (t)}{\varepsilon} = \frac{1}{\varepsilon} \left(\sum_j \hat{F}_2 \left(R \left(K_i (t), X_i (t) \right), \hat{X}_j (t) \right) \hat{K}_j (t) - K_i (t) \right) \tag{11}$$

Using the same token as in the derivation of (9), we obtain in the continuous approximation:

$$\frac{d}{dt} K_i (t) + \frac{1}{\varepsilon} \left(K_i (t) - \sum_j \hat{F}_2 \left(R \left(K_i (t), X_i (t) \right), \hat{X}_j (t) \right) \hat{K}_j (t) \right) = 0 \tag{12}$$

where $\frac{d}{dt} K_i (t)$ stands for $K_i (t + 1) - K_i (t)$.

4.4 Capital allocation dynamics

Investors choose to allocate their capital within sectors, and may modify their portfolio according to the returns of the sector or firms they invest in. This is modelled by a move along the sectors' space in the direction of the gradient of $R (K_i, X_i)$. The move of $\hat{X}_j$ is described by the dynamic equation:

$$\frac{d}{dt} \hat{X}_j - \frac{1}{\sum_i \delta \left(X_i - \hat{X}_j \right)} \sum_i \left(\nabla_{\hat{X}} F_0 \left(R \left(K_i, \hat{X}_j \right) \right) + \nu \nabla_{\hat{X}} F_1 \left(\bar{R} \left(K_i, \hat{X}_j \right) \right) \right) = 0 \tag{13}$$

where the factor $\sum_i \delta \left(X_i - \hat{X}_j \right)$ is the agents' density in the sector $\hat{X}_j$, so that the more competitors in a sector, the slower the move.

In equation (13), the term $\nabla_{\hat{X}} F_0 \left(R \left(K_i, \hat{X}_j \right) \right)$ is the tendency of investors to invest in sectors with the highest returns. This term induces a move in the direction defined by the gradient of a function F_0 of long-term returns.

The term $\nu \nabla_{\hat{X}} F_1 \left(\bar{R} \left(K_i, \hat{X}_j \right) \right)$ describes the investors' preference for stocks with the highest price-dividend ratio.

Ultimately, note that unlike K and $\hat{K}$, X and $\hat{X}$ are not a strictly standard economic variables, and that their dynamics should thus be ascribed an ad-hoc form.

5 Field formalism: general method

In the above, we have detailed a standard, classical microeconomic framework. We will now present a general method to translate such a framework into a field model.

To do so, we must first consider the types of agents in the model, and rewrite their dynamics as the minimization equations of some initial functions, in the same way as, for instance, consumption dynamics could be derived from an utility function.

Since each type of agent in our framework is described by two dynamic equations, there are four minimization functions to find. These minimization functions will be translated into four functionals of two independent fields[3], one for producers, the real economy, and one for investors, financial markets. The sum of the four functionals is the "field action functional" that describes the whole system in terms of fields[4].

5.1 Minimization functions

In standard economic frameworks, each type of agent is characterized by one or more dynamic equations. Some of these dynamic equations can result from a minimization, while others do not. The translation into fields is built upon the functions which, once minimized, yields the system's dynamics. Two cases arise.

When the dynamics are constructed from minimizations, it suffices to use the function which, minimized, gave the equation of the dynamics. These functions, related to the probabilistic interpretation, represent the deviation between the trajectory of an agent and an average or optimal trajectory. They are therefore directly linked to the number of agents in the system.

However, dynamics may not always result from a minimization. In this second case, we must ad-hoc reconstruct functions whose minimization would restore the dynamic equations. Such functions refer to the probabilistic interpretation of the system. They are not unique. When modelling heterogeneous agents, quadratic functions allow to translate the quadratic deviation from the mean trajectory of agents subject to idiosyncratic shocks. This quadratic deviation represents the variance of these shocks and is directly related to the probability of deviating from an average trajectory (see GLW). The construction of these quadratic functions is straightforward.

In general, we assume agents are described by vectors $\mathbf{A}_i(t)$ of arbitrary dimension, where $\mathbf{A}_i(t)$ satisfies a dynamic equation characterizing agent i:

$$\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t) ...\right) = 0 \tag{14}$$

This type of equation, involving the whole set of other agents, is characteristic of models with a large number of interacting agents. Squaring the lhs of (14), then summing over the whole set of agents to describe the full system, yields the associated minimization function:

$$\sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t) ...\right)\right)^2 \tag{15}$$

However, some minimization functions, such as (4), may include an additional term without any time derivative. To account for these additional terms, a more general minimization function writes, for some function g:

$$\begin{aligned} s\left(\{\mathbf{A}_l\}, \left\{\hat{\mathbf{A}}_l\right\}, t\right) &= \sum_i \left(\frac{d\mathbf{A}_i(t)}{dt} - \sum_{j,k,l...} f\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t) ...\right)\right)^2 \\ &\quad + \sum_i \sum_{j,k,l...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t) ...\right) \end{aligned} \tag{16}$$

Ultimately, the integral $\int s\left(\{\mathbf{A}_l\}, \left\{\hat{\mathbf{A}}_l\right\}, t\right) dt$ is the sum of all agents' squared deviation from the average dynamics within the system[5]. A generalisation of equation (16), in which agents interact at different times, and its translation in term of field is presented in appendix 1.

[3] The term functional refers to a function of a function, i.e. a function whose argument is itself a function.

[4] Details about the probabilistic step will be given as a reminder along the text and in appendix 1.

[5] The function s is related to the probabilistic approach, which associates the probability $\exp(-s)$ to the state of the system defined by the $A_i(t), A_j(t), ..$ see appendix 1 for an account.

5.2 Translation of minimization functions into fields functionals

The translation can itself be divided into two relatively simple processes, but varies slightly depending on the type of terms that appear in the various minimization functions.

5.2.1 Terms without temporal derivative

In (16), the terms that include indexed variables but no temporal derivative terms are the easiest to translate. They are of the form:

$$\sum_i \sum_{j,k,l,m...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t) ...\right)$$

These terms describe the whole set of interactions both among and between two groups of agents. Here, agents are characterized by their variables $\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t)...$ and $\hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...$ respectively, for instance in our model firms and investors.

In the field translation, agents of type $\mathbf{A}_i(t)$ and $\hat{\mathbf{A}}_l(t)$ are described by a field $\Psi(\mathbf{A})$ and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$, respectively.

In a first step, the variables indexed i such as $\mathbf{A}_i(t)$ are replaced by variables $\mathbf{A}$ in the expression of g. The variables indexed j, k, l, m..., such as $\mathbf{A}_j(t)$, $\mathbf{A}_k(t)$, $\hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t)...$ are replaced by $\mathbf{A}', \mathbf{A}''$, $\hat{\mathbf{A}}$, $\hat{\mathbf{A}}'$, and so on for all the indices in the function. This yields the expression:

$$\sum_i \sum_{j,k,l,m...} g\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right)$$

In a second step, each sum is replaced by a weighted integration symbol:

$$\begin{aligned}
\sum_i &\rightarrow \int |\Psi(\mathbf{A})|^2 d\mathbf{A}, \ \sum_j \rightarrow \int |\Psi(\mathbf{A}')|^2 d\mathbf{A}', \ \sum_k \rightarrow \int |\Psi(\mathbf{A}'')|^2 d\mathbf{A}'' \\
\sum_l &\rightarrow \int \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 d\hat{\mathbf{A}}, \ \sum_m \rightarrow \int \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}}'
\end{aligned}$$

which leads to the translation:

$$\begin{aligned}
&\sum_i \sum_j \sum_{j,k...} g\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t) ...\right) \\
\rightarrow &\int g\left(\mathbf{A}, \mathbf{A}', \mathbf{A}'', \hat{\mathbf{A}}, \hat{\mathbf{A}}'...\right) |\Psi(\mathbf{A})|^2 |\Psi(\mathbf{A}')|^2 |\Psi(\mathbf{A}'')|^2 \times ...d\mathbf{A}d\mathbf{A}'d\mathbf{A}''... \\
&\times \left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2 \left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 \times ...d\hat{\mathbf{A}}d\hat{\mathbf{A}}'...
\end{aligned} \tag{17}$$

where the dots stand for the products of square fields and integration symbols needed.

5.2.2 Terms with temporal derivative

The terms in (16) that imply a variable temporal derivative are of the form:

$$\sum_i \left(\frac{d\mathbf{A}_i^{(\alpha)}(t)}{dt} - \sum_{j,k,l,m...} f^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t) ...\right)\right)^2 \tag{18}$$

This particular form represents the dynamics of the α-th coordinate of a variable $\mathbf{A}_i(t)$ as a function of the other agents.

The method of translation is similar to the above, but the time derivative adds an additional operation. In a first step, we translate the terms without derivative inside the parenthesis:

$$\sum_{j,k,l,m...} f^{(\alpha)}\left(\mathbf{A}_i(t), \mathbf{A}_j(t), \mathbf{A}_k(t), \hat{\mathbf{A}}_l(t), \hat{\mathbf{A}}_m(t) ...\right) \tag{19}$$

This type of term has already been translated in the previous paragraph, but since there is no sum over i in (19), there should be no integral over $\mathbf{A}$, nor factor $|\Psi(\mathbf{A})|^2$.

The translation of (19) is therefore, as before:

$$\int f^{(\alpha)}\left(\mathbf{A},\mathbf{A}',\mathbf{A}'',\hat{\mathbf{A}},\hat{\mathbf{A}}'...\right)|\Psi(\mathbf{A}')|^2|\Psi(\mathbf{A}'')|^2 d\mathbf{A}'d\mathbf{A}''\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2\left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}}d\hat{\mathbf{A}}' \tag{20}$$

A free variable $\mathbf{A}$ remains, which will be integrated later, when we account for the external sum $\sum_i$. We will call $\Lambda(\mathbf{A})$ the expression obtained:

$$\Lambda(\mathbf{A})=\int f^{(\alpha)}\left(\mathbf{A},\mathbf{A}',\mathbf{A}'',\hat{\mathbf{A}},\hat{\mathbf{A}}'...\right)|\Psi(\mathbf{A}')|^2|\Psi(\mathbf{A}'')|^2 d\mathbf{A}'d\mathbf{A}''\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2\left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2 d\hat{\mathbf{A}}d\hat{\mathbf{A}}' \tag{21}$$

In a second step, we account for the derivative in time by using field gradients. To do so, and as a rule, we replace :

$$\sum_i\left(\frac{d\mathbf{A}_i^{(\alpha)}(t)}{dt}-\sum_j\sum_{j,k...}f^{(\alpha)}\left(\mathbf{A}_i(t),\mathbf{A}_j(t),\mathbf{A}_k(t),\hat{\mathbf{A}}_l(t),\hat{\mathbf{A}}_m(t)...\right)\right)^2 \tag{22}$$

by:

$$\int\Psi^{\dagger}(\mathbf{A})\left(-\nabla_{\mathbf{A}^{(\alpha)}}\left(\frac{\sigma^2_{\mathbf{A}^{(\alpha)}}}{2}\nabla_{\mathbf{A}^{(\alpha)}}+\Lambda(\mathbf{A})\right)\right)\Psi(\mathbf{A})\,d\mathbf{A} \tag{23}$$

The variance $\sigma^2_{\mathbf{A}^{(\alpha)}}$ reflects the probabilistic nature of the model which is hidden behind the field formalism. This variance represents the characteristic level of uncertainty of the system's dynamics. It is a parameter of the model. Note also that in (23), the integral over $\mathbf{A}$ reappears at the end, along with the square of the field $|\Psi(\mathbf{A})|^2$. This square is split into two terms, $\Psi^{\dagger}(\mathbf{A})$ and $\Psi(\mathbf{A})$, with a gradient operator inserted in between.

5.3 Gathering terms: the action functional

The field description is ultimately obtained by summing all the terms translated above and introducing a time dependency. This sum is called the action functional. It is the sum of terms of the form (17) and (23), and is denoted $S\left(\Psi,\Psi^{\dagger}\right)$.

For example, in a system with two types of agents described by two fields $\Psi(\mathbf{A})$and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$, the action functional has the form:

$$\begin{aligned}S\left(\Psi,\Psi^{\dagger}\right) &= \int\Psi^{\dagger}(\mathbf{A})\left(-\nabla_{\mathbf{A}^{(\alpha)}}\left(\frac{\sigma^2_{\mathbf{A}^{(\alpha)}}}{2}\nabla_{\mathbf{A}^{(\alpha)}}+\Lambda_1(\mathbf{A})\right)\right)\Psi(\mathbf{A})\,d\mathbf{A} \\ &+\int\hat{\Psi}^{\dagger}\left(\hat{\mathbf{A}}\right)\left(-\nabla_{\hat{\mathbf{A}}^{(\alpha)}}\left(\frac{\sigma^2_{\hat{\mathbf{A}}^{(\alpha)}}}{2}\nabla_{\hat{\mathbf{A}}^{(\alpha)}}+\Lambda_2(\hat{\mathbf{A}})\right)\right)\hat{\Psi}\left(\hat{\mathbf{A}}\right)d\hat{\mathbf{A}} \\ &+\sum_m\int g_m\left(\mathbf{A},\mathbf{A}',\mathbf{A}'',\hat{\mathbf{A}},\hat{\mathbf{A}}'...\right)|\Psi(\mathbf{A})|^2|\Psi(\mathbf{A}')|^2|\Psi(\mathbf{A}'')|^2\times...d\mathbf{A}d\mathbf{A}'d\mathbf{A}''... \\ &\times\left|\hat{\Psi}\left(\hat{\mathbf{A}}\right)\right|^2\left|\hat{\Psi}\left(\hat{\mathbf{A}}'\right)\right|^2\times...d\hat{\mathbf{A}}d\hat{\mathbf{A}}'...\end{aligned} \tag{24}$$

where the sequence of functions g_m describes the various types of interactions in the system.

5.3.1 Introducing time in the model

So far, no time variable was included in this model. We now introduce one, written θ to distinguish it from the classical model variables. We thus replace:

$$\begin{aligned}\Psi(\mathbf{A}) &\rightarrow \Psi(\mathbf{A},\theta) \\ \hat{\Psi}\left(\hat{\mathbf{A}}\right) &\rightarrow \hat{\Psi}\left(\hat{\mathbf{A}},\theta\right)\end{aligned}$$

and introduce an additional contribution to $S\left(\Psi,\Psi^{\dagger}\right)$ (see GLW), so that the full action functional then becomes:

$$\begin{aligned}&S\left(\Psi,\Psi^{\dagger}\right)+\Psi^{\dagger}\left(\mathbf{A},\theta\right)\left(-\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-1\right)\right)\Psi\left(\mathbf{A},\theta\right)\\&+\hat{\Psi}^{\dagger}\left(\hat{\mathbf{A}},\theta\right)\left(-\nabla_{\theta}\left(\frac{\sigma_{\theta}^{2}}{2}\nabla_{\theta}-1\right)\right)\hat{\Psi}\left(\hat{\mathbf{A}},\theta\right)+\alpha\left|\Psi\left(\mathbf{A},\theta\right)\right|^{2}+\alpha\left|\hat{\Psi}\left(\hat{\mathbf{A}},\theta\right)\right|^{2}\end{aligned} \tag{25}$$

where σ_{θ}^{2} is a variance term accounting for delays in interactions, and $\frac{1}{\alpha}$ is a time scale describing the average time span of interactions between agents. In practice, $\sigma_{\theta}^{2}<<1$, and $\alpha<<1$.

5.3.2 Simplification for static background fields

Equation (25) is necessary to describe some time-dependent processes, but including this time-variable in the model is not always necessary. This paper, for instance, is solely interested in the background fields of the system, which refers to a long-run, stationary-type of equilibrium. As such, introducing a time variable in the model can be avoided, and we can simply consider static fields, $\Psi\left(\mathbf{A}\right)$ and $\hat{\Psi}\left(\hat{\mathbf{A}}\right)$.

Later on, some time-dependent modifications in the background fields will be introduced. However, these modifications will reflect modifications in the parameters, that must not be confused with the short-term θ dependency.

6 Application to the framework

The translation of a dynamic framework into a field theoretic model presented in the previous section is now applied to the microeconomic framework of section 4.

6.1 Minimization functions

In our model, the dynamics of the variable X_i comes from the minimization of the function:

$$\left(\frac{dX_i}{dt}-\nabla_X R\left(K_i,X_i\right)H\left(K_i\right)\right)^2+\tau\sum_j\delta\left(X_i-X_j\right)$$

We simply re-use this function. Since we are interested in the whole system, we will sum over the whole set of agents, which yields the minimization function for the capital allocation dynamics:

$$\sum_i\left(\frac{dX_i}{dt}-\nabla_X R\left(K_i,X_i\right)H\left(K_i\right)\right)^2+\sum_i\tau\sum_j\delta\left(X_i-X_j\right) \tag{26}$$

The dynamics of K_i, $\hat{K}_i$ et $\hat{X}_i$ are not the result of a minimization. However, their associated quadratic functions (15) can easily be found. These functions are therefore:

The minimization function for physical capital K_i:

$$\sum_i\left(\frac{d}{dt}K_i+\frac{1}{\varepsilon}\left(K_i-\sum_j\hat{F}_2\left(R\left(K_i\left(t\right),X_i\left(t\right)\right),\hat{X}_j\left(t\right)\right)\hat{K}_j\left(t\right)\right)\right)^2 \tag{27}$$

The minimization function for the financial capital $\hat{K}_i$:

$$\sum_j\left(\frac{d}{dt}\hat{K}_j-\frac{1}{\varepsilon}\left(\sum_i\left(r_i+F_1\left(\frac{R\left(K_i,X_i\right)}{\sum_l\delta\left(X_l-X_i\right)R\left(K_l,X_l\right)},\frac{\dot{K}_i\left(t\right)}{K_i\left(t\right)}\right)\right)\hat{F}_2\left(R\left(K_i\left(t\right),X_i\left(t\right)\right),\hat{X}_j\left(t\right)\right)\hat{K}_j\right)\right)^2 \tag{28}$$

The minimization function for financial capital allocation $\hat{X}_i$:

$$\sum_i \left(\frac{d}{dt}\hat{X}_j - \frac{1}{\sum_i \delta\left(X_i - \hat{X}_j\right)} \sum_i \left(\nabla_{\hat{X}} F_0 \left(R\left(K_i, \hat{X}_j\right)\right) + \nu \nabla_{\hat{X}} F_1 \left(\bar{R}\left(K_i, \hat{X}_j\right)\right)\right)\right)^2 \tag{29}$$

6.2 Translation in terms of fields

We apply the general method developed above and translate the minimization functions (26), (27), (28) and (29) in terms of fields. We start with producers, and translate first (26) and (27).

6.2.1 The Real Economy

In both capital allocation dynamics (26) and capital accumulation dynamics (27), time derivatives appear. However, one of them, equation (26), includes time-independent terms and is thus of the form (16), the other, equation (26) is of the type (15). Based on the translation rules, appendix 1.3 computes the translation of the various minimization functions. We find:

Translation of the minimization function: Physical capital allocation Let us start by translating in terms of fields the expression (26). Using the translation (21) of (19)-type term, (26) translates into:

$$\begin{aligned} S_1 &= -\int \Psi^{\dagger}(K,X) \nabla_X \left(\frac{\sigma_X^2}{2} \nabla_X - \nabla_X R(K,X) H(K) \right) \Psi(K,X) \, dK dX \\ &\quad + \tau \int \left| \Psi(K',X) \right|^2 \left| \Psi(K,X) \right|^2 dK' dK dX \end{aligned} \tag{30}$$

Translation of the minimization function: Physical capital We can now turn to the translation of the second equation (27). Once again, we use the translation (21) of (19)-type term and the translated writes:

$$S_2 = -\int \Psi^{\dagger}(K,X) \nabla_K \left(\frac{\sigma_K^2}{2} \nabla_K + \frac{1}{\varepsilon} \left(K - \int \hat{F}_2 \left(R(K,X), \hat{X} \right) \hat{K} \left| \hat{\Psi}\left(\hat{K},\hat{X}\right) \right|^2 d\hat{K} d\hat{X} \right) \right) \Psi(K,X) \tag{31}$$

with:

$$\hat{F}_2\left(R(K,X), \hat{X}\right) = \frac{F_2(R(K,X))\, G\left(X - \hat{X}\right)}{\int F_2(R(K',X'))\, G\left(X' - \hat{X}\right) \left|\Psi(K',X')\right|^2 d(K',X')} \tag{32}$$

6.2.2 Financial markets

The functions to be translated are those of the financial capital dynamics (28) and of the financial capital allocation (29). Both expressions include a time derivative and are thus of type (18). As for the real economy, the application of the translation rules is straightforward.

Translation of the minimization function: Financial capital dynamics We consider the function (28), which translates, using the general translation formula of expression (22) in (23), into:

$$\begin{aligned} S_3 &= -\int \hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right) \nabla_{\hat{K}} \left(\frac{\sigma_{\hat{K}}^2}{2} \nabla_{\hat{K}} - \frac{\hat{K}}{\varepsilon} \int \left(r(K,X) - \gamma \frac{\int K' \left\| \Psi(K',X) \right\|^2}{K} \right.\right. \\ &\quad \left.\left. + F_1\left(\bar{R}(K,X), \Gamma(K,X)\right)\right) \hat{F}_2\left(R(K,X), \hat{X}\right) \left\| \Psi(K,X) \right\|^2 d(K,X) \right) \hat{\Psi}\left(\hat{K},\hat{X}\right) \end{aligned} \tag{33}$$

where:

$$\bar{R}(K,X) = \frac{R(K,X)}{\int R(K',X') \|\Psi(K',X')\|^2 d(K',X')} \tag{34}$$

$$\Gamma(K,X) = \frac{\int \hat{F}_2\left(R(K,X),\hat{X}\right)\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\hat{K}d\hat{X}}{K} - 1 \tag{35}$$

Translation of the minimization function: Financial capital allocation The translation of the function for financial capital allocation (29) follows the previous pattern. We obtain:

$$\begin{aligned} S_4 = & -\int \hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right) \\ & \times\left(\nabla_{\hat{X}}\sigma_{\hat{X}}^2\nabla_{\hat{X}} - \int\left(\frac{\nabla_{\hat{X}}F_0\left(R\left(K,\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_1\left(\bar{R}(K,X),\Gamma(K,X)\right)}{\int\left\|\Psi\left(K',\hat{X}\right)\right\|^2 dK'}\right)\left\|\Psi\left(K,\hat{X}\right)\right\|^2 dK\right)\hat{\Psi}\left(\hat{K},\hat{X}\right) \end{aligned} \tag{36}$$

6.3 Gathering contributions: the action functional

Once these translations are performed, the action functional of the system is described by the sum of all contributions (30),(31),(33),(36):

$$S = S_1 + S_2 + S_3 + S_4$$

At this point, we can introduce a simplification and assume that investors invest in only one sector. This translates into the following condition:

$$G\left(X-\hat{X}\right) = \delta\left(X-\hat{X}\right) \tag{37}$$

This simplification does not reduce the generality of our model: actually, an investor acting in several sectors could be modelled as an aggregation of several investors. Nor does it mean that investors should be static, since they can still move from one sector to another.

We write a compact form for the action functional S:

$$\begin{aligned} S = & -\int \Psi^{\dagger}(K,X)\left(\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X - \nabla_X R(K,X)H(K)\right) - \tau\left(\int\left|\Psi(K',X)\right|^2 dK'\right)\right. \\ & \left.+\nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K + u\left(K,X,\Psi,\hat{\Psi}\right)\right)\right)\Psi(K,X)\,dKdX \\ & -\int \hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} - \hat{K}f\left(\hat{X},\Psi,\hat{\Psi}\right)\right) + \nabla_{\hat{X}}\left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}} - g\left(K,X,\Psi,\hat{\Psi}\right)\right)\right)\hat{\Psi}\left(\hat{K},\hat{X}\right) \end{aligned} \tag{38}$$

where each line corresponds to one S_i and where, to simplify, we have defined:

$$u\left(K,X,\Psi,\hat{\Psi}\right) = \frac{1}{\varepsilon}\left(K - \int \hat{F}_2(R(K,X))\hat{K}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 d\hat{K}\right) \tag{39}$$

$$f\left(\hat{X},\Psi,\hat{\Psi}\right) = \frac{1}{\varepsilon}\int\left(r(K,X) - \frac{\gamma\int K'\left|\Psi(K,X)\right|^2}{K} + F_1\left(\bar{R}(K,X),\Gamma(K,X)\right)\right) \tag{40}$$

$$\times\hat{F}_2(R(K,X))\left|\Psi\left(K,\hat{X}\right)\right|^2 dK \tag{41}$$

$$g\left(K,\hat{X},\Psi,\hat{\Psi}\right) = \int\frac{\nabla_{\hat{X}}F_0\left(R\left(K,\hat{X}\right)\right)+\nu\nabla_{\hat{X}}F_1\left(\bar{R}\left(K,\hat{X}\right),\Gamma(K,X)\right)}{\int\left|\Psi\left(K',\hat{X}\right)\right|^2 dK'}\left|\Psi\left(K,\hat{X}\right)\right|^2 dK \tag{42}$$

The expression for $\bar{R}(K,X)$ is still given by (34). Under our assumption, the functions $\hat{F}_2$ and Γ become:

$$\hat{F}_2(R(K,X)) = \frac{F_2(R(K,X))}{\int F_2(R(K',X))|\Psi(K',X)|^2 dK'} \tag{43}$$

$$\Gamma(K,X) = \frac{\int \hat{F}_2(R(K,X))\hat{K}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 d\hat{K}}{K} - 1 \tag{44}$$

Recall that function $H(K_X)$ encompasses the determinants of the firms' mobility across the sector space. We will specify this function below as a function of expected long term-returns and capital.

Function u describes the evolution of capital of a firm, located at X. This dynamics depends on the relative value of a function F_2 that is itself a function of the firms' expected returns $R(K,X)$. Investors allocate their capital based on their expectations of the firms' long-term returns.

Function f describes the returns of investors located at $\hat{X}$, and investing in sector X a capital K. These returns depend on short-term dividends $r(K,X)$, the field-equivalent cost of capital $\frac{\gamma \int K' \|\Psi(K,X)\|^2}{K}$, and a function F_1 that depends on firms' expected long-term stock valuations. These valuations themselves depend on the relative attractivity of a firm expected long-term returns vis-a-vis its competitors.

Function g describes investors' shifts across the sectors' space. They are driven by the gradient of expected long-term returns and stocks valuations, who themselves depend on the firms' relative expected long-term returns.

Recall that we depart here from the general formalism: we do not introduce a time variable in the present model. Indeed, as mentioned earlier, our purpose is to find collective, or characteristic, configurations of the system that, as such, can be considered static. It is only when we will derive these configurations that a macro time scale will be introduced to study how the evolution of the background states through time.

7 Use of the field model

Now that we have found the field action functional S, we can use field theory to study the system of agents. This can be done at two levels: the individual and collective level.

At the individual level, the field formalism allows to compute agents' individual dynamics in the state defined by the background fields, through the transition functions of the system. This study is left for a subsequent work.

At the collective level, the background fields that describe the system can be computed. These background fields are the particular functions, $\Psi(K,X)$ and $\hat{\Psi}\left(\hat{K},\hat{X}\right)$, and their adjoints fields $\Psi^{\dagger}(K,X)$ and $\hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)$, that minimize the functional S. Once the background fields obtained, the associated density of firms and investors, per sector for a given capital K, can be computed. They are given by:

$$|\Psi(K,X)|^2 = \Psi^{\dagger}(K,X)\Psi(K,X) \tag{45}$$

and:

$$\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 = \hat{\Psi}^{\dagger}\left(\hat{K},\hat{X}\right)\hat{\Psi}\left(\hat{K},\hat{X}\right) \tag{46}$$

With these two density functions at hand, we can compute various average quantities in the collective state.

The density of producers $\|\Psi(X)\|^2$ and investors $\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2$ in sectors, which are computed using the formula:

$$\|\Psi(X)\|^2 \equiv \int |\Psi(K,X)|^2 dK \tag{47}$$

$$\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2 \equiv \int \left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^2 d\hat{K} \tag{48}$$

The total invested capital $\hat{K}_X$ in sector X, which is defined by a partial average:

$$\hat{K}_{\hat{X}} = \int \hat{K}\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2 d\hat{K} = \int \hat{K}\left|\hat{\Psi}\left(\hat{X}\right)\right|^2 d\hat{K} \tag{49}$$

and the average invested capital per firm in sector X, which is defined by:

$$K_X = \frac{\int \hat{K} \left|\hat{\Psi}\left(\hat{K}, X\right)\right|^2 d\hat{K}}{\|\Psi(X)\|^2} \tag{50}$$

Note that, given our assumptions, the total physical capital is equal to the total capital invested:

$$\int K |\Psi(K, X)|^2 dK = \int \hat{K} \left|\hat{\Psi}\left(\hat{K}, \hat{X}\right)\right|^2 d\hat{K}$$

so that K_X is also equal to the average physical capital per firm for sector X, i.e. :

$$K_X = \frac{\int K |\Psi(K, X)|^2 dK}{\|\Psi(X)\|^2} \tag{51}$$

In the following, we will use both expressions (50) or (51) alternately for K_X.

Ultimately, the distributions of invested capital per investor and of capital per firm, given a collective state and a sector X, are $\frac{\left|\hat{\Psi}\left(\hat{K},X\right)\right|^2}{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}$ and $\frac{|\Psi(K,X)|^2}{\|\Psi(X)\|^2}$, respectively.

Gathering equations (47), (48) and (50), each collective state is singularly determined by the collection of data that characterizes each sector: the number of firms for each sector, the number of investors for each sector, the average capital for each sector and the density of distribution of capital in each sector. All the above quantities allow to study the capital allocation among sectors as well as its dependency in system parameters such as expected long-term return, short-term return, or any parameters involved in the model. This "static" point of view, will be extended by introducing some fluctuations in the expectations, leading to a dynamic of the average capital at the macro-level. In the following, we solve the system for the background fields and compute the average associated quantities.

Part II: Resolution

The initial framework has been translated into a proper field formalism. We can now solve the model. The average capital per sector (defined in (50) and (49)) depends on the densities of agents $|\Psi(K, X)|^2$ and $\left|\hat{\Psi}\left(\hat{K}, X\right)\right|^2$ defined in (45) and (46). To compute these densities, we must first find the configurations $\Psi(K, X)$, $\Psi^\dagger(K, X)$ and $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$, $\hat{\Psi}^\dagger\left(\hat{K}, \hat{X}\right)$ that minimize the field action S.

This paper studies the influence of investment and financial allocation on the dynamics of the real economy. To do so, we must express quantities relevant to the producers' side, such as the density of agents and average capital as functions of financial quantities.

As a consequence, the order of resolution will be the following: we first minimize the (K, X) part of the fields action (38), i.e. $S_1 + S_2$, to find the real economy background fields $\Psi(K, X)$ and $\Psi^\dagger(K, X)$ and the density of firms $|\Psi(K, X)|^2$ as functions of the financial sectors' background fields $\Psi^\dagger(K, X)$ and $\hat{\Psi}^\dagger\left(\hat{K}, \hat{X}\right)$ and investors' variables. Then, in a second time, we minimize $S_3 + S_4$, and find the minimal configuration of the investors' field $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$ and $\hat{\Psi}^\dagger\left(\hat{K}, \hat{X}\right)$.

8 Density of producers

8.1 Minimization of $S_1 + S_2$

To compute[6] the field of the real economy $\Psi(K,X)$ as a function of the field of the financial sector $\hat{\Psi}\left(\hat{K},\hat{X}\right)$. We first minimize the (K,X) part of equation (38):

$$\begin{aligned} S_1 + S_2 &= -\int \Psi^{\dagger}(K,X)\left(\nabla_X\left(\frac{\sigma_X^2}{2}\nabla_X - \nabla_X R(K,X) H(K)\right) - \tau\left(\int \left|\Psi(K',X)\right|^2 dK'\right)\right. \\ &\left. + \nabla_K\left(\frac{\sigma_K^2}{2}\nabla_K + u\left(K,X,\Psi,\hat{\Psi}\right)\right)\right)\Psi(K,X)\, dK dX \end{aligned} \tag{52}$$

For relatively slow fluctuations in X, and up to an exponential change of variable in the fields, we show in appendix 2.1 that the background fields $\Psi(K,X)$ and $\Psi^{\dagger}(K,X)$ decompose as a product:

$$\Psi(K,X) = \Psi^{\dagger}(K,X) = \Psi(X)\,\Psi_1(K - K_X) \tag{53}$$

where K_X, the average invested capital per firm in sector X, is given by (51) and the functions $\Psi(X)$ and $\Psi_1(K - K_X)$ satisfy the following differential equations:

$$\begin{aligned} 0 &= \left(-\frac{\sigma_X^2}{2}\nabla_X^2 + \frac{\left(\nabla_X R(X) H(K_X)\right)^2}{2\sigma_X^2} + \frac{\nabla_X^2 R(K,X)}{2} H(K) + 2\tau\left|\Psi(X)\right|^2\right)\Psi(X) \\ &+ D\left(\|\Psi\|^2\right)\left(\int \|\Psi(X)\|^2 - N\right) + \int \mu(X)\|\Psi(X)\|^2 \end{aligned} \tag{54}$$

for $\Psi(X)$, and:

$$0 = -\nabla_K^2 \Psi_1(K - K_X) + \left(K - \frac{F_2(R(K,X))K_X}{F_2(R(K_X,X))}\right)^2 \Psi_1(K - K_X) + \gamma(X)\Psi_1(K - K_X) \tag{55}$$

for $\Psi_1(K - K_X)$.

The constants $D\left(\|\Psi\|^2\right)$, $\mu(X)$ and $\gamma(X)$ arising in (54) and (55) are Lagrange multipliers[7], that implement the constraints:

$$\int \|\Psi(X)\|^2 = N,\ \|\Psi(X)\|^2 \geqslant 0,\ \|\Psi_1(K - K_X)\|^2 = 1$$

where N is the total number of firms of the system.

The idea that the background field can be decomposed as a product in (53) has a simple interpretation. In the sectors' space, motion is slower than capital accumulation: capital accumulates as a function of the position X of the sector, through the capital allocated in this sector, K_X. This is translated through the decomposition of $\Psi(K,X)$ into two factors $\Psi(X)$ and $\Psi_1(K - K_X)$ in (53).

8.2 Determination of $\Psi_1(K - K_X)$

The function $\Psi_1(K - K_X)$, involved in the definitions (53) of the background fields $\Psi(K,X)$ describes the fluctuations of capital in a given sector X around an average value K_X. It is computed in appendix 2.1.2:

$$\Psi_1(K - K_X) = \mathcal{N}\exp\left(-\left(K - \frac{F_2(R(K,X))K_X}{F_2(R(K_X,X))}\right)^2\right) \tag{56}$$

where $\mathcal{N}$ is a normalization factor. The capital accumulated by a firm in a sector X is centered around the average capital K_X in this sector, weighted by a factor $\frac{F_2(R(K,X))}{F_2(R(K_X,X))}$. This factor depends on the firm's expected long-term return. It is relative to the average expected long-term return of the whole sector X described by the function $F_2(R(K_X,X))$[8].

[6] For detailed computations of this subsection, see appendix 2.

[7] Incidentally, note that, to keep track of the dependency of the Lagrange multiplier in $\|\Psi\|^2$ in the above, we have chosen the notation $D\left(\|\Psi\|^2\right)$.

[8] See discussion below equation (6).

8.3 Determination of $\Psi(X)$ and $\|\Psi(X)\|^2$

Equation (54) can be solved for the X-dependent part of the background field $\Psi(X)$[9]. From this solution, we can deduce the density of firms $\|\Psi(X)\|^2$ in sector X. However, when fluctuations in capital allocation σ_X^2 are small, we can express directly $\|\Psi(X)\|^2$ as a function of the financial variables.

This density is given by:

$$\|\Psi(X)\|^2 = \frac{D\left(\|\Psi\|^2\right)}{2\tau} - \frac{1}{4\tau}\left((\nabla_X R(X))^2 + \frac{\sigma_X^2 \nabla_X^2 R(K_X, X)}{H(K_X)}\right)\left(1 - \frac{H'\left(\hat{K}_X\right) K_X}{H\left(\hat{K}_X\right)}\right) H^2(K_X) \qquad (57)$$

provided that the rhs of (57) is positive; otherwise $\|\Psi(X)\|^2 = 0$.

The Lagrange multiplier $D\left(\|\Psi\|^2\right)$ is obtained by integration of (54) and yields:

$$ND\left(\|\Psi\|^2\right) = 2\tau \int |\Psi(X)|^4 + \frac{1}{2}\int (\nabla_X R(X) H(K_X))^2 \|\Psi(X)\|^2 \qquad (58)$$

Formula (57) will be used extensively in the sequel to compute K_X, the average physical capital per firm in sector X.

9 Density of investors

We have computed the background fields for producers, $\Psi(K, X)$ and $\Psi^\dagger(X, K)$, and the producers' density by minimizing $S_1 + S_2$. We can now compute the background fields $\hat{\Psi}\left(\hat{K}, \hat{X}\right)$ and $\hat{\Psi}^\dagger\left(\hat{K}, \hat{X}\right)$ for investors along with the density of investors $\left|\hat{\Psi}\left(\hat{X}, \hat{K}\right)\right|^2$ by minimizing $S_3 + S_4$.

9.1 Rewriting $S_3 + S_4$

We first rewrite the field action $S_3 + S_4$ by inserting the density of producers $\|\Psi(X)\|^2$, formula (57), The expression for $S_3 + S_4$, reduces to:

$$S_3 + S_4 = -\int \hat{\Psi}^\dagger\left(\hat{K}, \hat{X}\right)\left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2}{2}\nabla_{\hat{K}} - \hat{K} f\left(\hat{X}\right)\right) + \nabla_{\hat{X}}\left(\frac{\sigma_{\hat{X}}^2}{2}\nabla_{\hat{X}} - g\left(\hat{X}\right)\right)\right)\hat{\Psi}\left(\hat{K}, \hat{X}\right) \qquad (59)$$

(see appendix 3.1.1) where $f\left(K_{\hat{X}}, \hat{X}\right)$ is the short-term return:

$$f\left(K_{\hat{X}}, \hat{X}\right) = \frac{1}{\varepsilon}\left(r\left(K_{\hat{X}}, \hat{X}\right) - \gamma\left\|\Psi\left(\hat{X}\right)\right\|^2 + F_1\left(\bar{R}\left(K_{\hat{X}}, \hat{X}\right)\right)\right) \qquad (60)$$

and $g\left(K_{\hat{X}}, \hat{X}\right)$ depends on long-term returns:

$$g\left(K_{\hat{X}}, \hat{X}\right) = \left(\nabla_{\hat{X}} F_0\left(R\left(K_{\hat{X}}, \hat{X}\right)\right) + \nu \nabla_{\hat{X}} F_1\left(\bar{R}\left(K_{\hat{X}}, \hat{X}\right)\right)\right) \qquad (61)$$

with:

$$\bar{R}\left(K_{\hat{X}}, \hat{X}\right) = \frac{R\left(K_{\hat{X}}, \hat{X}\right)}{\int R\left(K'_{X'}, X'\right)\|\Psi(X')\|^2 dX'} \qquad (62)$$

and:

$$F_1\left(\bar{R}\left(K_{\hat{X}}, \hat{X}\right)\right) = F_1\left(\bar{R}\left(K_{\hat{X}}, \hat{X}\right), \Gamma = 0\right) \qquad (63)$$

In the sequel, any function $h\left(K_{\hat{X}}, \hat{X}\right)$ and its partial derivatives $h\left(K_{\hat{X}}, \hat{X}\right)$, will be written $h\left(\hat{X}\right)$, $\nabla_{K_{\hat{X}}} h\left(\hat{X}\right)$ and $\nabla_{\hat{X}} h\left(\hat{X}\right)$ respectively.

[9] A method of resolution of (54) and two examples for particular forms of the function $H(K)$ are presented in appendix 2.2.

9.2 Minimization of $S_3 + S_4$

The minimization of $S_3 + S_4$ (59) is computed using a change of variable:

$$\hat{\Psi} \to \exp\left(\frac{1}{\sigma_{\hat{X}}^2}\int g\left(\hat{X}\right)d\hat{X} + \frac{\hat{K}^2}{\sigma_{\hat{K}}^2}f\left(\hat{X}\right)\right)\hat{\Psi}$$

(see appendix 3.1.2) which yields the equation for $\hat{\Psi}$:

$$0 = \left(\frac{\sigma_{\hat{X}}^2\nabla_{\hat{X}}^2}{2} - \frac{\left(g\left(\hat{X}\right)\right)^2}{2\sigma_{\hat{X}}^2} - \frac{\nabla_{\hat{X}}g\left(\hat{X}\right)}{2}\right)\hat{\Psi} + \left(\nabla_{\hat{K}}\left(\frac{\sigma_{\hat{K}}^2\nabla_{\hat{K}}}{2} - \hat{K}f\left(\hat{X}\right)\right) - F\left(\hat{X}\right)\hat{K} - \hat{\lambda}\right)\hat{\Psi} \tag{64}$$

and the equation for $\hat{\Psi}^{\dagger}$:

$$0 = \left(\frac{\sigma_{\hat{X}}^2\nabla_{\hat{X}}^2}{2} - \frac{\left(g\left(\hat{X}\right)\right)^2}{2\sigma_{\hat{X}}^2} - \frac{\nabla_{\hat{X}}g\left(\hat{X}\right)}{2}\right)\hat{\Psi}^{\dagger} + \left(\left(\frac{\sigma_{\hat{K}}^2\nabla_{\hat{K}}}{2} + \hat{K}f\left(\hat{X}\right)\right)\nabla_{\hat{K}} - F\left(\hat{X}\right)\hat{K} - \hat{\lambda}\right)\hat{\Psi}^{\dagger} \tag{65}$$

with:

$$F\left(\hat{X}\right) = \nabla_{K_{\hat{X}}}\left(\frac{\left(g\left(\hat{X}\right)\right)^2}{2\sigma_{\hat{X}}^2} + \frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X}\right) + f\left(\hat{X}\right)\right)\frac{\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2}{\left\|\Psi\left(\hat{X}\right)\right\|^2} + \frac{\left\langle\hat{K}^2\right\rangle_{\hat{X}}\nabla_{K_{\hat{X}}}f^2\left(\hat{X}\right)}{\sigma_{\hat{K}}^2\left\|\Psi\left(\hat{X}\right)\right\|^2} \tag{66}$$

where $\left\langle\hat{K}^2\right\rangle_{\hat{X}}$ denotes the average of $\hat{K}^2$ in sector $\hat{X}$ (see appendix 3.1.2) and $\left\|\hat{\Psi}\left(\hat{X}\right)\right\|^2 = \int\left|\hat{\Psi}\left(\hat{X},\hat{K}\right)\right|^2 d\hat{K}$.

A Lagrange multiplier $\hat{\lambda}$ has been included in the system of equations (64) and (65) to implement the constraint for $\hat{\Psi}$ and $\hat{\Psi}^{\dagger}$:

$$\int\left|\hat{\Psi}\left(\hat{X},\hat{K}\right)\right|^2 d\hat{X}d\hat{K} = \hat{N} \tag{67}$$

Incidentally, note that the function $F\left(\hat{X},K_{\hat{X}}\right)$ arising in the minimization equations (64) and (210) describes the impact of individual variations on the collective state (the field $\hat{\Psi}$). It can be neglected in first approximation.

9.3 Background field and density of investors

Appendix 3.1.3 computes the solutions of equations (64) and (210) for the investors' background fields. We find an infinite number of solutions for $\hat{\Psi}_{\hat{\lambda}}$ and $\hat{\Psi}_{\hat{\lambda}}^{\dagger}$ parametrized by $\hat{\lambda} \in \mathbb{R}$, which translates the fact that $S_3 + S_4$ has an infinite number of local minima.

However, appendix 3.1.4.2 shows that the eigenvalue $\left|\hat{\lambda}\right|$ has a lower bound M[10] defined by:

$$M = \max_{\hat{X}}\left(A\left(\hat{X}\right)\right) \tag{68}$$

where:

$$A\left(\hat{X}\right) = \frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + f\left(\hat{X}\right) + \frac{1}{2}\sqrt{f^2\left(\hat{X}\right)} + \nabla_{\hat{X}}g\left(\hat{X}\right) - \frac{\sigma_{\hat{K}}^2F^2\left(\hat{X}\right)}{2f^2\left(\hat{X}\right)} \tag{69}$$

and that $\hat{\Psi}_{-M}$ is the global minimum of $S_3 + S_4$. The background fields are thus $\hat{\Psi}_{-M}$ and its adjoint $\hat{\Psi}_{-M}^{\dagger}$.

[10] This lower bound is reminiscent of the fact that the Lagrange multiplier λ is the eigenvalue of the second order operator arising in equation (65), and that this operator is bounded from below.

For these background fields, the density of agents with capital $\hat{K}$ invested in sector $\hat{X}$ is:

$$\left|\hat{\Psi}_{-M}\left(\hat{K},\hat{X}\right)\right|^{2}=\hat{\Psi}_{-M}^{\dagger}\left(\hat{X},\hat{K}\right)\hat{\Psi}_{-M}\left(\hat{K},\hat{X}\right)$$

We find:

$$\left|\hat{\Psi}_{-M}\left(\hat{K},\hat{X}\right)\right|^{2}\simeq C\left(\bar{p}\right)\exp\left(-\frac{\sigma_{X}^{2}\hat{K}^{4}\left(f'\left(X\right)\right)^{2}}{96\sigma_{\hat{K}}^{2}\left|f\left(\hat{X}\right)\right|}\right)D_{p\left(\hat{X}\right)}^{2}\left(\left(\frac{\left|f\left(\hat{X}\right)\right|}{\sigma_{\hat{K}}^{2}}\right)^{\frac{1}{2}}\left(\hat{K}+\frac{\sigma_{\hat{K}}^{2}F\left(\hat{X}\right)}{f^{2}\left(\hat{X}\right)}\right)\right) \tag{70}$$

where D_p is the parabolic cylinder function with parameter $p\left(\hat{X}\right)$ and:

$$p\left(\hat{X}\right)=\frac{M-A\left(\hat{X}\right)}{\sqrt{f^{2}\left(\hat{X}\right)}} \tag{71}$$

The constant $C\left(\bar{p}\right)$ ensures that the constraint (67) is satisfied. Its expression is given in appendix 3.1.3.

Section 14.1 will show that $p\left(\hat{X}\right)$ encompasses the relative expected returns of sector X vis-à-vis its neighbouring sectors.

10 Average capital invested per firm per sector

10.1 General form of the average capital equation

Now that the densities for both producers and investors are computed, we can determine the average capital invested per firm in sector $\hat{X}$, i.e. $K_{\hat{X}}$.

First, we rewrite the defining equation of $K_{\hat{X}}$ (50) as:

$$K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^{2}=\int\hat{K}\left|\hat{\Psi}\left(\hat{K},\hat{X}\right)\right|^{2}d\hat{K} \tag{72}$$

and evaluate this equation for the background field (70):

$$K_{X}\left\|\Psi\left(X\right)\right\|^{2}=\int\hat{K}\left|\hat{\Psi}_{-M}\left(\hat{K},\hat{X}\right)\right|^{2}d\hat{K} \tag{73}$$

Equation (73) allows to find the average capital $K_{\hat{X}}$. Actually, both the densities of agents $\left\|\Psi\left(\hat{X}\right)\right\|^{2}$ and $\left|\hat{\Psi}_{-M}\left(\hat{K},\hat{X}\right)\right|^{2}$, equations (57) and (70), are functions of $K_{\hat{X}}$, so that equation (73) is itself an equation for $K_{\hat{X}}$.

10.2 Final form for the average capital equation

From this general equation, we can find the average capital at point $\hat{X}$. Appendix 3.1.4.2 computes the integral (73) using the financial background field (70).

In the sequel, we will write $p\left(\hat{X}\right)$ defined in (71) as:

$$p\equiv p\left(\hat{X}\right) \tag{74}$$

Equation (73) becomes ultimately:

$$K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^{2}\left|f\left(\hat{X}\right)\right|=C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\hat{\Gamma}\left(p+\frac{1}{2}\right) \tag{75}$$

with:

$$\hat{\Gamma}\left(p+\frac{1}{2}\right) = \exp\left(-\frac{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(p+\frac{1}{2}\right)^{2}\left(f'\left(X\right)\right)^{2}}{96\left|f\left(\hat{X}\right)\right|^{3}}\right) \times\left(\frac{\Gamma\left(-\frac{p+1}{2}\right)\Gamma\left(\frac{1-p}{2}\right)-\left(\Gamma\left(-\frac{p}{2}\right)\right)^{2}}{2^{p+2}\Gamma\left(-p-1\right)\Gamma\left(-p\right)}+p\frac{\Gamma\left(-\frac{p}{2}\right)\Gamma\left(\frac{2-p}{2}\right)-\left(\Gamma\left(-\frac{p-1}{2}\right)\right)^{2}}{2^{p+1}\Gamma\left(-p\right)\Gamma\left(-p+1\right)}\right) \tag{76}$$

where Γ is the Gamma function.

This final form of the capital equation, (75), will be central to our following computations. However, it involves some functions, such as f, that have a general form, and functions of the unknown variable $K_{\hat{X}}$ (see for instance equation (60)). Thus, it cannot, in general, be solved analytically.

11 Average capital across sectors in a given environment

We just stated that the final form of the capital equation, (75), cannot be solved analytically, except for some particular cases[11]. However, several approaches can be used to study the behaviour of its solutions or approximate its solutions.

A first approach, the most general, studies, for each sector $\hat{X}$, the variation of average capital per firm $K_{\hat{X}}$ with respect to any parameter of the system. This is done by studying the differential form of the capital equation (75) while keeping very general forms for the parameter-functions f and g. In particular, the influence of the local environment on a sector can be computed. It is depicted by the variation of $K_{\hat{X}}$ with respect to the sector's relative expected returns. This approach reveals stable and unstable equilibria in the system. It is developed in section 11.1. However, it does not yield the sectors' precise levels of capital.

A second approach studies the capital equation (75) by using an expansion around particular solutions. It confirms the existence of stable and unstable equilibria, that correspond to multiple solutions to the capital equation in a given sector. To put it differently, for a given set of the parameter-functions, the background fields $\hat{\Psi}_p\left(\hat{K},\hat{X}\right)$ are not unique. Depending on initial configurations, an infinite number of collective state may arise[12]. This approach is presented in section 11.2.

A third approach provides approximate solutions to the capital equation (75) for standard forms of the parameter-functions. The existence of multiple solutions is confirmed, along with the associated stability analysis. This final approach is presented in section 11.3.

Combined, these three approaches confirm and complete with each other.

11.1 First approach: differential form of the average capital equation

One way to better understand equation (75) is to study its differential form.

Assume at point $\hat{X}$ of the system, a variation $\delta Y\left(\hat{X}\right)$ for any parameter, in which the parameter $Y\left(\hat{X}\right)$ can be either $R\left(X\right)$, its gradient, or any parameter arising in the definition of $f\left(\hat{X}\right)$and $g\left(\hat{X}\right)$. This variation $\delta Y\left(\hat{X}\right)$ induces in turn a variation $\delta K_{\hat{X}}$ in average capital expressed by differentiating (75):

$$\delta K_{\hat{X}} = \left(-\left(\frac{\partial\ln f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}+\frac{\partial\ln\left|\Psi\left(\hat{X},K_{\hat{X}}\right)\right|^{2}}{\partial K_{\hat{X}}}+l\left(\hat{X},K_{\hat{X}}\right)\right)+k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}\right)K_{\hat{X}}\delta K_{\hat{X}} + \frac{\partial}{\partial Y\left(\hat{X}\right)}\left(\frac{\sigma_{\hat{K}}^{2}C\left(\bar{p}\right)2\hat{\Gamma}\left(p+\frac{1}{2}\right)}{\left|f\left(\hat{X},K_{\hat{X}}\right)\right|\left\|\Psi\left(\hat{X},K_{\hat{X}}\right)\right\|^{2}}\right)\delta Y\left(\hat{X}\right) \tag{77}$$

[11] These particular cases will be studied in the following sections.
[12] This point will be developed in section 14.

where the coefficients $l\left(\hat{X},K_{\hat{X}}\right)$ and $k\left(p\right)$ are computed in appendix 3.2.1. The parameter $l\left(\hat{X},K_{\hat{X}}\right)$ accounts for the variation of the short-term returns across sectors, while $k\left(p\right)$ describes the impact of relative returns variations across sectors.

Equation (77) will be used to compute the dependency of average capital per firm in sector $\hat{X}$, i.e. $K_{\hat{X}}$, as a function of any parameter $Y(\hat{X})$, and more fundamentally to investigate the stability of the solutions of (75) with respect to the variations in parameters.

11.1.1 Local stability

The differential form given by equation (77), computes the effect of a variation $\delta Y\left(\hat{X}\right)$ in the parameters on the average capital $K_{\hat{X}}$. Moreover, equation (77) can be understood as the fixed-point equation of a dynamical system of the following mechanism: each variation $\delta Y\left(\hat{X}\right)$ in the parameters impacts directly the average capital through the second term in the RHS of (77). In a second step, the variation $\delta K_{\hat{X}}$ impacts the various functions implied in (75), and indirectly modifies $K_{\hat{X}}$ through the first term in the rhs of (77)[13].

What matters here is the condition of stability. We show that the fixed point is stable when:

$$\left|k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}-\left(\frac{\partial\ln f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}+\frac{\partial\ln\left|\Psi\left(\hat{X},K_{\hat{X}}\right)\right|^{2}}{\partial K_{\hat{X}}}+l\left(\hat{X},K_{\hat{X}}\right)\right)\right|<1 \tag{78a}$$

and unstable otherwise.

Thus, two types of solutions emerge for the average capital per firm $K_{\hat{X}}$. The stable solutions $K_{\hat{X}}$ can be considered as the potential equilibrium averages for sector $\hat{X}$. However unstable solutions must rather be considered as thresholds: when $K_{\hat{X}}$ is driven away from this threshold, it may either converge toward a stable solution of (75), or diverge towards 0 or infinity.

Remark *At first sight, introducing the dynamical system may seem artificial. It is nonetheless coherent in the context of our field model: the variation in average capital induced by a change in parameter reveals a shift $\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)$ in the background state $\hat{\Psi}\left(\hat{K},\hat{X}\right)$. The new configuration $\hat{\Psi}\left(\hat{K},\hat{X}\right)+\delta\hat{\Psi}\left(\hat{K},\hat{X}\right)$ may not be a minimum of the action functional. We must therefore determine whether the system will settle on a background state, slightly modified with a different $K_{\hat{X}}$, or be driven towards an altogether different equilibrium. To this end, we will study the dynamics equation for $K_{\hat{X}}$ in section 12.*

11.1.2 Dependency in the parameters

Once the notion of stability understood, we can use equation (77) to compute the impact of the variation of any parameter $Y(\hat{X})$ on $\delta K_{\hat{X}}$. Two applications are of particular interest to us.

Dependency in relative expected returns The main application of equation (77) is to consider the dependency of the average capital $K_{\hat{X}}$ in the parameter $p(\hat{X})$ defined in (71). Section 14 will show that this parameter encompasses sector $\hat{X}$ relative expected returns vis-à-vis its neighbours.

Using (77), we show (see appendix 3.2.1) that the variation of $K_{\hat{X}}$ with respect to $p(\hat{X})$ depends on the notion of equilibrium stability defined in section 11.1.1.

For a stable equilibrium where the expected return $f\left(\hat{X}\right)$ is positive[14], we find that:

$$\frac{\delta K_{\hat{X}}}{\delta p\left(\hat{X}\right)}>0 \tag{79}$$

[13] The computations and formula for the dynamics' fixed points are given in appendix 3.2.1.

[14] which is the case of interest for us (see section 11.3.3)

so that $p(\hat{X})$ writes as:

$$p(\hat{X}) = \frac{M - \left(\frac{(g(\hat{X}))^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}\right) - \frac{\sigma_{\hat{K}}^2 F^2(\hat{X})}{2f^2(\hat{X})}\right)}{f\left(\hat{X}\right)} - \frac{3}{2} \tag{80}$$

The definitions (61) and (62) show that $g\left(\hat{X}\right)$ and $\nabla_{\hat{X}} g\left(\hat{X}\right)$ are proportional to $\nabla_{\hat{X}} R\left(\hat{X}\right)$ and $\nabla_{\hat{X}}^2 R\left(\hat{X}\right)$ respectively. Thus, when the expected long-term return $R\left(\hat{X}\right)$ is a maximum, $p(\hat{X})$ is maximal too[15]: under a stable equilibrium, capital accumulation is maximal in sectors where the expected long-term return $R\left(\hat{X}\right)$ is maximal.

On the other hand, when the equilibrium is unstable we have:

$$\frac{\delta K_{\hat{X}}}{\delta p\left(\hat{X}\right)} < 0 \tag{81}$$

Actually, the capital $K_{\hat{X}}$ is minimal for $R\left(\hat{X}\right)$ maximal. Actually, as seen above, in the instability range, the average capital $K_{\hat{X}}$ acts as a threshold. When, due to variations in the system's parameters, the average capital per firm is shifted above the threshold $K_{\hat{X}}$, capital will either move to the next stable equilibrium, possibly zero, or tend to infinity. Our results show that when the expected long-term return of a sector increases, the threshold $K_{\hat{X}}$ decreases, which favours capital accumulation.

Dependency in short term returns A second use of equation (77) is to consider $Y(\hat{X})$ as any parameter-function involved in the definition of $f\left(\hat{X}, K_{\hat{X}}\right)$ that may condition either real short-term returns or the price-dividend ratio. We show in appendix 3.2.1 that again the result depends on the stability of the solution.

Around a stable equilibrium, in most cases:

$$\frac{\delta K_{\hat{X}}}{\delta f\left(\hat{X}\right)} > 0$$

A higher short-term return, decomposed as a sum of dividend and price variation, induces a higher average capital. This effect is magnified for larger levels of capital: the third approach will confirm that, in most cases, the return $f\left(\hat{X}\right)$ is asymptotically a constant $c << 1$ when capital is high: $K_{\hat{X}} >> 1$.

Turning now to the case of an unstable equilibrium, we find:

$$\frac{\delta K_{\hat{X}}}{\delta f\left(\hat{X}\right)} < 0$$

In the instability range, and due to this very instability, an increase in returns $f\left(\hat{X}\right)$ reduces the threshold of capital accumulation for low levels of capital. When short-term returns $f\left(\hat{X}\right)$ increase, a lower average capital will trigger capital accumulation towards an equilibrium. Otherwise, when average capital $K_{\hat{X}}$ is below this threshold, it will converge toward 0.

11.2 Second approach: particular solutions of the capital equation and expansion around these solutions

The second approach to equation (75) is to find the average capital at some particular points $\hat{X}$, and then by first order expansion, the solutions in the neighbourhood of these particular points. We choose as particular

[15] See also section 14.1 for more details.

points $\hat{X}$ those such that $A\left(\hat{X}\right)$ defined in (69) is maximal. At these points $\left(\hat{X}_M, K_{\hat{X}_M}\right)$, we have[16]:

$$A\left(\hat{X}_M\right) = M = \max_{\hat{X}} A\left(\hat{X}\right) \tag{82}$$

and $p = 0$, given (74).

11.2.1 Particular solutions for capital when $p = 0$

For $p = 0$ equation (75) at points $\left(\hat{X}_M, K_{\hat{X}_M}\right)$ reduces to:

$$K_{\hat{X},M}\left|f\left(\hat{X}, K_{\hat{X},M}\right)\right|\left\|\Psi\left(\hat{X}, K_{\hat{X},M}\right)\right\|^2 \simeq \sigma_{\hat{K}}^2 C\left(\bar{p}\right)\exp\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(f'\left(X_M, K_{\hat{X},M}\right)\right)^2}{384\left|f\left(\hat{X}_M, K_{\hat{X},M}\right)\right|^3}\right) \tag{83}$$

Given that $\left\|\Psi\left(\hat{X}_M, K_{\hat{X},M}\right)\right\|^2$ is decreasing in $K_{\hat{X}}$ (see (253)), and assuming $f\left(\hat{X}\right)$ is decreasing too, as is usual for marginal decreasing returns, equation (75) has two solutions.

For some particular values of the parameters, an approximate form can be found for these solutions. Here, we will merely consider a power law for $f\left(\hat{X}\right)$:

$$f\left(\hat{X}\right) \simeq B\left(X\right) K_{\hat{X}}^{\alpha} \tag{84}$$

The parameter $B\left(X\right)$ is the productivity in sector X, and equation (84) shows that the return $f\left(\hat{X}\right)$ is increasing in $B\left(X\right)$.

The stable case corresponds to an intermediate level of capital, $K_{\hat{X}}^{\alpha} << D$. In such a case, given the density of producers (57), we can assume that this density satisfies $\left\|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right\|^2 \simeq D$. The solution to equation (75) is then:

$$K_{\hat{X}}^{\alpha} = \left(\frac{DB\left(X\right)}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}\right)^{-\frac{\alpha}{\alpha+1}} \exp\left(W_0\left(-\frac{\sigma_X^2\sigma_{\hat{K}}^2\left(B'\left(X\right)\right)^2\alpha}{384\left(B\left(X\right)\right)^3\left(\alpha+1\right)}\left(\frac{DB\left(X\right)}{C\left(\bar{p}\right)\sigma_{\hat{K}}^2}\right)^{\frac{\alpha}{\alpha+1}}\right)\right) \tag{85}$$

where W_0 is the Lambert W function.

For $B\left(X\right) << 1$, we can check that $K_{\hat{X}}^{\alpha}$ is increasing with $B\left(X\right)$, i.e. with short-term returns $f\left(\hat{X}\right)$[17], which confirms the results found in the first approach: in the stable case, capital equilibrium increases with short-term returns $f\left(\hat{X}\right)$.

The unstable case corresponds to a higher level of capital. Given the density of producers (57), this case amounts to consider, in first approximation, that capital K_X is concentrated among a small group of agents, that is $\left|\Psi\left(\hat{X}, K_{\hat{X}}\right)\right|^2 << 1$. Considering a power law for $H^2\left(K_X\right)$:

$$H^2\left(K_X\right) = K_X^{\alpha}$$

[16] See equation (68).
[17] See equation (84).

the solution (83) can be written as:

$$\begin{aligned}
K_{\hat{X}}^{\alpha} \simeq & \frac{2D}{\left(\nabla_X R(X)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R(X)}{H(K_X)}} \\
& - \left(\frac{\left(\nabla_X R(X)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R(X)}{H(K_X)}}{2D} \right)^{\frac{1}{\alpha}} \frac{\sigma_{\hat{K}}^2 C(\bar{p})}{DB(X)} \\
& \times \exp\left(-\frac{\sigma_X^2 \sigma_{\hat{K}}^2 \left(B'(X)\right)^2}{768 D \left(B(X)\right)^3} \left(\left(\nabla_X R(X)\right)^2 + \sigma_X^2 \frac{\nabla_X^2 R(X)}{H(K_X)} \right) \right)
\end{aligned} \tag{86}$$

at the first order in D, plus corrections of order $\frac{1}{D}$, with:

$$f\left(\hat{X}\right) \simeq B(X) K_{\hat{X}}^{\alpha}$$

The (in)stability analysis of the previous approach applies. In the range $B(X) << 1$, when $f\left(\hat{X}\right)$ increases, or which is equivalent, $B(X)$ increases, average capital must reduce to preserve the possibility of unstable equilibria. Likewise, equilibrium capital is higher when expected returns $R(X)$ are minimal. When expected returns increase, the threshold defined by the unstable equilibrium decreases.

11.2.2 Expansion around particular solutions for $p = 0$

To better understand the behaviour of the solutions of the average capital equation (75), we expand this equation around the points $\left(\hat{X}, K_{\hat{X},M}\right)$ that solve equation (75). Appendix 3.2.2.2 computes this expansion at the second order around $\hat{X}_M$ and $K_{\hat{X},M}$. This yields the form of the solutions of (75) in the vincinity of the points $\left(\hat{X}, K_{\hat{X},M}\right)$. We recover the existence of stable and unstable solutions. Their dependency in the parameters of the system confirms the results of the first approach. The details are left to the appendix.

11.3 Third approach: solving for standard parameter functions

A third approach computes the approximate solutions of (75) for the average capital per firm per sector X. To do so, we choose some general forms for the three parameter-functions arising in the definition of the action functional: f that defines short-term returns, that include dividend and expected long-term price variations, and is given by equation (60); g that describes investors' mobility in the sector space, given by (61), and the function $H(K_X)$ involved in the firms' background field, that describes firms' moves in the sectors space and is given by equation (57).

Once these parameter-functions chosen, the approximate solutions of (75) for average capital per firm per sector can be found. We have already seen in the second approach that this equation has in general several solutions. To find them, we must consider several relevant ranges for average capital, namely a very large level of capital, $K_X >> 1$, a very low one, $K_X << 1$, and an intermediate range $\infty > K_X > 1$. We will derive the solutions for K_X within these various ranges. Details of the computations are given in appendix 3.2.3.

11.3.1 Choice of parameter functions

Our choices for the parameter functions f, g and $H^2(K_X)$ are the following.

Firms' intersectoral moves $H^2(K_X)$ We can choose for $H^2(K_X)$ a power function of K_X, so that equation (57) rewrites:

$$\|\Psi(X)\|^2 = D - L(X)\left(\nabla_X R(X)\right)^2 K_X^{\eta} \tag{87}$$

with $L(X)$ given in the appendix.

Short-term returns f To determine the function f, we must first assume a form for $r(K,X)$, the physical capital marginal returns, and for F_1, the function that measures the impact of expected long-term return on investment choices.

We assume Cobb-Douglas production functions, i.e. $B(X)K^{\alpha}$ with $B(X)$ a productivity factor. We also choose the expected long-term return F_1 to be an increasing function of the arctan type, so that investments increase linearly with expected returns and capital for small-capitalized firms, but is bounded for large values of capital.

Under these assumption, the short-term return can be written in a compact form as:

$$f\left(\hat{X},\Psi,\hat{\Psi}\right)=B_1\left(\hat{X}\right)K_{\hat{X}}^{\alpha-1}+B_2\left(\hat{X}\right)K_{\hat{X}}^{\alpha}-C\left(\hat{X}\right) \tag{88}$$

The coefficients $B_1\left(\hat{X}\right)$, $B_2\left(\hat{X}\right)$ and $C\left(\hat{X}\right)$ are given in the appendix 3.2.3.

investors' mobility in the sector space g To determine the form of the investors' mobility in the sector space g, given by (61), we must first choose a form for F_0, the investors' mobility towards higher long-term returns[18].

Here again, we choose an arctan type function of the expected long-term return, so that the velocity in the sectors' space g increases with capital, and is bounded and maximal when $K_{\hat{X}}^{\alpha}\rightarrow\infty$.

Appendix 3.2.2 shows that $g\left(\hat{X},\Psi,\hat{\Psi}\right)$ can be written:

$$g\left(\hat{X},\Psi,\hat{\Psi}\right)\nabla_{\hat{X}}R\left(\hat{X}\right)A\left(\hat{X}\right)K_{\hat{X}}^{\alpha} \tag{89}$$

where the function $A\left(\hat{X}\right)$ is given in appendix 3.2.2.

11.3.2 Solutions for the average capital

Now that the particular functions have been chosen, we can find approximate solutions to (75) in several ranges of sector X's average capital: Very large and stable capitalization ($K_X >>> 1$), Very large and unstable, i.e. bubble-like, capitalization ($K_X >>> 1$), large capitalization stable or unstable ($K_X >>> 1$), the intermediate case of mid-capitalization ($\infty >> K_X > 1$) and ultimately small capitalization ($K_X << 1$), Besides, we only consider positive short-term returns[19], $f>0$.

We consider the several type of solutions separately.

Case 1 *Very large and stable capitalization,* $K_{\hat{X}} >>> 1$

When returns are either slowing or increasing in $\hat{X}$, i.e. $\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^2\neq 0$, a solution the capital equation (75) may exist with $K_{\hat{X}} >>> 1$. In this case, only a small number of firms are present in the sector. Indeed, in such a case, the competition-deterent factor $L\left(\hat{X}\right)$ in (87) is very large, and we can assume, in first approximation, that:

$$\left\|\Psi\left(\hat{X}\right)\right\|^2 << 1 \tag{90}$$

A sector in which average capital is very large implies a very high competition, that act as a barrier to the entry of other firms. In this case, we can show that $f\left(\hat{X}\right)\simeq c$, for some constant c. Appendix 3.2.3.2 solves

[18] See section 4.4.

[19] Solutions for negative returns, $f<0$, are discussed below.

equation (75) given these assumptions. The average capital is given by:

$$K_{\hat{X}}^{\alpha} \simeq \frac{D}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{2}} - \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\sqrt{\frac{M-c}{c}}}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{2\left(1-\frac{1}{\alpha}\right)} D^{\frac{1}{\alpha}} c} - \frac{d}{R\left(\hat{X}\right)} \frac{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}} C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\left(\sqrt{\frac{M-c}{c}} + \frac{\frac{M}{c}+\nabla_{\hat{X}}^{2} R\left(\hat{X}\right)\frac{f}{d}}{2\sqrt{\frac{M-c}{c}}}\right)}{c^{2} D^{1+\frac{1}{\alpha}}\left(1 - \frac{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{\frac{2}{\alpha}} C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}}{c D^{1+\frac{1}{\alpha}}}\sqrt{\frac{M-c}{c}}\right)} \tag{91}$$

which shows that $K_{\hat{X}}^{\alpha}$ is increasing in $f\left(\hat{X}\right)$ and $R\left(\hat{X}\right)$ for $K_{\hat{X}}^{\alpha}$ large, $f\left(\hat{X}\right) \simeq c << 1$ and $D >> 1$. The analysis of section 11.1.3, shows that this corresponds to a stable local equilibrium.

Case 2 *Very large and unstable, i.e. bubble-like, capitalization,* $K_{\hat{X}} >>> 1$

This case arises when the expected long term returns is a local maximum, i.e. when $\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^{2} \to 0$ and $\nabla_{X}^{2} R\left(K_{X}, X\right) < 0$[20]. This describes a sector with a large number of firms and very high level of capital. Actually, the number of firms given in (87) shows that:

$$\left\|\Psi\left(\hat{X}\right)\right\|^{2} > D >> 1 \tag{92}$$

and appendix 3.2.3.2 shows that the average capital is given by:

$$K_{\hat{X}} = \left(\frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}}{\left|\nabla_{X}^{2} R\left(X\right)\right| c}\Gamma\left(\frac{M - \nabla_{\hat{X}} g\left(\hat{X}\right)}{c}\right)\right)^{\frac{2}{3\alpha}} \tag{93}$$

where $f\left(\hat{X}\right) \simeq c << 1$ for some constant c and $D >> 1$.

The case (93) is unstable. Actually, in this case K_X is decreasing in $f\left(\hat{X}\right)$. When returns increase, an equilibrium arises only for a relatively low average capital. Otherwise, capital tends to accumulate infinitely. When the sector's expected returns are at a local maximum, the pattern of accumulation becomes unstable. Note that an equilibrium with $K_{\hat{X}} >>> 1$ is merely possible for $c << 1$. Otherwise, there is no equilibrium for $R\left(K_X, X\right)$ maximum.

Case 3 *Large capitalization,* $K_{\hat{X}} >> 1$

For a very large and stable capitalization, i.e. when average capital $K_{\hat{X}}$ is large but below a given threshold, we can assume in first approximation that the density of firms in sector X (87) becomes:

$$\left\|\Psi\left(X\right)\right\|^{2} \simeq D \tag{94}$$

Appendix 3.2.3.2 shows that average capital in sector X is :

$$K_{X}^{\alpha} \simeq \frac{C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\Gamma\left(\frac{M}{c}\right)}{Df\left(X\right)} + \frac{d}{f\left(X\right)R\left(X\right)}\left(1 + M\,\mathrm{Psi}\left(\frac{M}{c}\right)\left(1 + \frac{\nabla_{\hat{X}}^{2} R\left(X\right)}{M}\right)\right) \tag{95}$$

where $\mathrm{Psi}\left(x\right) = \frac{\Gamma'(x)}{\Gamma(x)}$, and d and c are some constant parameters. This solution only holds when $f\left(X\right) > 0$ and $\frac{C(\bar{p})\sigma_{\hat{K}}^{2}\Gamma\left(\frac{M}{c}\right)}{Df(X)} > 1$.

[20] The case $\nabla_{X}^{2} R\left(K_X, X\right) > 0$ i.e. a minimum for the expected long term return is studied in appendix 3.2.3.2 which shows that this equilibrium is unlikely and can be discarded.

Formula (95) shows that this dependency of $K_{\hat{X}}^{\alpha}$ in $R\left(\hat{X}\right)$ depends in turns on the sign of the second term in the rhs of (95).

When the condition:

$$1+M\,\mathrm{Psi}\left(\frac{M}{c}\right)\left(1+\frac{\nabla_{\hat{X}}^{2}R\left(\hat{X}\right)}{M}\right)>0$$

holds, average capital in sector $\hat{X}$, $K_{\hat{X}}^{\alpha}$, is a decreasing function of both returns $R\left(\hat{X}\right)$ and the short-term returns $f\left(\hat{X}\right)$. The stability analysis in section 11.1.3 thus implies that the solution (95) is unstable.

On the contrary, when:

$$1+M\,\mathrm{Psi}\left(\frac{M}{c}\right)\left(1+\frac{\nabla_{\hat{X}}^{2}R\left(\hat{X}\right)}{M}\right)<0$$

a stable equilibrium is possible. In this case, the average capital in sector X, $K_{\hat{X}}^{\alpha}$, is increasing with both returns $R\left(\hat{X}\right)$ and short-term returns $f\left(\hat{X}\right)$. This case arises when, for already maximum returns, $\nabla_{\hat{X}}^{2}R\left(\hat{X}\right)<<0$, a further increase in long-term returns $R\left(\hat{X}\right)$ occurs. This increases the number of firms $\left\Vert\Psi\left(\hat{X}\right)\right\Vert^{2}$ in the sector without impairing average capital per firm. Note that stable equilibrium is an extreme case of the next case, intermediate level of capital.

Case 4 *intermediate case, mid-capitalization* $\infty>>K_{\hat{X}}>1$

To solve equation (75) in this general case, we consider that $\sigma_{X}^{2}<<1$ and the following simplifying assumptions:

$$f\left(\hat{X}\right)\simeq B_{2}\left(X\right)K_{\hat{X}}^{\alpha} \tag{96}$$

and:

$$\left\Vert\Psi\left(\hat{X}\right)\right\Vert^{2}\simeq D$$

Eventually, appendix 3.2.3.2 shows that:

$$\begin{aligned}K_{\hat{X}}^{\alpha} &= \left(\frac{8C\left(\bar{p}\right)}{D}\sqrt{\frac{3\sigma_{\hat{K}}^{2}\left|B_{2}\left(X\right)\right|}{\sigma_{X}^{2}\left(B_{2}'\left(X\right)\right)^{2}}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)\right)^{\frac{2\alpha}{1+\alpha}} \\ &\quad\times\exp\left(-W_{0}\left(-\frac{48\alpha}{1+\alpha}\left(\sqrt{\frac{3\sigma_{\hat{K}}^{2}}{\sigma_{X}^{2}}}\frac{8C\left(\bar{p}\right)}{D}\right)^{\frac{2\alpha}{1+\alpha}}\frac{\left|B_{2}\left(X\right)\right|^{3+\frac{\alpha}{1+\alpha}}}{\sigma_{X}^{2}\sigma_{\hat{K}}^{2}\left(B_{2}'\left(X\right)\right)^{2+\frac{2\alpha}{1+\alpha}}}\left(\ln\left(\bar{p}+\frac{1}{2}\right)-1\right)^{2+\frac{\alpha}{1+\alpha}}\right)\right)\end{aligned} \tag{97}$$

where W_{0} is the Lambert W function and $\bar{p}$ a constant.

In first approximation, equation (97) implies that $K_{\hat{X}}^{\alpha}$ is an increasing function of $B_{2}\left(X\right)$. Given our simplifying assumption (96), average capital is higher in high short-term returns sectors.

Moreover, $K_{\hat{X}}^{\alpha}$ is a decreasing function of $\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^{2}$ and $\nabla_{\hat{X}}^{2}R\left(\hat{X}\right)$: capital accumulation is locally maximal when expected returns $R\left(\hat{X}\right)$ of sector $\hat{X}$ are at a local maxima, i.e. $\left(\nabla_{\hat{X}}R\left(\hat{X}\right)\right)^{2}=0$ and $\nabla_{\hat{X}}^{2}R\left(\hat{X}\right)<0$.

Thus, in the intermediate case, the average values $K_{\hat{X}}$ are stable. In addition, both short-term and long term returns matter in the intermediate range.

Case 5 *Small capitalization* $K_{\hat{X}}<<1$

When average physical capital per firm in sector $\hat{X}$ is very low, we can use our assumptions about $g\left(\hat{X}\right)$ equation (89), and assume that:

$$f\left(\hat{X}\right) \simeq B_1\left(\hat{X}\right) K_{\hat{X}}^{\alpha-1} >> 1,\ g\left(\hat{X}\right) \simeq 0 \tag{98}$$

and:

$$\left\|\Psi\left(\hat{X}\right)\right\|^2 \simeq D$$

For these conditions, the solution of (75) is locally stable. We show in appendix 3.2.3.2 that the solution for average capital is at the first order[21]:

$$K_{\hat{X}} = \left(\frac{C(\bar{p})\,\sigma_{\hat{K}}^2 \hat{\Gamma}(-1)}{DB_1\left(\hat{X}\right)}\right)^{\frac{1}{\alpha}} + \frac{\frac{C(\bar{p})\sigma_{\hat{K}}^2}{D}\hat{\Gamma}'(-1)\left(M - \left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}\right)\right)\right)}{B_1^{\frac{1}{\alpha}}\left(\hat{X}\right)\left(\frac{C(\bar{p})\sigma_{\hat{K}}^2\hat{\Gamma}(-1)}{D}\right)^{1-\frac{1}{\alpha}}} \tag{99}$$

Equation (99) shows that average capital $K_{\hat{X}}$ increases with $M - \left(\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} + \nabla_{\hat{X}} g\left(\hat{X}\right)\right)$: when expected long-term returns increase, more capital is allocated to the sector. Equation (98) also shows that average capital $K_{\hat{X}}$ is maximal when returns $R\left(\hat{X}\right)$ are at a local maximum, i.e. when $\frac{\left(g\left(\hat{X}\right)\right)^2}{\sigma_{\hat{X}}^2} = 0$ and $\nabla_{\hat{X}} g\left(\hat{X}, K_{\hat{X}}\right) < 0$.

Inversely, the same equations (99) and (98) show that average capital $K_{\hat{X}}$ is decreasing in $f\left(\hat{X}\right)$. The equilibrium is unstable. Recall that in this unstable equilibrium, $K_{\hat{X}}$ must be seen as a threshold. The rise in $f\left(\hat{X}\right)$ reduces the threshold $K_{\hat{X}}$, which favours capital accumulation and increases the average capital $K_{\hat{X}}$. Actually, when average capital is very low, i.e. $K_{\hat{X}} << 1$, which is the case studied here, marginal returns are high. Any increase in capital above the threshold $K_{\hat{X}}$, or any shift reducing the threshold, widely increases returns, which drives capital towards the next stable equilibrium, with higher $K_{\hat{X}}$.

This case is thus an exception: the dependency of $K_{\hat{X}}$ in $R\left(\hat{X}\right)$ is stable, but the dependency in $f\left(\hat{X}\right)$ is unstable. This saddle path type of instability may lead the sector, either towards a higher level of capital (case 4 below) or towards 0. where the sector disappears.

11.3.3 Remark: The case of negative short-term returns $f < 0$

In the four cases described above, we have only considered the case where a sector $\hat{X}$ short-term returns are positive $f\left(\hat{X}\right) > 0$. We can nonetheless extend our analysis to the case $f\left(\hat{X}\right) < 0$.

In such a case, the equilibria, whether stable or unstable, defined in cases 1, 2 with $K_{\hat{X}} >> 1$, and 4 with $K_{\hat{X}} > 1$, are still valid, and capital allocation relies on expectations of high long-term returns. If we consider that $f\left(\hat{X}\right) < 0$ is an extreme case, where expectations of large future profits must offset short-term losses. However, such equilibria become unsustainable when $R\left(\hat{X}\right)$ decreases to such an extent that it does not compensate for the loss $f\left(\hat{X}\right)$. Case 3, $K_{\hat{X}} < 1$ is the only case that is no longer possible when $f\left(\hat{X}\right) < 0$, since the returns that matter in this case are dividends. If they turn negative, the equilibrium is no longer sustainable.

12 Average capital across sectors in a dynamic environment

So far, we have determined and studied the dependency in parameters of average capital per firm and per sector. However parameters may vary over time, and so should average capital values. We thus introduce a macro time scale and design a dynamic model that involves average capital and time varying expectations in long-term returns.

[21] Given our hypotheses, $D >> 1$, which implies that $K_{\hat{X}} << 1$, as needed.

12.1 Dynamics for average capital and long-run expected returns

We consider the dynamics for $K_{\hat{X}}$ generated by modifications in parameters. Assuming that some time-dependent parameters modify expected long-term returns $R(X)$, average capital $K_{\hat{X}}$ becomes a function of the time variable θ. To find the evolution over time of the average physical capital per firm in sector $\hat{X}$, $K_{\hat{X}}$, can be found by defining the equation for $K_{\hat{X}}$, (75), and compute its variation with respect to θ, using the fact that the functions $\left\|\Psi\left(\hat{X}\right)\right\|^2$ and $\hat{\Gamma}\left(p+\frac{1}{2}\right)$ both depend on time θ through $K_{\hat{X}}$ and $R(X)$. The variations of these two functions with respect to the two dynamical variables $K_{\hat{X}}$ and $R(X)$ are computed in appendix 4.1. We show that, when $C(\bar{p})$ constant, the variation of (75) writes:

$$k\frac{\nabla_\theta K_{\hat{X}}}{K_{\hat{X}}}+l\frac{\nabla_\theta R\left(\hat{X}\right)}{R\left(\hat{X}\right)}-2m\frac{\nabla_{\hat{X}}\nabla_\theta R\left(\hat{X}\right)}{\nabla_{\hat{X}}R\left(\hat{X}\right)}+n\frac{\nabla_{\hat{X}}^2\nabla_\theta R\left(\hat{X}\right)}{\nabla_{\hat{X}}^2R\left(\hat{X}\right)}=-C_3\left(p,\hat{X}\right)\frac{\nabla_\theta r\left(\hat{X}\right)}{f\left(\hat{X}\right)} \tag{100}$$

where coefficients k, l, m and n are computed in appendix 4.1.

To make the system self-consistent, and since $K_{\hat{X}}$ already depends on R, we merely need to introduce an endogenous dynamics for R.

To do so, we assume that R depends on $K_{\hat{X}}, \hat{X}$ and $\nabla_\theta K_{\hat{X}}$, and that this dependency has the form of a diffusion process (see appendix 4.2). This leads to write R as a function $R\left(K_{\hat{X}},\hat{X},\nabla_\theta K_{\hat{X}}\right)$. The variation of R is of the form:

$$\begin{aligned}\nabla_\theta R\left(\theta,\hat{X}\right) &= a_0\left(\hat{X}\right)\nabla_\theta K_{\hat{X}}+b\left(\hat{X}\right)\nabla_{\hat{X}}^2\nabla_\theta K_{\hat{X}}+c\left(\hat{X}\right)\nabla_\theta\left(\nabla_\theta K_{\hat{X}}\right)+d\left(\hat{X}\right)\nabla_\theta^2\left(\nabla_\theta K_{\hat{X}}\right) \\ &\quad +f\left(\hat{X}\right)\nabla_{\hat{X}}^2\left(\nabla_\theta R\left(\theta,\hat{X}\right)\right)+h\left(\hat{X}\right)\nabla_\theta^2\left(\nabla_\theta R\left(\theta,\hat{X}\right)\right) \\ &\quad +u\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta\left(\nabla_\theta K_{\hat{X}}\right)+v\left(\hat{X}\right)\nabla_{\hat{X}}\nabla_\theta\left(\nabla_\theta R\left(\theta,\hat{X}\right)\right)\end{aligned} \tag{101}$$

We can also assume that the coefficients in the expansion are slowly varying, since they are obtained by computing averages.

The dynamics (101) corresponds to a diffusion process: expected returns in one sector depend on the variations of capital and returns in neighbouring sectors.

To find the intrinsic dynamics for $K_{\hat{X}}$, we assume that the exogenous variation $\frac{\nabla_\theta r\left(\hat{X}\right)}{r\left(K_{\hat{X}},\hat{X}\right)}$ is null, and that the system of equations (100) and (101) yields the dynamics for $\nabla_\theta K_{\hat{X}}$ and $\nabla_\theta R\left(\theta,\hat{X}\right)$. Approximating these dynamics to the first order in derivatives, we find in appendix 4.2 the following matricial equation:

$$0=M_1\begin{pmatrix}\nabla_\theta K_{\hat{X}}\\ \nabla_\theta R\end{pmatrix}-M_2\begin{pmatrix}\nabla_\theta K_{\hat{X}}\\ \nabla_\theta R\end{pmatrix}-M_3\begin{pmatrix}\nabla_\theta K_{\hat{X}}\\ \nabla_\theta R\end{pmatrix} \tag{102}$$

12.2 Oscillatory solutions

We look for oscillating solutions of (102) of the type:

$$\begin{pmatrix}\nabla_\theta K_{\hat{X}}\\ \nabla_\theta R\left(\hat{X}\right)\end{pmatrix}=\exp\left(i\Omega\left(\hat{X}\right)\theta+iG\left(\hat{X}\right)\hat{X}\right)\begin{pmatrix}\nabla_\theta K_0\\ \nabla_\theta R_0\end{pmatrix} \tag{103}$$

with slowly varying $G\left(\hat{X}\right)$ and $\Omega\left(\hat{X}\right)$. We are then led to the relation between $\Omega\left(\hat{X}\right)$ and $G\left(\hat{X}\right)$:

$$\begin{aligned}0 &= \frac{k}{K_{\hat{X}}}\left(1-ieG-ig\Omega\right)+\left(\frac{l}{R\left(\hat{X}\right)}-i\frac{2m}{\nabla_{\hat{X}}R\left(\hat{X}\right)}G\right)\left(a_0+iaG+ic\Omega\right) \\ &\quad -\frac{l}{R\left(\hat{X}\right)}\left(d\Omega^2+bG^2+u\Omega G\right)+\frac{k}{K_{\hat{X}}}\left(e\Omega^2+fG^2+v\Omega G\right)\end{aligned} \tag{104}$$

In the sequel, we will limit ourselves to the first order terms which yields the expression for Ω as a function of the parameters involved in (100) and (101). Appendix 4.3 computes the expression of Ω.

It also derives the condition of stability for the oscillations. When:

$$\frac{lc}{R\left(\hat{X}\right)}\left(\frac{k}{K_{\hat{X}}}+\frac{a_0 l}{R\left(\hat{X}\right)}\right)+\frac{4m^2 c a_0}{\left(\nabla_{\hat{X}} R\left(\hat{X}\right)\right)^2}G^2>0 \tag{105}$$

oscillations are dampened and return to the steady state. Otherwise, oscillations are diverging: the system settles on another steady state, i.e. another background state. Appendix 4.4 studies the condition (105) as a function of the parameter functions $f\left(\hat{X}\right)$ and $R\left(\hat{X}\right)$, the level of average capital $K_{\hat{X}}$, and the coefficients arising in the expectations formations. The results are presented in the next section.

Part III: Results

Since the multiple solutions of the average capital equation (75) may be stable or unstable, we will first discuss this notion of stability before detailing the determinants per sector of average capital, firms, and the investors' density. We then describe the three patterns of accumulation emerging for the system. Each pattern is characterized by an average capital for the sector and a density of firms. We will therefore detail the stability of each pattern and the transitions between patterns. To account for this core instability, we ultimately study the interactions between average capital per sector and some endogeneized long-term returns expectations.

13 Instability

In each sector, there may be several values of average capital solving (75). Average capital can be stable or unstable: when modified, a stable average capital naturally returns to its initial equilibrium, whereas an unstable one will stabilize durably at another equilibrium level. Unstable average capital does not constitute an equilibrium levels for sectors. It merely acts as an accumulation threshold. As such, it can always vary and be modified.

13.1 Determinants of instability

The average capital per firm per sector defined in equation (75) can be seen as the fixed point of a dynamic equation with varying parameters[22], (see section 11.1.1) whose (in)stability depends on the values of the sector's parameters.

Average capital is potentially unstable in a sector (see section 11.1.1 (78a)) if the following condition is met:

$$\left|B\left(\hat{X}\right)\right|\equiv\left|k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}-\left(\frac{\partial\ln f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}+\frac{\partial\ln\left|\Psi\left(\hat{X},K_{\hat{X}}\right)\right|^2}{\partial K_{\hat{X}}}+l\left(\hat{X},K_{\hat{X}}\right)\right)\right|>1 \tag{106}$$

Any variation in average capital $K_{\hat{X}}$ will drive the system away from its initial value. This instability depends on the four parameters of $\left|B\left(\hat{X}\right)\right|$.

Two of these parameters impact directly the stability: the relative variation of short-term returns, dividends and price fluctuations, $\frac{\partial\ln f\left(\hat{X},K_{\hat{X}}\right)}{\partial K_{\hat{X}}}$, and the variation in the number of firms moving in, or out of, sector $K_{\hat{X}}$, $\frac{\partial\ln\left|\Psi\left(\hat{X},K_{\hat{X}}\right)\right|^2}{\partial K_{\hat{X}}}$.

[22] The definitions of the parameters are given in appendix 3.2.1.

Two others act indirectly by a modification in the background field: the function $l\left(\hat{X},K_{\hat{X}}\right)$ that measures the indirect variation induced by a modification of the short-term returns $f\left(\hat{X}\right)$, and the modification of sector $\hat{X}$'s relative return, written $k\left(p\right)\frac{\partial p}{\partial K_{\hat{X}}}$, which depends on the shape of the returns around $\hat{X}$.

A modification in any of these parameters affects the system as a whole and may reshape the collective state through a change in the background field. Taken altogether, these modifications may magnify or dampen any change in a sector's average capital and, depending on the magnitude of $\left|B\left(\hat{X}\right)\right|$, determine the stability of the system.

13.2 Shifts in average capital

This instability has an implication in terms of shifts in average capital value. Actually, when average capital is instable in a sector, it acts as a potential threshold for the capital accumulation of individual firms in the sector: any small deviation of average capital above or below this unstable equilibrium value, will shift average firms of the sector above or below the sector's threshold. Firms are driven away towards an other equilibrium value and this motion will ultimately shift average capital towards this equilibrium. Thus, transitions from unstable to stable average capital arise over time: there is an intrinsic dynamics of average capital per sectors that is driven by instability and the exogenous variations of the system's parameters.

14 Determinants of patterns of capital accumulation

Depending on the stability of the equilibrium, several parameters command capital accumulation. We will first describe the determinants of capital accumulation, its patterns, and its dependency on parameters, before studying the density of firms and investors per sector.

14.1 Average capital per sector

14.1.1 Determinants

Average capital in sector $\hat{X}$ is determined by short-term returns, $f\left(\hat{X}\right)$ - dividends and price fluctuations - and by expected long-term returns, $R\left(\hat{X}\right)$, i.e. the growth prospects of the firm. Yet short- and long-term returns are not fully independent: the price fluctuations part of the short-term returns are driven by expected long-term returns.

Average capital in a sector also depends on the expected long-term returns in neighbouring sectors. This dependency is measured by the parameter $p\left(\hat{X}\right)$, defined in (71). Indeed, when the short-term returns $f\left(\hat{X}\right)$ are positive, which is the case here[23], $p(\hat{X})$ rewrites as:

$$p(\hat{X})=\frac{M-\left(\frac{\left(g(\hat{X})\right)^{2}}{\sigma_{\hat{X}}^{2}}+\nabla_{\hat{X}}g\left(\hat{X}\right)-\frac{\sigma_{\hat{K}}^{2}F^{2}\left(\hat{X}\right)}{2f^{2}\left(\hat{X}\right)}\right)}{f\left(\hat{X}\right)}-\frac{3}{2} \tag{107}$$

which is the parameter that enters directly into the differential equation for $K_{\hat{X}}$ (77).

This parameter $p(\hat{X})$ is composed of three terms, up to the normalization by the short-term return $f\left(\hat{X}\right)$ of sector $\hat{X}$.

The first term $\frac{\left(g(\hat{X})\right)^{2}}{\sigma_{\hat{X}}^{2}}$ is directly proportional to the gradient of expected long-term returns $\nabla R\left(\hat{X}\right)$[24], and the second term $\nabla_{\hat{X}}g\left(\hat{X}\right)$ is proportional to the second derivative $\nabla^{2}R\left(\hat{X}\right)$ of $R\left(\hat{X}\right)$. These two first

[23] See explanation in section 11.3.3.

[24] See the definition of the parameter function g, equation (61), and (62).

terms measure the variations of expected returns across sectors, i.e. the value of expected returns in sector $\hat{X}$ relative to its neighbours.

The last term $\frac{\sigma_{\hat{K}}^2 F^2(\hat{X},K_{\hat{X}})}{2f^2(\hat{X})}$ is a smoothing factor between neighbours' sectors. It can be neglected in the first approximation[25] and will be discussed in section 14.3.

The parameter $p(\hat{X})$ is maximal for sectors such that $\nabla R\left(\hat{X}\right) = 0$ and $\nabla^2 R\left(\hat{X}\right) < 0$. It is thus a local maximum when $R\left(\hat{X}\right)$ is itself a local maximum so that the parameter $p(\hat{X})$ describes the expected long-term returns of a sector relative to its neighbours: the higher $p\left(\hat{X}\right)$, the more attractive is sector $\hat{X}$ relative to its neighbours[26].

14.1.2 Impact of determinants depending on the stability

Taken altogether, the three parameters $R\left(\hat{X}\right)$, $f\left(\hat{X}\right)$, $p\left(\hat{X}\right)$ are the main determinants of average capital in sector $\hat{X}$. However, their influence on $K_{\hat{X}}$ will depend on the stability of the sector. In stable sectors, average capital values can be understood as equilibria. In unstable ones, they are potential thresholds for the capital accumulation of individual firms.

When sectors are stable, average capital is increasing in short-term returns $f\left(\hat{X}\right)$, expected long-term returns $R\left(\hat{X}\right)$, and in the sector's relative attractivity $p\left(\hat{X}\right)$, respectively. Actually, the higher the returns $f\left(\hat{X}\right)$ and $R\left(\hat{X}\right)$, the higher the capital accumulation. Moreover, any increase in relative returns $p\left(\hat{X}\right)$ attracts investors from neighbouring sectors and increases average capital in sector $\hat{X}$.

In unstable sectors, average capital is decreasing in these same variables, and any increase in short- or expected long-term, be they absolute or relative, returns reduces the amount of capital required to initiate the capital accumulation process for the individual firms.

14.2 Density of producers per sector

The way agents, firms, and investors shift across the sectors' space depends on various parameters.

The density of firms per sector (57) depends on expected long-term returns through the function:

$$V(X) = (\nabla_X R(X))^2 + \frac{\sigma_X^2 \nabla_X^2 R(X)}{H(K_X)}$$

where $\nabla_X R(X)$ is the gradient of expected long-term returns along the sectors space, and $\nabla_X^2 R(K_X, X)$ is the Laplacian, i.e. the generalisation of the second derivative of $R(K_X, X)$ with respect to the sectors' space.

The density of firms is a decreasing function of $V(X)$ (see (57)). As a consequence, we can deduce the following impact of the expected long-term return on firms density:

When expected returns are minimal, i.e. when the gradient of expected long-term returns along the sectors' space is null, and the Laplacian is positive, $\nabla_X R(X) = 0$ and $\nabla_X^2 R(K_X, X) > 0$, average capital is low, and the number of firms is relatively large. This describes sectors where a large number of small firms provide short-term returns through dividends.

When returns in sector X, $R(X)$, are at a local maximum, the gradient of expected long-term returns along the sectors space is null, $\nabla_X R(X) = 0$, but the Laplacian is negative, $\nabla_X^2 R(K_X, X) < 0$. The number of firms in the sector is maximal since firms seek the most profitable sectors to accumulate capital, These sectors exhibit both a large number of firms and a high level of capital K_X per firm, but this equilibrium is unstable. Incidentally, competition ensures that sectors with low or minimal expected returns are not completely depleted.

[25] See the discussion following equation (66).

[26] Note that the parameter $p\left(\hat{X}\right)$ is normalized by short-term returns. It computes the ratio of relative attractivity to short-term returns. This allows to consider these two variables separately.

When expected long-term returns are intermediate, i.e. when the gradient of $R(X)$ along the sectors space is non-null, $\nabla_X R(X) \neq 0$, the sector is "transitory". It is surrounded by neighbouring sectors, with both lower and higher expected returns. Firms head towards sectors with higher returns. The greater the discrepancy between neighbouring returns $\nabla_X R(X)$, the faster firms leave the sector.

14.3 Density of investors per sector

The average number of investors in sector $\hat{X}$ (formula (70)) is an increasing function of short-term returns $f\left(\hat{X}\right)$, and of the sector relative long-term attractivity p, (equation (74)). All else equal, an increase in short-term returns or an improvement of the sector's relative long-term attractivity increases the number of investors and, in turn, firms' disposable capital.

The density of investors in a given sector increases with the sector's relative attractivity $p\left(\hat{X}\right)$ defined in (71), which can be written as:

$$p\left(\hat{X}\right)=\frac{M-\left(\left(g\left(\hat{X}\right)\right)^2+\sigma_{\hat{X}}^2\left(f\left(\hat{X}\right)+\nabla_{\hat{X}}g\left(\hat{X}\right)\right)\right)}{\sigma_{\hat{X}}^2\sqrt{f^2\left(\hat{X}\right)}}+\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X}\right)}{2\sigma_{\hat{X}}^2\left(\sqrt{f^2\left(\hat{X}\right)}\right)^3}$$

The first term is the sector's relative attractivity towards its neighbours, normalized by its short-term returns $f\left(\hat{X}\right)$. The second term is a factor that smoothes differences between sectors. It is negatively correlated to the variations of the sectors' relative attractivity. Investors and capital will increase in sectors surrounded by significantly more attractive sectors, i.e. sectors with higher average capital and investors[27]: the whole system tends to reach stable configurations, and capital discrepancies are reduced between close neighbours[28].

15 Patterns of capital accumulation

The equation for average capital per firm per sector (75) has, in general, several solutions (see Section 11.2): these potential equilibria depend on the parameter functions $f\left(\hat{X}\right)$, $R\left(\hat{X}\right)$ and $p\left(\hat{X}\right)$, and on the firms' densities $\left\|\Psi\left(X,K_X\right)\right\|^2$, that themselves depend on expected long-term returns and their variations across sectors.

Each parameter influences the others. Their combinations induce several possible patterns of capital accumulation for each sector. A pattern combines both average capital, the density, or number, of firms, along with their associated values of long- and short-term returns.

These returns depend on the sector's intrinsic characteristics, but also on the average level of capital of the sector, and the returns of neighbouring sectors.

Within a given pattern some sectors may be unstable. Changes in parameters or unexpected shocks may drive the sector from one pattern to another.

Three patterns of capital accumulation arise. Depending on its parameters, any sector of a given pattern may be stable or unstable. Unstable sectors may shift from one pattern to another. Any deviation of average capital above or below an unstable equilibrium value drives firms away from this equilibrium and ultimately shifts average capital towards another equilibrium. These transitions provide bridges between patterns of capital. Due to a change in external conditions, sectors may move from one pattern to another.

[27] A close inspection of equation (66) shows that this term contains -squared- contributions of short-term returns, $f\left(\hat{X}\right)$, and the sector's relative attractivity: $\frac{\left(g\left(\hat{X}\right)\right)^2}{2\sigma_{\hat{X}}^2}+\frac{1}{2}\nabla_{\hat{X}}g\left(\hat{X}\right)$. These contributions are both proportional to the gradient of R with respect to $K_{\hat{X}}$. When this gradient is different from zero, i.e. when an increase in capital may improve either the sector's relative attractivity or short-term returns, the correction $\frac{\sigma_{\hat{K}}^2F^2\left(\hat{X}\right)}{2\sigma_{\hat{X}}^2\left(\sqrt{f^2\left(\hat{X}\right)}\right)^3}$ increases $\frac{A\left(\hat{X}\right)}{f\left(\hat{X}\right)}$, and in turn $K_{\hat{X}}$, in most cases.

[28] Derivation of the minimization equation in appendix 3.1.2 shows that the term $F\left(\hat{X}\right)$ arises as a backreaction of the whole system with respect to modifications at one point of the thread.

15.1 First pattern: low capital, high short-term returns driven by dividends only

These are sectors where growth prospects are subdued, with a relatively large number of low-capitalized firms. Because firms are small, marginal productivity is high and firms attract capital with short-term returns $f\left(\hat{X}\right)$ through dividends. These firms do not move towards sectors with higher growth prospects due to a lack of capital.

These sectors are stable to small fluctuations in growth prospects: any increase in $R\left(\hat{X}\right)$ shifts moderately investment and average capital.

However, they are unstable to short-term returns: any increase in $f\left(\hat{X}\right)$ induces higher dividends and attracts investors. Average capital accumulates in the sector and reaches a pattern 2 type stable equilibrium with a higher number of firms and average capital.

However, an adverse shock lowering $f\left(\hat{X}\right)$ increases the threshold of capital accumulation and drives the equilibrium towards 0. Producers remain in the sector, but with such a low capital on average, this very lack of capital prevents them to shift towards more attractive sectors in the long run (see appendix 3.3).

15.2 Second pattern: intermediate-to-high level of capital, short-term returns, long-term expectations

These sectors have moderate growth prospects, so that any increase in short-term, i.e. dividends and stock prices or long-term returns, increases their relative attractivity $p\left(\hat{X}\right)$ and attracts investors and capital. Locally, the higher the relative attractivity of the sector, the higher the capital accumulation. The relatively high number of firms in the sector is a decreasing function of average capital: competition favours higher average capital, and concentration of firms. This is the most standard pattern of capital allocation. It is stable to variations in average capital, except when average capital is high and the firms' density is low.

In this case, any deviation of average capital above its equilibrium increases the threshold and drives the sector backward to a stable pattern 2 equilibrium, i.e. a sector with a large number of average capitalized firms. The lower capital per firm reduces competition and attracts new firms into the sector.

On the contrary, any deviation of average capital below its equilibrium reduces the threshold and favours capital accumulation. The sector is driven towards a stable pattern 3 equilibrium, with a small number of very capitalized firms (see description of this pattern below).

15.3 Third pattern: high capital, long-term returns, and relative attractivity

These are sectors where growth prospects are extremely high. Capital accumulation is driven by expectations of long-term returns sustained by ever-higher levels of investment. These are the most attractive sectors. Two cases arise.

When expected long-term returns are not maximal, the sector stabilizes with very few firms with very high capital arises. This extension of pattern 2 corresponds to a few large oligopolistic groups.

When expected long-term returns are maximal, the sector's attractivity allows a large number of firms with high capital to coexist. All else equal, these firms could grow indefinitely, so that such equilibria are bound to be unstable (see section 11.1).This describes bubble-like, unstable sectors.

An adverse shock drives these unstable sectors towards a stable pattern 3: average capital is approximatively maintained, but an increase in competition evicts the less capitalized firms and the total number of firms is reduced to a small set.

On the contrary, a positive shock reduces the threshold of capital accumulation. Most firms can accumulate without bound, which attracts even higher capital. Capital accumulation is modified in all sectors, which may transform the whole economic landscape. Total available capital is reduced, which modifies the stability conditions for all sectors. Low-capitalized sectors may become unstable and disappear, whereas others may accumulate capital. All in all, the system may end with a reduced sectors space (see appendix 3.3 for technical details).

16 Global instability

The source of instability studied so far is a local one, and stems from solutions for average capital per firm per sector, equation (75). Here we point to a difference cause of instability, which is global and stems from the constraint imposed in the model on the total number of investors.

In our model, we have assumed a fixed number of agents spread across the sectors. This hypothesis binds the dynamics of the whole set of sectors. If this constraint were to be lifted, the sectors would be independent and their associated background field would stabilize the system: each sector could reach a stable configuration for the average capital, given the short-term and expected long-term returns.

In our system, however, the density of agents of a sector is dependent on the whole system's characteristics. Thus there can only be a global equilibrium for the system. Any change of parameter induces a perturbation $\delta\Psi\left(\hat{X}, K_{\hat{X}}\right)$ destabilizes the whole system as a whole: the equilibrium is globally unstable. The mechanism of this instability is detailed in appendix 3.3.1.

Relaxing the condition on the number of agents amounts to replace the average capital equation (75) by[29]:

$$K_{\hat{X}}\left\|\Psi\left(\hat{X}\right)\right\|^{2}\left|f\left(\hat{X}\right)\right|=C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\hat{\Gamma}\left(\frac{1}{2}\right)=C\left(\bar{p}\right)\sigma_{\hat{K}}^{2}\exp\left(-\frac{\sigma_{\hat{X}}^{2}\sigma_{\hat{K}}^{2}\left(f'\left(X\right)\right)^{2}}{384\left|f\left(\hat{X}\right)\right|^{3}}\right)\tag{108}$$

which is identical to (83) and has at least one locally stable solution (computed in (85) and (86)). The solutions of the modified average capital equation (108) do no longer directly depend on a sector's relative characteristics, but rather on the returns $f\left(\hat{X}\right)$ and on the density of firms in the sector, $\left\|\Psi\left(\hat{X}\right)\right\|^{2}$[30][31].

17 Patterns in a dynamic environment

In the previous paragraphs expected long-term returns $R\left(\hat{X}\right)$ were considered exogenous. To account for global instability propagating from one sector to the others, section 12 removed this assumption and introduced a dynamic interaction between average capital and long-term expected returns. This leads to the dynamic system (102) which propagates shocks in capital and expectations across the system (see appendix 4.3). The mechanism is the following.

Assume a shift, $\delta K_{\hat{X}}$ or $\delta R\left(\hat{X}\right)$ in the average capital or long-term return of a given sector $\hat{X}$. The interactions between average capital and expected long-term return induce volatility around the equilibrium values $K_{\hat{X}}$ and $R\left(\hat{X}\right)$. The fluctuation $\delta R\left(\hat{X}\right)$ directly impacts average capital and expected return in neighbouring sectors through the induced variation of relative expected returns, which initiates the propagation of the initial perturbation to the whole system.

This propagation is described by the oscillating solutions (103). For a given sector $\hat{X}$, the velocity of oscillations in average capital and expected returns are measured by the frequency $\Omega\left(\hat{X}\right)$ depending on the sector's characteristics, These oscillations may be dampening, also referred to as stable oscillations or widening, also referred to as unstable oscillations.

We have already given the condition for dampening oscillations in (105). Three main parameters determine which type of oscillations a sector may experience (see appendix 4.4).

- The elasticity of expected long-term returns with respect to variations of capital, i.e. c, that arises in equation (101). It determines two relevant forms of expectations:

  When $c > 0$, expectations are highly reactive to variations of capital, and expected long-term returns increase with any acceleration in capital accumulation. In such a case, expected long-term returns depend positively on the variations of average capital $K_{\hat{X}}$.

[29] Expression (268) is used to compute $\hat{\Gamma}\left(\frac{1}{2}\right)$.

[30] An intermediate situation between (75) and (108) could also be considered: it would be to assume a constant number of agents in some regions of the sector space.

[31] Alternatively, limiting the number of investors per sector can be achieved through some public regulation to maintain a constant flow of investment in the sector.

When $c < 0$, expectations are moderately reactive to variations in the capital, and expected long-term returns depend negatively on the variations of average capital $K_{\hat{X}}$.

- The neighbouring sectors' discrepancy in capital fluctuations at a given time, G. It arises in the oscillatory solutions (103) and measures the inhomogeneity between sectors.
- Last but not least, the sector average level of capital $K_{\hat{X}}$ impacts the type of fluctuations that the sector experiences.

The $K_{\hat{X}}$'s dependency of a sector's fluctuations allows us to present the results by type of capital.

17.1 Low average capital sectors, $K_{\hat{X}} << 1$

When average capital is very low in a sector, $K_{\hat{X}} << 1$, the sole relevant parameter to the fluctuations is the reactivity c of the expected return $R\left(\hat{X}\right)$ to an increase in capital (see appendix 4.4).

Two cases arise.

When the long-term return $R\left(\hat{X}\right)$ strongly reacts to capital fluctuations, i.e. when $c > 0$, oscillations are unstable. When $R\left(\hat{X}\right)$ only reacts mildly, i.e. when $c < 0$, oscillations are stable.

Indeed, in the first case, expected long-term returns and average capital variations are positively correlated, which creates an amplification in the dynamics of these two variables: an increase in capital induces an increase in expected returns that in turn amplifies the increase in capital.

In the second case, expected long-term returns and average capital variations are negatively correlated. A stabilization occurs with dampening oscillations.

These results show that for expectations mildly reactive to variations of capital, some equilibria with relatively low capital are possible and resilient to oscillations in expectations, a niche effect may exist for some sectors.

17.2 High average capital sectors, $K_{\hat{X}} >> 1$

In sectors with a high level of capital, whether the equilibrium is stable or unstable, here again, only reactivity of expectations to increase in capital, i.e. c, matters. Oscillations are dampening for $c > 0$ and explosive for $c < 0$.

Indeed, highly reactive expectations, $c > 0$, tend to amplify fluctuations of capital and return expectations.

In the stable case, fluctuations that would otherwise be destabilizing for sectors with low capital may stabilize or maintain sectors with both stable and high levels of capital. A large reactivity between expectations and capital will allow for an intrinsic high level of capital to consolidate. Fluctuations will moderately impact these high-capitalized sectors: for instance, considering an initial increase in returns only, i.e. $\delta R\left(\hat{X}\right) > 0$, will induce a net outflow of capital towards less capitalized sectors with an higher increase in relative returns, while a decreasing return, i.e. $\delta R\left(\hat{X}\right) < 0$, will induce a net inflow of capital dampening the sector's fluctuations.

In the unstable case, the specific equilibrium $K_{\hat{X}} >> 1$, induces the stabilization as follows: an initial increase in $K_{\hat{X}}$ increases the expected long-term returns, while at the same time, the negative correlation between variations in investment and expected return lowers the average capital. Thus, an increase in capital $K_{\hat{X}}$ improves the sector profitability, which in turn, lowers the capital threshold in this sector,i.e. the potential equilibrium level of capital. To put it differently, an initial increase in the average capital amplifies the expected return, which reduces $K_{\hat{X}}$ and offsets the initial increase in capital.

When expectations are mildly reactive, i.e. $c < 0$, the mechanism of dampening oscillations that arises for $c > 0$ is impaired. In the unstable case, for instance, an initial increase in the threshold $K_{\hat{X}}$, impacts only moderately the sector expected returns, and does not offset the initial increase in capital.

17.3 Intermediate average capital sectors $\infty > K_{\hat{X}} > 1$

In sectors with intermediate capital, the analysis is mixed: oscillation in a sector depends both on the reactivity of expectations, c, to an increase in capital, and G, the discrepancy between sectors (see appendix 4.4.2.).

When expectations are midly reactive, $c < 0$, and the discrepancy between neighbouring sectors is moderate, $G << 1$, oscillations are dampening for sectors with relatively low average capital. The analysis of the first case applies to the extent that indeed some homogeneity in capital between the neighbouring sectors exists.

When expectations are strongly reactive, $c > 0$, and discrepency between neighbouring sectors is large, $G >> 1$, oscillations are dampening for sectors with relatively high average capital. The analysis of the second case applies to a locally dominating sector.

17.4 The role of expectations in average capital fluctuations

For each sector, the threshold between dampening and explosive oscillations depends on the parameters of the system. Mildly reactive expectations only favour low- to high-capital sectors (patterns 1 and 2), while very reactive expectations only favour very high-capital sectors (pattern 3). In this latter case, oscillations will be felt as relatively weak, in values, for high capital sectors and this leads to a reallocation of capital towards these sectors: for instance, a very high capital sector whose neighbours experience a decrease in expected long-term returns will benefit from their outflow of capital, and this sector will be stabilized.

Recall that in extreme cases of pattern 3, both maximal capital and returns act as thresholds that repel low-capital firms and propel high-capital firms to ever higher accumulation. Oscillations in these thresholds generate a high global instability: a constantly oscillating threshold crowds firms out of the sector.

To conclude, the dynamics for average capital and expected returns merely reflect fluctuations in the background fields i.e. the collective states. These fluctuations may destabilize the patterns in some sectors and ultimately switch the collective state and modify the patterns' landscape.

18 Synthesis

Let us now synthesize our results. They can be regrouped along four main topics

18.1 Capital allocation

Capital allocation by producers and investors differ and interact: these interactions impact the form of the collective state and the average values of capital per sector. The main determinants of allocation are:

- short-term returns, which are composed of dividends, driven by marginal productivity, and variations in stock prices, driven by expectations of long-term returns.
- Expected long-term returns that describe growth prospects.
- The sector's relative attractivity, which measures the growth prospects of a sector relative to its neighbouring sectors.

Firms tend to allocate their capital in sectors with relatively higher long-term returns at a speed that depends on their capital endowment. However, they can be crowded out by competitors. The higher the firm's capital, the higher the power to overcome competitors. Eventually, firms with the highest capital concentrate in sectors that have the highest expected long-term returns, while the rest locate in neighbours sectors, and possibly least expected return sectors.

Financial capital allocation depends on short-term returns, dividends and price fluctuations, and expected long-term returns. However, since price fluctuations are driven by expected long-term returns, short and long-term returns are not independent. The financial capital allocation also depends on the sector's relative attractivity, which measures the expected returns of a sector relative to its neighbours. However financial capital is volatile. High short-term returns are an incentive, but the relative attractivity of sectors lures investors. Financial capital allocation thus depends on the ratio of sectors' relative attractivity to short-term returns. Since this ratio depends on expectations, it is subject to fluctuations, which in turn impact the collective state.

18.2 Patterns of accumulation and firms density

Three stationary[32] patterns of capital accumulation may emerge for each sector. A pattern is characterized by the combination of the firms' average capitalization, the number or density of firms in the sector, and the type of returns these firms may provide to their investors. The emergence of a given pattern depends on the parameters of this sector:

The first pattern associates a large number of low-capitalised firms. Dividends are determinant in this pattern; the lack of capital, combined with the prospects of competition with better-capitalized firms prevent firms to shift to neighbouring sectors.

The second pattern associates a relatively high number of average-to-high capitalised firms and a combination of short and long-term returns. This combination lures intermediate-to-high capital investors in the sector.

In the third pattern, high expectations about long-term returns generate massive inputs of capital toward highly-capitalized firms. In this pattern, firms with the highest expected returns could theoretically accumulate endlessly. Actually, this accumulation is stopped by the amount of available capital.

In each pattern, some sectors are stable, others are unstable. Transitions between patterns occur through exogenous shocks. In pattern 1, some sectors may disappear, in pattern 3, some may grow endlessly and the large amounts of capital they drive may modify the whole system's landscape.

18.3 Multiplicity of potential collective states

The multiple combinations of various accumulation patterns in each sector yield an infinite number of possible collective states. It does not follow that all combinations are possible: sector patterns depend on the relative attractivity of both the sector and its neighbours'. There are also constraints: for instance, massive inflows of capital are needed for the emergence of the third pattern, which is only driven by high expected long-term returns, while niche effects merely occur for relatively high-productivity firms. However, from relatively homogeneous levels of capital to largely heterogeneous patterns of accumulation between sectors, a potentially infinite range of collective states may exist.

18.4 Oscillations and switches in collective states

The existence of multiple collective states has a dynamic implication. When parameters vary, a given collective state may switch to another: a change in expectations or parameters may, for instance, induce variations in average capital, and in turn, induce changes in sectors' patterns of capital accumulation. To study these possible switches, we introduced a dynamic interaction between average capital and expected long-term returns, now endogenized. The main characteristic of this dynamic interaction depends both on the patterns of accumulation and the way expectations are formed.

Two types of expectations are relevant: highly reactive expectations, i.e. expectations of long-term returns that react positively to any variation in the capital, and moderately reactive expectations, i.e. expectations reacting negatively to the same variation.

In this dynamics system, average capital and expectations present some oscillatory patterns that may dampen equilibria or drive them towards other equilibria. Expectations highly reactive to capital variations stabilize high-capital configurations. They drive low-to-moderate capital sectors towards zero or higher capital, depending on their initial conditions. Inversely, expectations moderately reactive to capital variations stabilize low-to-moderate capital configurations, and drive high-capital sectors towards lower capital equilibria.

[32] The values of average capital are stationary results: agents accumulate and shift from sectors to other ones, but, in average, the density of firms and average capital per firm per sector are constant.

Amplifying oscillations may modify some sectors' pattern: the ensuing reallocation of capital across the whole sectors' space may initiate a transition in collective states. The mechanism of transition and its implications are discussed below.

19 Discussion

We have shown how statistical field theory can describe a microeconomic framework in terms of collective states of sectors composed of a large number of firms.

Each collective state is modeled by a background field that encodes the data characterizing each sector: number of firms, number of investors, average capital, and density of distribution of capital. These collective states do not change at the slightest exogenous variation of one of these data: they are characterized by theoretical averages over long-term periods, not instantaneous empirical averages. Nor are collective states arbitrary: they directly result from the agents' interactions, are obtained through minimization conditions, and are the most likely stable states of the system. Other states exist, but they are unstable.

The collective states describe the possible background states of the economy that eventually condition the agents' individual dynamics. They depend on the parameters of the model, short-term and long-term returns, relative attractiveness of the sector, and any parameter conditioning these three quantities. Their multiplicity stems from the multiple possibilities of patterns in each sector. For instance, pattern-3, stable and unstable, are more present in the US than in the UK stock markets, where patterns 1 and 2 dominate.

A particular collective state can be described by its distribution into patterns of capital accumulation - type 1, 2, or 3 - across sectors. Each sector has its own pattern of accumulation, and the distribution in patterns is directly conditioned by the economic constraints imposed on the system. Type-3 patterns appear in sectors locally more attractive in the long term. It is this relative attractivity that determines the sector's capital. Patterns 1 or 2, which are relatively less attractive sectors, lure in the capital with dividends and expected returns. Besides, sectors are connected and benefit from the relative attractiveness of their neighbours: this smoothing effect between sectors materialises in mergers and acquisitions.

The selection of a particular collective state and its sectoral patterns is ultimately determined by exogenous conditions. Structural changes, such as an extra-loose monetary policy or the choice of a pension system are external conditions that modify collective states.

Collective states are not static. Their dynamics depend on the form of short-term and long-term return functions, that are exogenous, and more broadly on a whole landscape of technological and economic conditions. But as a system, they also present an internal dynamics. We have considered these two types of variations in the paper.

First, exogenous modifications in the parameters change collective states. Any modification in expectations or, more generally, structural changes in economic and/or monetary conditions, may change expected returns and in turn the collective state. Unstable type-3 sectors are particularly sensitive to these changes in long-term growth, inflation, and interest rates. Higher expectations in these sectors attract investment, which in turn increases expectations. This seemingly endless expected growth spirals until the outlook flattens or deteriorates. This would be the growth model of a company whose ever-broadening range of products fuels higher expected long-term returns and stock price increases. Type-1 and -2 sectors attract capital through dividends and, although only partially and for high capital type-2 sectors, expected returns. Under higher expectations, these sectors are relatively less attractive than the nearby type-3 sector. They may nonetheless survive in the long-term provided their short-term returns and dividends are high enough. This may be done by cutting costs or investment, at the expense of future growth. Moreover, advert signaling may emerge: an increase in dividends can be interpreted as faltering growth prospects. Conversely, any increase in long-term uncertainty impact expected returns and drive sector-3 capital towards other patterns. External shocks, inflation, and monetary policy impact expectations, reduce long-term investment and either drive capital out of sectors 3 to sector 1 or 2 or favour other pattern-3 sectors.

Second, any deviation of capital from its collective state equilibrium value may initiate oscillations in the entire system. A temporary deviation from the given collective state implies an unstable redistribution of capital, growth prospects and returns, and generates intersectoral capital reallocation and global oscillations. These oscillations can either dampen or drive the system toward a new collective state.

There are thus potential transitions between collective states, which occur at a slower, larger timescale than that of market fluctuations. In the long run, when transitions occur, both sectors' averages and patterns may have changed: pattern 2 may morph in, say, pattern 3 stable or unstable, or sectors may simply disappear. Concretely, any significant modification in average capital in a sector could induce oscillations and initiate a transition.

Moreover, once endogenous expectations are introduced, they react to variations in the capital: collective states of mixed 1-2-3 patterns are difficult to maintain. Highly reactive expectations favour pattern 3: expected returns magnify capital accumulation at the expense of other patterns. Mildly reactive expectations favour patterns 1 and 2: their oscillations, which are actually induced by uncertainties, dampen. Type-3 sectors on the contrary experience strong fluctuations in capital : attracting capital is less effective with fading expectations. The threshold in capital accumulation shifts upwards and the least-profitable firms are ousted from the sector. The recent evolution in performances between value and growth investment strategies exemplifies these shifts in investors' sentiment between expected growth and real returns. In periods of uncertainties, fluctuations affect capital accumulation in growth sectors and today's tech companies, and strengthen more dividend-driven investments. Note however that the most profitable and best-capitalized firms, that remain above the threshold, maintain relatively high levels of capital. Here our versatile notion of firms[33] proves convenient: any firm that accumulates enough capital to be able to buy back, in periods of volatility, its own stocks is actually acting as an autonomous investor. When volatility is high, the most likely investors for the best-capitalized firms are, first and foremost, the best-capitalized companies themselves. They react, so to speak, as pools of closely held investors. In other words, provided firms have high enough capital, they can always cushion the impact of price fluctuations and adverse shocks through buybacks. Similarly, they also could choose to acquire companies in their sector or in neighbouring sectors.

Fluctuations in financial expectations impose their pace on the real economy. Expected returns are both exogenous and endogenous. Being exogenous, they may change quickly. Expected returns, should theoretically reflect long-term perspectives, but rely actually on short-term sentiments: any incoming information, change in the global economic outlook, or adverse shock modifies long-term expectations and shifts the capital from sector to sector.

But expected returns are also endogenous. Because they are expectations, they react, either highly or mildly, to changes within the system. When high levels of capital seek to maximize returns, expectations may react strongly to capital changes. Expectations that are both highly sensitive to exogenous conditions and highly reactive to variations in capital induce large fluctuations of capital in the system. Creating or inflating expectations attracts capital, at times unduly. When this cannot be done, the sole remaining tool to reduce capital outflows is dividend policy, which may be done at the expense of the labour force, capital expenditures, and future growth.

20 Conclusion

We have studied the impact of financial capital on physical capital allocation and shown that collective states distribute sectors into several patterns of accumulation. All else equal, sectors with the highest expected returns and capital may, through expectations, indefinitely attract capital at the expense of other sectors. This expansion is nevertheless unstable since adverse changes in expectations drive capital away.

At a macro timescale, the system can be globally described by oscillations between average capital and expected long-term returns, depending on the sectors' patterns. These oscillations, which can be either dampening or explosive, may change sectors patterns and explain switches from one collective state to another. Markets, supposedly the most efficient resource allocation mechanism, add, in a context of uncertainties, their fluctuations to those of the real economy. This should render the role of Central Banks, or any kind of regulation, crucial to the good functioning of these markets.

[33] We modeled a single company as a set of independent firms. Similarly, the notion of sector merely refers to a group of entities with similar activities.